Variable internal stores (1 nutrient)

In[1]:=
```
<< EcoEvo`
```

Out[1]=
```
EcoEvo Package Version 1.7.2 (September 1, 2023)
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```

In[17]:=
```
SetModel[{
  Aux[R] → {Equation :→ a (Rin - R) - v[R] n},
  Pop[pop] → {
    Component[Q] :→ {Equation :→ v[R] - μ[Q] Q, Type → "Intensive"},
    Component[n] :→ {Equation :→ (μ[Q] - m) n}
   },
  Parameters :→ {a > 0, Rin > 0, vmax > 0, H > 0, μ∞ > 0, Qmin > 0, m > 0(*,v,μ*)}
 }]
```

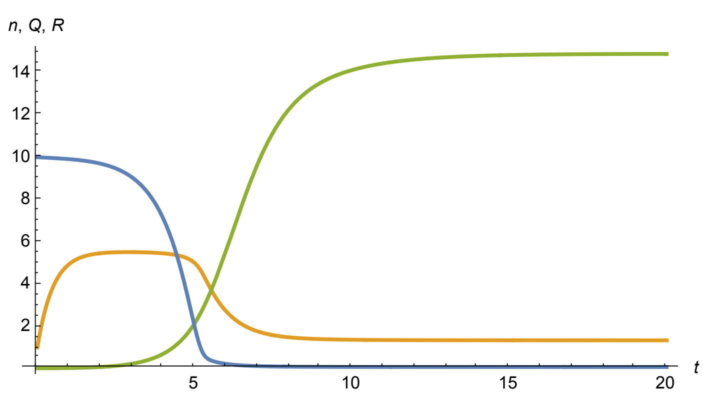
In[18]:=
```
v[R_] := vmax R / (R + H);
μ[Q_] := μ∞ (1 - Qmin / Q);
```

In[20]:=
```
a = 1;
Rin = 10;
vmax = 10;
H = 1;
μ∞ = 2;
Qmin = 1;
m = 0.5;
```

In[27]:=
```
sol = EcoSim[{R → Rin, Q → Qmin, n → 0.01}, 20];
```

In[28]:=
```
PlotDynamics[sol, AspectRatio → 1 / 2]
FinalSlice[sol]
```

Out[28]=



Out[29]=
```
{Q → 1.33348, n → 14.8879, R → 0.0714542}
```

Some analytical calculations:

In[44]:=
```
ClearParameters
```

In[45]:=
```
(* QSS Q *)
qss = SolveEcoEq[{Q}, QSS → True]
```

Out[45]=

$$\left\{\left\{Q \to \frac{R\,\text{vmax} + H\,\text{Qmin}\,\mu\infty + \text{Qmin}\,R\,\mu\infty}{(H + R)\,\mu\infty}\right\}\right\}$$

In[46]:=
```
(* growth rate in empty system *)
Inv[]
```

Out[46]=

$$-m + \frac{R\,\text{vmax}\,\mu\infty}{H\,\text{Qmin}\,\mu\infty + R\,(\text{vmax} + \text{Qmin}\,\mu\infty)}$$

In[47]:=
```
(* equilibria *)
eq = SolveEcoEq[]
```

Out[47]=

$$\left\{\left\{Q \to \frac{\text{Rin}\,\text{vmax} + H\,\text{Qmin}\,\mu\infty + \text{Qmin}\,\text{Rin}\,\mu\infty}{(H + \text{Rin})\,\mu\infty},\ n \to 0,\ R \to \text{Rin}\right\},\right.$$
$$\left\{Q \to \frac{\text{Qmin}\,\mu\infty}{-m + \mu\infty},\ n \to -\frac{a\,(m - \mu\infty)\,(m\,\text{Rin}\,\text{vmax} + H\,m\,\text{Qmin}\,\mu\infty + m\,\text{Qmin}\,\text{Rin}\,\mu\infty - \text{Rin}\,\text{vmax}\,\mu\infty)}{m\,\text{Qmin}\,\mu\infty\,(m\,\text{vmax} + m\,\text{Qmin}\,\mu\infty - \text{vmax}\,\mu\infty)},\right.$$
$$\left.\left. R \to -\frac{H\,m\,\text{Qmin}\,\mu\infty}{m\,\text{vmax} + m\,\text{Qmin}\,\mu\infty - \text{vmax}\,\mu\infty}\right\}\right\}$$

In[72]:=
```
a = 1;
Rin = 10;
vmax = 10;
H = 1;
μ∞ = 2;
Qmin = 1.;
m = 0.5;

eq
```

Out[79]=

```
{{Q → 5.54545, n → 0, R → 10}, {Q → 1.33333, n → 14.8929, R → 0.0714286}}
```

The first equilibrium matches the QSS Q during exponential growth.

Variable internal stores (2 nutrient)

In[ ]:=
```
SetModel[{
  Aux[R1] → {Equation :→ a (Rin1 - R1) - v1[R1] n},
  Aux[R2] → {Equation :→ a (Rin2 - R2) - v2[R2] n},
  Pop[pop] → {
    Component[Q1] :→ {Equation :→ v1[R1] - μ[Q1, Q2] Q1, Type → "Intensive"},
    Component[Q2] :→ {Equation :→ v2[R2] - μ[Q1, Q2] Q2, Type → "Intensive"},
    Component[n] :→ {Equation :→ (μ[Q1, Q2] - m) n}
   },
  Parameters :→ {a > 0, Rin1 > 0, Rin2 > 0, vmax1 > 0, vmax2,
    H1 > 0, H2 > 0, μ∞ > 0, Qmin1 > 0, Qmin2 > 0, m > 0, μ, μ1, μ2, v1, v2}
 }]
```

In[ ]:=
```
EcoEqns[] // TableForm
```

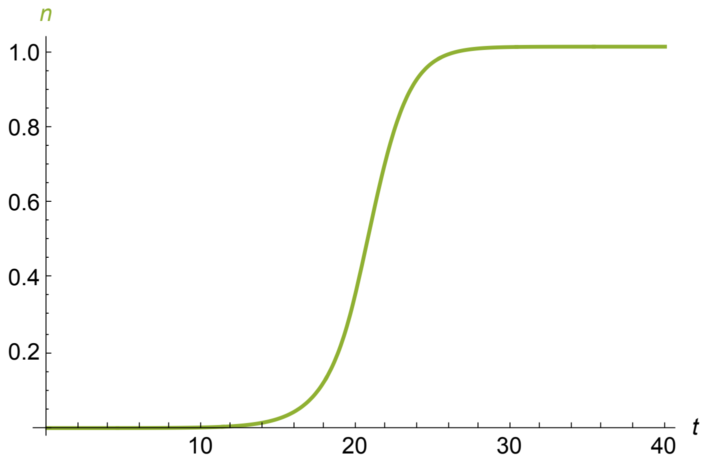
Out[ ]//TableForm=
```
Q1′ == v1[R1] - Q1 μ[Q1, Q2]
Q2′ == v2[R2] - Q2 μ[Q1, Q2]
n′ == n (-m + μ[Q1, Q2])
R1′ == a (-R1 + Rin1) - n v1[R1]
R2′ == a (-R2 + Rin2) - n v2[R2]
```

In[ ]:=
```
v1[R1_] := vmax1 R1 / (R1 + H1);
v2[R2_] := vmax2 R2 / (R2 + H2);
μ1[Q1_] := μ∞ (1 - Qmin1 / Q1);
μ2[Q2_] := μ∞ (1 - Qmin2 / Q2);
μ[Q1_, Q2_] := Min[μ1[Q1], μ2[Q2]];
```

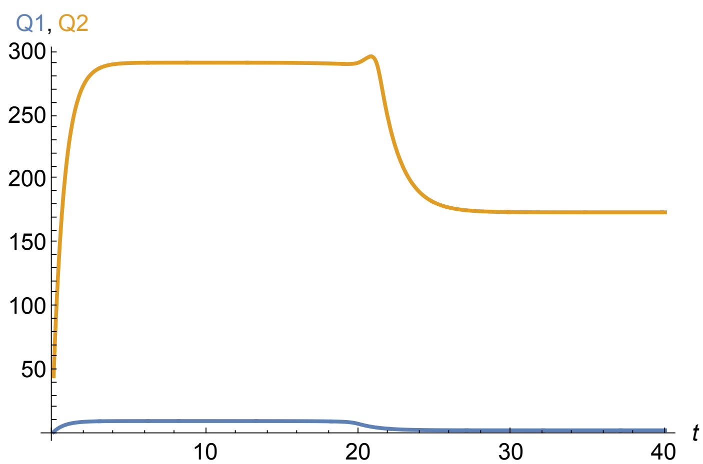
In[ ]:=
```
(* after Klausmeier et al. 2004 L&O *)
a = 0.59;
Rin1 = 3; (* P *)
Rin2 = 180; (* N *)
vmax1 = 12.3;
vmax2 = 341;
H1 = 0.2;
H2 = 5.6;
μ∞ = 1.35;
Qmin1 = 1.64;
Qmin2 = 45.4;
m := a;
```

In[ ]:=
```
(* Fig 1 *)
sol = EcoSim[{R1 → Rin1, R2 → Rin2, Q1 → Qmin1, Q2 → Qmin2, n → 10^-5}, 40];
PlotDynamics[sol, {n}]
PlotDynamics[sol, {Q1, Q2}]
PlotDynamics[sol, {Q2 / Q1}, PlotRange → {0, 60}]
PlotTrajectory[sol, {R1, R2}]
```

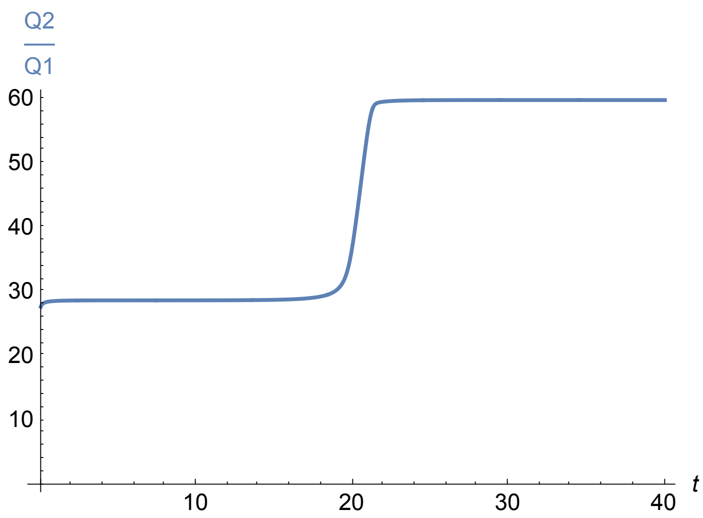
Out[ ]=



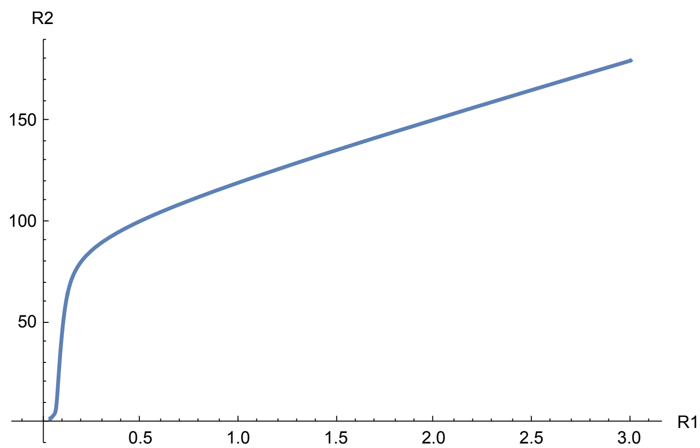
Out[ ]=



Out[ ]=



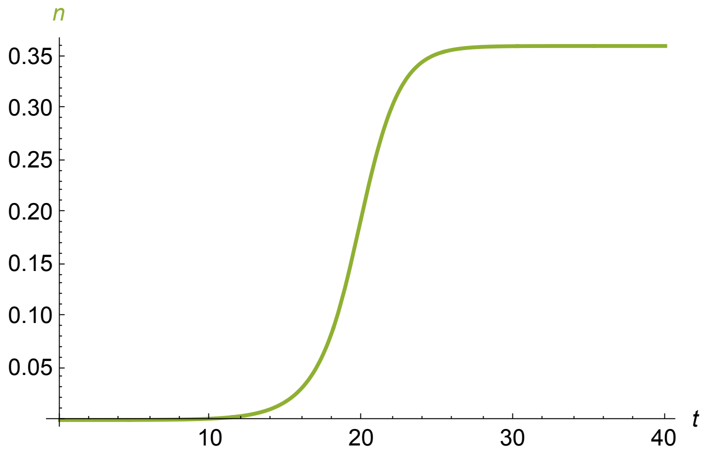
Out[ ]=



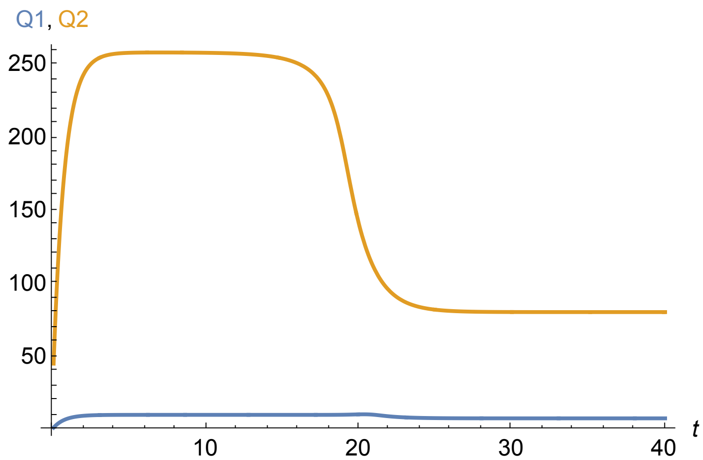
In[ ]:=

```
(* Fig 2 *)
Rin2 = 30;
sol = EcoSim[{R1 → Rin1, R2 → Rin2, Q1 → Qmin1, Q2 → Qmin2, n → 10^-5}, 40];
PlotDynamics[sol, {n}]
PlotDynamics[sol, {Q1, Q2}]
PlotDynamics[sol, {Q2 / Q1}, PlotRange → {0, 30}]
PlotTrajectory[sol, {R1, R2}]
```

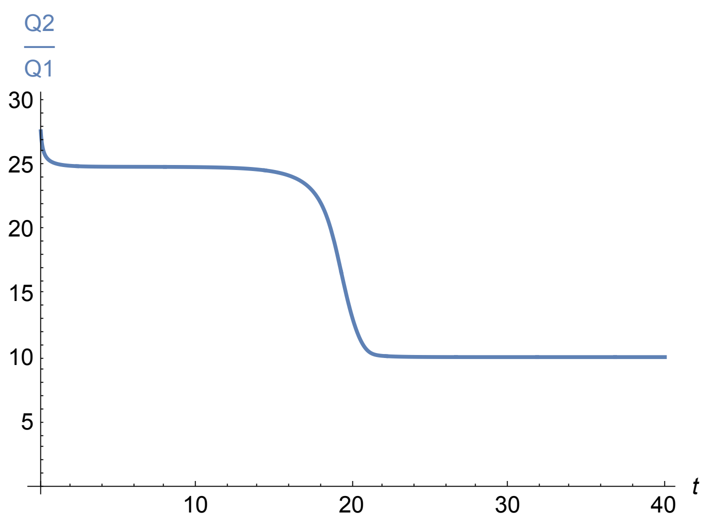
Out[ ]=


n
0.35
0.30
0.25
0.20
0.15
0.10
0.05
10
20
30
40
t

Out[ ]=


Q1, Q2
250
200
150
100
50
10
20
30
40
t

Out[ ]=


Q2/Q1
30
25
20
15
10
5
10
20
30
40
t

Out[ ]=

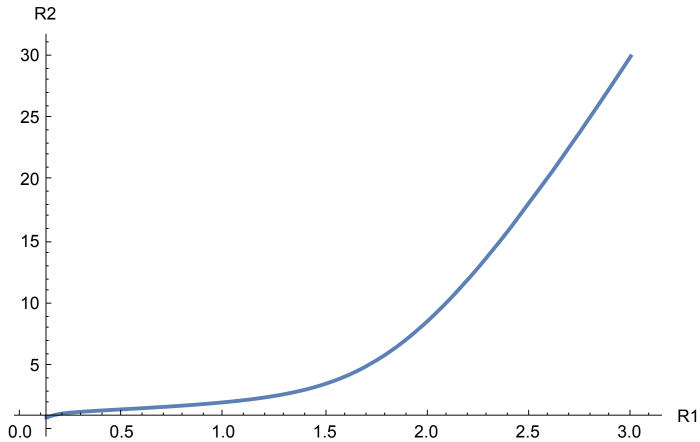


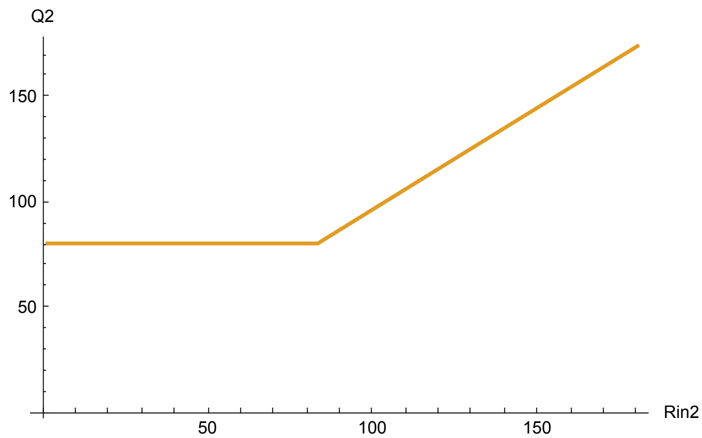
Fig 3

In[ ]:=
```
μ[Q1_, Q2_] := NMin[μ1[Q1], μ2[Q2]];
```

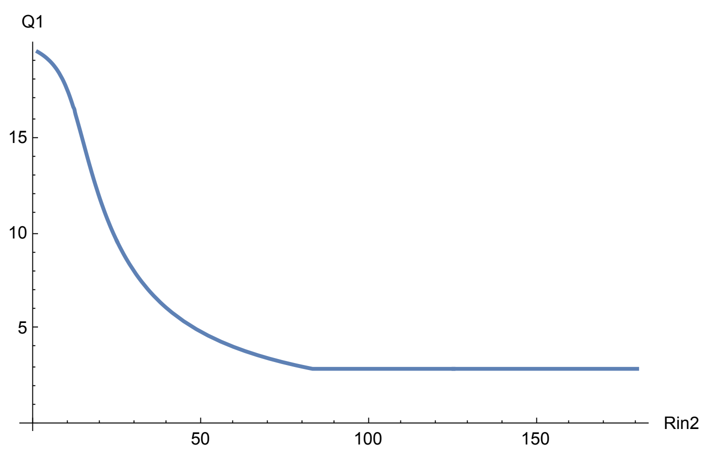
In[ ]:=
```
a = 0.59; Rin2 = 30;
sol = EcoSim[{R1 → Rin1, R2 → Rin2, Q1 → Qmin1, Q2 → Qmin2, n → 10^-5}, 100];
```

In[ ]:=
```
Clear[Rin2]
tr = TrackEcoEq[FinalSlice[sol], Rin2 → 30, {Rin2, 1, 180}, SMax → 1000];
```

In[∘]:=
```
PlotEcoEq[tr, {Q2}]
PlotEcoEq[tr, {Q1}]
```
Out[∘]=



Out[∘]=



In[∘]:=
```
Plot[Q2 / Q1 /. tr, {Rin2, 0, 180}, PlotRange → All]
```
Out[∘]=

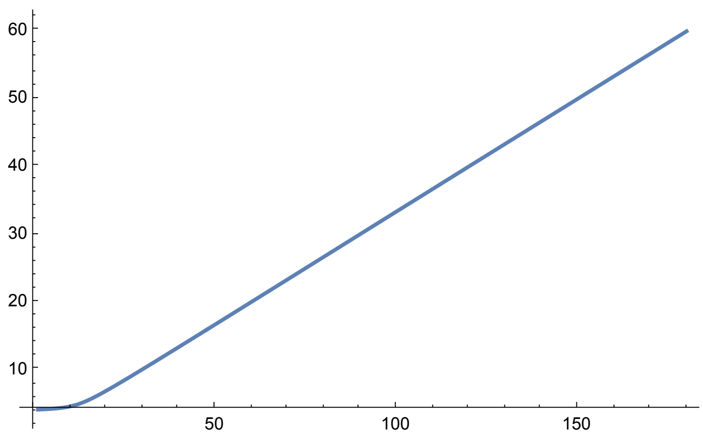


Fig 4A

In[ ]:=
```
a = 0.59; Rin2 = 10 Rin1;
sol = EcoSim[{R1 → Rin1, R2 → Rin2, Q1 → Qmin1, Q2 → Qmin2, n → 10^-5}, 100];
Clear[a];
tr10 = TrackEcoEq[FinalSlice[sol], a → 0.59, {a, 0, 1.11}, SMax → 1000];
```

TrackEq: Stability change at *a*=6.07921956164102\`*^-7, eigenvalues={−37.592, −37.2216, −1.35, −1.39216×10^{-6}, 0}.

In[ ]:=
```
a = 0.59; Rin2 = 27.7 Rin1;
sol = EcoSim[{R1 → Rin1, R2 → Rin2, Q1 → Qmin1, Q2 → Qmin2, n → 10^-5}, 100];
Clear[a];
tr27 = TrackEcoEq[FinalSlice[sol], a → 0.59, {a, 0, 1.11}, SMax → 1000];
```

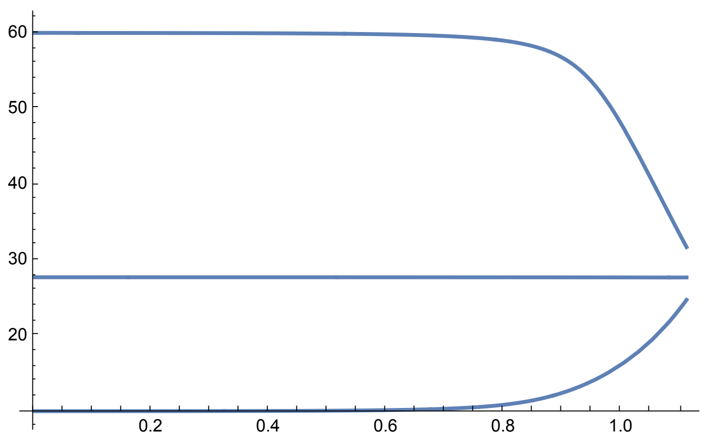
TrackEq: Stability change at *a*=4.816176668596621\`*^-6, eigenvalues={−104.166, −103.138, −1.35, −4.93329×10^{-6}, 0}.

In[ ]:=
```
a = 0.59; Rin2 = 60 Rin1;
sol = EcoSim[{R1 → Rin1, R2 → Rin2, Q1 → Qmin1, Q2 → Qmin2, n → 10^-5}, 100];
Clear[a];
tr60 = TrackEcoEq[FinalSlice[sol], a → 0.59, {a, 0, 1.11}, SMax → 1000];
```

TrackEq: Stability change at *a*=6.9051770563086226\`*^-6, eigenvalues={−104.025, −102.998, −1.35, −7.80318×10^{-6}, 0}.

In[ ]:=
```
Plot[Q2 / Q1 /. {tr10[[1]], tr27[[1]], tr60[[1]]}, {a, 0, 1.2}, PlotRange → All]
```

Out[ ]=



Piecewise

Fig 5

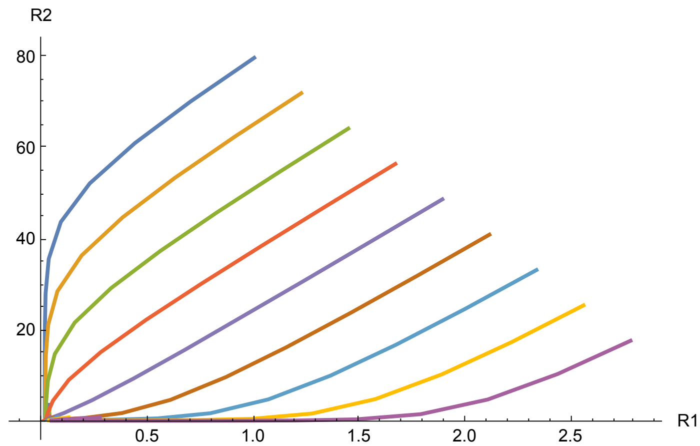
In[ ]:=
```
a = 0.59;
```

In[ ]:=
```
Rin1s = Subdivide[1, 3, 9];
Rin2s = Subdivide[80, 10, 9];
```

In[ ]:=
```
sols = Table[
    Rin1 = Rin1s[[i]]; Rin2 = Rin2s[[i]];
    EcoSim[{n → 1, Q1 → Qmin1, Q2 → Qmin2, R1 → Rin1, R2 → Rin2}, 100]
    , {i, 1, 9, 1}];
```

In[ ]:=
```
PlotTrajectory[sols, {R1, R2}]
```

Out[ ]=



Temperature-response curves