In[ ]:=
```
(* requires EcoEvo package <https://
 github.com/cklausme/EcoEvo> -- run this once (ever) to install *)
PacletInstall[
 "https://github.com/cklausme/EcoEvo/raw/master/Paclets/EcoEvo-1.7.2.paclet"]
```

Out[ ]=

PacletObject[ Name: EcoEvo
Version: 1.7.2]

Set model

In[ ]:=
```
<< EcoEvo`
```

Out[ ]=
```
EcoEvo Package Version 1.7.2 (September 1, 2023)
Christopher A. Klausmeier <christopher.klausmeier@gmail.com>
```

In[ ]:=
```
SetModel[{
   Pop[n1] → {Equation :→ (r1 - α11 n1 - α12 n2) n1},
   Pop[n2] → {Equation :→ (r2 - α21 n1 - α22 n2) n2},
   Parameters :→ {r1 > 0, r2 > 0, α11 > 0, α12 > 0, α21 > 0, α22 > 0}
  }];
```

Analysis

Solve for equilibria:

In[ ]:=
```
eq = SolveEcoEq[]
```

Out[ ]=

$$\left\{\{n1 \to 0,\ n2 \to 0\},\ \left\{n1 \to \frac{r1}{\alpha 11},\ n2 \to 0\right\},\ \left\{n1 \to 0,\ n2 \to \frac{r2}{\alpha 22}\right\},\ \left\{n1 \to -\frac{-r2\,\alpha 12 + r1\,\alpha 22}{\alpha 12\,\alpha 21 - \alpha 11\,\alpha 22},\ n2 \to -\frac{r2\,\alpha 11 - r1\,\alpha 21}{\alpha 12\,\alpha 21 - \alpha 11\,\alpha 22}\right\}\right\}$$

Jacobian matrix:

In[ ]:=
```
EcoJacobian[] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} r1 - 2\,n1\,\alpha 11 - n2\,\alpha 12 & -n1\,\alpha 12 \\ -n2\,\alpha 21 & r2 - n1\,\alpha 21 - 2\,n2\,\alpha 22 \end{pmatrix}$$

Calculate eigenvalues at each equilibrium:

In[•]:=
```
EcoEigenvalues[eq[[1]]]
```
Out[•]=
$$\{r1, r2\}$$

In[•]:=
```
Simplify[EcoEigenvalues[eq[[2]]]]
```
Out[•]=
$$\left\{-r1, r2 - \frac{r1\,\alpha21}{\alpha11}\right\}$$

In[•]:=
```
Simplify[EcoEigenvalues[eq[[3]]]]
```
Out[•]=
$$\left\{-r2, r1 - \frac{r2\,\alpha12}{\alpha22}\right\}$$

In[•]:=
```
Simplify[EcoEigenvalues[eq[[4]]]]
```
Out[•]=
$$\left\{-\frac{1}{2\,\alpha12\,\alpha21 - 2\,\alpha11\,\alpha22}\right.$$
$$\left(r2\,\alpha11\,\alpha12 - r1\,\alpha11\,\alpha22 - r2\,\alpha11\,\alpha22 + r1\,\alpha21\,\alpha22 + \sqrt{\left(4\,(r2\,\alpha11 - r1\,\alpha21)\,(r2\,\alpha12 - r1\,\alpha22)\right.\right.}$$
$$\left.\left.(-\alpha12\,\alpha21 + \alpha11\,\alpha22) + (r2\,\alpha11\,(\alpha12 - \alpha22) + r1\,(-\alpha11 + \alpha21)\,\alpha22)^2\right)\right),$$
$$\frac{1}{2\,\alpha12\,\alpha21 - 2\,\alpha11\,\alpha22}\left(-r2\,\alpha11\,\alpha12 + r1\,\alpha11\,\alpha22 + r2\,\alpha11\,\alpha22 - r1\,\alpha21\,\alpha22 + \right.$$
$$\sqrt{\left(4\,(r2\,\alpha11 - r1\,\alpha21)\,(r2\,\alpha12 - r1\,\alpha22)\,(-\alpha12\,\alpha21 + \alpha11\,\alpha22) + \right.}$$
$$\left.\left.\left.(r2\,\alpha11\,(\alpha12 - \alpha22) + r1\,(-\alpha11 + \alpha21)\,\alpha22)^2\right)\right)\right\}$$

Stability conditions:

In[•]:=
```
EcoStableQ[eq[[1]]]
```
Out[•]=
```
False
```

In[•]:=
```
FullSimplify[EcoStableQ[eq[[2]]]]
```
Out[•]=
$$\begin{cases} \text{True} & r1\,\alpha21 > r2\,\alpha11 \\ \text{False} & r1\,\alpha21 < r2\,\alpha11 \;||\; r1\,(\alpha11 + \alpha21) < r2\,\alpha11 \\ \text{Indeterminate} & \text{True} \end{cases}$$

In[•]:=
```
FullSimplify[EcoStableQ[eq[[3]]]]
```
Out[•]=
$$\begin{cases} \text{True} & r2\,\alpha12 > r1\,\alpha22 \\ \text{False} & r1\,\alpha22 > r2\,(\alpha12 + \alpha22) \;||\; r2\,\alpha12 < r1\,\alpha22 \\ \text{Indeterminate} & \text{True} \end{cases}$$

In[•]:=
```
FullSimplify[EcoStableQ[eq[[4]]]]
```

Out[•]=

$$\begin{cases} \text{True} & (\alpha12\,\alpha21-\alpha11\,\alpha22)\,(\text{r2}\,\alpha11\,(\alpha12-\alpha22)+\text{r1}\,(-\alpha11+\alpha21)\,\alpha22)>0\;\&\& \\ & (\text{r2}\,\alpha11-\text{r1}\,\alpha21)\,(\text{r2}\,\alpha12-\text{r1}\,\alpha22)\,(-\alpha12\,\alpha21+\alpha11\,\alpha22)<0 \\ \text{False} & (\alpha12\,\alpha21-\alpha11\,\alpha22)\,(\text{r2}\,\alpha11\,(\alpha12-\alpha22)+\text{r1}\,(-\alpha11+\alpha21)\,\alpha22)<0\;|| \\ & (\text{r2}\,\alpha11-\text{r1}\,\alpha21)\,(\text{r2}\,\alpha12-\text{r1}\,\alpha22)\,(-\alpha12\,\alpha21+\alpha11\,\alpha22)>0 \\ \text{Indeterminate} & \text{True} \end{cases}$$

Invasion rates:

In[•]:=
```
λ21 = Inv[eq[[2]], n2]
```

Out[•]=

$$\text{r2}-\frac{\text{r1}\,\alpha21}{\alpha11}$$

In[•]:=
```
λ12 = Inv[eq[[3]], n1]
```

Out[•]=

$$\text{r1}-\frac{\text{r2}\,\alpha12}{\alpha22}$$

Numerical examples

Case I - 1 outcompetes 2

In[•]:=
```
r1 = 1.2; r2 = 1;
α11 = 1; α22 = 1;
α12 = 1; α21 = 1;
```

In[•]:=
```
eq = SolveEcoEq[]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[•]=
```
{{n1 → 0, n2 → 0}, {n1 → 1.2, n2 → 0}, {n1 → 0, n2 → 1.}}
```

In[•]:=
```
EcoEigenvalues[eq]
```

Out[•]=
```
{{1.2, 1.}, {-1.2, -0.2}, {-1., 0.2}}
```

In[•]:=
```
NumberedGridForm[eq, EcoEigenvalues[eq], Header → True]
```

Out[•]=

#	eq	EcoEigenvalues[eq]
1	{n1 → 0, n2 → 0}	{1.2, 1.}
2	{n1 → 1.2, n2 → 0}	{-1.2, -0.2}
3	{n1 → 0, n2 → 1.}	{-1., 0.2}

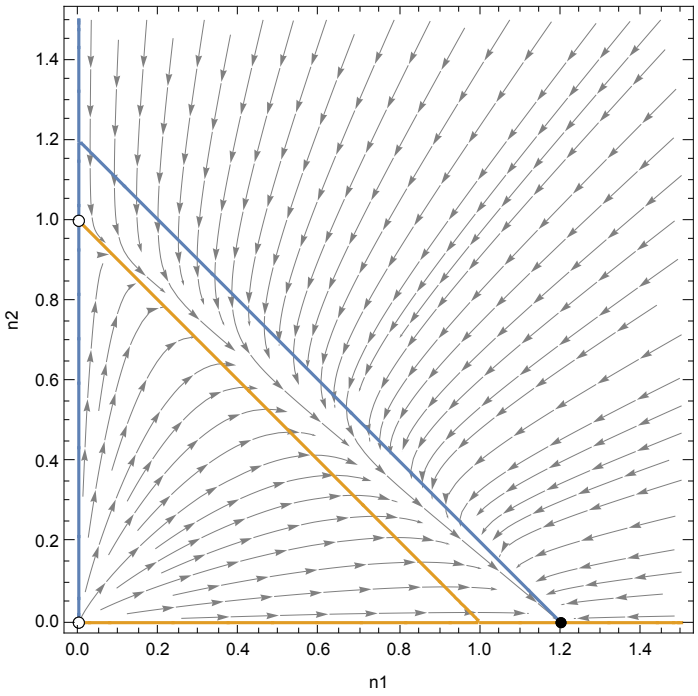
In[ ]:=
```
λ12 = Inv[eq[[3]], n1]
λ21 = Inv[eq[[2]], n2]
```

Out[ ]=
```
0.2
```

Out[ ]=
```
-0.2
```

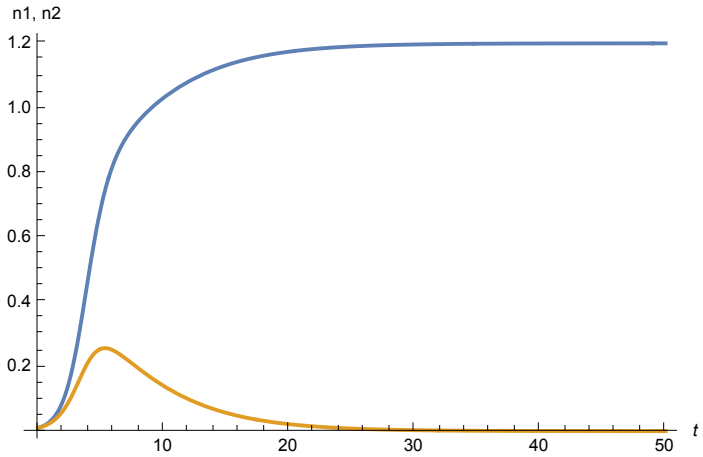
In[ ]:=
```
pp = PlotEcoPhasePlane[{n1, 0, 1.5}, {n2, 0, 1.5}]
```

Out[ ]=



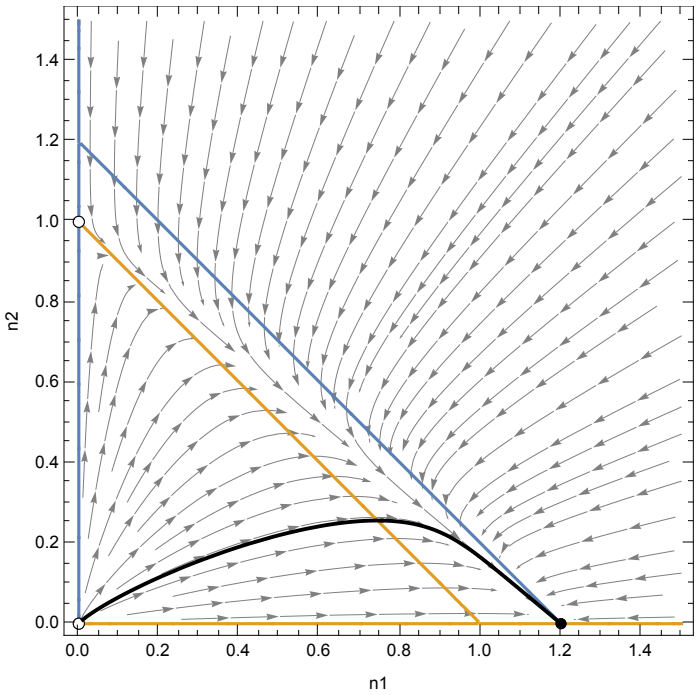
In[ ]:=
```
sol = EcoSim[{n1 → 0.01, n2 → 0.01}, 50];
PlotDynamics[sol]
```

Out[ ]=



In[•]:=
```
Show[pp, PlotTrajectory[sol, PlotStyle → Black]]
```

Out[•]=



Case III - 1 & 2 coexist

In[•]:=
```
r1 = 1; r2 = 1;
α11 = 1; α22 = 1;
α12 = 0.5; α21 = 0.5;
```

In[•]:=
```
eq = SolveEcoEq[];
NumberedGridForm[eq, EcoEigenvalues[eq], Header → True]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[•]=

#	eq	EcoEigenvalues[eq]
1	{n1 → 0, n2 → 0}	{1., 1.}
2	{n1 → 1., n2 → 0}	{-1., 0.5}
3	{n1 → 0, n2 → 1.}	{-1., 0.5}
4	{n1 → 0.666667, n2 → 0.666667}	{-1., -0.333333}

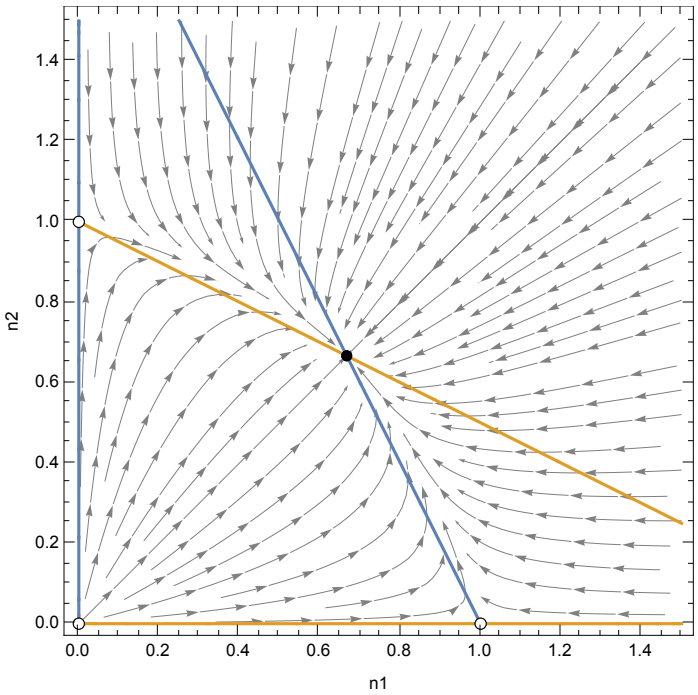
In[•]:=
```
λ12 = Inv[eq[[3]], n1]
λ21 = Inv[eq[[2]], n2]
```

Out[•]=
```
0.5
```

Out[•]=
```
0.5
```

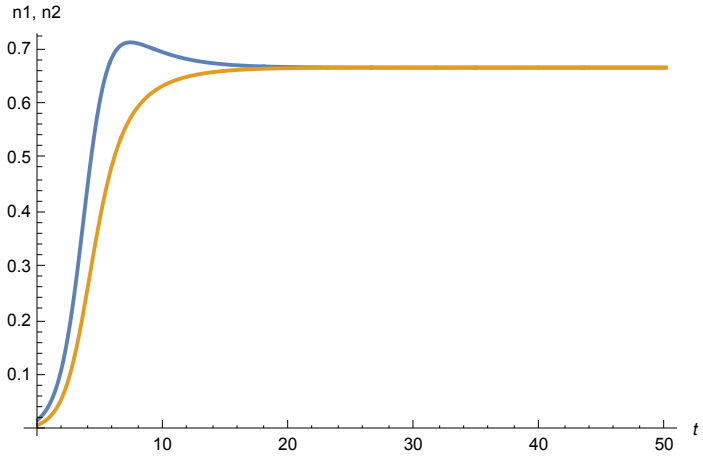
In[ ]:=
```
pp = PlotEcoPhasePlane[{n1, 0, 1.5}, {n2, 0, 1.5}]
```

Out[ ]=



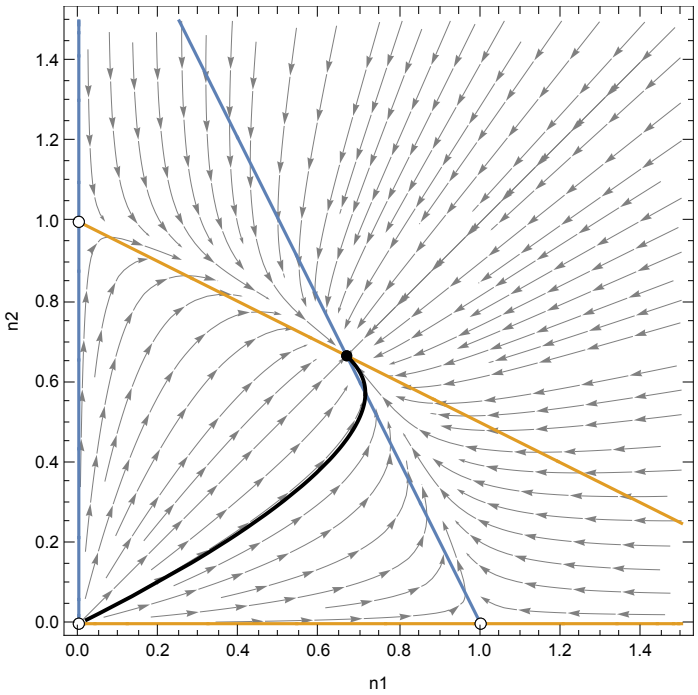
In[ ]:=
```
sol = EcoSim[{n1 → 0.02, n2 → 0.01}, 50];
PlotDynamics[sol]
```

Out[ ]=



In[•]:=
```
Show[pp, PlotTrajectory[sol, PlotStyle → Black]]
```

Out[•]=



Case IV - 1 or 2 wins (founder control)

In[•]:=
```
r1 = 1; r2 = 1;
α11 = 1; α22 = 1;
α12 = 2; α21 = 2;
```

In[•]:=
```
eq = SolveEcoEq[];
NumberedGridForm[eq, EcoEigenvalues[eq], Header → True]
```

Out[•]=

⌗	eq	EcoEigenvalues[eq]
1	{n1 → 0, n2 → 0}	{1, 1}
2	{n1 → 1, n2 → 0}	{−1, −1}
3	{n1 → 0, n2 → 1}	{−1, −1}
4	$\{n1 \to \frac{1}{3}, n2 \to \frac{1}{3}\}$	$\{-1, \frac{1}{3}\}$

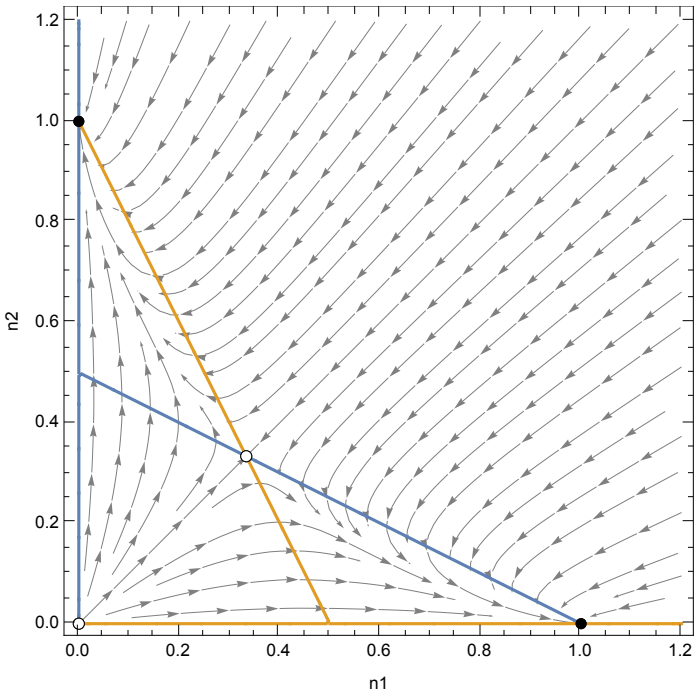
In[•]:=
```
λ12 = Inv[eq[[3]], n1]
λ21 = Inv[eq[[2]], n2]
```

Out[•]=
```
-1
```

Out[•]=
```
-1
```

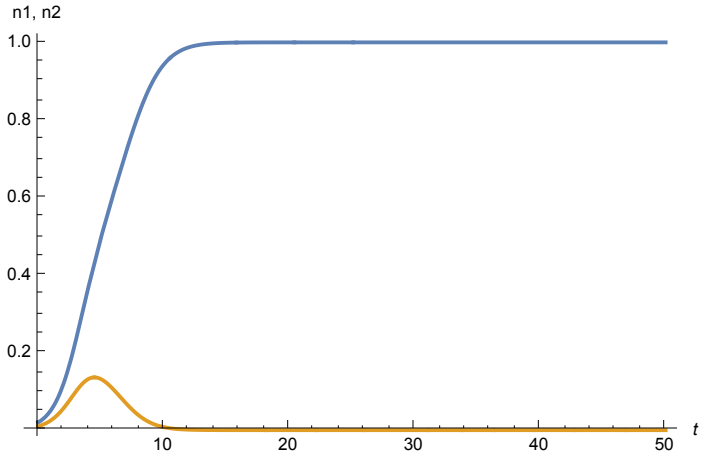
In[ ]:=
```
pp = PlotEcoPhasePlane[{n1, 0, 1.2}, {n2, 0, 1.2}]
```

Out[ ]=



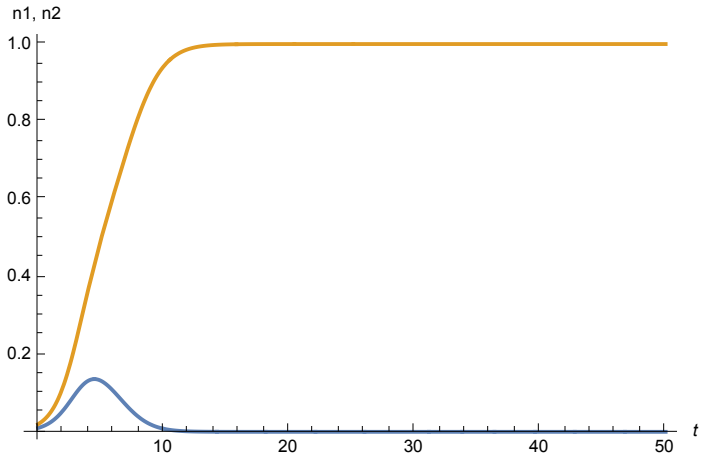
In[ ]:=
```
sol = EcoSim[{n1 → 0.02, n2 → 0.01}, 50];
PlotDynamics[sol]
```

Out[ ]=



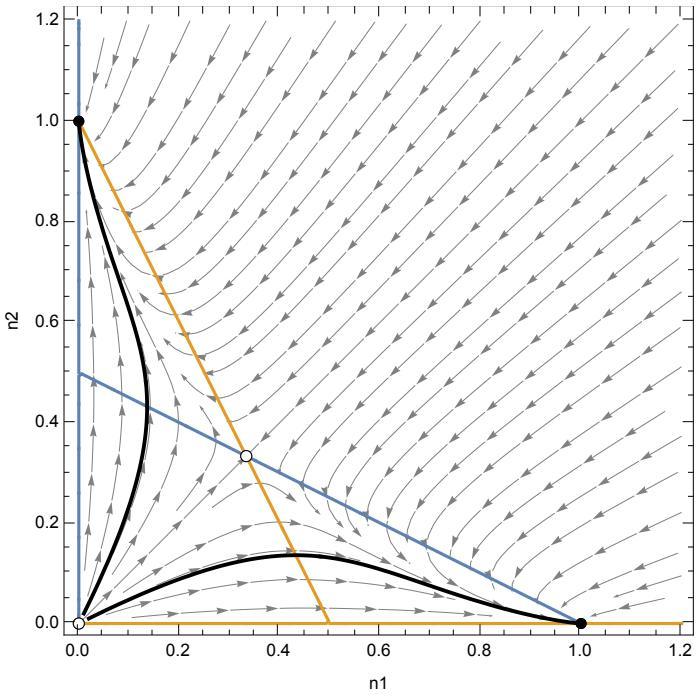
In[ ]:=
```
sol2 = EcoSim[{n1 → 0.01, n2 → 0.02}, 50];
PlotDynamics[sol2]
```

Out[ ]=



In[ ]:=
```
Show[pp, PlotTrajectory[{sol, sol2}, PlotStyle → Black]]
```

Out[ ]=



Case 0 - neutrality

In[ ]:=
```
r1 = 1; r2 = 1;
α11 = 1; α22 = 1;
α12 = 1; α21 = 1;
```

In[•]:=
```
eq = SolveEcoEq[];
NumberedGridForm[eq, EcoEigenvalues[eq], Header → True]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=

#	eq	EcoEigenvalues[eq]
1	{n1 → 0, n2 → 0}	{1, 1}
2	{n2 → 1 − n1}	{−1, 0}

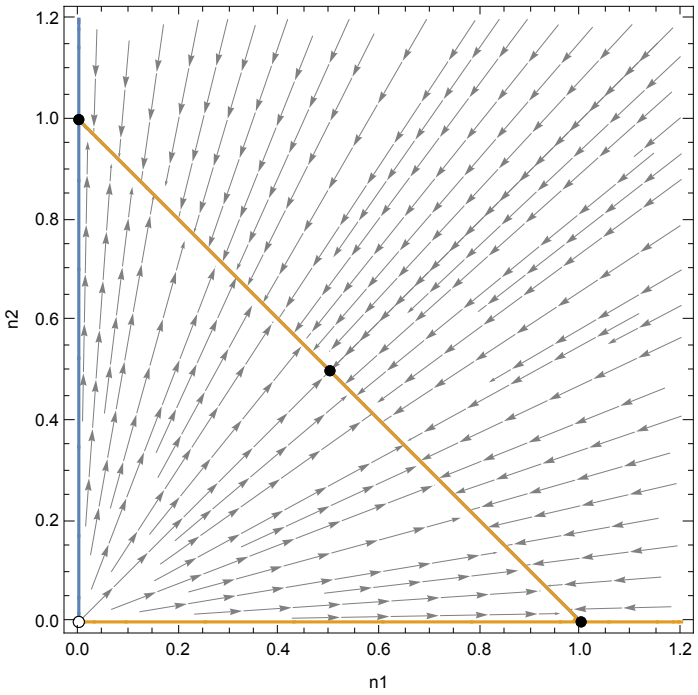
In[•]:=
```
λ12 = Inv[{n1 → 0, n2 → r2 / α22}, n1]
λ21 = Inv[{n1 → r1 / α11, n2 → 0}, n2]
```

Out[•]=
```
0
```

Out[•]=
```
0
```

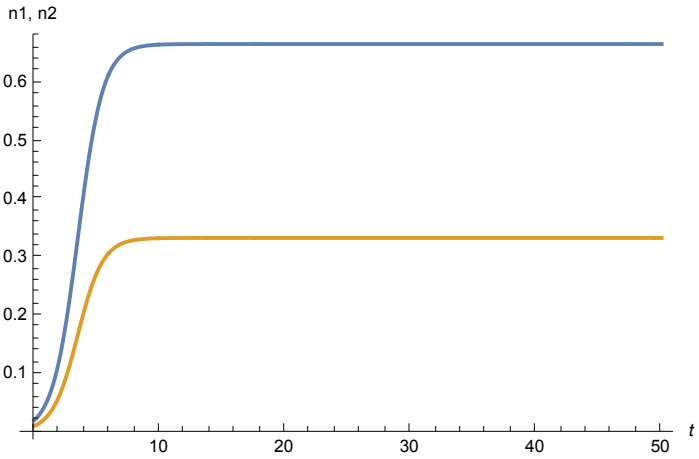
In[•]:=
```
pp = PlotEcoPhasePlane[{n1, 0, 1.2}, {n2, 0, 1.2}]
```

Out[•]=



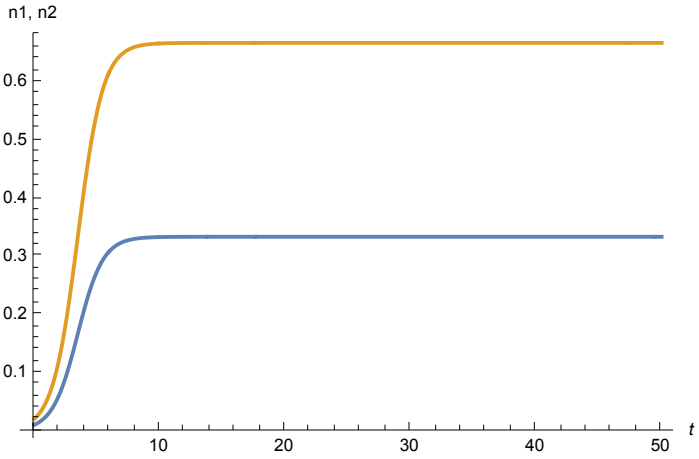
In[ ]:=
```
sol = EcoSim[{n1 → 0.02, n2 → 0.01}, 50];
PlotDynamics[sol]
```

Out[ ]=



In[ ]:=
```
sol2 = EcoSim[{n1 → 0.01, n2 → 0.02}, 50];
PlotDynamics[sol2]
```

Out[ ]=



In[ ]:=
```
Show[pp, PlotTrajectory[{sol, sol2}, PlotStyle → Black]]
```

Out[ ]=