Solution for Homework 10

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Exercise 3.6.1 Suppose G is a group, $N, K < G$ with N normal in G, $N \cap K = \{e\}$, and $G = NK$. Prove that $G/N \cong K$.

By Second Isomorphism Theorem, we have $KN/N \cong K/(K \cap N)$. By Corollary 3.6.10, $NK = KN$. Hence, $G = KN$. Then $G/N \cong K/(K \cap N)$.

Since $K \cap N = \{e\}$, we have $K/(K \cap N) = K/\{e\} = \{k\{e\} \mid k \in K\}$.

Consider $\phi : K \to K/(K \cap N)$.

If $\phi(k_1) = \phi(k_2)$ for some $k_1, k_2 \in K$, we have $k_1\{e\} = k_2\{e\}$, which implies $\{k_1\} = \{k_2\}$. Hence, $k_1 = k_2$. Therefore, ϕ is injective.

By definition of ϕ, for all $k\{e\}$, there exists $k \in K$. Hence, ϕ is surjective.

We conclude ϕ is bijective.

Let $k_1, k_2 \in K$, then we have

$$\phi(k_1k_2) = k_1k_2\{e\} = \{k_1k_2e\} = \{k_1ek_2e\} = \{k_1e\}\{k_2e\} = (k_1\{e\})(k_2\{e\}) = \phi(k_1)\phi(k_2).$$

Hence, ϕ is a isomorphism.

Since $G/N \cong K/(K \cap N)$, there exists some $\psi : G/N \to K/(K \cap N)$ such that ψ is an isomorphism.

By Proposition 3.3.17, $\phi^{-1} : K/(K \cap N) \to K$ is also isomorphism. Then $\psi \circ \phi^{-1} : G/N \to K$ is isomorphism.

Hence, $G/N \cong K$.

Exercise 3.6.6 Suppose G is a finite group and N, $H < G$ with N normal in G. Prove

$$|HN| = \frac{|H||N|}{|H \cap N|}$$

By Second Isomorphism Theorem, we have $HN/N \cong H/(H \cap N)$.

Since G is a finite group, H is also a finite group and

$$|HN/N| = |H/(H \cap N)|.$$

By Corollary 3.6.10., HN is a subgroup of G. For an arbitrary $n \in N$, $n \in \{n\} = \{en\}$, where $e \in H$. Hence, $n \in HN$. Then $N \subset HN$. Since both N and HN are groups, N is a subgroup of HN.

By Lagrange's Theorem, we have $|HN| = [HN : N]|N|$.

Clearly, $H \cap N \subset H$.

- For all $g, h \in H \cap N$, $g, h \in H, N$. Since H and N are both subgroups, $gh \in H, N$. Hence, $gh \in H \cap N$.
- For all $g \in H \cap N$, since H and N are both subgroups, $g^{-1} \in H, g^{-1} \in N$. Then $g^{-1} \in H \cap N$.

We conclude $H \cap N$ is a subgroup of H. Then $|H| = [H : H \cap N]|H \cap N|$.

By definition, $[HN : N] = |HN/N|, [H : H \cap N] = |H/(H \cap N)|$.

Then we have

$$\frac{|HN|}{|H|} = \frac{|N|}{|H \cap N|}.$$

We conclude

$$|HN| = \frac{|H||N|}{|H \cap N|}.$$

Exercise 3.6.4 Suppose that G_1, G_2 are groups, $N_1 \lhd G_1$, $N_2 \lhd G_2$ are normal subgroups. Prove that $N_1 \times N_2 \lhd G_1 \times G_2$, and $(G_1 \times G_2)/(N_1 \times N_2) \cong G_1/N_1 \times G_2/N_2$.

Let (g_1, g_2) be an arbitrary element in $G_1 \times G_2$.

For $m \in (g_1, g_2)N_1 \times N_2(g_1, g_2)^{-1}$, $m = (g_1, g_2)(n_1, n_2)(g_1, g_2)^{-1}$ for some $(n_1, n_2) \in N_1 \times N_2$. By Proposition 3.1.7, we have $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$.

Then we have

$$m = (g_1, g_2)(n_1, n_2)(g_1^{-1}, g_2^{-1}) = (g_1 n_1 g_1^{-1}, g_2 n_2 g_2^{-1}).$$

Since both N_1 and N_2 are normal, we have $g_1 n_1 g_1^{-1} \in N_1$ and $g_2 n_2 g_2^{-1} \in N_2$.

Hence, $m \in N_1 \times N_2$.

Since (g_1, g_2) is arbitrary, $(g_1, g_2)N_1 \times N_2(g_1, g_2)^{-1} \subset N_1 \times N_2$. By Lemma 3.3.12., $N_1 \times N_2$ is normal.

Consider $\phi : G_1 \times G_2 \to G_1/N_1 \times G_2/N_2$ be $\phi(g_1, g_2) = (g_1 N_1, g_2 N_2)$. Then for some $(g_1, g_2), (h_1, h_2) \in G_1 \times G_2$, we have

$$\begin{aligned}
\phi((g_1, g_2)(h_1, h_2)) &= \phi((g_1 h_1, g_2, h_2)) \\
&= (g_1 h_1 N_1, g_2 h_2 N_2) \\
&= (g_1 N_1 h_1 N_1 N_1, g_2 N_2 h_2 N_2 N_2) \\
&= (g_1 N_1 h_1 N_1, g_2 N_2 h_2 N_2) \\
&= (g_1 N_1, g_2 N_2)(h_1 N_1, h_2 N_2) \\
&= \phi(g_1, g_2)\phi(h_1, h_2)
\end{aligned}$$

Then ϕ is homomorphism.

By definition, for all $(g_1 N_1, g_2 N_2) \in G_1/N_1 \times G_2/N_2$, there exists $(g_1, g_2) \in G_1 \times G_2$ such that $\phi(g_1, g_2) = (g_1 N_1, g_2 N_2)$. Hence, ϕ is surjective.

By Theorem 3.6.5., $G_1 \times G_2/\ker(\phi) \cong G_1/N_1 \times G_2/N_2$.

Let $(n_1, n_2) \in N_1 \times N_2$, then $\phi(n_1, n_2) = (n_1 N_1, n_2 N_2) = (N_1, N_2)$. Since (N_1, N_2) is the identity in $G_1/N_1 \times G_2/N_2$, $(n_1, n_2) \in \ker(\phi)$. Hence, $N_1 \times N_2 \subset \ker(\phi)$. Let $(m_1, m_2) \in \ker(\phi)$, then $\phi(m_1, m_2) = (m_1 N_1, m_2 N_2) = (N_1, N_2)$, which implies $m_1 \in N_1, m_2 \in N_2$. Therefore, $(m_1, m_2) \in N_1 \times N_2$. Then $\ker(\phi) \subset N_1 \times N_2$. Hence, $\ker(\phi) = N_1 \times N_2$.

We conclude $(G_1 \times G_2)/(N_1 \times N_2) \cong G_1/N_1 \times G_2/N_2$.

Exercise 3.7.2 Suppose G is a finite group, $H, K \lhd G$ are normal subgroups, $\gcd(|H|, |K|) = 1$, and $|G| = |H||K|$. Prove that $G \cong H \times K$.

Clearly, $H \cap K \subset K$.

- For all $g, k \in H \cap K$, $g, k \in H, K$. Since H and K are both subgroups, $gk \in H, K$. Hence, $gk \in H \cap N$.
- For all $g \in H \cap K$, since H and K are both subgroups, $g^{-1} \in H, g^{-1} \in K$. Then $g^{-1} \in H \cap K$.

We conclude $H \cap K$ is a subgroup of K.

Similarly, $H \cap K$ is a subgroup of H.

By Lagrange's Theorem, we have $|H| = [H : H \cap K]|H \cap K|$ and $|K| = [K : H \cap K]|H \cap K|$.

Hence, $|H \cap K| \Big| |H|$ and $|H \cap K| \Big| |K|$. By the property of greatest common divisor, $|H \cap K| \Big| 1$. Hence, $H \cap K = \{e\}$.

By Exercise 3.6.6., we have $|HK| = \dfrac{|H||K|}{|H \cap K|} = |H||K| = |G|$.

By Corollary 3.6.10., HK is a subgroup of G. Then by Lagrange's Theorem, $|G| = [G : HK]|HK|$. Hence, $[G : HK] = 1$. By Corollary 3.5.4., G/HK is a partition of G and partition G into only one group. Therefore, $G = HK$.

By Proposition 3.7.1., we conclude $G \cong H \times K$.

Exercise 3.7.6 Prove that for all $n \geq 2$, we have $S_n \cong A_n \rtimes Z_2$, where S_n is the symmetric group and A_n the alternating subgroup (consisting of the even permutations).

Consider a cyclic group $K = \langle(1\ 2)\rangle = \{(1), (1\ 2)\}$. By Corollary 3.5.8., the order of K is 2, which is a prime integer, then $K \cong \mathbb{Z}_2$. Since $K \subset S_n$ and K is a cyclic group, $K < S_n$, for $n \geq 2$.

We have $A_n = \ker(\epsilon) \triangleleft S_n$, where ϵ is the sign homomorphism. Since $(1\ 2)$ is a 2-cycle, $\epsilon(1\ 2) = (-1)^1 = -1$. Then $(1\ 2) \notin A_n$. Hence, $K \cap A_n = (1) = \{e\}$.

From the description of the elements in S_n, we clearly have $A_n K = S_n$. by Proposition 3.7.7, we have

$$S_n \cong A_n \rtimes K \cong A_n \rtimes \mathbb{Z}_2.$$