Introduction to Discrete Mathematics

Target Audience

1. Under Graduate students who have discrete mathematics in their syllabus.
2. Students who are preparing for gate and other competitive exams.
3. Students who want to learn competitive programming.
4. Everyone who want to learn discrete mathematics as a whole or maybe a small subset of their subject.

Why study Discrete Mathematics?

1. It develops your mathematical thinking.
2. Improves your problem solving ability.
3. Discrete mathematics is important to survive in subjects like: compiler design, databases, computer security, operating system, automata theory etc.
4. Many problems can be solve using discrete mathematics.

For Example:

1. Sorting the list of integers.
2. Finding the shortest path from your home to your friends home.
3. Drawing a graph with two conditions.
   3.1. You are not allowed to lift your pen.
   3.2. you are not allowed to repeat edges.
4. How many different combinations of passwords are possible with just 8 alphanumeric characters.
5. Encrypt a message and deliver it to your friend and you don't want anybody to read that message except your friend.

What is Discrete Mathematics?

1. Discrete mathematics is the study of discrete objects.

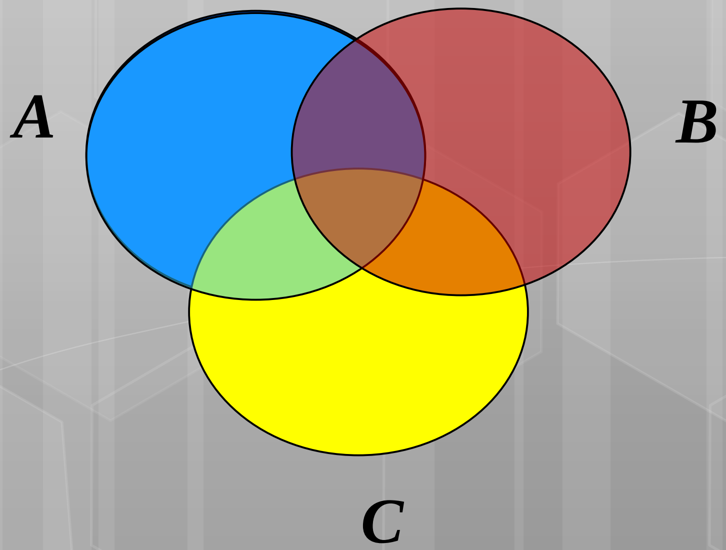
   Discrete means “distinct or not connected.”

2. It is not a branch of mathematics. It is rather a description of set of branches that have one common property that they are “discrete” and not “continuous”.

Syllabus

1. Set Theory
2. Function
3. Relation
4. Propositional Logic
5. Graph Theory
6. Proof Techniques and Counting
7. Group Theory


Thank you!



Set Theory

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Defining Sets

Definition: A **set** is an unordered collection of objects.

The objects in a set are called the **elements** or **members** of the set S, and we say S **contains** its elements.

The notation $a \in A$ denotes that a is an element of the set A.
If a is not a member of A, write $a \notin A$

We can define a set by directly listing all its elements.

e.g. S = {2, 3, 5, 7, 11, 13, 17, 19},

S = {CSC1130, CSC2110, ERG2020, MAT2510}

After we define a set, the set is a single mathematical object, and it can be an element of another set.

e.g. S = {{1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}}

Defining Sets by Properties

It is inconvenient, and sometimes impossible, to define a set by listing all its elements.

Alternatively, we can define by a set by describing the *properties* that its elements should satisfy.

We use the notation $\{x \in A \mid P(x)\}$

to define the set as the set of elements, x, in *A* such that x satisfies property *P*.

e.g. $\{x \mid x \text{ is a prime number and } x < 1000\}$

$\{x \mid x \text{ is a real number and } -2 < x < 5\}$

Example of Sets

Well known sets:

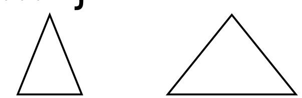
- the set of all real numbers, $\mathbb{R}$
- the set of all complex numbers, $\mathbb{C}$
- the set of all integers, $\mathbb{Z}$
- the set of all positive integers $\mathbb{Z}^+$

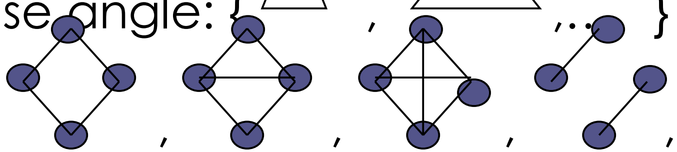
Other examples:

- empty set, $\emptyset = \{\}$ the set with no elements.

The set of all polynomials with degree at most three: {1, x, x^2, x^3, $2x+3x^2$,…}.

The set of all n-bit strings: {000…0, 000…1, …, 111…1}

The set of all triangles without an obtuse angle: {  ,…}

The set of all graphs with four nodes: {  ,…}

Membership

Order, number of occurrence is not important.

e.g. {a,b,c} = {c,b,a} = {a,a,b,c,b}

The most basic question in set theory is whether an element is in a set.

$x \in A$ x is an element of A

x is in A

$x \notin A$ x is not an element of A

x is not in A

e.g.

Recall that Z is the set of all integers. So $7 \in Z$ and $2/3 \notin Z$.

Let P be the set of all prime numbers. Then $97 \in P$ and $321 \notin P$

Let Q be the set of all rational numbers. Then $0.5 \in Q$ and $\sqrt{2} \notin Q$

(will prove later)

Numerical Sets (Well-Defined)

- Set of even numbers:

  {..., -4, -2, 0, 2, 4, ...}
- Set of odd numbers:

  {..., -3, -1, 1, 3, ...}
- Set of prime numbers:

  {2, 3, 5, 7, 11, 13, 17, ...}
- Positive multiples of 3 that are less than 10:

  {3, 6, 9}

Numerical Sets (Not Well-Defined)

- There can also be sets of numbers that have no common property, they are just defined that way. For example:

- {2, 3, 6, 828, 3839, 8827}
- {4, 5, 6, 10, 21}
- {2, 949, 48282, 428859, 119484}
- {111, 8888, 001922, 98373773}

Examples of Sets (Well-defined)

- A set of even numbers between 1 and 15

  A= {2, 4, 6, 8, 10, 12,14}

- B set of multiple of 5 between 8 and 28

  B= {10, 15, 20, 25}

- **Note:** A set may be denoted by a capital letter as shown in above examples.

Your Task☺

- **Which of the following are well-defined sets?**

1. All the colors in the rainbow.
2. All the points that lie on a straight line.
3. All the honest members in the family.
4. All the consonants of the English alphabet.
5. All the tall boys of the school.
6. All the hardworking teachers in a school.
7. All the prime numbers less than 100.
8. All the letters in the word GEOMETRY.

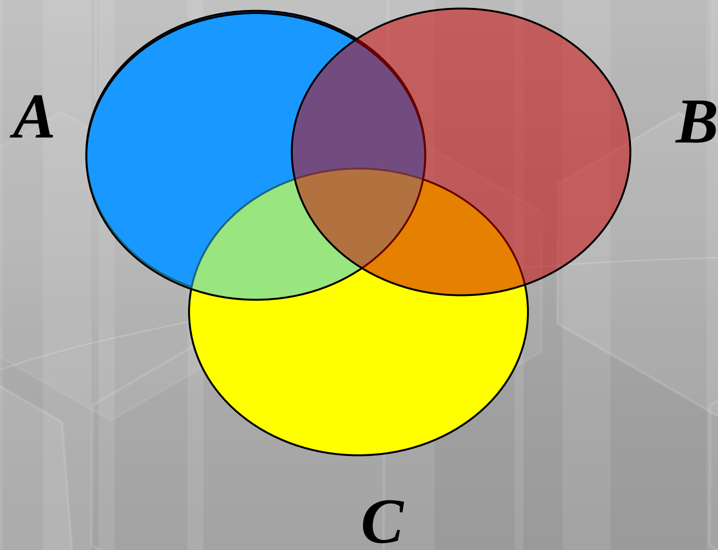
Your Task☺

- **Which of the following are well-defined sets?**

1. All the colors in the rainbow.
2. All the points that lie on a straight line.
3. All the honest members in the family.
4. All the consonants of the English alphabet.
5. All the tall boys of the school.
6. All the hardworking teachers in a school.
7. All the prime numbers less than 100.
8. All the letters in the word GEOMETRY.

Answers: 1, 2, 4, 7, 8 are well-defined.


Thank you!



Set Theory

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Cardinality of Sets

- The cardinality of a set S, denoted $|S|$, is the number of elements in S. If the set has an infinite number of elements, then its cardinality is ∞.
- The cardinality of a set A is denoted by $|A|$.
- 1: If $A = \phi$, then $|A| = 0$.
- 2: If A has exactly n elements, then
- $|A| = n$.
- Note that n is a nonnegative number.
- 3: If A is an infinite set, then $|A| = \infty$.

Size of a Set

In this course we mostly focus on finite sets.

> Definition: The **size** of a set S, denoted by **|S|**,
> is defined as the number of elements contained in S.

e.g. if S = {2, 3, 5, 7, 11, 13, 17, 19}, then |S|=8.

if S = {CSC1130, CSC2110, ERG2020, MAT2510}, then |S|=4.

if S = {{1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}}, then |S|=6.

Subsets

- A set A is a subset of a set B, if all the elements of A are contained in/members of the larger set B.
- set A is a subset of B if and only if every element of A is also an element of B.
- We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
- The empty set({ } or ϕ) is a subset of every set.

Subsets

- **Example 1:**
- If, B = {3, 5, 6, 8, 9, 10, 11, 13}
- And, A = {5, 11, 13}
- Then, A is a subset of B.
- A subset of this is {5, 11, 13}. Another subset is {3, 5} or even another is {3}, etc.
- But {1, 6} is not a subset, since it has an element (1) which is not in the parent set.
- That is, $A \subseteq B$ (where $\subseteq$ means 'is a subset of').
- A is a subset of B if and only if every element of A is in B.

Subsets

- **Example 2:**
- If, F = {1, 2, 3}
- And, G = {1, 2}
- Then, G is a subset of F.

Not a Subsets Example

- If, A = {1, 2, 3}
- And, B = {1, 2, 3, 4, 5}
- Then, B is not a subset of A.
- That is, $B \not\subseteq A$ (Where $\not\subseteq$ means 'not is a subset of')

- **Note:** Every set is a subset of itself. The empty set is a subset of every set.

Your Task☺

- A = {3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$?
- **Answer:** Yes

- A = {3, 3, 3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$?
- **Answer:** Yes

- A = {1, 2, 3}, B = {2, 3, 4}, $A \subseteq B$?
- **Answer:** No

Number of Subsets

- If, M = {a, b, c}
- Then, the subsets of M are:
- {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, {}
- Therefore, the number of subsets, S = 8
- And the formula, $S = 2^n$
- Where, S is the number of sets And, n is the number of elements of the set.
- Is the formula used to calculate the number of subsets of a given set.
- So from above, M = {a, b, c}
- $S = 2^n$, $2^3 = 2 \times 2 \times 2 = 8$

Proper Subset

- A is a **proper** subset of B if and only if every element in A is also in B, and there exists **at least one element** in B that is **not** in A.
- **Example 1:**
- A = {1, 2, 3} is a subset of B = {1, 2, 3}, but is not a proper subset of {1, 2, 3}.
- **Example 2:**
- A = {1, 2, 3} is a proper subset of B = {1, 2, 3, 4} because the element 4 is not in the first set.
- Hence $A \subset B$

Subset and Proper Subset

Difference Between Subset and Proper Subset

- If every member of one set is also a member of a second set, then the first set is said to be a subset of the second set. Usually, it turns out that the first set is smaller than the second, but not always. The definition of "subset" allows the possibility that the first set is the same as (equal to) the second set. But a "proper subset" must be smaller than the second set.
- The set {2,3,5,7} is a subset of {2,3,5,7}.
- The set {2,3,5,7} is NOT a proper subset of {2,3,5,7}.
- The set {2,3,5} is a proper subset of {2,3,5,7}.

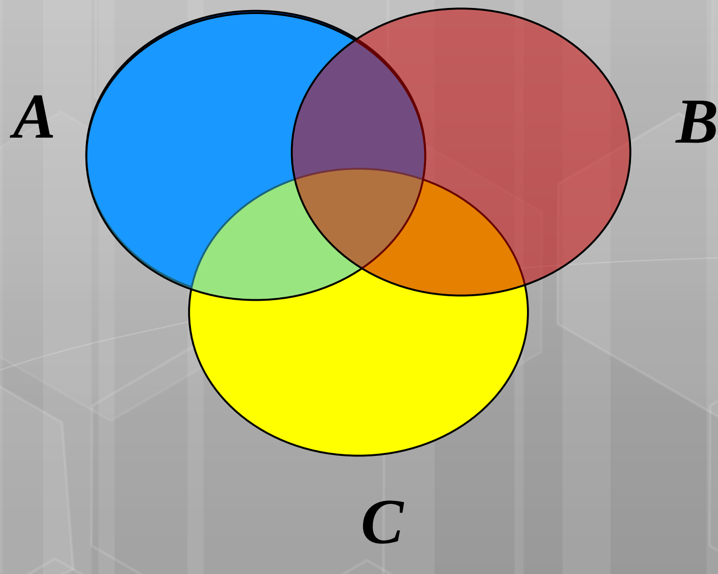
Your Task☺

- **Determine whether each of the following statements is true or false.**
  - $x \in \{x\}$
  - $\{x\} \subseteq \{x\}$
  - $\{x\} \in \{x\}$
  - $\{x\} \in \{\{x\}\}$
  - $\varnothing \subseteq \{x\}$
  - $\varnothing \in \{x\}$

Solution

- **Determine whether each of the following statements is true or false.**
  - $x \in \{x\}$ TRUE
    - (Because x is the member of the singleton set { x })
  - $\{x\} \subseteq \{x\}$ TRUE
    - (Because Every set is the subset of itself. Note that every Set has necessarily two subsets $\varnothing$ and the Set itself, these two subset are known as Improper subsets and any other subset is called Proper Subset)
  - $\{x\} \in \{x\}$ FALSE
    - (Because { x} is not the member of {x}) Similarly other
  - $\{x\} \in \{\{x\}\}$ TRUE
  - $\varnothing \subseteq \{x\}$ TRUE
  - $\varnothing \in \{x\}$ FALSE


Thank you!



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Representation of a Set

- Sets can be represented in two ways :

- **Roster or Tabular Form**
- **Descriptive Form**
- **Set Builder Notation**

Tabular Form

- **Tabular Form:**
  - Listing all the elements of a set, separated by commas and enclosed within braces or curly brackets{}.

- **Examples:**
  - In the following examples we write the sets in Tabular Form.
  - A = {1, 2, 3, 4, 5} is the set of first five **Natural Numbers**.
  - B = {2, 4, 6, 8, ..., 50} is the set of **Even numbers** up to 50
  - C = {1, 3, 5, 7, 9, ...} is the set of **positive odd numbers.**

Descriptive Form

- **Descriptive Form:**
  - Stating in words the elements of a set.
- **Examples**:
  - Now we will write the same examples which we write in Tabular Form ,in the Descriptive Form.
  - A = set of first five Natural Numbers.(is the Descriptive Form)
  - B = set of positive even integers less or equal to fifty. (is the Descriptive Form)
  - C = {1, 3, 5, 7, 9, ...}   (is the Tabular Form)
  - C = set of positive odd integers. (is the Descriptive Form)

Set Builder Form

- **Set Builder Form:**
  - Writing in symbolic form the common characteristics shared by all the elements of the set.
- **Examples:**
  - Now we will write the same examples which we write in Tabular as well as Descriptive Form ,in Set Builder Form .
  - $A = \{x \in N \mid x <= 5\}$ (is the Set Builder Form)
  - $B = \{x \in E \mid 0 < x <= 50\}$ (is the Set Builder Form)
  - $C = \{x \in O \mid 0 < x \}$ (is the Set Builder Form)

Your Task☺

- **Write the following sets in the set builder form.**

- (a) A = {2, 4, 6, 8}
- (b) B = {3, 9, 27, 81}
- (c) C = {1, 4, 9, 16, 25}
- (d) D = {1, 3, 5, ...}
- (e) E = {a, e, i, o, u}

Answers of the Previous Questions

- **Write the following sets in the set builder form.**

- (a) $\{x : x \text{ is even and } x \leq 8$
- (b) $\{x : x = 3n, n \in N, n \leq 4\}$
- (c) $\{x : x = n^2, n \leq 5, n \in N\}$
- (d) $\{x : x \text{ is odd}\}$
- (e) $\{x : x = \text{Vowels in English alphabets}\}$

Your Task☺

- **Write the following sets in the roster form.**

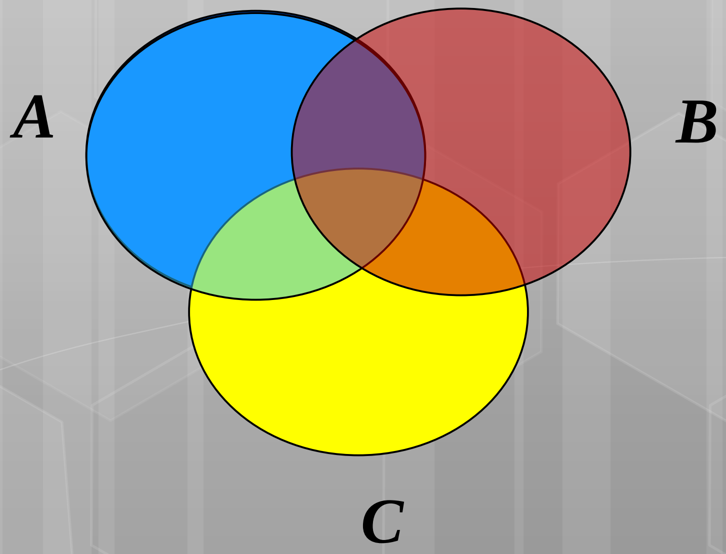
- (a) $A = \{x : x \in W, x \leq 5\}$
- (b) B = {The set all even numbers less than 12)
- (c) $C = \{x : x$ is divisible by $12\}$
- (d) D = {The set of first seven natural numbers}
- (e) E = {The set of whole numbers less than 5}

Answers of the Previous Questions

- **Write the following sets in the roster form.**

- (a) {0, 1, 2, 3, 4, 5}
- (b) {2, 4, 6, 8. 10}
- (c) {12, 24, 36, ……}
- (d) {1, 2, 3, 4, 5, 6, 7}
- (e) {0, 1, 2, 3, 4}


Thank you!



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Finite and infinite Sets

- **Finite Set:** A finite set is one in which it is possible to list and count all the members of the set.
- **Example: 1,** D = {days of week}
- D = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}
- So, n(D) = 7 which is countable, so it is finite set.
- **Example: 2,** A = {1, 2, 3, 4, 5}
- The set 'A' has 5 elements and so it is finite set.
- **Example: 3,** F = {-3, -2, -1, 0, 1, 2, 3}
- The set 'F' has countable number of elements so it is also a finite set.

Finite and infinite Sets

- **Infinite Set:** An infinite set is one in which it is not possible to list and count all the members of the set.
- **Example: 1**
- E = {even numbers greater than 9}
- E = {10, 12, 14, 16...}
- Here n(E) = infinite
- **Example: 2**
- G = { whole numbers greater than 2000 }
- G = { 2001, 2002, 2003, 2004, ...}
- Here n(G) = infinite

Your Task☺

- **Classify the following as finite and infinite sets.**

a. A = {x : x ∈ N and x is even}

b. B = {x : x ∈ N and x is composite}

c. C = {x : x ∈ N and $3x - 2 = 0$}

d. D = {x : x ∈ N and $x^2 = 9$ }

e. E = {The set of numbers which are multiple of 3}

f. F = {The set of letters in English alphabets}

g. G = {The set of persons living in a house}

h. H = {x : x ∈ P, P is a number}

i. I = {The set of fractions with numerator 3}

Answer of the Previous Questions

a. Infinite
b. Infinite
c. Finite
d. Finite
e. Infinite
f. Finite
g. Finite
h. Infinite
i. Infinite

Empty Set / Null Set

- An empty set is a set which has no members.
- **Example:**
- If, H = {the number of dinosaurs on earth}
- Then, H is an empty set.
- That is, H = {}
- **Note:** An empty set is denoted by the symbol {} or ϕ.
- **Note** the subtlety in $\phi \neq \{\phi\}$
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set

Singleton Set or Unit Set

- A singleton is a set that contains exactly one element.
- Sometimes, it is known as unit set.
- The singleton containing only the element a can be written {a}.
- Note that ϕ is empty set and {ϕ} is not empty set but it is a singleton set.

- Singleton set or unit set contains only
- one element. A singleton set is denoted
- by {s}.
- **Example** : $S = \{x \mid x \in N, 7 < x < 9\}$

Your Task☺

- **Identify the following as null set or singleton set.**

a. $A = \{x : x \in N, 1 < x < 2\}$

b. P = {Point of intersection of two lines}

c. C = {x : x is an even prime number greater than 2}

d. Q = {x : x is an even prime number}

e. $E = \{x : x^2 = 9, x \text{ is even}\}$

f. $B = \{0\}$

g. D = {The set of largest 1 digit number}

h. F = {The set of triangles having 4 sides}

i. H = {The set of even numbers not divisible by 2}

Answers:
a. Null
b. Singleton
c. Null
d. Singleton
e. Null
f. Singleton
g. Singleton
h. Null
i. Null

Equal and Equivalent Sets

- **Equal Set:** Two sets are equal if they both have the same members.
- **Example 1:** if $A = \{1, 2, 3\}$
- And $B = \{1, 2, 3\}$
- Then $A = B$, that is both sets are equal.
- **Example 2:** if $C = \{1, 2, 5\}$
- And $D = \{5, 1, 2\}$
- Then $C = D$, that is both sets are equal.
- **Note:** The order in which the members of a set are written does not matter.

Equal and Equivalent Sets

- **Equivalent Set:** Two sets are equivalent if they have the same number of elements.
- **Example 1:** if F = {2, 4, 6, 8, 10}
- And G = {10, 20, 30, 40, 50}
- Then n(F)=n(G), that is, sets F and G are equivalent.

- **Example 2:** if A = {1, 2, 3}
- And B = {a, b, c}
- Then n(A)=n(B), that is, sets A and B are equivalent.

Equal and Equivalent Sets

- **Note:** When each member of a set matches one and only one member of the other set, there is a 1-1 correspondence between the two sets.
- For Example:

$$\begin{array}{ccc} 1 & 2 & 3 \\ \updownarrow & \updownarrow & \updownarrow \\ a & b & c \end{array}$$

- Sets that cannot be paired in a 1-1 correspondence are called **non-equivalents sets**.

Your Task☺

- **Which of the following pairs of sets are equivalent or equal?**

a. $A = \{x : x \in N, x \leq 6\}$

$B = \{x : x \in W, 1 \leq x \leq 6\}$

b. P = {The set of letters in word "plane"}

Q = {The set of letters in word "plain"}

c. X = {The set of color in the rainbow}

Y = {The set of days in a week}

d. $M = \{4, 8, 12, 16\}$

$N = \{8, 12, 4, 16\}$

Answers:

a. Equal Sets
b. Equivalent Sets
c. Equivalent Sets
d. Equal Sets

One More Task☺

- **Is this is equal set?**
- {2,3,5,7} , {2,2,3,5,3,7}

✓ Equal

- And-----------------:
- {2,3,5,7} , {2,3}

✓ Not Equal

Note that:

- Equal sets are always equivalent.
- Equivalent sets may not be equal

Examples

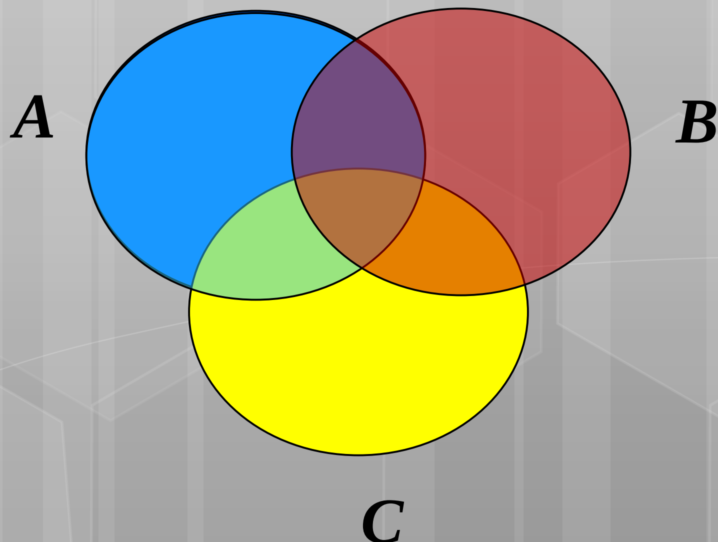
- Let A be the set of odd positive integers less than 10. Then $|A|=5$.
- Let S be the set of letters in the English alphabet. Then $|S|=26$.
- Let P be the set of infinite numbers.

  Then $|P|=\infty$.
  - The cardinality of the empty set is $|\varnothing|=0$
  - The sets $\mathcal{N}, \mathcal{Z}, \mathcal{Q}, \mathcal{R}$ are all infinite

Cardinality of Sets

- A = {Mercedes, BMW, Porsche}, $|A| = 3$
- B = { x: x is an odd number divisible by 2}, $|B| = 0$
- $C = \{1, \{2, 3\}, \{4, 5\}, 6\}$ $|C| = 4$
- D = {x: x is a counting number <10}, $|D| = 9$
- $E = \emptyset$ $|E| = 0$
- F = {Letter in the words BANANA}, $|F| = 3$
- $G = \{ x \in \mathbf{N} \mid x \leq 7000 \}$, $|G| = 7000$


Thank you!



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Power Sets

- A Power Set is a set of all the subsets of a set.
- The power set of S is denoted by P(S).
- **Notation:**
- The number of members of a set is often written as |S|, so we can write:

$$|P(S)| = 2^n$$

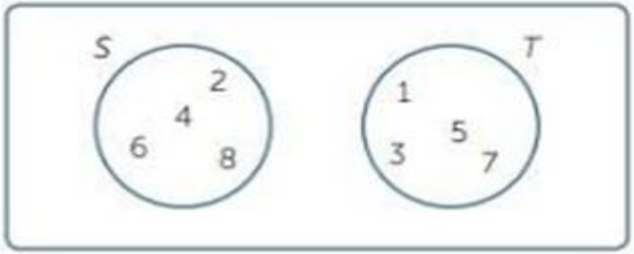
Power Sets

- **A= { a, b, }**
- The power set of A is $2^4 = 16$
- P(A)={}, {a}, {b}, {a, b}, {c}, {a, c}, {b, c}, {a, b, c}, {d}, {a, d}, {b, d}, {a, b, d}, {c, d}, {a, c, d}, {b, c, d}, {a, b, c, d}.
- **B={1, 2, 3}**
- The power set of B is $2^3 = 8$
- P(B)={}, {1}, {2}, {1, 2}, {3}, {1, 3}, {2, 3}, {1, 2, 3}

Disjoint Sets

- **Disjoint sets:**
- Two sets are called disjoint if they have no elements in common.

- **For Example:**
- The sets S = {2, 4, 6, 8} and T = {1, 3, 5, 7} are disjoint.



Disjoint Sets

- Another way to define disjoint sets is to say that their intersection is the empty set,
- Two sets A and B are disjoint if $A \cap B = \{\ \}$.
- In the example above,
- $S \cap T = \{\ \}$ because no number lies in both sets.
- The overlapping region of two circles represents the intersection of the two sets.
- When two sets are disjoint, we can draw the two circles without any overlap.

Your Task ☺

- **Which of the following sets are disjoint or overlapping:**

a. A = {The set of boys in the school}

B = {The set of girls in the school}

b. P = {The set of letters in the English alphabets}

Q = {The set of vowels in the English alphabets}

c. X = {x : x is an odd number, $x < 9$}

Y = {x : x is an even number, $x < 10$}

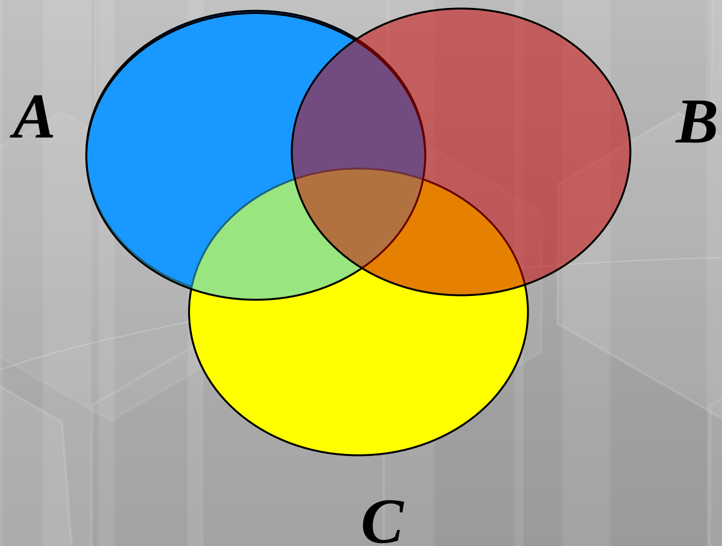
d. E = {9, 99, 999}

F = {1, 10, 100}

Answers:

a. Disjoint Sets
b. Overlapping Sets
c. Disjoint Sets
d. Disjoint Sets


Thank you!



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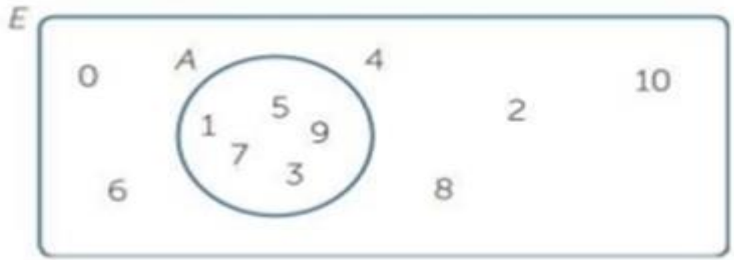
Venn Diagrams

- Diagrams make mathematics easier because they help us to see the whole situation at a glance. The English mathematician John Venn (1834–1923) began using diagrams to represent sets. His diagrams are now called Venn diagrams.
- In most problems involving sets, it is convenient to choose a larger set that contains all of the elements in all of the sets being considered. This larger set is called the universal set, and is usually given the symbol E. In a Venn diagram, the universal set is generally drawn as a large rectangle, and then other sets are represented by circles within this rectangle.

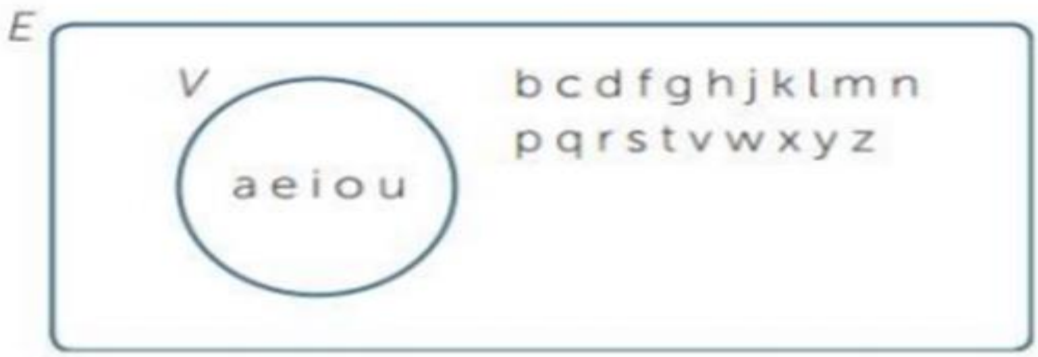
Venn Diagrams

- In the Venn diagram below, the universal set is $E = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$, and each of these numbers has been placed somewhere within the rectangle.



- The region inside the circle represents the set A of odd whole numbers between 0 and 10. Thus we place the numbers 1, 3, 5, 7 and 9 inside the circle, because $A = \{ 1, 3, 5, 7, 9 \}$.Outside the circle we place the other numbers 0, 2, 4, 6, 8 and 10 that are in E but not in A

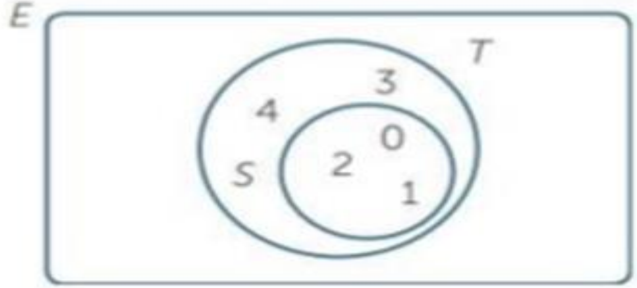
Venn Diagrams



- For example, if V = { vowels }, we could choose the universal set as E = { letters of the alphabet } and all the letters of the alphabet would then need to be placed somewhere within the rectangle, as shown below.

Representing Subsets on a Venn diagram

- When we know that *S* is a subset of *T*, we place the circle representing *S* inside the circle representing *T*. For example, let $S = \{ 0, 1, 2 \}$, and $T = \{ 0, 1, 2, 3, 4 \}$. Then *S* is a subset of *T*, as illustrated in the Venn diagram below.



- Make sure that 5, 6, 7, 8, 9 and 10 are placed outside both circles

Cartesian Product

Definition: The Cartesian Product of two sets A and B, denoted by A $\times$ B is the set of ordered pairs (a,b) where a ∈ A and b ∈ B .

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Definition: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B.

Cartesian Product

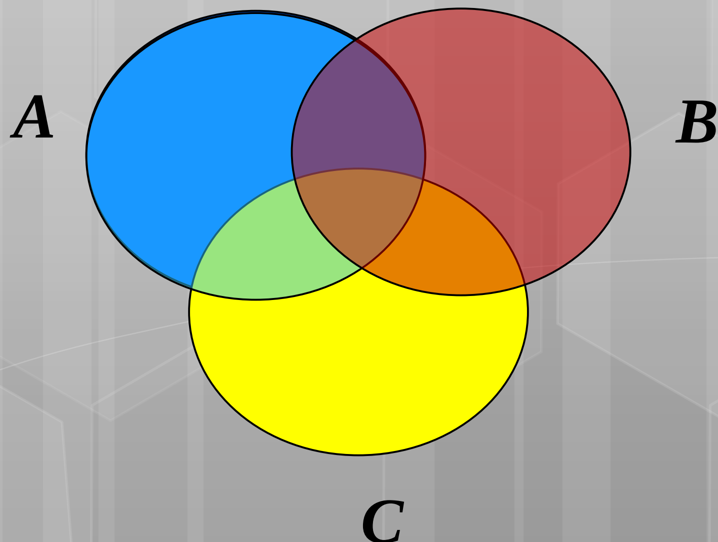
Definition: The Cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ **=** {(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

Thank you!



Set Theory

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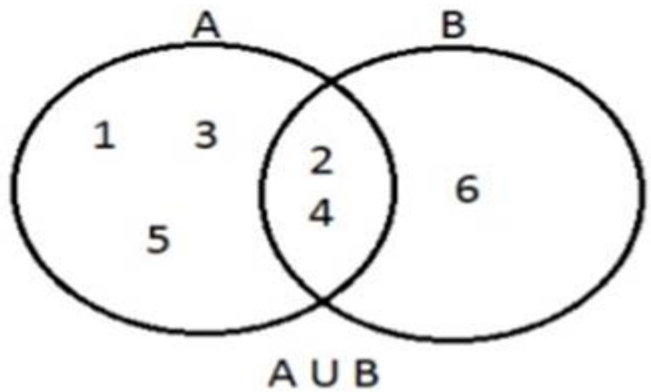
Union of Sets

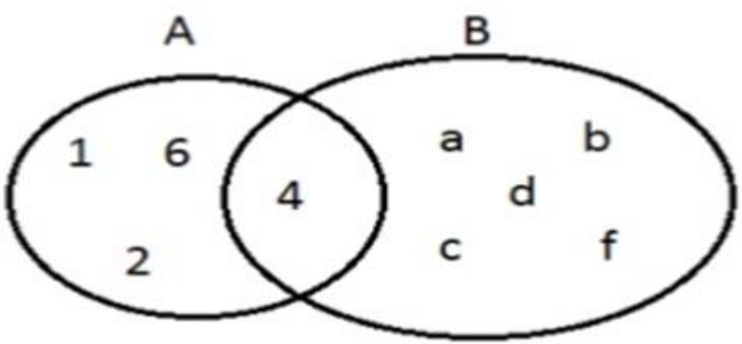
- Combining all the elements of any two sets is called the Union of those sets.
- Union of two sets A and B is obtained by combining all the members of the sets and is represented as A U B
- **Examples of Union of Sets**
- If A = {1, 2, 3, 4, 5} and
- B = {2, 4, 6},
- Then the union of these sets is A U B = {1, 2, 3, 4, 5, 6}

Union of Sets



Examples of Union of Sets

- A = {1, 2, 4, 6} and B = {4, a, b, c, d, f}
- Then the union of these sets is A U B = {1, 2, 4, 6, a, b, c, d, f}



Union of Sets

- **Examples of Union of Sets**
- A = {x / x is a number bigger than 4 and smaller than 8}
- B = {x / x is a positive number smaller than 7}
- A = {5, 6, 7} and B = {1, 2, 3, 4, 5, 6}
- A ∪ B = {1, 2, 3, 4, 5, 6, 7}
- Or A ∪ B = {x / x is a number bigger than 0 and smaller than 8}

Union of Sets

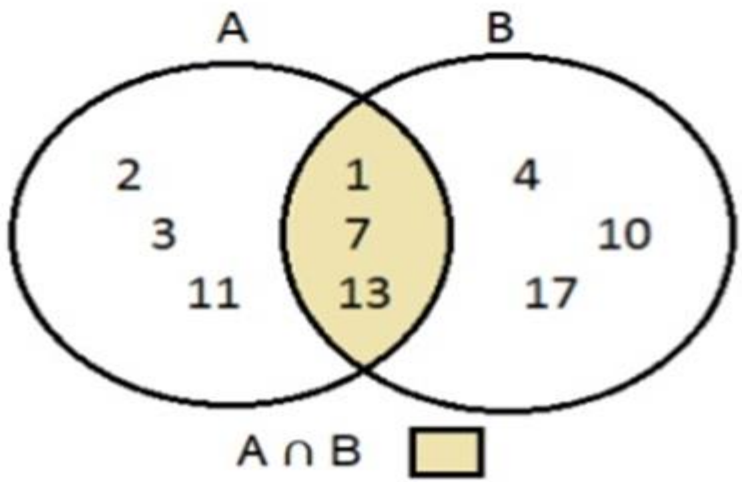
- **Examples of Union of Sets**
- A = {#, %, $}
- B = { }
- Then, A ∪ B = {#, %, $}

- **Examples of Union of Sets**
- N = {-5, -4, 0, 6, 8} and O = {-4, 0, 8, 9}
- Then, N ∪ O = {-5, -4, 0, 6, 8, 9}

Intersection of Sets

- Intersection of Sets is defined as the grouping up of the common elements of two or more sets.
- It is denoted by the symbol ∩
- **Example of Intersection of Sets**
- When Set A = {1, 2, 3, 7, 11, 13} and Set B = {1, 4, 7, 10, 13, 17},
- A ∩ B is all the common elements of the set A and B.
- Therefore, A ∩ B = {1, 7, 13}.
- This can be shown by using Venn diagram as:

Intersection of Sets

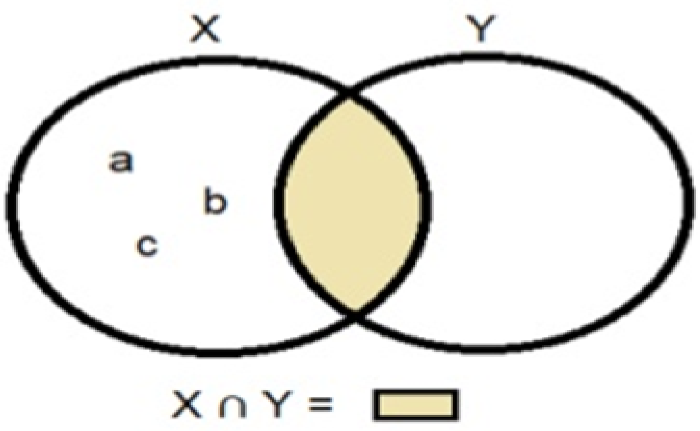


Intersection of Sets

- **Example of Intersection of Sets**
- If A = {a, b, c, d} and B = {1, a, 2, b}.
- A n B is all the common elements of the set A and B.
- Therefore, $A \cap B = \{a, b\}$.

More Example

- If X = {a, b, c} and Y = {ϕ}. Find intersection of two given sets X and Y.
- **Solution:**
- $X \cap Y = \{\ \}$



$X \cap Y = \{\ \}$ or ϕ (an empty set). Non empty sets which have no members in common are called "**disjoint sets**".

Your Task☺

- If set A = {4, 6, 8, 10, 12}, set B = {3, 6, 9, 12, 15, 18} and set C = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

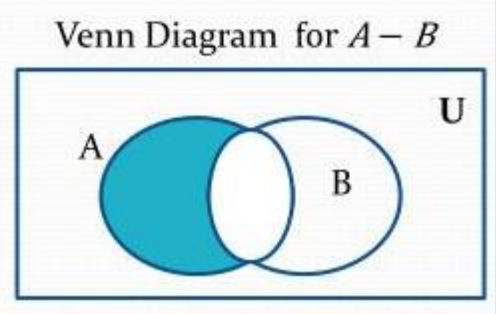
- (i) Find the intersection of sets A and B.
- (ii) Find the intersection of two set B and C.
- (iii) Find the intersection of the given sets A and C.

Solution

- (i) Intersection of sets A and B is $A \cap B$
- Set of all the elements which are common to both set A and set B is {6, 12}.
- (ii) Intersection of two set B and C is $B \cap C$
- Set of all the elements which are common to both set B and set C is {3, 6, 9}.
- (iii) Intersection of the given sets A and C is $A \cap C$
- Set of all the elements which are common to both set A and set C is {4, 6, 8, 10}.

Difference of Sets

- The difference set of any two sets A and B. is the set of the members of set A which is not the members of set B.
- **Example of Difference of Sets**
- A = {0, 1, 2, 3}
- B = {2, 3}
- The difference set is {0, 1}.
- We can write it as A – B or A \ B. We say: 'A difference B'.
- B – A or B \ A = { }



Your Task☺

- A = {1, 2, 3} and B = {4, 5, 6}.

- Find the difference between the two sets:
- (i) A and B
- (ii) B and A

Solution

- The two sets are disjoint as they do not have any elements in common.
- (i) $A - B = \{1, 2, 3\} = A$
- (ii) $B - A = \{4, 5, 6\} = B$

More Task☺

- **Given three sets P, Q and R such that:**
- P = {x : x is a natural number between 10 and 16},
- Q = {y : y is a even number between 8 and 20} and
- R = {7, 9, 11, 14, 18, 20}
-
- (i) Find the difference of two sets P and Q
- (ii) Find Q - R
- (iii) Find R - P
- (iv) Find Q – P

Solution

- According to the given statements:
- P = {11, 12, 13, 14, 15}
- Q = {10, 12, 14, 16, 18}
- R = {7, 9, 11, 14, 18, 20}
- (i) P – Q = {Those elements of set P which are not in set Q}

  = {11, 13, 15}
- (ii) Q – R = {Those elements of set Q not belonging to set R}

  = {10, 12, 16}
- (iii) R – P = {Those elements of set R which are not in set P}

  = {7, 9, 18, 20}
- (iv) Q – P = {Those elements of set Q not belonging to set P}

  = {10, 16, 18}

Universal Set

- The universal set is the set of all elements that are considered in a specific theory. We'll note the universal set with U.
- We'll choose as universal set: U = {6,7,8,9,15,16,17,18,20,21}.
- We have to determine the sets:
- M = {x / x are the multiple of 3}
- N = {x / x are the multiple of 5}
- The elements of M and N have to be chosen from the universal set U.
- To determine M, we'll identify the multiples of 3 from U: {6,9,15,18,21}
- M = {6,9,15,18,21}
- To determine N, we'll identify the multiples of 5 from U: {15,20}.
- N = {15,20}

Complement of a Set

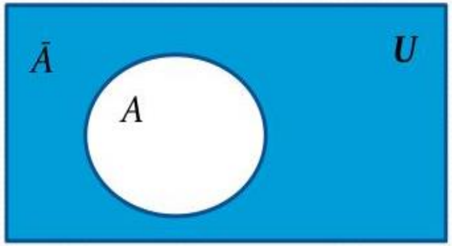
- If a set A is a subset of a given Universal Se U, Then the difference U – A or U \ A id the complement of A.
- We write U – A or U \ A = A′. We say A complement.
- **Example:**
- If, U = {3, 5, 7, 9, 11, 13, 15, 17, 19}
- And, A = {5, 11, 17, 19}
- Then, U — A = A′ = {3, 7, 9, 13, 15}

Complement of a Set

- Where, A′ is "the complement of A".
- The union of A and A′ is the Universal set.
- U = A ∪ A′ = {5, 11, 17, 19} ∪ {3, 7, 9, 13, 15}
- U = A ∪ A′ = {3, 5, 7, 9, 11, 13, 15, 17, 19}

- The intersection of A and A′ is an empty set.
- A ∩ A′ = { } or ϕ

Venn Diagram for Complement

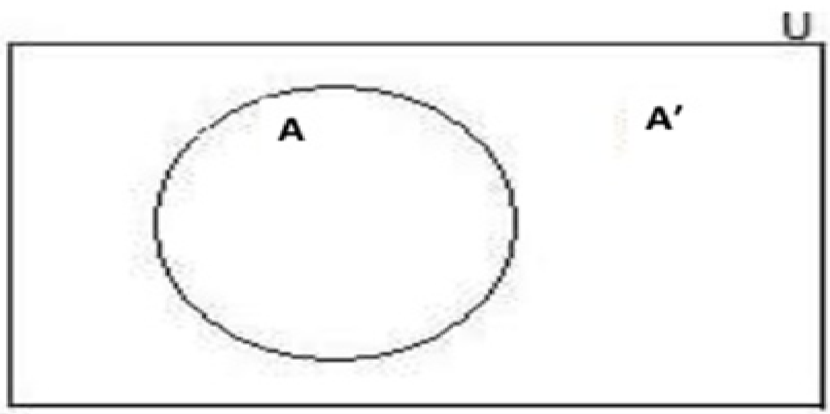


Your Task ☺

- **Find the complement of A in U**
- A = { x / x is a number bigger than 4 and smaller than 8}
- U = { x / x is a positive number smaller than 7}
- A = { 5, 6, 7} and U = { 1, 2, 3, 4, 5, 6}
- A′ = { 1, 2, 3, 4}
- Or A′ = { x / x is a number bigger than 1 and smaller than 5 }

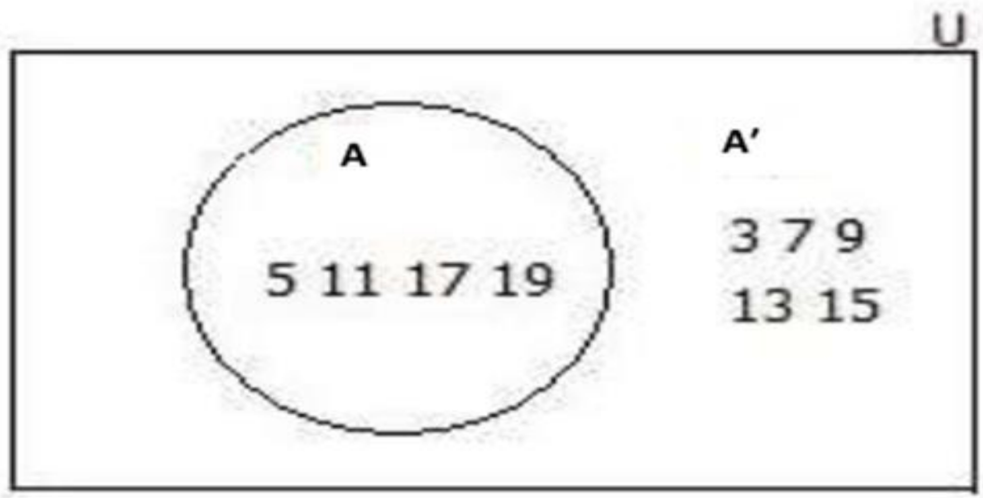
Example- Using Venn diagram

- Sets are represented by a drawing called a Venn diagram, in which a rectangle is used to represent a universal set, U, and circles inside the rectangle, used to represent subsets.



Example- Using Venn diagram

- Using the previous above, below is a Venn diagram showing A’.



Note Some Points:

- The Complement of a universal set is an empty set.
- **For Example:**

$$U = \{1, 2, 3, 4\}$$
$$U' = \phi$$

- The Complement of an empty set is a universal set.
- **For Example:**

$$U = \{1, 2, 3, 4\}$$
$$A = \{\,\}$$

Then $A' = \{1, 2, 3, 4\}$ or we simply say U.

Note Some Points:

- The set and its complement are disjoint sets.
- **For Example:**

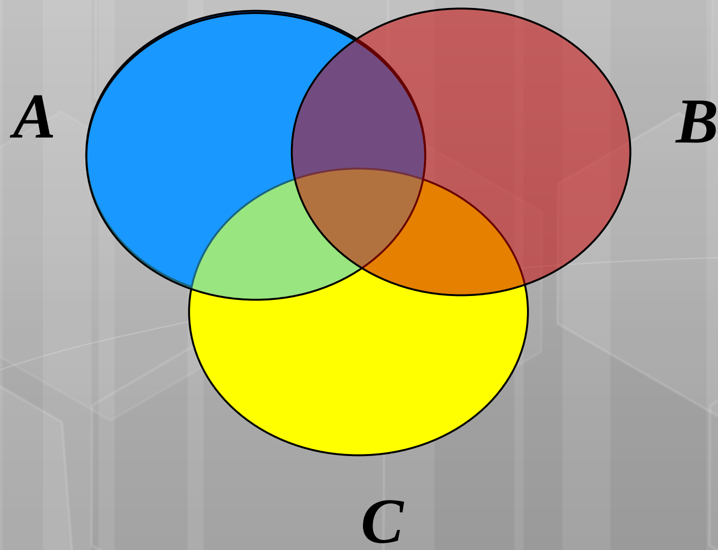
$$U = \{1, 2, 3, 4\}$$
$$A = \{1, 2\}$$
$$A' = \{3, 4\}$$

- And $A \cap A' = \{\ \}$ or ϕ

Set identities

$A \cup \varnothing = A$ $A \cap U = A$	Identity Law	$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination law
$A \cup A = A$ $A \cap A = A$	Idempotent Law	$(A^c)^c = A$	Complementation Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Law	$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$	De Morgan's Law
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Law	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Law
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Law	$A \cup A^c = U$ $A \cap A^c = \varnothing$	Complement Law


Thank you!



Set Theory

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Multi-set

Definition: These are unordered collection of elements where an element can occur as a member more than once. The notation $\{m_1.a_1 , m_2.a_2 \ldots\ldots m_r.a_r \}$ denotes the multi-set with element a_1 occurring m_1 times, element a_2 occurring m_2 times and so on. The numbers m_i , $i = \{1, 2,\ldots..r\}$ *are called multiplicities of the element* a_i , $i = \{1, 2,\ldots..r\}$

Multi-set of prime factors of a number n

$$120 = 2^3 3^1 5^1$$

which gives the multiset {2, 2, 2, 3, 5}.

Operations on Multi-set

1. Union:

For example, if A = {2, 3, 4, 4}, B = {1, 4, 3, 3}
then A ∪ B = {1, 2, 3, 3, 4, 4}.

2. Intersection:

For example, if A = {3, 3, 3, 4, 4}, B = {1, 4, 3, 3}
then A ∩ B = {3, 3, 4}.

Operations on Multi-set

3. Addition/Sum/Merge:

For example, if A = {1, 1, 2, 2, 4, 4, 4}, B = {1, 2, 3, 3}
then A + B = {1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4}.

4. Difference:

For example, if A = {3, 3, 3, 4, 4}, B = {1, 4, 3, 3}
then A - B = {3, 4}.

Exercise: Suppose that A is the multiset that has as its elements the types of computer equipment needed by one department of a university and the multiplicities are the number of pieces of each type needed, and B is the analogous multiset for a second department of the university. For instance, A could be the multiset {107 · personal computers, 44 · routers, 6 · servers} and B could be the multiset {14 · personal computers, 6 · routers, 2 · mainframes}.

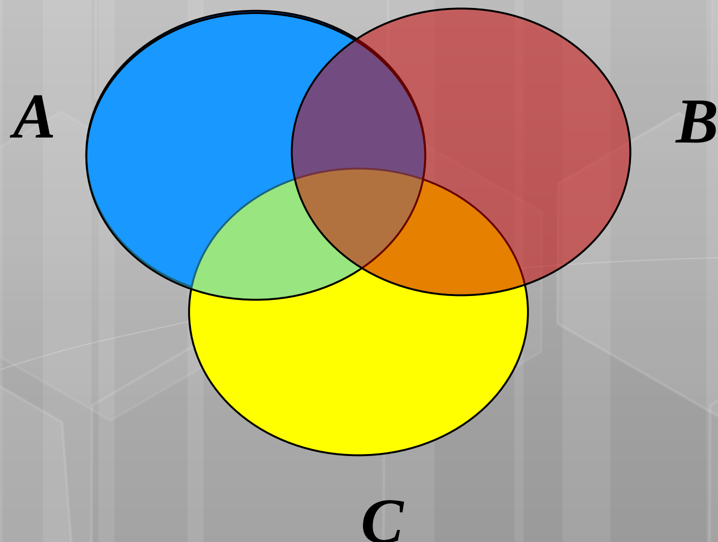
a) What combination of A and B represents the equipment the university should buy assuming both departments use the same equipment?

b) What combination of A and B represents the equipment that will be used by both departments if both departments use the same equipment?

c) What combination of A and B represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment?

d) What combination of A and B represents the equipment that the university should purchase if the departments do not share equipment?


Thank you!



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Fuzzy Set Operation

Given X to be the universe of discourse and Ã and Ḃ to be fuzzy sets with $\mu_A(x)$ and $\mu_B(x)$ are their respective membership function, the fuzzy set operations are as follows:

Union:

$$\mu_{A \cup B}(x) = \max\left(\mu_A(x), \mu_B(x)\right)$$

Intersection:

$$\mu_{A \cap B}(x) = \min\left(\mu_A(x), \mu_B(x)\right)$$

Complement:

$$\mu_A(x) = 1 - \mu_A(x)$$

Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$ $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

Union:

$$A \cup B = \{(x_1,0.8),(x_2,0.7),(x_3,1)\}$$

Because

$$\mu_{A \cup B}(x_1) = \max(\mu_A(x_1), \mu_B(x_1))$$
$$= \max(0.5,0.8)$$
$$= 0.8$$
$$\mu_{A \cup B}(x_2) = 0.7 \text{ and } \mu_{A \cup B}(x_3) = 1$$

Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$ $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

Intersection:

$$A \cap B = \{(x_1,0.5),(x_2,0.2),(x_3,0)\}$$

Because

$$\mu_{A\cap B}(x_1) = \min\ (\mu_A(x_1),\ \mu_B(x_1))$$
$$= \max(0.5,0.8)$$
$$= 0.5$$

$$\mu_{A\cap B}(x_2) = 0.2 \ \text{ and } \mu_{A\cap B}(x_3) = 0$$

Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$

Complement:

$$A^c = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1)\}$$

Because

$$\begin{aligned} \mu_A(x_1) &= 1 - \mu_A(x_1) \\ &= 1 - 0.5 \\ &= 0.5 \end{aligned}$$

$$\mu_A(x_2) = 0.3 \quad \text{and} \quad \mu_A(x_3) = 1$$

Consider the fuzzy set A

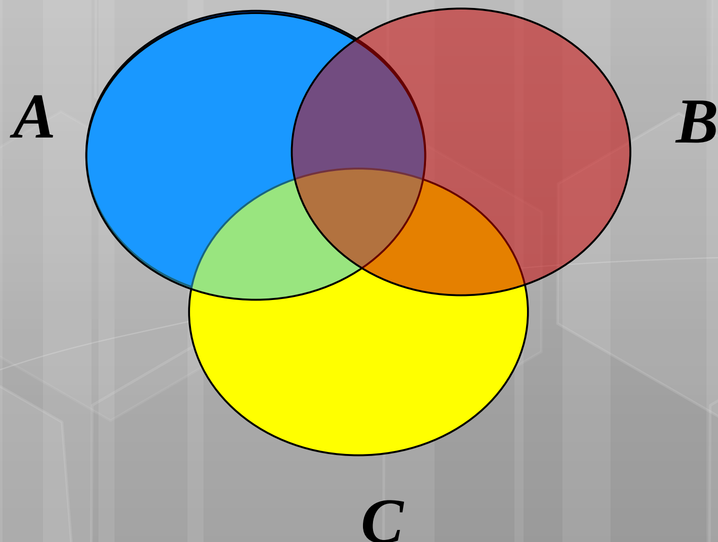
$A = \{(1, 0), (2, 0), (3, 0.2), (4, 0.5), (5, 0.8), (6, 1)\}$

Ans:

$\text{Card}(A) = |A| = 0 + 0 + 0.2 + 0.5 + 0.8 + 1 = 2.5$

$\text{Relcard}(A) = ||A|| = \dfrac{2.5}{6} \approx 0.417$


Thank you!



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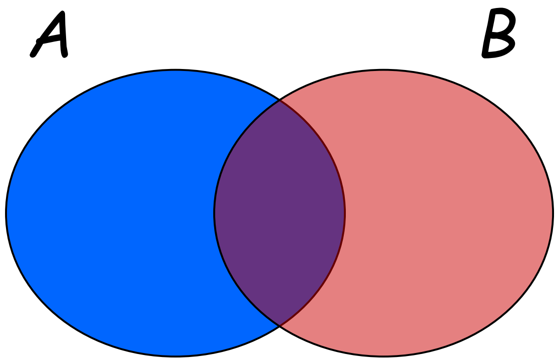
Inclusion-Exclusion (2 sets)

For two arbitrary sets *A* and *B*

$$|A \cup B| = |A| + |B| - |A \cap B|$$



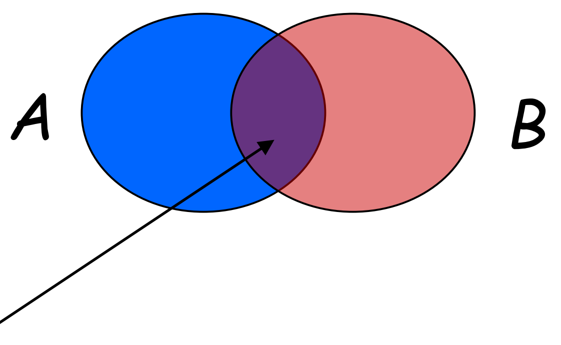
Inclusion-Exclusion (2 sets)

Let S be the set of integers from 1 through 1000 that are multiples of 3 or multiples of 5.

Let A be the set of integers from 1 to 1000 that are multiples of 3.

Let B be the set of integers from 1 to 1000 that are multiples of 5.

It is clear that S is the union of A and B, but notice that A and B are not disjoint.



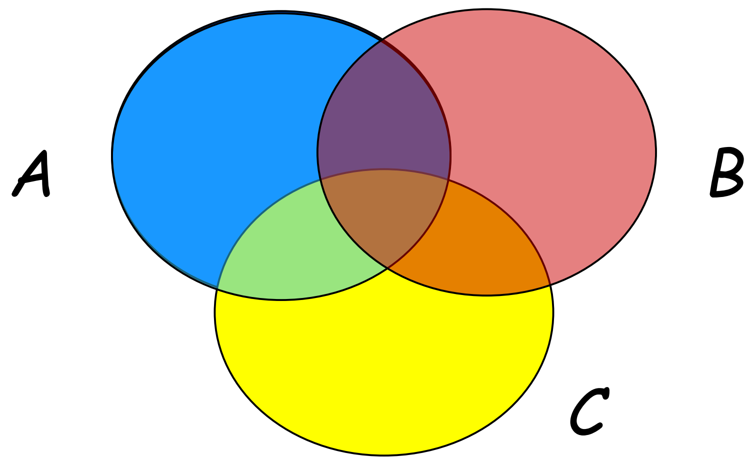
$|A| = 1000/3 = 333$ $|B| = 1000/5 = 200$

$A \cap B$ is the set of integers that are multiples of 15, and so $|A \cap B| = 1000/15 = 66$

So, by the inclusion-exclusion principle, we have $|S| = |A| + |B| - |A \cap B| = 467$.

Inclusion-Exclusion (3 sets)

$$
\begin{aligned}
|A \cup B \cup C| = |A| + |B| + |C| \\
- |A \cap B| - |A \cap C| - |B \cap C| \\
+ |A \cap B \cap C|
\end{aligned}
$$



Inclusion-Exclusion (3 sets)

From a total of 50 students:

$|A| \rightarrow$ 30 know Java

$|B| \rightarrow$ 18 know C++

$|C| \rightarrow$ 26 know C#

$|A \cap B| \rightarrow$ 9 know both Java and C++

$|A \cap C| \rightarrow$ 16 know both Java and C#

$|B \cap C| \rightarrow$ 8 know both C++ and C#

$|A \cup B \cup C| \rightarrow$ 47 know at least one language.

How many know none?

How many know all? $\leftarrow |A \cap B \cap C|$

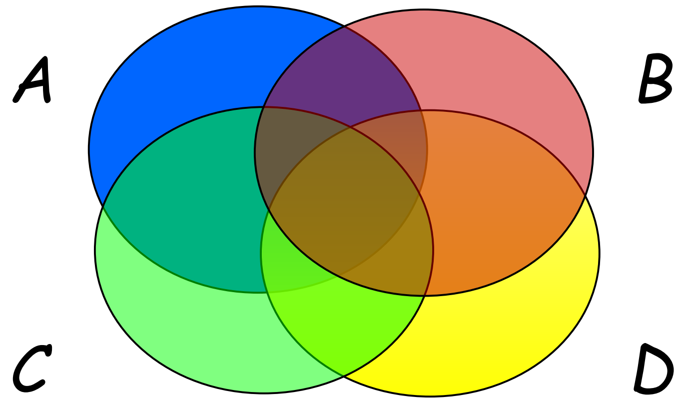
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$47 = 30 + 18 + 26 - 9 - 16 - 8 + |A \cap B \cap C|$$

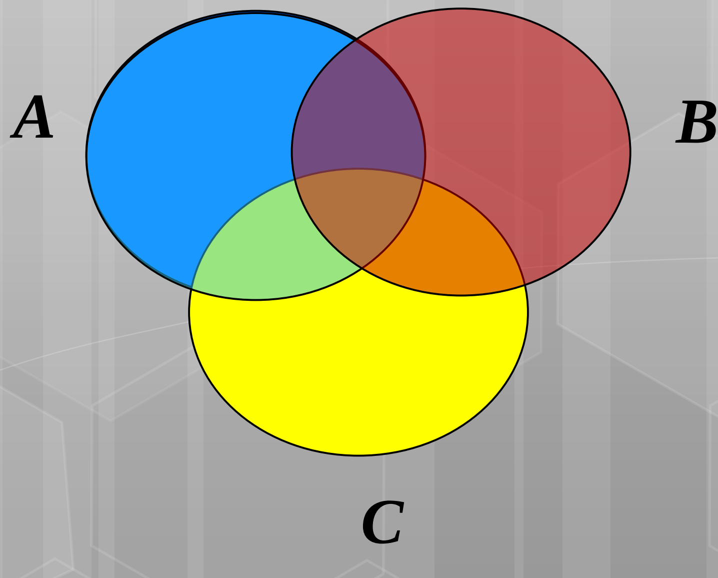
$$|A \cap B \cap C| = 6$$

Inclusion-Exclusion (4 sets)

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$
$$- |A \cap B \cap C \cap D|$$




Thank you!



Set Theory

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Partitioning of a Set

Partition of a set, say *S*, is a collection of *n* disjoint subsets, say $P_1, P_2, \ldots P_n$ that satisfies the following three conditions −

- P_i does not contain the empty set.

$$[P_i \neq \{\emptyset\}\ for\ all\ 0 < i \leq n]$$

- The union of the subsets must equal the entire original set.

$$[P_1 \cup P_2 \cup \cdots \cup P_n = S]$$

- The intersection of any two distinct sets is empty.

$$[P_a \cap P_b = \{\emptyset\},\ for\ a \neq b\ where\ n \geq a,\ b \geq 0]$$

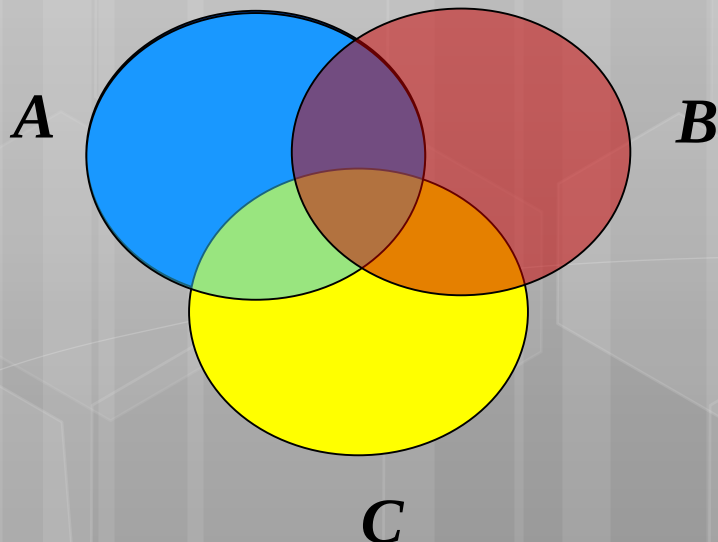
Covering on Set A

It is defined as a set on non-empty subsets Ai, whose union leads to the original set A and which are need not be pairwise disjoint. Here are the two conditions that are to be satisfied:

$$\bigcup_{i \in n} A_i = A$$

$$A_i \cap A_j \neq \phi \; for \; each \; (i, j) \in n \, ; i \neq j$$


Thank you!



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Set identities

$A \cup \varnothing = A$ $A \cap U = A$	Identity Law	$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination law
$A \cup A = A$ $A \cap A = A$	Idempotent Law	$(A^c)^c = A$	Complementation Law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Law	$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$	De Morgan's Law
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Law	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Law
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Law	$A \cup A^c = U$ $A \cap A^c = \varnothing$	Complement Law

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Proof of Second De Morgan Law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \land (x \in B))$	defn. of intersection
$\neg(x \in A) \lor \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \lor x \notin B$	defn. of negation
$x \in \overline{A} \lor x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x | x \notin A \cap B\} & \text{by defn. of complement} \\
 & = & \{x | \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
 & = & \{x | \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
 & = & \{x | \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
 & & & \text{for Prop Logic} \\
 & = & \{x | x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
 & = & \{x | x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
 & = & \{x | x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
 & = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0


Thank you!

Growth of Functions

Algorithm

- An algorithm is a finite set of precise instructions for performing a computation or for solving a problem.
- Algorithms can be described using English language or pseudocode etc.

Evaluating algorithms

- One of the important criteria in evaluating algorithms is the time it takes to complete a job.
- To have a meaningful comparison of algorithms, the estimate of computation time:
  - must be independent of the programming language, compiler, and computer used;
  - must reflect on the size of the problem being solved;
  - and must not depend on specific instances of the problem being solved.
- The quantities often used for the estimate are the **worst case execution time, and average execution time of an algorithm**, and they are represented by the number of some key operations executed to perform the required computation.

Algorithm example

- **Example: Algorithm for Sequential Search**
- Algorithm **SeqSearch**(L, n, x)
- L is an array with n entries indexed 1, .., n, and x is the key to be searched for in L.
- Output: if x is in L , then output its index, else output 0.

1. *index := 1;*
2. *while (index <=n and L[index]* $\neq$ *x)*
3. *index := index + 1 ;*
4. *if (index > n) , then index := 0*
5. *return index .*

Asymptotic Efficiency

- When we look at input sizes large enough to make only the order of growth of the running time relevant, we are studying the **asymptotic efficiency** of algorithms.
- How the running time of an algorithm increases with the size of the input in the limit, as the size of the input increases without bound.
- An algorithm that is asymptotically more efficient will be the best choice

Asymptotic notation

- The notation used to describe the asymptotic running time of an algorithm is defined in terms of functions whose domains are the set of natural numbers.
- This notation refers to how the problem scales as the problem gets larger.
- Main concern is with algorithms for large problems, i.e. how the performance scales as the problem size approaches infinity.

Different asymptotic notations

1. O-notation
2. Ω-notation
3. Θ-notation

Asymptotic Notations:-

They are Mathematical way of representing the time complexity. They are used in prior Analysis.

prior analysis:- We don't execute the Algorithm.

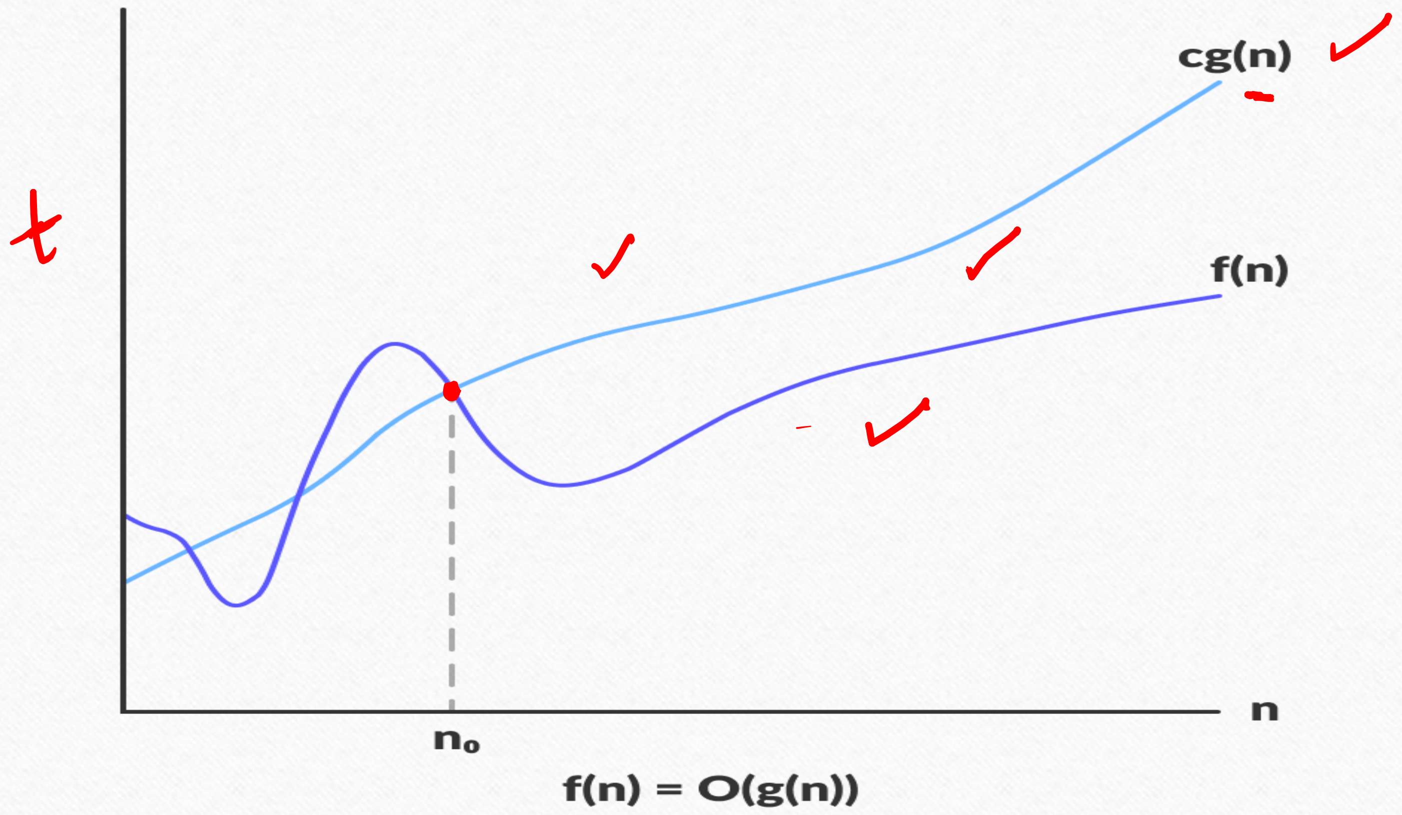
So there are 3- types of Notations:-

① O - Notation:- When we have only asymptotically upper bound of the Algorithm, then we use O - Notation.

$$f(n) = O(g(n))$$ ✓

$$0 \le f(n) \le c\,g(n) \quad \forall n \ge n_0$$

c, n_0 are +ve Constant



$\Rightarrow f(n) = O(g(n))$

$0 \le f(n) \le c\,g(n)$

$\forall n \ge n_0$

c, n_0 are +ve constants.

① Ex

$f(n) = 3n+2$ $\qquad g(n) = n$

Can we say that $f(n) = O(g(n))$

$$f(n) \le c(g(n))$$

$$3n+2 \le cn \checkmark$$

$n_0 \ge 1$

$c = 4 \quad n = 1$

$c = 4 \checkmark$

$n \ge n_0$

$5 \le 4 \cdot 1 \qquad F$

$n_0 \ge 2 \qquad T$

$f(n) = O(g(n))$

$c = 4 \quad n = 2$

$6+2 \le 4 \cdot 2$

$8 \le 8 \checkmark \qquad T$

Ex 2:- $n^3 + n + 5 \le c\,n^3$ $\qquad f(n) = O(g(n))$

$c = 7 \qquad n_0 = 1$

Hence $f(n) = O(g(n))$

Big-O Notation is used to Capture all upper Bound.

But we prefer the least upper bound.

$f(n)$ is $O(n^3) \rightarrow O(n^3)$

$O(n^4)$

$O(n^5)$

$\vdots$

Ex 3:-

Prove $3^n \neq O(2^n)$

Sol^n:- Proof by Contradiction

Suppose $3^n = O(2^n)$

$$3^n \le c \cdot 2^n$$

Dividing both sides by 2^n

$$(1.5)^n \le c$$

This is a contradiction. Because c is a constant. It can not depend on the value of n. So our assumption is wrong.

Hence proved.

② Ω Notation:-

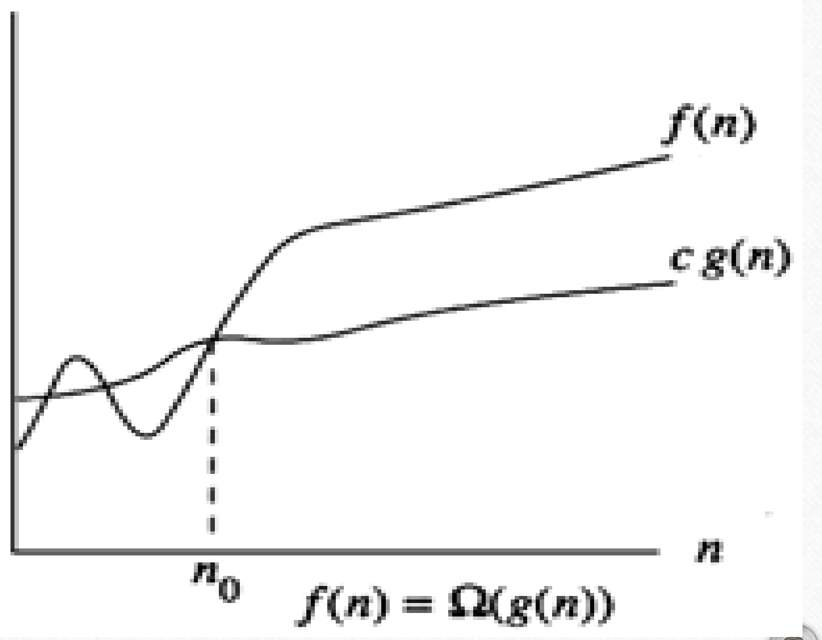
It provides Asymptotically lower bound on the function.

$$f(n) = \Omega(g(n))$$

$$f(n) \geq c\,g(n) \geq 0$$

$$n \geq n_0$$

c, n_0 are +ve Constants.



Ex 1

$f(n) = 3n+2 \qquad g(n) = n$

$f(n) = \Omega(g(n))$

$f(n) \geq c\, g(n)$

$3n+2 \geq cn$

$c = 1 \quad n_0 \geq 1$

$f(n) = \Omega(g(n))$

Ex2:-

$$n^3+n+5 \geq c \cdot n^3$$

$c=1 \quad n_0=1$

$7 \geq 1$

$c=1 \quad n_0=2$

$15 \geq 8$

$c=1 \quad \forall n_0 \geq 1$

Hence it is $\Omega(n^3)$

Ex 3:-

Prove $3n^2+2 \neq \Omega(n^3)$

$$3n^2+2 \geq c \cdot n^3$$

Dividing both sides by n^3, we get

$$\frac{3n^2+2}{n^3} \geq c$$

This is a Contradiction. Because value of c is a constant. But in this case it is dependent on n. So this is true.

$$3n^2+2 \neq \Omega(n^3)$$

③ Θ-Notation :-

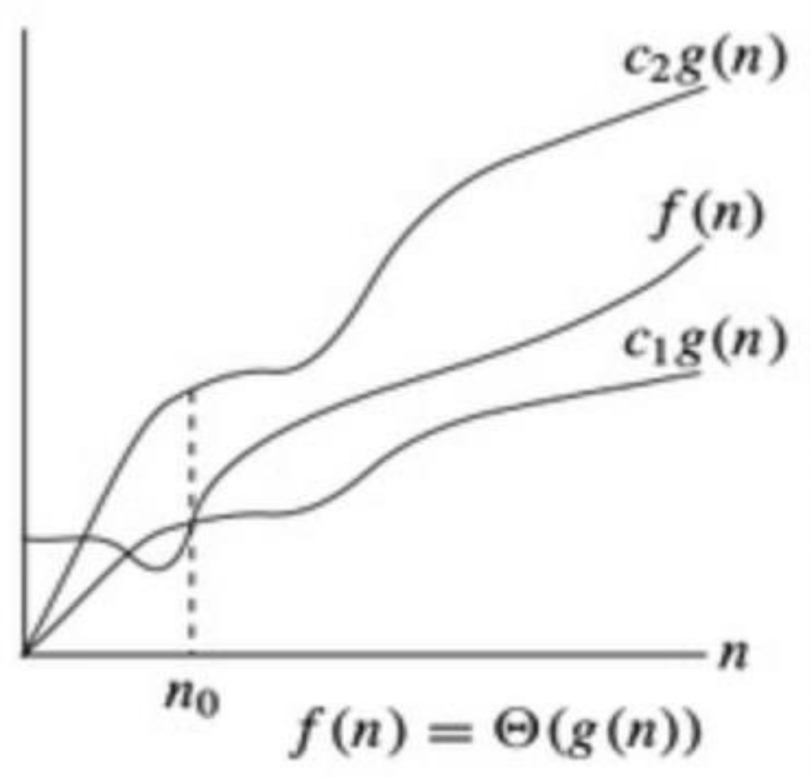
When our function $f(n)$ is

$$f(n) = \Theta(g(n))$$

$$c_1(g(n)) \leq f(n) \leq c_2(g(n))$$

$$\forall\, n \geq n_0 \quad \&$$

c, n_0 +ve constants.



Θ - Notation Can be used to denote tight bounds of the Algorithm.

$$f(n) = \Theta(g(n))$$

$$f(n) = \Omega(g(n))$$

&

$$f(n) = O(g(n))$$

Then it is $f(n) = \Theta(g(n))$

Ex 1:-

$$f(n) = 3n+2 \qquad g(n) = n$$

$$f(n) \leq c_2\, g(n)$$

$$3n+2 \leq c_2\, n$$

$$c_2 = 4 \qquad n_0 \geq 1$$

$$f(n) \geq c_1\, g(n)$$

$$3n+2 \geq n$$

$$c_1 = 1 \qquad n_0 \geq 1$$

Θ is also called Asymptotically equal.

$$f(n) = \Theta(g(n))$$

Ex 2:- $10n^2 + 4n + 2 = O(n^2)$

① $O(n^2)$

$$10n^2 + 4n + 2 \le c_1 \cdot n^2 \qquad \forall n \ge n_0$$

$$10 + 4 + 2 \le 20 \cdot 1 \qquad n_0 = 1, \ c_1 = 20$$

$$16 \le 20 \quad T \qquad n_0 = 2, \ c_1 = 20$$

$$50 \le 80 \quad T$$

So this is $O(n^2)$.

② $\Omega(n^2)$

$$10n^2+4n+2 \geq c_2 n^2$$

$16 \geq 10$ T $c_2=10,\ n_0=1$

T $c_2=10,\ n_0=2$

for $c_1=20,\ n_0=1$

Both the functions are true.

So we can say that this is $\Theta(n^2)$.

Complexity of Algorithms

Complexity

- An Algorithm can be analyzed on two accounts space and time:
- Memory Space: Space occupied by program code and the associated data structures.
- **CPU Time:** Time spent by the algorithm to solve the problem.

Counting operations

- Instead of measuring the actual timing, we count the number of operations
  - Operations: arithmetic, assignment, comparison, etc.
- Counting an algorithm's operations is a way to assess its efficiency.
  - An algorithm's execution time is related to the number of operations it requires

Example: Counting Operations

```
for (int i = 1; i <= n; i++){
    perform 100 operations; // A
        for (int j = 1; j <= n; j++){
        perform 2 operations; // B
    }
}
```

Asymptotic Analysis

Asymptotic analysis is an analysis of algorithms that focuses on :

- Analyzing problems of large input size.
- Consider only the leading term of the formula.
- Ignore the coefficient of the leading term

Why Choose Leading Term?

- Lower order terms contribute lesser to the overall cost as the input grows larger
- Example

  $f(n) = 2n^2 + 100n$

- $f(1000) = 2(1000)^2 + 100(1000) = \mathbf{2{,}0}00{,}000 + \mathbf{1}00{,}000$
- $f(100000) = 2(100000)^2 + 100(100000) = \mathbf{20{,}000}{,}000{,}000 + \mathbf{10}{,}000{,}000$
  Hence, lower order terms can be ignored.

Examples: Leading Terms

- $a(n) = n + 4$

  Leading term:

- $b(n) = 240n + 2n^2$

  Leading term:

- $c(n) = n\lg(n) + \lg(n) + n \lg(\lg(n))$

  Leading term:

Upper Bound: Big-Oh Notation

$T(n)=O(g(n))$ is defined as: $T(n) <= c*g(n)$ where $c>0$

- From the above relation we can say that for a large value of n, the function 'g' provides an upper bound on the growth rate 'T.

Order of growth:

- $O(1) < O(\log_k n) < O(n) < O(n\log n) < O(n^2) < O(n^3) < O(2^n)$

Running Time Calculations

Rule 1:

- Simple program statements are assumed to take a constant amount of time which is **O(1)** i.e. Not dependent upon n.

Example:

- One arithmetic operation (eg., +, *)
- One assignment
- One test (e.g. x==0)
- One read(accessing an element from an array)

Running Time Calculations

Rule 2: **Loops**

- The running time of a loop is at most the running time of the statements inside the loop (including tests) times the number of iterations of the loop.

Example;

for (i = 0; i < N; i++) {

statement(s) of O(1)

}

Running Time Calculations

Rule 3 - **Nested loops**

- The total running time of a statement inside a group of nested loops is the running time of the statement multiplied by the product of the sizes of all the loops
- Example:

```
for (i = 0; i < N; i++) {
for (j = 0; j < M; j++) {
statement(s) of O(1)
}
}
```

Running Time Calculations

Rule 4: Conditional Statements

- The running time of a conditional statement is never more than the running time of the test plus the largest of the running times of the various blocks of conditionally executed statements.

Rule 5:Consecutive statements

- These just add
- Only the maximum is the one that counts

Example

Parameters: A finite-length list, L, of positive integers.

Returns: The sum of the integers in the list.

```
{ sum := 0;
for each x in L
{ sum := sum + x;
}
 return sum;
 }
```

Complexity of Algorithms

Example 1

```
A( )
{ int a=0;
    for(int i=0; i<m; i++)
    {--}
    for(int j=0; j<n; j++)
    {--}
}
```

Example 2

```
A( )
{ int a=0;
    for(int i=0; i<m; i++)
        for(int j=0; j<n; j++)
            {--}
}
```

Example 3

```
A( )
{ int a=0;
    for(int i=1; i<=n; i=i*2)
                    {--}
}
```

Example 4

```
A( )
{ int a=0;
    for(int i=1;i<=n;i++)
        for(int j=1;j<=n;j=j+2)
            {--}
}
```

Example 5

```
A( )
{ int a=0;
    for(int i=1;i<=n;i++)
        for(int j=1;j<n;j=j*2)
            for(int k=1;k<=n;k++)
                {--}
}
```

Example 6

```
A( )
{for(int i=n; i>0; i/=2)
        for(int j=1; j<=i; j++)
            {--}
}
```

Example 7

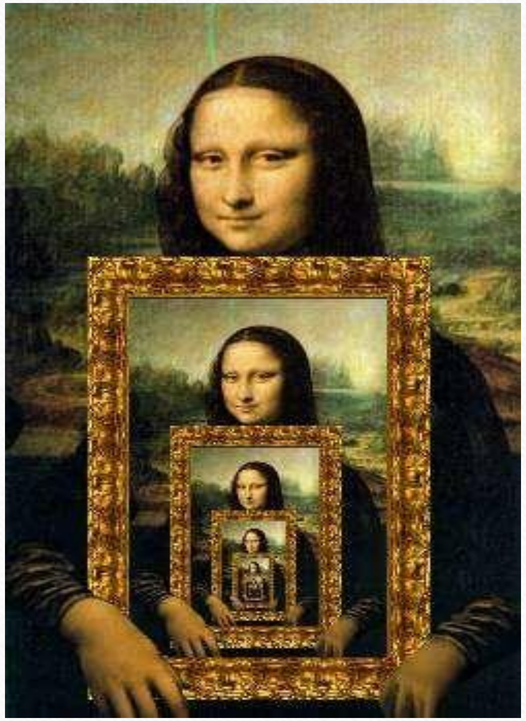
```
A( )
{for(i=1;i<=n;i++)
    for(j=1;j<=i;j++)
        for(k=1;k<=100;k++)
            {-- }
}
```

Recursive Functions

Recursion

- Sometimes it is possible to define an object (function, sequence, algorithm, structure) in terms of itself. This process is called **recursion.**
- An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

Recursive Examples



Recursive definition

- There are two parts:
  - Basic case (basis): the most primitive case(s) of the entity are defined without reference to the entity.
  - Recursive (inductive) case: new cases of the entity are defined in terms of simpler cases of the entity.
- Recursive sequences
  - A sequence is an ordered list of objects, which is potentially infinite.
  - A sequence is defined recursively by explicitly naming the first value (or the first few values) in the sequence and then define later values in the sequence in terms of earlier values.

Examples

- A recursively defined sequence:
  - S(1) =2 …… basis case
  - S(n) = 2 * S(n-1), for n >= 2 …… recursive case
  - what does the sequence look like?

  - T(1) =1 ...... basis case
  - T(n) = T(n-1) + 3, for n >= 2 …… recursive case
  - what does the sequence look like?

Recursive Definitions of Important Functions

- Some important functions/sequences defined recursively

1. Factorial function:

2. Fibonacci numbers:

Recursively defined functions

- **Example:** Assume a recursive function on positive integers:
  - $f(0) = 3$
  - $f(n+1) = 2f(n) + 3$
- **What is the value of $f(2)$?**

Example

- Give a recursive definition of the following sets of objects:

  1, 2, 4, 7, 11, 16, 22, …

Solution:

Ackermann function

- The Ackermann function is a classic example of a recursive function. It grows very quickly in value, as does the size of its call tree.
- The Ackermann function is:

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

Example

McCarthy 91 function

- The McCarthy 91 function is a recursive function, defined by computer scientist John McCarthy.
- The McCarthy 91 function is defined as

$$M(n) = \begin{cases} n - 10, & \text{if } n > 100 \\ M(M(n+11)), & \text{if } n \le 100 \end{cases}$$

Example

Thank You

Introduction to Functions(I)


Activity One

Function definition

- Let A and B be nonempty sets.
- A *function* f from A to B is an assignment of exactly one element of B to each element of A.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
- If f is a function from A to B, we write
  - $f : A \rightarrow B$.

Domain and Range

- If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f.
- If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b.
- The *range*, or *image*, of f is the set of all images of elements of A.
- Also, if f is a function from A to B, we say that f *maps* A to B.

Equal functions

- Two functions **f** and **g** are **equal** when
- they have the same domain **or** domain of f = domain of g
- have the same co-domain **or** co-domain of f = co-domain of g
- Map each element of their common domain to the same element in their common co-domain.

or

$f(x) = g(x)$ for every x belonging to their common domain.

f A={1,2} and B={3,6} and two functions $f: A \to B$ and $g: A \to B$ are defined respectively as :
$f(x) = x^2 + 2 \text{ and } g(x) = 3x$
Find whether $f = g$

Summary

- Concept of Functions
- Domain and Range of functions
- Equal Functions

Introduction to Functions(II)

Real valued/Integer valued functions

- A function is called **real-valued** if its codomain is the set of real numbers.
- A function is called **integer-valued** if its codomain is the set of integers.
- Two real-valued functions or two integer-valued functions with the same domain can be added, as well as multiplied.

Function addition/multiplication

- Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by
- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$,
- $(f_1 f_2)(x) = f_1(x)\, f_2(x)$.

Example

Question: Let f_1 and f_2 be functions from **R** to **R** such that

$f_1(x) = x^2$ and $f_2(x) = x - x^2$.

What are the functions $f_1 + f_2$ and $f_1 f_2$?

Answer:

Image of a subset

- Let f be a function from A to B and let S be a subset of A.
- The *image* of S under the function f is the subset of B that consists of the images of the elements of S.
- The image of S is denoted by $f(S)$ where
- $f(S) = \{t \mid \exists s \in S \, (t = f(s))\}$.

Example

- Question: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. Consider subset $S = \{b, c, d\}$.

  What is image of S?

  Answer:

Summary

- Concept of Functions
- Domain and Range of functions
- Equal Functions
- Function addition/multiplication
- Image of a subset

Types of Function

Types of Function

- A function can be of three types:

1. One-to-One function (Injective function)
2. Onto function (Surjective function)
3. One-to-One correspondence (Bijective function)

One-to-One function

- A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.
- A function is said to be *injective* if it is one-to-one.

Example 1

- Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Example 2

- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Onto function

- A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
- A function f is called *surjective* if it is onto.

Example 1

- Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Example 2

- Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

One-to-One correspondence

- The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto.
- A function f is called *bijective* if it is one to one and onto.

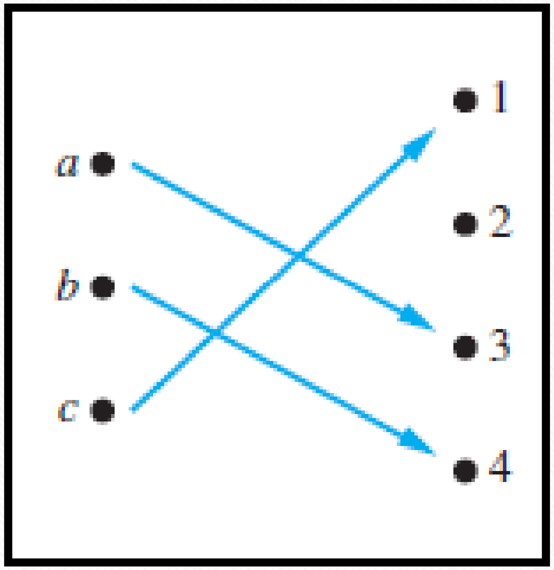
Example 1

- Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?

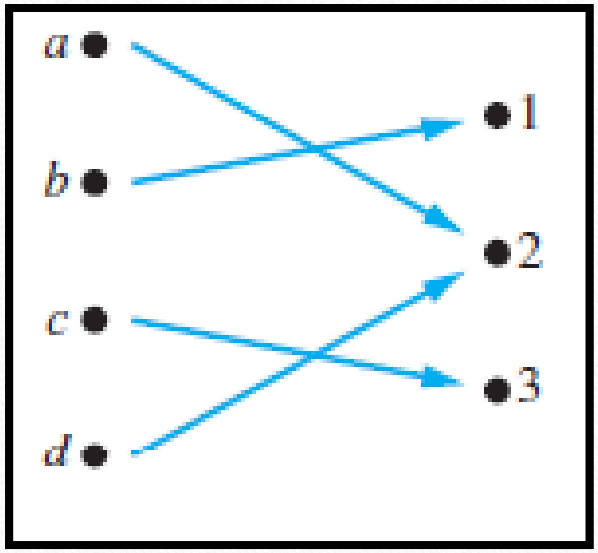
Example 2

- Is the function $f(x) = x^2$ from the set of integers to the set of integers bijective?

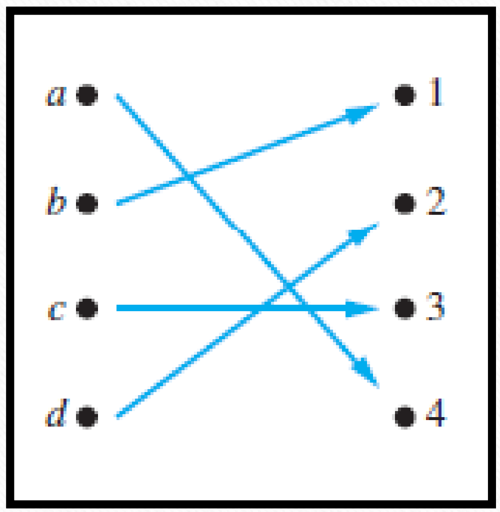
Examples of Different Types of Correspondences



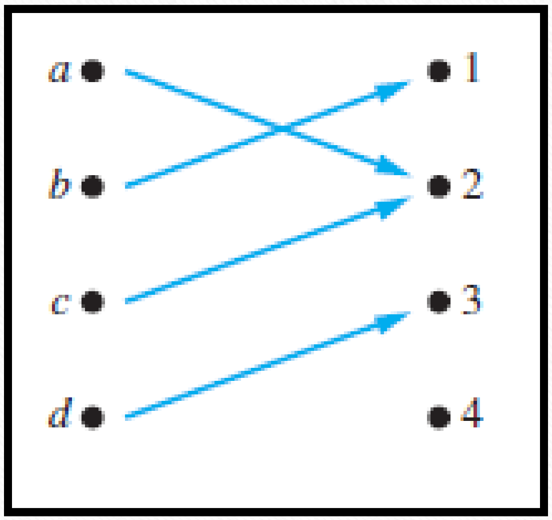
Examples of Different Types of Correspondences



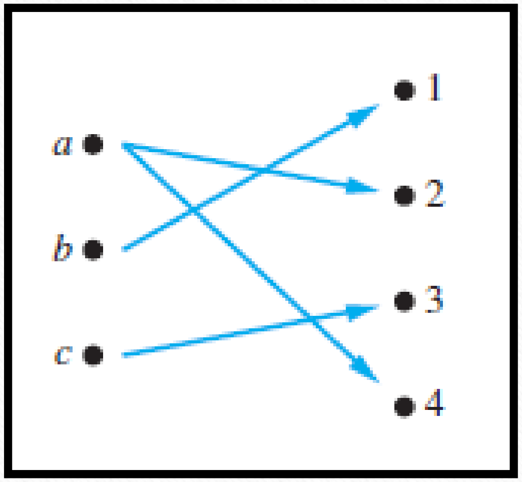
Examples of Different Types of Correspondences



Examples of Different Types of Correspondences



Examples of Different Types of Correspondences



Inverse Function and Compositions of Functions

Inverse function

- Let f be a one-to-one correspondence from the set A to the set B.
- The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.
- The inverse function of f is denoted by f^{-1}.
- $f^{-1}(b) = a$ when $f(a) = b$.
- A one-to-one correspondence is called **invertible** as we can define an inverse of it.
- A function is **not invertible** if it is not a one-to-one correspondence.

Function f^{-1} Is the Inverse of Function f

Example 1

- Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Example 2

- Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible?

Example 3

- Find the inverse function of $F(x) = x^3+1$.

Example 4

- Find the inverse function of $F(x) = \dfrac{x-3}{2}$.

Example 5

- Find the inverse function of $F(x) = \dfrac{3x+2}{4x-1}$.

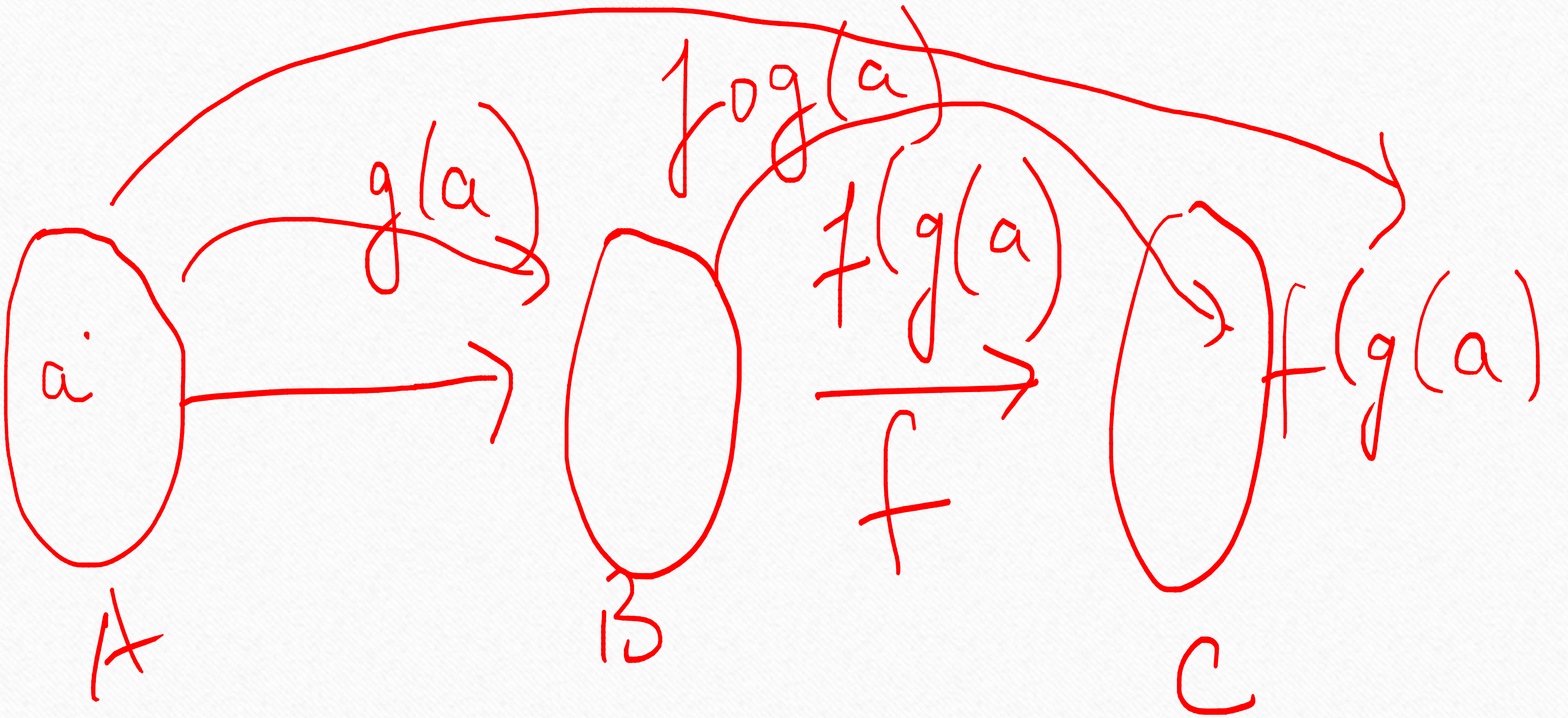
Example 6

- Let f is a function from R to R given by $f(x) = x^2+1$. Find $f^{-1}(-5)$.

Composition of functions

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C.
- The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by
  - $(f \circ g)(a) = f(g(a))$.

omposition of the Functions f and g.



Example 1

- Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Example 2

- Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x+3$ and $g(x) = 3x+2$. What is the composition of f and g? What is the composition of g and f?

Floor and Ceil Function

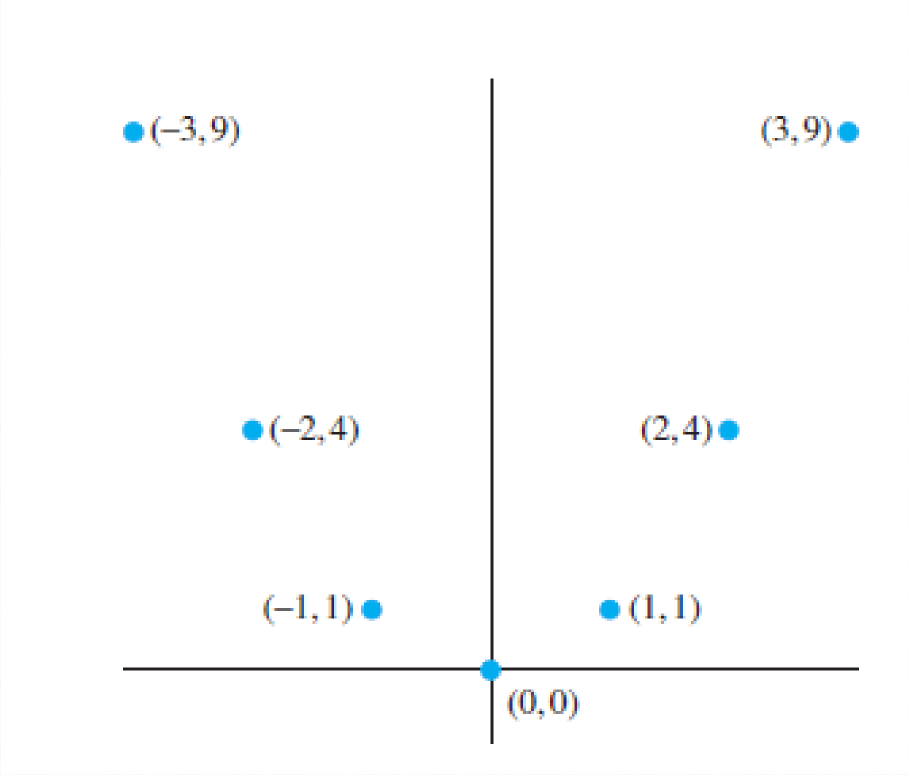
Graphs of Functions

- Let f be a function from the set A to the set B.
- The *graph* of the function f is the set of ordered pairs $\{(a, b) \mid a \in A$ and $f(a) = b\}$.

Example

- Display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers.

The graph of f is the set of ordered pairs of the form $(x, f(x)) = (x, x^2)$, where x is an integer

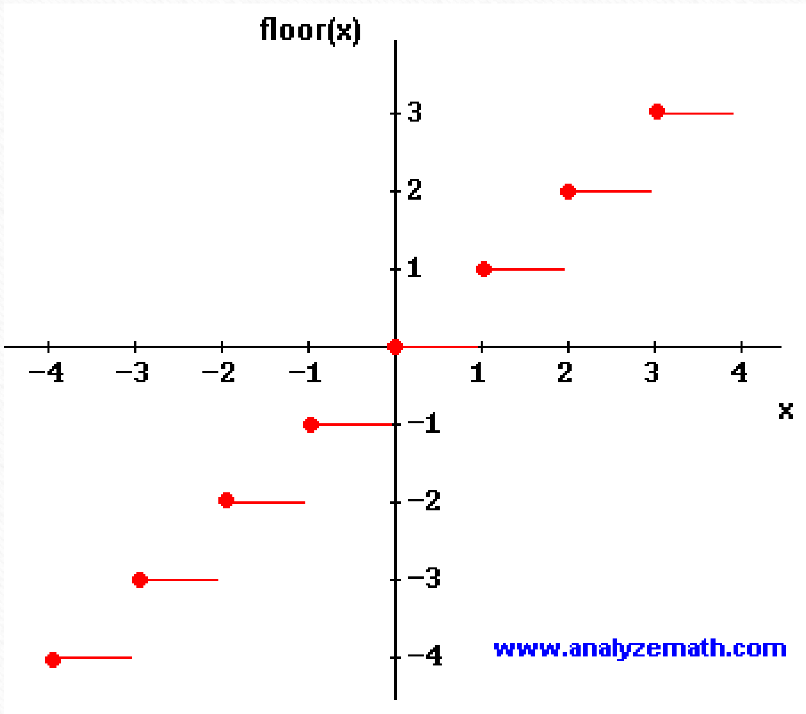


The Graph of $f(x) = x^2$ from Z to Z.

Floor function

- The ***floor function*** assigns to the real number x the largest integer that is less than or equal to x.
- The value of the floor function at x is denoted by $\lfloor x \rfloor$.

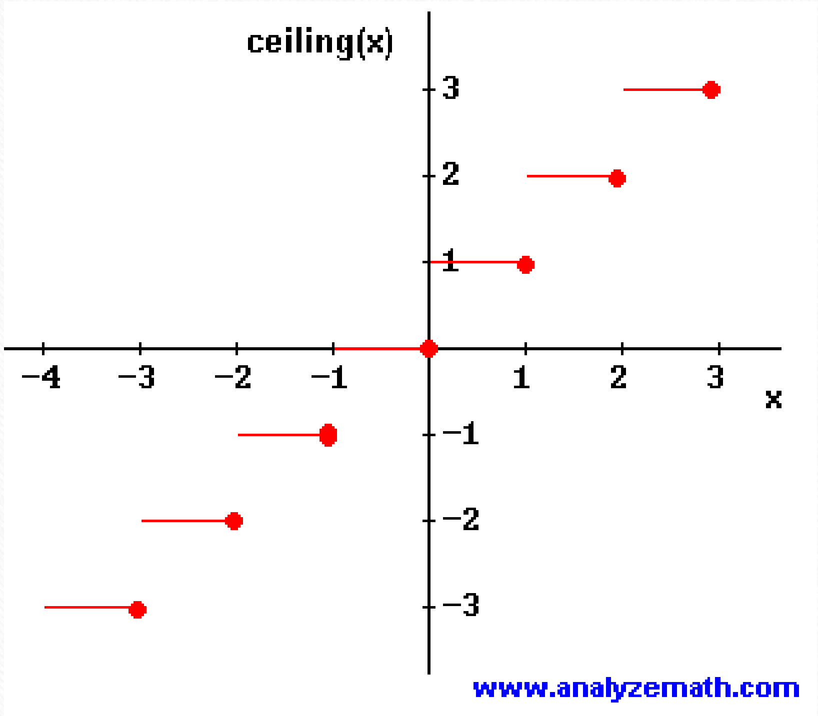
Graph of floor function



Ceil function

- The ***ceiling function*** assigns to the real number x the smallest integer that is greater than or equal to x.
- The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Graph of ceil function



Example 1

- Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

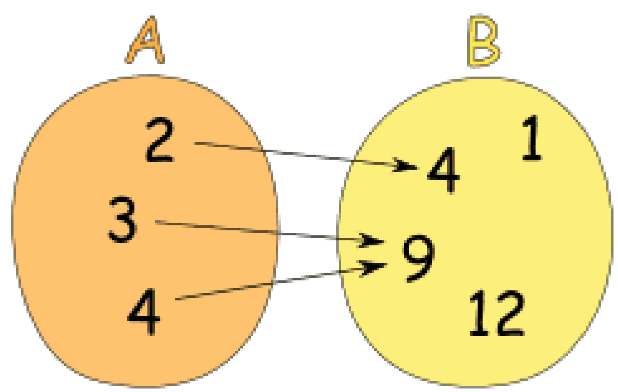
Example 2

- In asynchronous transfer mode, data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?

Thank You

Practice on Basics of Functions

Question 1

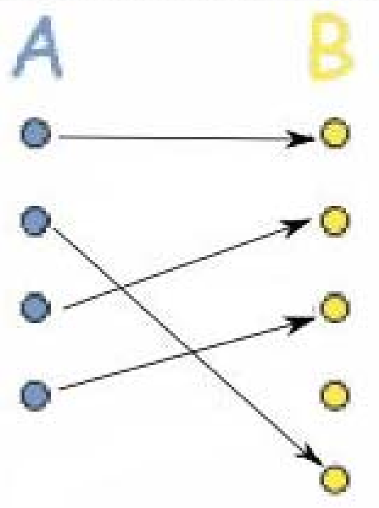


For the function illustrated above, what is the range?

Question 2

What is the domain for the function $f(x) = \dfrac{(x-2)(x-4)}{(x-1)(x-3)}$?

Question 3



The function from set A to set B is

Question 4

Which of the following functions is NOT injective (one-to-one)?

A $f(x) = x^3 + 4$ from R to R

B $f(x) = x^3 + 4$ from N to N

C $f(x) = x^2 + 4$ from R to R

D $f(x) = x^2 + 4$ from N to N

Question 5

If X = Floor(X) = Ceil(X) then :

a) X is a fractional number

b) X is a Integer

c) X is less than 1

d) none of the mentioned

Question 6

Suppose that $f(x) = 3x - 8$

a) Is f^{-1} a function?

b) Find the inverse function of f.

c) Compute $f(f^{-1}(7))$ and $f^{-1}(f(7))$

Thank You

HASHING FUNCTIONS

Hashing Function

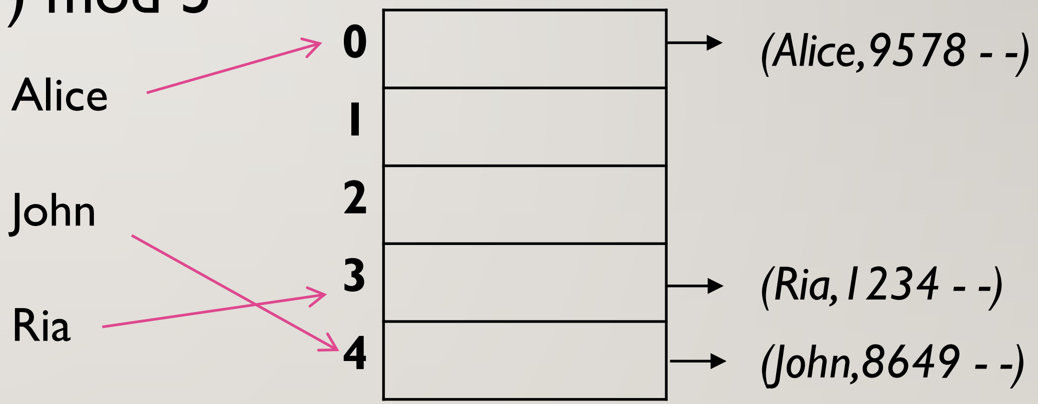
- Hashing function is a function which is applied on a key by which it produces an integer, which can be used as an address in hash table.
- A simple hashing function: $h(k) = k \bmod m$

Properties of Hashing Functions

- Easy to compute
- Uniform distribution
- Less collisions

Hash Table: Example

- **Example:** phone book with table size N = 5
- **hash function** h(w) = (length of the word w) mod 5



- **Problem:** collisions
- Where to store Joe (collides with Ria)

Collisions

- Collisions occur when different elements are mapped to the same cell.
- Keys k_1, k_2 with $h(k_1) = h(k_2)$ are said to collide

What should we do now?

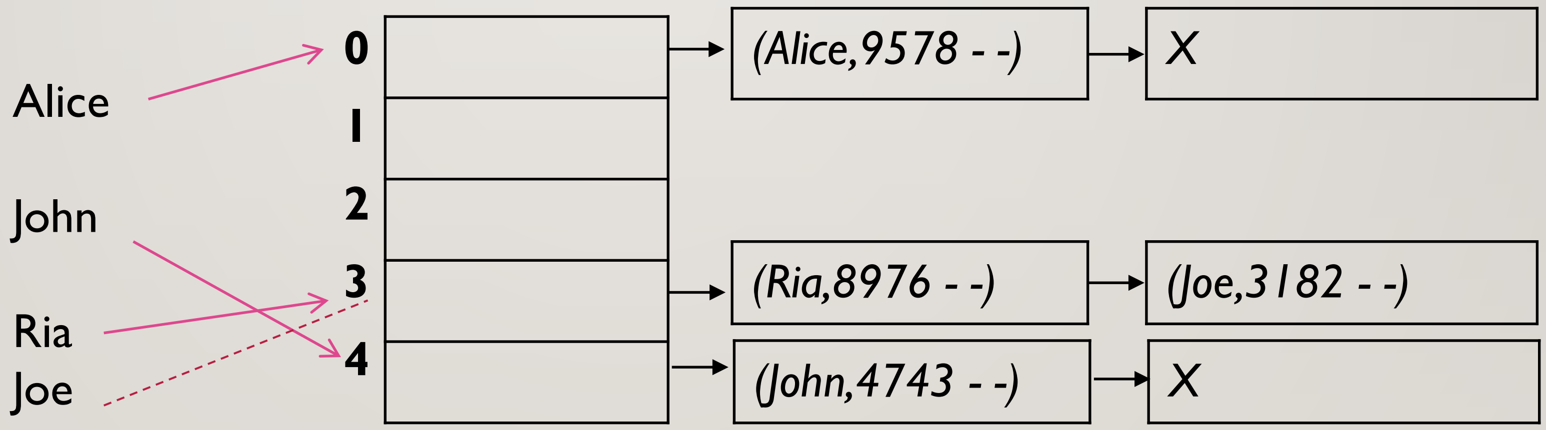
- Find a better hashing algorithm
- Use a bigger table
- Need a system to deal with collisions

Resolving Collisions

- Two different methods for collision resolution:
  - **Separate Chaining**: Use a dictionary data structure (such as a linked list) to store multiple items that hash to the same slot.
  - **Closed Hashing (or *Open Addressing*)**: search for empty slots using a second function and store item in first empty slot that is found.

Separate Chaining

- Each cell of the hash table points to a linked list of elements that are mapped to this cell.
- Simple, but requires additional memory outside of the table



Closed Hashing or Open Addressing

- Open addressing does not introduce a new structure.
- If a collision occurs then we look for availability in the next spot generated by an algorithm.
- There are many implementations of open addressing, using different strategies for where to probe next:

1. Linear Probing
2. Quadratic Probing
3. Double Hashing

Contd..

- Given an item X, try cells $h_0(X), h_1(X), h_2(X), \ldots, h_i(X)$
  - $h_i(X) = (Hash(X) + F(i)) \bmod \mathit{TableSize}$
  - $F(0) = 0$
- F is the *collision resolution* function. Some possibilities:
  - Linear: $F(i) = i$
  - Quadratic: $F(i) = i^2$
  - Double Hashing: $F(i) = i*Hash_2(X)$

Linear Probing Example

	insert(14) $14\%7 = 0$	insert(8) $8\%7 = 1$	insert(21) $21\%7 = 0$	insert(2) $2\%7 = 2$
0	14	14	14	14
1		8	8	8
2			21	21
3				2
4				
5				
6				

Quadratic Probing Example

	insert(14) $14\%7 = 0$	insert(8) $8\%7 = 1$	insert(21) $21\%7 = 0$	insert(2) $2\%7 = 2$
0	14	14	14	14
1		8	8	8
2				2
3				
4			21	21
5				
6				

Double Hashing

- *Double hashing can be done using :*
  (hash1(key) + i * hash2(key)) % TABLE_SIZE
- First hash function is typically

hash1(key) = key % TABLE_SIZE

- A popular second hash function is :

hash2(key) = PRIME – (key % PRIME)

where PRIME is a prime smaller than the TABLE_SIZE.

Double Hashing Example

Index	insert(19) $19\%13 = 6$	insert(27) $27\%13 = 1$	insert(36) $36\%13 = 10$	insert(10) $10\%13 = 10$
0				
1		27	27	27
2				
3				
4				
5				10
6	19	19	19	19
7				
8				
9				
10			36	36
11				
12				

Collision 2

Let Hash2(key)=7-(key % 7)

Hash1(10)=10%13=10 (Collision1)
Hash 2(10)=7-(10%7)=4

(Hash1(10)+1*Hash2(10))%13=1(Collision 2)
(Hash1(10)+2*Hash2(10))%13=5

Collision 1

PRACTICE QUESTIONS ON GROWTH OF FUNCTIONS

Q1. Give a big-O notation to estimate the sum of the first n positive integers.

Q2. Give a big-O estimate for the factorial function.

Q3. Give a big-O estimate for the following function:

$$f(n) = 3n \log (n!) + (n^2 + 3) \log n$$

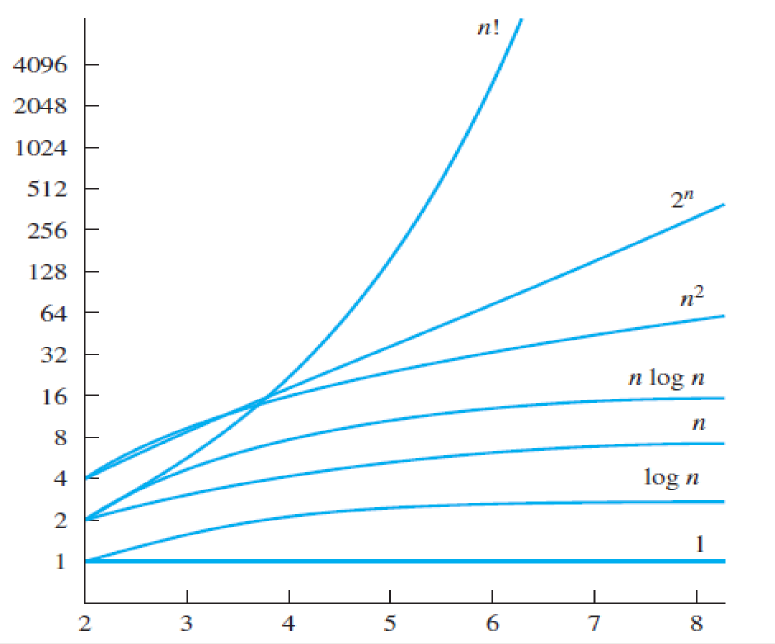
Q4. Give a big-O estimate for the following function:

$$f(x) = (x + 1) \log (x^2 + 1) + 3x^2$$

As mentioned before, big-*O notation is used to estimate the number of operations needed to* solve a problem using a specified procedure or algorithm. The functions used in these estimates often include the following:

1, $\log n$, n, $n \log n$, n^2, 2^n, $n!$

Using calculus it can be shown that each function in the list is smaller than the succeeding function, in the sense that the ratio of a function and the succeeding function tends to zero as *n grows without bound. Figure displays the graphs of these functions, using a scale for* the values of the functions that doubles for each successive marking on the graph. That is, the vertical scale in this graph is logarithmic.

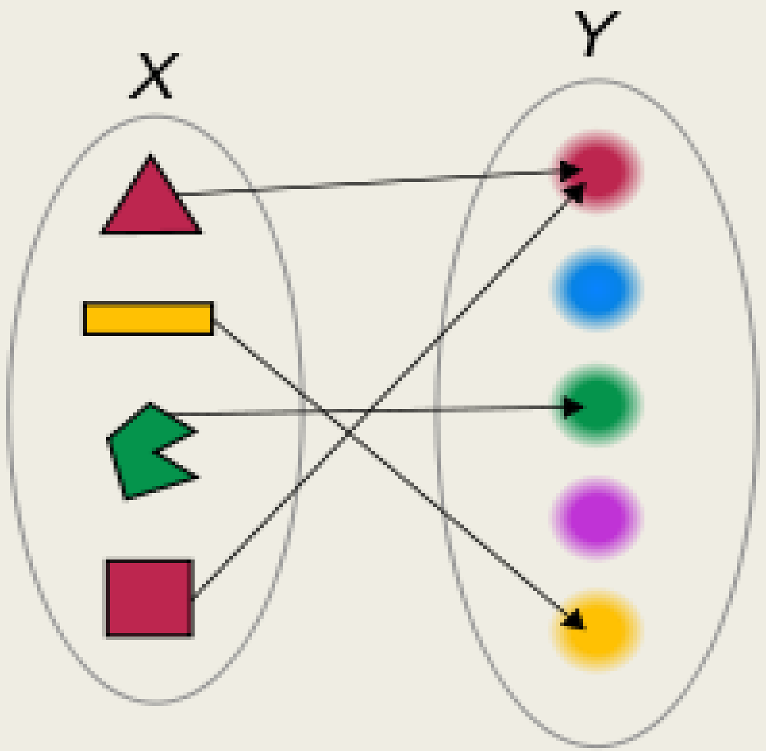


A Display of the Growth of Functions Commonly Used in Big-*O Estimates.*

Thank You

APPLICATIONS OF FUNCTIONS

Function



Applications of functions

- Aadhaar card
- Every Indian citizen holds a unique 12-digit aadhaar number printed on aadhaar card. Only one aadhaar number is allotted to one individual. So, this is one-to-one mapping/function.

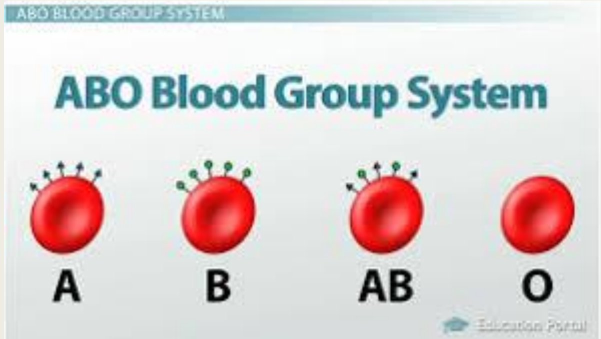
Applications of functions (Contd…)

- ATM

ATM debit card is mapped to a single Savings / current bank account. Only one bank account is linked with a debit card. This is an example of One-to-one mapping/function.

Applications of functions (Contd…)

- Blood group mapping
- Every human being has one of the four blood groups – A, B, AB and O. So, blood group of all Human beings can be mapped to only these four blood groups. This is an example of many-to-one mapping/function.

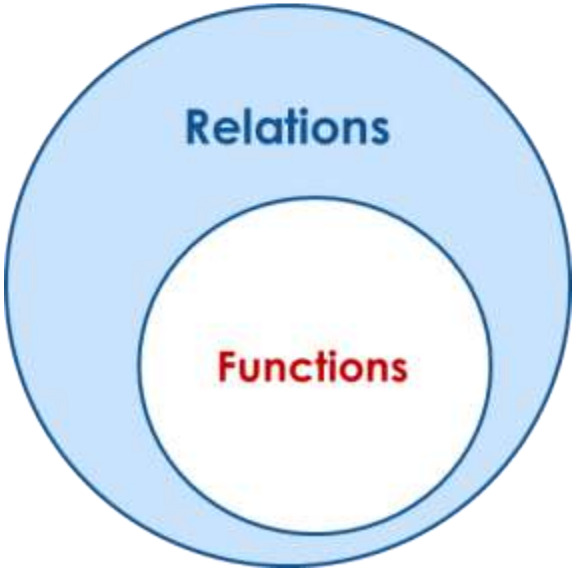


Applications of functions (Contd…)

- Recursive functions
- Palindrome checker
  - *Reverse strings can be obtained from recursive functions.*
- Decimal to binary conversion
  - *Decimal numbers are repeatedly divided by 2 in this process, which can be done easily by recursive functions.*
- Balanced parenthesis checker
  - *Opening and closing of parenthesis can be matched using recursive functions.*

Activity time

Assume you have an Amazon.in coupon of getting 50 Rs. off on a particular item and also, the item you think of buying is on 20% off. What do you think is better : taking Rs. 50 off first and then applying discount or applying discount first and then taking Rs. 50 off?



Relations

Contents

What is a Relation?

In discrete mathematics, relation is a way of showing a relationship between any two sets.

- Relationship between any program and its variable.
- Relationship between pair of cities linked by railway in a network.

Necessity for studying Relation

- Relational Database model is based on the concept of relation.

Cartesian Product

- Given two sets A and B, their ***cartesian product*** $A \times B$, is defined as

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

Ordered Pairs

- The elements of $A \times B$ are called ***ordered pairs*** with the elements of A as the first entry and elements of B as the second entry.

- Order matters

Special Case:

$$A^2 = A \times A = \{(a_1, a_2) \mid a_1, a_2 \in A\}$$

Similarly,

$$A^n = A \times A \times \cdots \times A (n\ times) = \{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in A\}$$

Relations

Relation is the subset of the cartesian product of the sets.

n-ary Relation

- Let $\{A_1, A_2, \ldots, A_n\}$ be n sets.
- An n-ary relation R on $A_1 \times A_2 \times \ldots \times A_n$ is a subset of $A_1 \times A_2 \times \ldots \times A_n$.
- If $A_i = \mathrm{A}\,; \forall i$, then R is called the *n-ary relation on A*.

Empty and Universal Relation

- If $R = \emptyset$, then R is called the ***empty*** or ***void relation***.

- If $R = A_1 \times A_2 \times \ldots \times A_n$, then R is called the ***universal relation***.

Definition (Binary Relation)

- Given two sets A and B, a relation between A and B is a subset of $A \times B$.
- If R is a relation on $A \times B$ (i.e., $R \subseteq A \times B$) and $(a, b) \in R$, we say "a is related to b".
- It can also be written as aRb.

Example:

Let $\boldsymbol{A} = \{\boldsymbol{a}, \boldsymbol{b}\}$ and $\boldsymbol{B} = \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$

$\boldsymbol{R} = \{(\boldsymbol{a}, \mathbf{3}), (\boldsymbol{b}, \mathbf{2}), (\boldsymbol{b}, \mathbf{4})\}$ is a relation from A to B.

Binary Relation on a set

- A binary relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is a **subset of $\boldsymbol{A} \times \boldsymbol{A}$**.

Examples:

1. "Taller -than " is a relation on people.

   $(a, b) \in$ "Taller -than" if person a is taller than person b.

2. "$\geq$" is a relation on real set **R**.

   "$\geq$" $= \{(x, y) \in \mathbf{R} \mid x, y \in \mathbf{R}, x \geq y\}$

Examples (Cont..)

3. Let $\boldsymbol{A = \{1, 2, 3, 4, 5, 6\}}$.

If $R = \{(a, b) |\ a\ divides\ b)\}$ is a relation from A to B then ordered pairs in the relation R are

{(1,1,), (1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)}

Examples (Cont..)

Let A={1, 2, 3}

$$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

- Here, $A \times A$ is an universal relation on A.
- ∅ is an empty relation on A.

Examples (Cont..)

"=" $= \Delta = \{(1,1),(2,2),(3,3)\}$

"<" $= \{(1,2),(1,3),(2,3)\}$

">" $= \{(2,1),(3,1),(3,2)\}$

"≤" $= \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

"≥" $= \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

"|" $= \{(1,1),(1,2),(1,3),(2,2),(3,3)\}$

"multiple of" $= \{(1,1),(2,1),(2,2),(3,1),(3,3)\}$

Representing Relations

Relations can be represented in two ways:

- Matrix
- Graph

Representation of Relations as Matrix

- If R is a relation on set $A = \{a_1, a_2, \dots, a_n\}$ and $|A| = n$, then it can be represented as $n \times n$ Boolean Matrix M_R.

M_R can be defined as:

$$M_R = [m_{ij}]_{n\times n}$$

where, $m_{ij} = \begin{cases} 0 & ; if\ (a_i, a_j) \notin R \\ 1 & ; if\ (a_i, a_j) \in R \end{cases}$

Examples

- Let A={1, 2, 3}
- Let $R = \{(1,1),(1,2),(2,1),(2,3),(3,2),(3,3)\}$ be a relation on A.

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Examples (Cont..)

- Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

- Because R consists of those ordered pairs with $a_{ij} = 1$, it follows that:

$$\boldsymbol{R} = \{(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{3}),(\mathbf{2},\mathbf{4}),(\mathbf{2},\mathbf{5}),(\mathbf{3},\mathbf{1}),(\mathbf{3},\mathbf{3}),(\mathbf{3},\mathbf{5})\}.$$

Representation of Relations as a Digraph (Directed Graph)

- The graph of a relation R over A is a directed graph with nodes corresponding to the elements of A. There is an edge from node x to y if and only if $(x, y) \in R$.
- An edge of the form (x, x) is called a self-loop.

Examples

- Let A={1, 2, 3}
- Let $R_1 = \{(1,2), (1,3), (2,3)\}$ be a < relation on A.

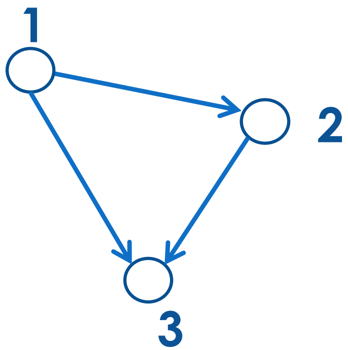


Figure 1

Examples (Cont..)

- Let $A = \{1,2,3\}$ and $B = \{a, b\}$
- Let $R_2 = \{(1, a), (1, b), (2, a), (3, b)\}$ be a relation from A to B.

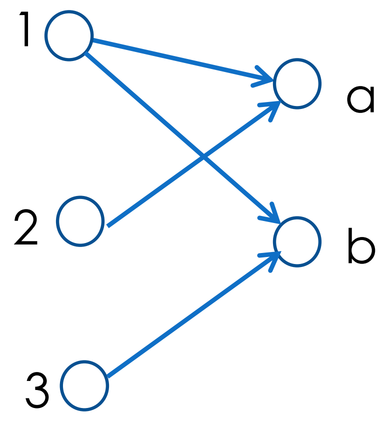


Figure 2

Domain and Range

Domain of Relation R= set of all first co-ordinates

Range of Relation R= set of all second co-ordinates

Example

"<"= $\{(1,2),(1,3),(2,3)\}$ on $A = \{1,2,3\}$

- Domain of "<"= $\{1,2\}$
- Range of "<"= $\{2,3\}$

Equality of Two relations

- Let R_1 be an n-ary relation on $A_1 \times A_2 \times \ldots \times A_n$.
- Let R_2 be an m-ary relation on $B_1 \times B_2 \times \cdots \times B_m$.

- Then, $\boldsymbol{R_1 = R_2}$

  If and only if

- **n=m**
- $\boldsymbol{A_i = B_i};\ \forall i,\ 1 \le i \le n$
- and, $\boldsymbol{R_1}$ **&** $\boldsymbol{R_2}$ **are equal set of ordered pairs.**

Example

- Let $A = \{a, b\}, B = \{1, 2\}, C = \{1, 2, 3\}$

- Let $R_1 = \{(a, 1), (b, 2)\}$ is a relation on $A \times B$
- Let $R_2 = \{(a, 1), (b, 2)\}$ is a relation on $A \times C$

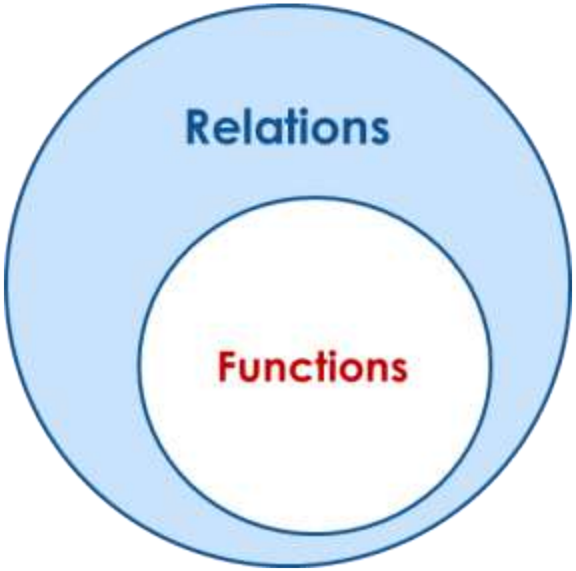
$$R_1 = R_2?$$

No

How many number of relations are there on a set A having n elements?

$$2^{n^2}$$


Thank you !!!
...



Relations

Contents

Properties of Relations

- Reflexive
- Symmetric
- Transitive
- Irreflexive
- Asymmetric
- Antisymmetric

Reflexive Relations

- R is ***reflexive*** iff $(x, x) \in R$ for every element $x \in A$.

Examples

1. Let A={1, 2, 3}

   Suppose $R_1 = \{(1,1), (2,2), (2,3)\}$ be a relation on A.

   Is R_1 reflexive?

   No

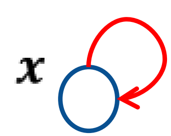
2. $=, A \times A, \leq, \geq, |, multiple\ of$ Reflexive? Yes
3. $\emptyset, <, >$ Reflexive? No

Reflexive Relation in Matrix and Graph

- If R is a **reflexive** relation, all the elements on the main diagonal of M_R are equal to 1.

$$M_R = \begin{bmatrix} 1 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & 1 \end{bmatrix}$$

- A loop must be present at all vertices in the graph.



Symmetric Relations

- R is ***symmetric*** iff $(y, x) \in R$ whenever $(x, y) \in R$ for all $x, y \in A$.

Examples

1. Let A={1, 2, 3}

   Suppose R_2= {(1,2), (2,1), (2,3)} be a relation on A.

   Is R_1 Symmetric?

   No

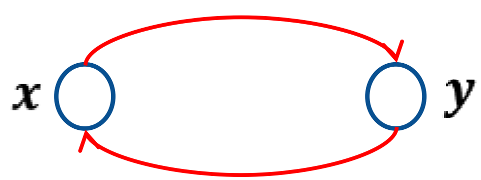
2. "sibling-of" is symmetric, but "sister-of" is not.
3. $A \times A, \emptyset, =$ Symmetric? Yes
4. $<, >, \leq, \geq, |,$ *multiple of* Symmetric? No

Symmetric Relation in Matrix and Graph

- R is a symmetric relation if and only if $m_{ji} = 1$, whenever $m_{ij} = 1$.

$$M_R = \begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix}$$

- If (x, y) is an edge in the graph, then there must be an edge (y, x) also.



Transitive Relations

- A relation R on a set A is called **transitive** if whenever $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$, for all $x, y, z \in A$.

Examples

1. Let A={1, 2, 3}

   Suppose $R_3 = \{(1,3), (3,1)\}$ be a relation on A.

   Is R_3 Transitive?

   No

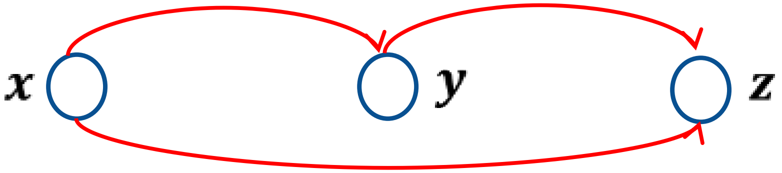
2. $A \times A,\ \emptyset, =, <, >, \leq,\ \geq,\ |,\ multiple\ of$

   Transitive?

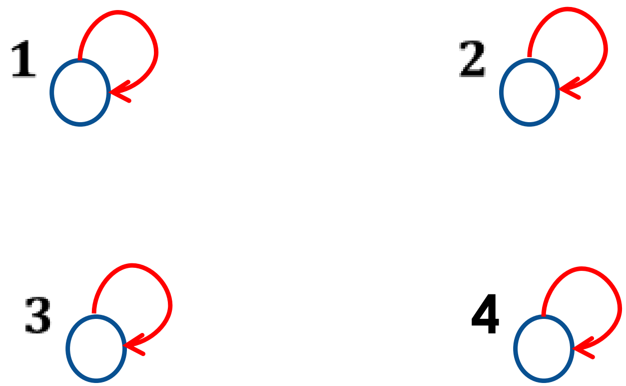
   Yes

Transitive Relations in Graph

- R is transitive iff in its graph, for any three nodes x, y and z such that there is an edge (x, y) and (y, z), there exists an edge (x, z).



Examples

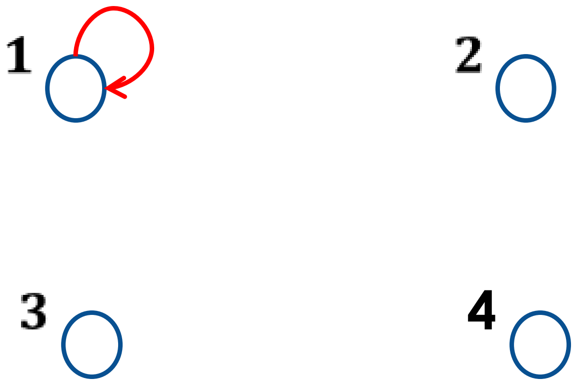


- Reflexive?
  Yes
- Symmetric?
  Yes
- Transitive?
  Yes

Equality Relation on A= {1, 2, 3, 4}

$$M_R = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Examples (Cont..)



- Reflexive?
  No
- Symmetric?
  Yes
- Transitive?
  Yes

Examples (Cont..)

- Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- Reflexive?

  Yes

- Symmetric?

  Yes

How many number of Reflexive Relations are there on set A having n elements?

$$2^{n(n-1)}$$

Relations

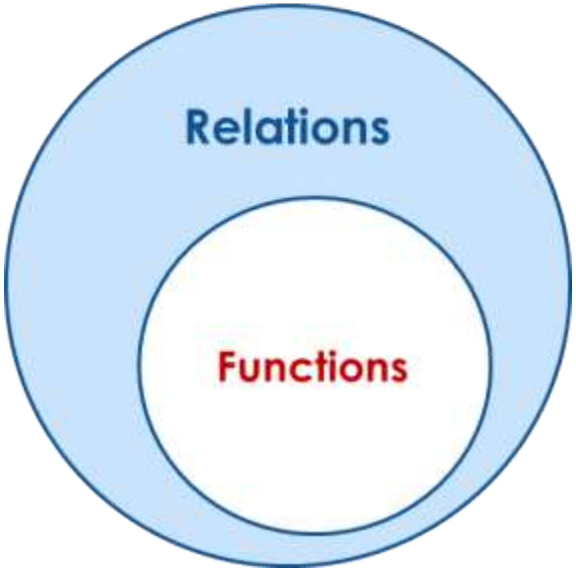
How many number of Symmetric Relations are there on set A having n elements?

$$2^{n(n+1)/2}$$

How many number of Transitive Relations are there on set A having n elements?

No closed form found


Thank you!!!
...



Relations

Properties of Relations

- **Reflexive**
- **Symmetric**
- **Transitive**
- Irreflexive
- Asymmetric
- Antisymmetric

Irreflexive Relations

- *R* is **irreflexive** iff (x, x) ∉ *R* for every element x ∈ A.
- No Reflexive ordered pair should belong to the relation.

Examples

1. Let A={1, 2, 3}

   Suppose R_1= {(1,1), (2,2), (2,3)} be a relation on A.

   Is R_1 Irreflexive?

   No

2. ∅, <, > Irreflexive? Yes
3. $\Delta, A \times A, \leq, \geq, |, \textit{multiple of}$ Irreflexive? No

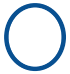
Irreflexive Relation in Matrix and Graph

- If R is an irreflexive relation, all the elements on the main diagonal of M_R are equal to 0.

$$M_R = \begin{bmatrix} 0 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & 0 \end{bmatrix}$$

- No vertex should contain self-loop in the graph.

x



Asymmetric Relations

- A relation R on a set A such that for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \notin R$, is called ***asymmetric*.**

Examples

Let A={1, 2, 3}

1. Suppose $R_2 = \{(1,2)\}$ be a relation on A.

Is R_2 Asymmetric?

Yes

2. Suppose $R_3 = \{(1, 3), (3,1), (2,3)\}$ be another relation on A.

Is R_3 Asymmetric?

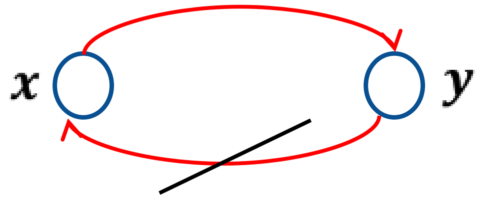
No

3. $\emptyset, <, >$ Asymmetric? Yes

4. $A \times A, \leq, \geq, |, \textit{multiple of}$ Asymmetric? No

Asymmetric Relations in Graph

- If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.
- There must also be no self loop.



Antisymmetric Relations

- A relation R on a set A such that for all $x, y \in A$, if $(x, y) \in R$ and if $(y, x) \in R$, then $x = y$, is called ***antisymmetric.***

If $x \neq y$ and if (x, y) is present, then (y, x) should not be present there.

Examples

1. Let A={1, 2, 3}

   Suppose $R_1 = \{(1,2), (2,1), (2,3)\}$ be a relation on A.

   Is R_1 Antisymmetric?

   No

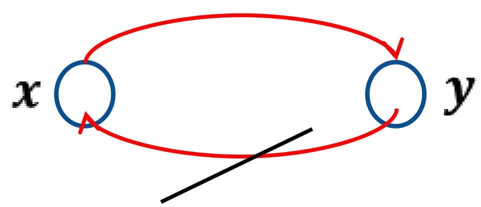
2. $\emptyset, \Delta, <, >, \leq, \geq, |, \textit{multiple of}$ Antisymmetric? Yes
3. $A \times A$ Antisymmetric? No

Antisymmetric Relation in Matrix and Graph

- R is a antisymmetric relation if and only if $m_{ji} = 0$, or $m_{ij} = 0$, when $i \neq j$.

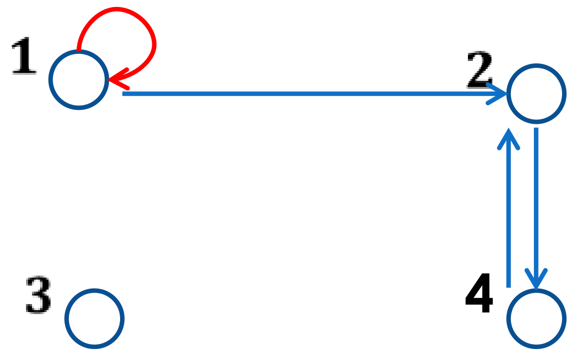
$$M_R = \begin{bmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{bmatrix}$$

- If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.



Self-loops can be there.

Example



- **Reflexive?**
  No
- **Symmetric?**
  No
- **Transitive?**
  No
- **Irreflexive?**
  No
- **Antisymmetric?**
  No
- **Asymmetric ?**
  No

Some Points to remember

- There can be a relation which is neither reflexive nor irreflexive.

Example

1. Let A={1, 2, 3}

   Suppose R_3= {(1,1), (2,2), (2,3)} be a relation on A.

Neither Reflexive nor Irreflexive

Some Points to remember (Cont..)

- There can be a relation which is both symmetric and antisymmetric.

Example:

1. Let A = {1, 2, 3}

   Suppose R_4= {(1,1), (2,2), (3,3)} be a relation on A.

   both symmetric and antisymmetric

Some Points to remember (Cont..)

- There can be a relation which is neither symmetric nor antisymmetric.

Example

Let A={1, 2, 3}

Suppose R_5= {(1,2), (2,3), (3,2)} be a relation on A.

Neither Symmetric nor Antisymmetric

Some Points to remember (Cont..)

- Every asymmetric relation is antisymmetric but every antisymmetric relation need not be asymmetric.

Example:

Let A = {1, 2, 3}

1. Suppose R_6= {(1,2)} be a relation on A.

   both asymmetric and antisymmetric

2. Suppose R_7= {(1,1), (1,2)} be a relation on A.

   Antisymmetric but not asymmetric

Relations

How many number of Irreflexive Relations are there on set A having n elements?

$$2^{n(n-1)}$$

Relations

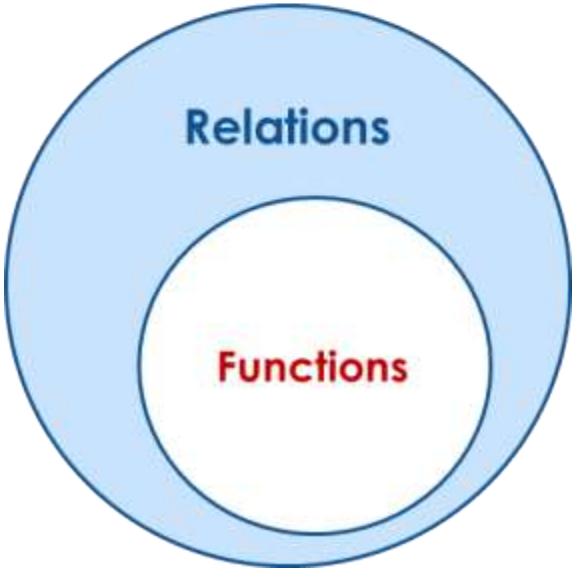
How many number of Asymmetric Relations are there on set A having n elements?

$$3^{n(n-1)/2}$$

How many number of Antisymmetric Relations are there on set A having n elements?

$$2^n 3^{n(n-1)/2}$$


Thank you !!!



Relations

Contents

Inverse Relation

- If $R \subseteq \mathrm{A} \times B$ then $R^{-1} \subseteq \mathrm{B} \times A$, and is defined as:

$$R^{-1} = \{(\mathrm{b}, \mathrm{a}) | (\mathrm{a}, \mathrm{b}) \in R\}$$

R	R^{-1}
$<$	$>$
$\leq$	$\geq$
divides	*multiple of*
subset	*superset*

Example

- Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$
  Suppose $R = \{(1,3), (1,5), (2,4), (3,5)\}$

$$R^{-1} = \{(3,1), (5,1), (4,2), (5,3)\}$$

Complementary Relations

- Let R be a relation from A to B, then complementary relation R^C is defined as:

$$R^C = \{(a,b) | (a,b) \notin R \; and \; (a,b) \in A \times B\}$$

Example

- Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

  Suppose $R = \{(1,3), (1,5), (2,4), (3,5)\}$

$$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5))\}$$

$$R^C = \{(1,4), (2,3), (3,3), (3,4), (2,5)\}$$

Combining Relation

- Given two relations R_1 and R_2, these can be combined by using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

- $R_1 \cup R_2 = \{(a,b) |\ (a,b) \in R_1 or\ (a,b) \in R_2\ or\ both\}$
- $R_1 \cap R_2 = \{(a,b) |\ (a,b) \in R_1\ and\ (a,b) \in R_2\}$
- $R_1 - R_2 = \{(a,b) |\ (a,b) \in R_1\ and\ (a,b) \notin R_2\}$
- $R_2 - R_1 = \{(a,b) |\ (a,b) \in R_2\ and\ (a,b) \notin R_1\}$

Example

- Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$
  Suppose $R = \{(1,3), (1,5), (2,4), (3,5)\}$,
  $R_1 = \{(1,4), (2,3), (2,5), (3,3), (3,5)\}$ and
  $R_2 = \{(1,3), (1,4), (2,3), (3,4), (3,5)\}$

- $R_1 \cup R_2 = \{(1,3), (1,4), (2,3), (2,5), (3,3), (3,4), (3,5)\}$
- $R_1 \cap R_2 = \{(1,4), (2,3), (3,5)\}$
- $R_1 - R_2 = \{(2,5), (3,3)\}$
- $R_2 - R_1 = \{(1,3), (3,4)\}$

Results

- Let R, R_1 and R_2 be relations on A.

R, R_1 and R_2 are	R^{-1}	$R_1 \cap R_2$	$R_1 \cup R_2$
Reflexive	Yes	Yes	Yes
Irreflexive	Yes	Yes	Yes
Symmetric	Yes	Yes	Yes
Asymmetric	Yes	Yes	Need not be, but cannot be assured
Antisymmetric	Yes	Yes	Need not be, but cannot be assured
Transitive	Yes	Yes	Need not be, but cannot be assured

Examples

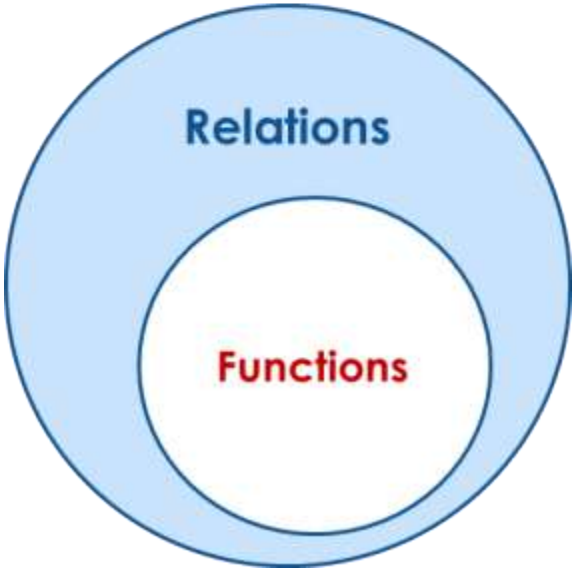
$A = \{1,2,3\}$

$R_1 = \{(1,2)\}$

$R_2 = \{(2,1)\}$

$R_1 \cup R_2 = \{(1,2), (2,1)\}$


Thank you!!!
...



Relations

Contents

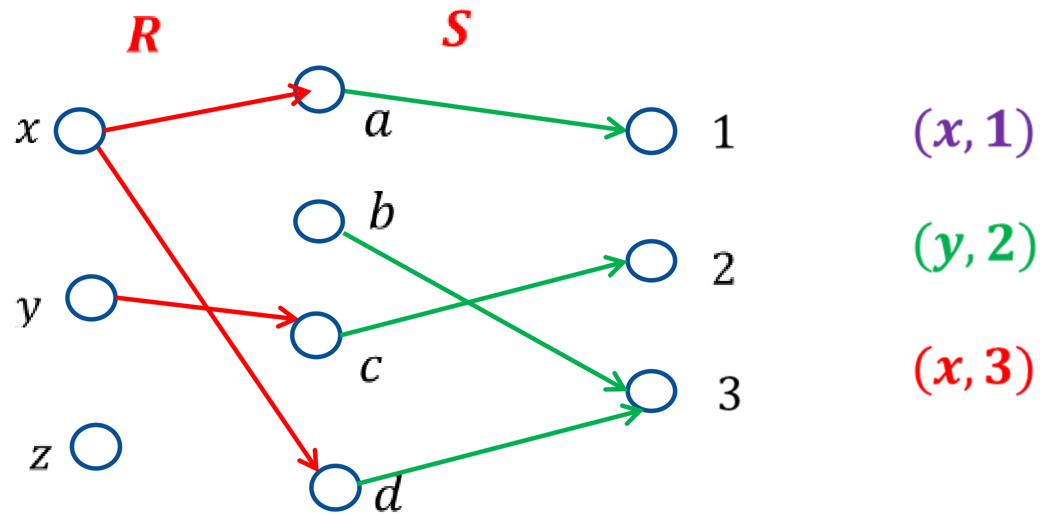
Composition of Relations

- If $R \subseteq \mathrm{A} \times B$ and $S \subseteq B \times C$ are two relations, then the composition (or composite) of S with R is a relation from A to C and is defined as:

$$SoR = \{\{(a,c)|\exists b \in B \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$$

Representing the Composition of Relations

- Let $A = \{x, y, z\}$, $B = \{a, b, c, d\}$ and $C = \{1, 2, 3\}$.
- Suppose $R = \{(x, a), (x, d), (y, c)\}$ be a relation from A to B.
- Suppose $S = \{(a, 1), (b, 3), (c, 2), (d, 3)\}$ be a relation from B to C.



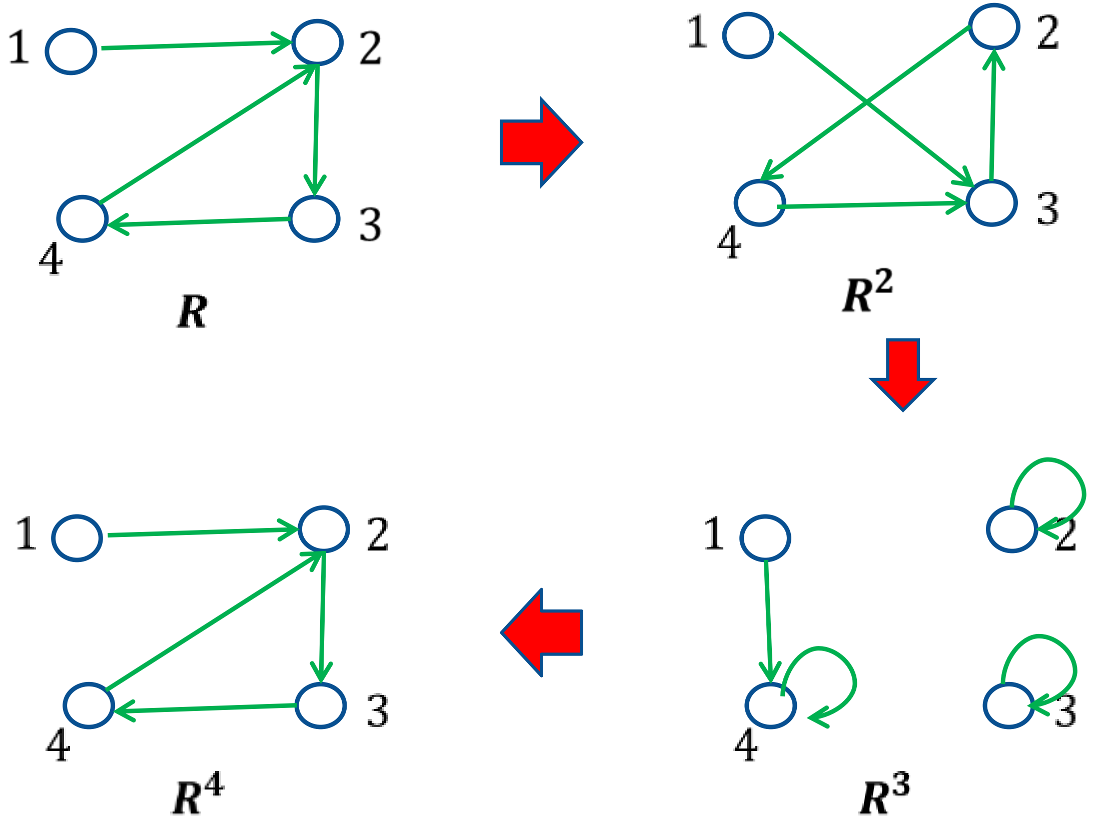
$\boldsymbol{SoR = \{(x, 1), (y, 2), (x, 3)\}}$

Power of Relations

- If $R \subseteq A \times A$, then

$$R^2 = RoR$$
$$R^3 = R^2 oR$$
$$\vdots$$
$$R^n = R^{n-1} oR$$

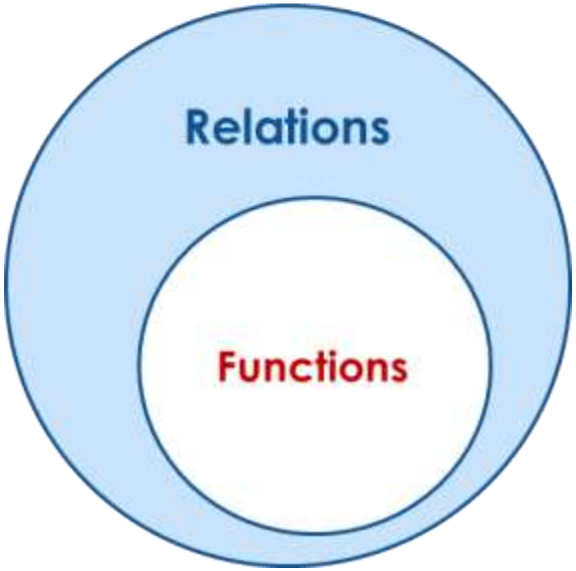
Example



R = {1, 2, 3, 4}
R ⊆ A X A

The pair (a, b) is in R^n if there is a path of length n from a to b in R.


Thank you!!! ...



Relations

Contents

Equivalence Relation

- Let R be a relation on set A, then R is called equivalence relation if it is:

- Reflexive
- Symmetric
- Transitive

Examples

- Let $A = \{1, 2, 3\}$

1. $\emptyset$ $\boldsymbol{i.e.}$ **Empty Relation on $\boldsymbol{A}$**

Reflexive?

Symmetric?

Transitive?

Not an Equivalence Relation

2. $\Delta = \{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3})\}$

Reflexive?

Symmetric?

Transitive?

Equivalence Relation on $\boldsymbol{A}$

Smallest Equivalence Relation on $\boldsymbol{A}$

Examples (Cont..)

- Let $A = \{1, 2, 3\}$

3. Universal Relation on A i.e. $A \times A$

Reflexive?

Symmetric?

Transitive?

Equivalence Relation

Largest Equivalence Relation on A

4. Let $A = \{1, 2, 3, 4\}$

$$\boldsymbol{R} = \{(\mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{2}), (\mathbf{2}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3}), (\mathbf{3}, \mathbf{4}), (\mathbf{4}, \mathbf{3}), (\mathbf{4}, \mathbf{4})\}$$

Reflexive?

Symmetric?

Transitive?

Equivalence Relation on A

Relations

If R_1 *and* R_2 are two equivalence relations on A, then which of the following is always true?

I. $R_1 \cap R_2$ is an Equivalence Relation.

II. $R_1 \cup R_2$ is an Equivalence Relation.

(a) Only I

(b) Only II

(c) Both are true

(d) Both are false

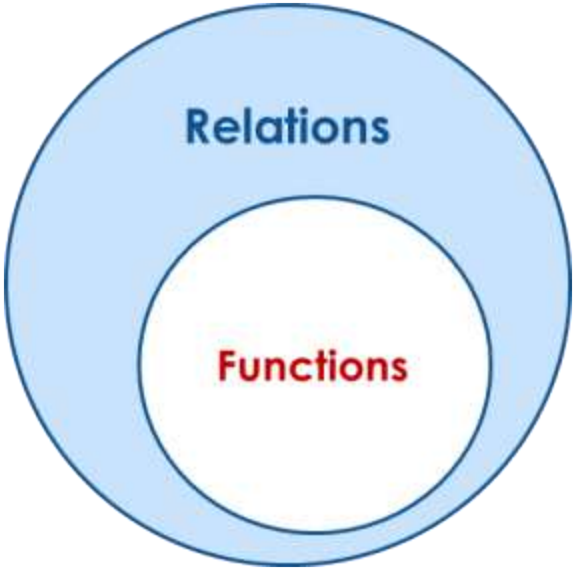
Exercise

- Let R be a relation defined on *set of integers* as:

$$xRy \text{ iff } x + y \text{ is even}$$

Is R an equivalence relation?


Thank you!!!
...



Relations

Contents

Equivalence Class

- Let R be an equivalence relation on A, and $a \in A$.
- The equivalence class of a, denoted as $[a]$ or $\overline{a}$, is defined as:

$$\overline{a} = [a] = \{b \in A | (a, b) \in R\}$$

Examples

- Let $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$ on $A = \{1, 2, 3, 4\}$

First Check whether R is an equivalence relation on A or not.

Reflexive?

Symmetric?

Transitive?

Equivalence Classes:-

$[1] = \{1, 2\}$
$[2] = \{1, 2\}$
$[3] = \{3, 4\}$
$[4] = \{3, 4\}$

Examples

- Let $R = \{(1,1), (2,2), (3,3), (4,4)\}$ on $A = \{1, 2, 3, 4\}$

Equivalence Classes:-

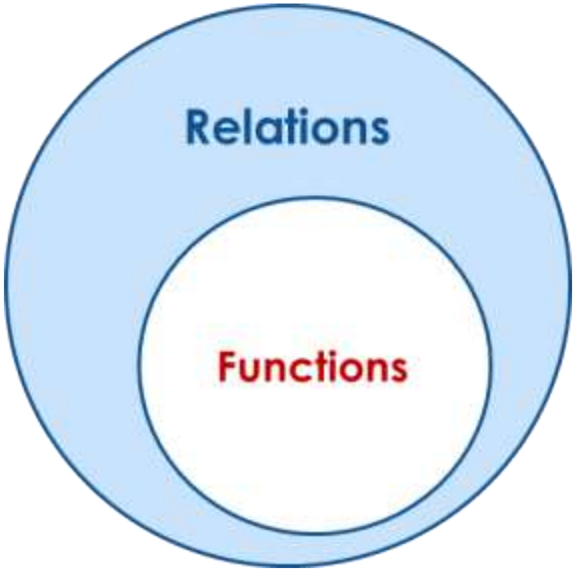
$[1] = \{1\}$
$[2] = \{2\}$
$[3] = \{3\}$
$[4] = \{4\}$

Properties

- Let R be an equivalence relation on A.

1. $a \in [a]$
2. If $b \in [a]$ then $a \in [b]$
3. If $b \in [a]$ then $[a] = [b]$
4. $[a] = [b]$ or $[a] \cap [b] = \emptyset$


Thank you !!!



Relations

Contents

Equivalence Relation to Partition

- Let $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$ on $A = \{1, 2, 3, 4\}$.
- R is an equivalence relation on A.

Equivalence Classes:-

$[1] = \{1, 2\} = [2]$

$[3] = \{3, 4\} = [4]$

- Partition $P = \{\{1, 2\}, \{3, 4\}\}$

Partition to Equivalence Relation

- Let $A = \{1, 2, 3, 4\}$ be a set and $P = \{\{1, 3\}, \{2, 4\}\}$ be a partition on A.
- Find Equivalence relation on A.

The parts of partition are distinct equivalence classes.

$\{1, 3\} \rightarrow \{(1, 1), (1,3), (3,1), (3,3)\}$
$\{2, 4\} \rightarrow \{(2, 2), (2,4), (4,2), (4,4)\}$

Therefore, the equivalence relation on A is:

- $\boldsymbol{R = \{(1, 1), (1, 3), (3, 1), (3, 3), (2, 2), (2, 4), (4, 2), (4, 4)\}}$

Result

- There is a one-to-one correspondence between partitions of A and Equivalence Relation on A.

- Therefore, if $|A| = n$, then

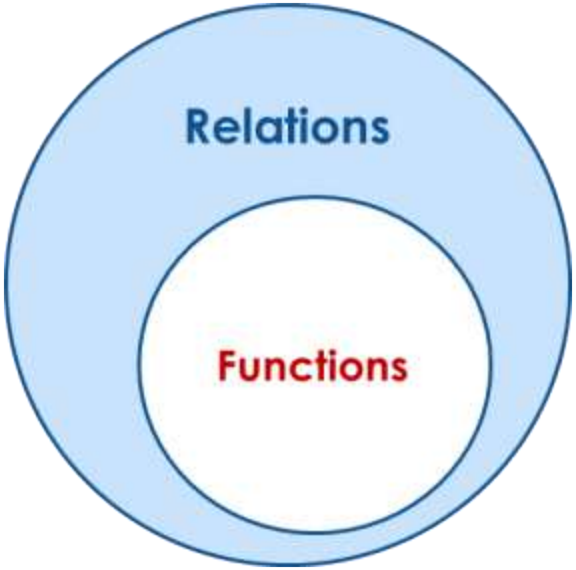
Number of Partitions of A= Number of Equivalence Relations on A
$= B_n$ (Bell Number)

Bell Number:

$$B_n = \sum_{k=0}^{n-1} n - 1_{C_k} B_k$$

where, $B_0 = 1$


Thank you!!!
...



Relations

Contents

Closure of Relations

- Reflexive Closure
- Symmetric Closure
- Transitive Closure

Reflexive Closure

- A relation is called **reflexive closure R_r** of relation R if:

1) It is reflexive.
2) It contains R.
3) It is the minimal relation satisfying conditions (1) and (2).

Examples

1. Let A={1, 2, 3}

   Suppose $R_1 = \{(1,1), (2,2), (2,3)\}$ be a relation on A.

$$R_r = \{(1,1), (2,2), (2,3), (3,3)\}$$

Examples (Cont..)

2. R is a relation defined on set of positive integers such that aRb if $a < b$.

Reflexive Closure?

Result:

- $\boldsymbol{R_r = R \cup \Delta}$
- $\boldsymbol{R_r = R}$ **iff $\boldsymbol{R}$ is Reflexive.**

Symmetric Closure

- A relation is called **symmetric closure R_s** of relation R if:

1) It is symmetric.
2) It contains R.
3) It is the minimal relation satisfying conditions (1) and (2).

Examples

- Let A={1, 2, 3}

  Suppose $R_1 = \{(1,1), (2,2), (2,3)\}$ be a relation on A.

$$R_s = \{((1,1), (2,2), (2,3), (3,2))\}$$

Result

- $R_s = R \cup R^{-1}$
- $R_r = R$ iff R is Symmetric.

Transitive Closure

- A relation is called Transitive closure R^* of relation R if:

1) It is transitive.
2) It contains R.
3) It is the minimal relation satisfying conditions (1) and (2).

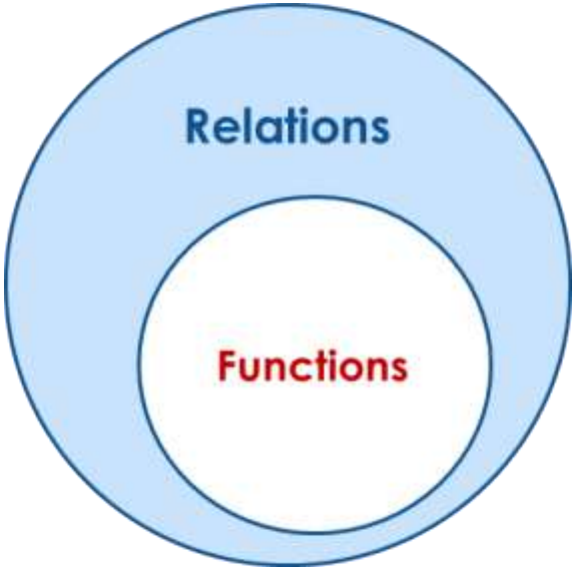
Result

1. Let $|A| = n$,

   $then, R^* = R^1 \cup R^2 \cup \cdots \cup R^n$

2. $R\ is\ transitive\ iff\ R^* = R$


Thank you!!!



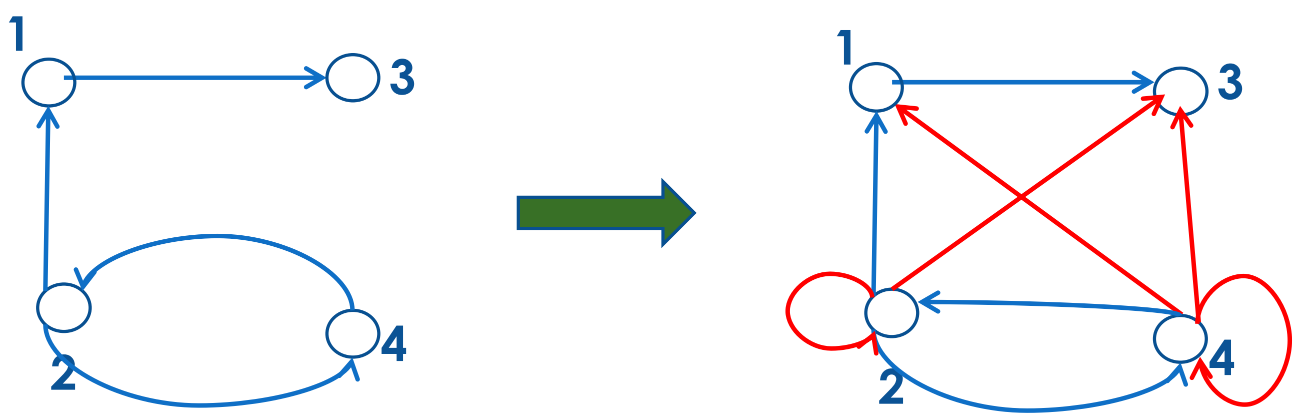
Relations

Contents

Warshall's Algorithm

- Computes the transitive closure of a relation

Example of transitive closure:



Warshall's Algorithm (Cont..)

- Main concept: a path exists between two vertices i, j, iff
  - there is an edge from i to j; or
  - there is a path from i to j going through vertex 1; or
  - there is a path from i to j going through vertex 1 and/or 2; or
  - there is a path from i to j going through vertex 1, 2, and/or 3; or
  - ...
  - there is a path from i to j going through any of the other vertices

Warshall's Algorithm (Cont..)

- On the k^{th} iteration, the algorithm determine if a path exists between two vertices i, j using vertices among $1, \ldots, k$ allowed as intermediate

$$\boldsymbol{W^{(k)}[i,j]} = \begin{cases} \boldsymbol{W^{(k-1)}[i,j]} \\ \boldsymbol{or} \\ \left(\boldsymbol{W^{(k-1)}[i,k]}\right) \boldsymbol{and} \left(\boldsymbol{W^{(k-1)}[k,j]}\right) \end{cases}$$

Warshall's Algorithm (Cont..)

- Recurrence relating elements $W^{(k)}$ to elements of $W^{(k-1)}$ is:

  $$\boldsymbol{W^{(k)}[i,j] = W^{(k-1)}[i,j]} \text{ or } \left(\boldsymbol{W^{(k-1)}[i,k] \; and \; W^{(k-1)}[k,j]}\right)$$

- It implies the following rules for generating $W^{(k)}$ from $W^{(k-1)}$ is:

1. If an element in row i and column j is 1 in $W^{(k-1)}$, it remains 1 in $W^{(k)}$.

2. If an element in row i and column j is 0 in $W^{(k-1)}$, it has to be changed to 1 in $W^{(k)}$ if and only if the element in its row i and column k and the element in its row k and column j are both $1's$ in $W^{(k-1)}$.

Warshall's Algorithm (Cont..)

- The procedure for computing $W^{(k)}$ from $W^{(k-1)}$ is as follows:

1. First transfer all $1's$ in $W^{(k-1)}$ to $W^{(k)}$.

2. List the locations $p_1, p_2, \ldots$, in column k of $W^{(k-1)}$, where the entry is 1, and the locations $q_1, q_2, \ldots$, in row k of $W^{(k-1)}$, where the entry is 1.

3. Put $1's$ in all the positions (p_i, q_i) of $W^{(k)}$ (if they are not already there).

Warshall's Algorithm (Cont..)

$$W^{(k-1)} = \begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} i \\ k \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & \end{bmatrix} \end{array} \Rightarrow W^{(k)} = \begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} i \\ k \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix} \end{array}$$

Figure 1: Step for Changing zeros in Warshall's Algorithm

Example

1. Find transitive closure of relation R represented by following matrix (using Warshall's algorithm):

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Solution 1:

$$W^{(0)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$W^{(1)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$W^{(2)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$W^{(3)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$W^{(4)} = W^{(3)} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Answer is :

$$M_{R^*} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$R^* = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)\}$$


Thank you!!! ...