Solution for Homework 1

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March 22, 2021

Exercise 1.1.1 Given functions $\sigma : A \to B$ and $\tau : B \to C$, prove that if $\tau \circ \sigma$ is injective,then so is σ.

Proof. $\sigma : A \to B, \tau : B \to C \implies \tau \circ \sigma : A \to C$.

$\tau \circ \sigma$ is injective $\implies \forall a_1, a_2 \in A$, if $\tau \circ \sigma(a_1) = \tau \circ \sigma(a_2)$, then $a_1 = a_2$. $\implies \forall a_1, a_2 \in A$ if $\tau(\sigma(a_1)) = \tau(\sigma(a_2))$, then $a_1 = a_2$

$\forall a_1, a_2 \in A$, $\sigma(a_1) = \sigma(a_2) \implies \tau(\sigma(a_1)) = \tau(\sigma(a_2)) \implies a_1 = a_2$

We conclude that $\forall a_1, a_2 \in A$, if $\sigma(a_1) = \sigma(a_2)$, then $a_1 = a_2$. Hence, σ is injective. $\square$

Exercise 1.2.2 Decide which of the following are equivalence relations on the set of natural numbers N. For those that are, prove it. For those that are not, explain why.

(i) $x \sim y$ if $\|x - y\| \leq 3$

- $\forall x \in \mathbb{N}$, $\|x - x\| = 0 \leq 3 \implies x \sim x \implies$ **reflexive**
- $\forall x, y \in \mathbb{N}$, $\|x - y\| = \|-(x - y)\| = \|y - x\| \leq 3$
  Hence, $x \sim y \implies y \sim x$. $x \sim y$ is **symmetric**
- Suppose $x = 5, y = 4, z = 1$, then $\|x - y\| = 1 \leq 3$, $\|y - z\| = 3 \leq 3$
  However, $\|x - z\| = 4 > 3$
  Hence, there exists some $x, y, z \in \mathbb{N}$ such that $x \sim y, y \sim z$ but $x \not\sim z \implies$ **Not transitive**

We conclude that $x \sim y$ is not a equivalence relation.

(ii) $x \sim y$ if $\|x - y\| \geq 3$

- $\forall x \in \mathbb{N}$, $\|x - x\| = 0 < 3 \implies$ **Not reflexive**

We conclude that $x \sim y$ is not a equivalence relation.

(iii) $x \sim y$ if x and y have the same digit in the 1's place (expressed in base 10).

- $\forall x \in \mathbb{N}$, x and x have the same digit in the 1's place $\implies$
  $x \sim x \implies$ **reflexive**
- $\forall x \in \mathbb{N}$, x and y have the same digit in the 1's place
  $\implies$ y and x have the same digit in the 1's place
  Hence, $x \sim y \implies y \sim x$. $x \sim y$ is **symmetric**.
- $\forall x, y, z \in \mathbb{N}$, x and y have the same digit in the 1's place,
  y and z have the same digit in the 1's place
  $\implies$ x and z have the same digit in the 1's place
  Hence, $x \sim y, y \sim z \implies x \sim z \implies$ **transitive**

We conclude that $x \sim y$ is a equivalence relation.

(iv) $x \sim y$ if $x \geq y$

- $\forall x \in \mathbb{N}$, $x \geq x \implies x \sim x \implies$ **reflexive**

- $\forall x \in \mathbb{N}$ (for example, $2 \geq 1 \not\Rightarrow 1 \geq 2$), $x \geq y \not\Rightarrow y \geq x$
  Hence, $x \sim y \not\Rightarrow y \sim x$. $x \sim y$ is **not symmetric**.

We conclude that $x \sim y$ is not a equivalence relation.

Exercise 1.2.3 Prove that for any $f : X \to Y$, the relation $\sim_f$ defined in Example 1.2.6 is an equivalence relation. Show that for any $x \in X$, the equivalence class of x is precisely $[x] = f^{-1}(f(x))$.

- $\forall x \in X, f(x) = f(x) \implies x \sim_f x \implies$ **reflexive**
- $\forall x, y \in X, f(x) = f(y) \implies f(y) = f(x)$
  Hence, $x \sim_f y \implies y \sim_f x \implies$ **symmetric**
- $\forall x, y, z \in X, f(x) = f(y), f(y) = f(z) \implies f(x) = f(z)$
  Hence, $x \sim_f y, y \sim_f z \implies x \sim_f z \implies$ **transitive**

We conclude that $x \sim_f y$ is a equivalence relation.
For part 2, $[x] = \{y \in X \mid y \sim_f x\} = \{y \in X \mid f(y) = f(x)\}$ and
$f^{-1}(f(x)) = \{y \in X \mid f(y) = f(x)\}$.
Hence, $[x] = f^{-1}(f(x))$.

Exercise 1.2.4 Prove Proposition 1.2.12: If $\sim$ is an equivalence relation on X, then the function $\pi : X \to X/\sim$ defined by $\pi(x) = [x]$ is a surjective map, and the equivalence relation $\sim_\pi$ determined by π is precisely $\sim$.

Proof. By definition, $\forall\, [x] \in X \to X/\sim$, there exists some $x \in X$ such that $\pi(x) = [x]$. Hence, π is a surjective map.
$\pi(x) = \pi(y) \iff [x] = [y] \iff x \sim y$
By definition, $\pi(x) = \pi(y) \implies x \sim_\pi y$
We conclude that $x \sim_\pi y \iff x \sim y$, which means $\sim_\pi$ is $\sim$. □

Exercise 1.3.1 Let $\sigma \in S_8$ be the permutation given by the 2×8 matrix

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 2 & 7 & 6 & 8 & 5 \end{pmatrix}$$

Express $\sigma, \sigma^2, \sigma^3$, and σ^{-1} in disjoint cycle notation.

- $1 \mapsto \sigma(1) = 4 \mapsto \sigma(4) = 2 \mapsto \sigma(2) = 1$
  $5 \mapsto \sigma(5) = 7 \mapsto \sigma(7) = 8 \mapsto \sigma(8) = 5$
  Hence, $\sigma = (1\,4\,2)(5\,7\,8)$
- $1 \mapsto \sigma^2(1) = \sigma(4) = 2 \mapsto \sigma^2(2) = \sigma(1) = 4 \mapsto \sigma^2(4) = \sigma(2) = 1$
  $5 \mapsto \sigma^2(5) = \sigma(7) = 8 \mapsto \sigma^2(8) = \sigma(5) = 7 \mapsto \sigma^2(7) = \sigma(8) = 5$
  Hence, $\sigma = (2\,4\,1)(8\,7\,5)$
- $1 \mapsto \sigma^3(1) = \sigma(2) = 1$
  $2 \mapsto \sigma^3(2) = \sigma(4) = 2$
  $4 \mapsto \sigma^3(4) = \sigma(1) = 4$
  $5 \mapsto \sigma^3(5) = \sigma(8) = 5$
  $7 \mapsto \sigma^3(7) = \sigma(5) = 7$
  $8 \mapsto \sigma^3(8) = \sigma(7) = 8$
  Hence, $\sigma = (1)(2)(4)(3)(5)(7)(8)(6)$

- Let $\sigma = (1\,4\,2)(5\,7\,8)$. We obtain σ^{-1} by reversing each of the cycles:

$$\sigma^{-1} = (2\,4\,1)(8\,7\,5)$$

Exercise 1.3.2 Consider $\sigma = (3\,4\,8)(5\,7\,6\,9)$ and $\tau = (1\,9\,3\,5)(2\,7\,4)$ in S_9 expressed in disjoint cycle notation. Compute $\sigma \circ \tau$ and $\tau \circ \sigma$, expressing both in disjoint cycle notation.

- $1 \mapsto \sigma(\tau(1)) = \sigma(9) = 5 \mapsto \sigma(\tau(5)) = \sigma(1) = 1$
  $2 \mapsto \sigma(\tau(2)) = \sigma(7) = 6 \mapsto \sigma(\tau(6)) = \sigma(6) = 9 \mapsto \sigma(\tau(9)) = \sigma(3) = 4 \mapsto \sigma(\tau(4)) = \sigma(2) = 2$
  $3 \mapsto \sigma(\tau(3)) = \sigma(5) = 7 \mapsto \sigma(\tau(7)) = \sigma(4) = 8 \mapsto \sigma(\tau(8)) = \sigma(8) = 3$
  Hence, $\sigma \circ \tau = (1\,5)(2\,6\,9\,4)(3\,7\,8)$

- $1 \mapsto \tau(\sigma(1)) = \tau(1) = 9 \mapsto \tau(\sigma(9)) = \tau(5) = 1$
  $2 \mapsto \tau(\sigma(2)) = \tau(2) = 7 \mapsto \tau(\sigma(7)) = \tau(6) = 6 \mapsto \tau(\sigma(6)) = \tau(9) = 3 \mapsto \tau(\sigma(3)) = \tau(4) = 2$
  $4 \mapsto \tau(\sigma(4)) = \tau(8) = 8 \mapsto \tau(\sigma(8)) = \tau(3) = 5 \mapsto \tau(\sigma(5)) = \tau(7) = 4$
  Hence, $\sigma \circ \tau = (1\,9)(2\,7\,6\,3)(4\,8\,5)$