STAT 410

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$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{pmatrix} \quad – \quad n\text{-dimensional random vector}$$

$$E(\mathbf{X}) = \boldsymbol{\mu}_{\mathbf{X}} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \dots \\ \mu_n \end{pmatrix}$$

Covariance matrix:

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \dots & \dots & & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{pmatrix}$$

$$\sigma_{ij} = \operatorname{Cov}(X_i, X_j)$$

$$\boldsymbol{\Sigma}_{\mathbf{X}} = E[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T]$$

$\boldsymbol{\Sigma}_{\mathbf{X}}$ – symmetric, nonnegative-definite

$n = 1$

$$\mathbf{X} = (X)$$

$$\boldsymbol{\mu}_{\mathbf{X}} = (\mu)$$

$$\boldsymbol{\Sigma}_{\mathbf{X}} = (\sigma^2)$$

$n = 2$

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \qquad \boldsymbol{\mu}_{\mathbf{X}} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{pmatrix} \operatorname{Var}(X_1) & \operatorname{Cov}(X_1, X_2) \\ \operatorname{Cov}(X_1, X_2) & \operatorname{Var}(X_2) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\,\sigma_1\sigma_2 \\ \rho\,\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

Let $\mathbf{Y} = \mathbf{A}\,\mathbf{X} + \mathbf{b}$, where $\mathbf{A}$ – $m \times n$ (non-random) matrix,

$\mathbf{b} \in \mathbf{R}^m$ – (non-random) vector

Then $E(\mathbf{Y}) = \boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{A}\,\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{b}$ $\qquad \boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{A}\,\boldsymbol{\Sigma}_{\mathbf{X}}\,\mathbf{A}^T$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$$

$$a_1 X_1 + a_2 X_2 + \dots + a_n X_n = \mathbf{a}^T \mathbf{X}$$

$$E(\mathbf{a}^T \mathbf{X}) = \mathbf{a}^T \boldsymbol{\mu}_X \qquad \operatorname{Var}(\mathbf{a}^T \mathbf{X}) = \mathbf{a}^T \boldsymbol{\Sigma}_{\mathbf{X}}\,\mathbf{a}$$

1. Consider a random vector $\vec{\mathbf{X}} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ with mean $E(\vec{\mathbf{X}}) = \vec{\boldsymbol{\mu}} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$

and variance-covariance matrix $\operatorname{Cov}(\vec{\mathbf{X}}) = \boldsymbol{\Sigma} = \begin{pmatrix} 9 & 2 & -3 \\ 2 & 4 & -2 \\ -3 & -2 & 16 \end{pmatrix}$.

Then $\operatorname{Var}(X_1) = 9$, $\operatorname{Var}(X_2) = 4$, $\operatorname{Var}(X_3) = 16$,

$$\rho_{12} = \frac{2}{\sqrt{9}\cdot\sqrt{4}} = \frac{1}{3}, \qquad \rho_{13} = \frac{-3}{\sqrt{9}\cdot\sqrt{16}} = -\frac{1}{4}, \qquad \rho_{23} = \frac{-2}{\sqrt{4}\cdot\sqrt{16}} = -\frac{1}{4}.$$

Consider $2X_1 - 3X_2 - X_3 = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$.

$$E(2X_1 - 3X_2 - X_3) = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix}\vec{\boldsymbol{\mu}} = 2\mu_1 - 3\mu_2 - \mu_3 = 2\cdot 5 - 3\cdot 1 - 1\cdot 3 = 4.$$

$$\operatorname{Var}(2X_1 - 3X_2 - X_3) = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix}\begin{pmatrix} 9 & 2 & -3 \\ 2 & 4 & -2 \\ -3 & -2 & 16 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & -6 & -16 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 64.$$

OR

$$\operatorname{Var}(2X_1 - 3X_2 - X_3) = 4\operatorname{Var}(X_1) + 9\operatorname{Var}(X_2) + \operatorname{Var}(X_3)$$

$$- 12\operatorname{Cov}(X_1, X_2) - 4\operatorname{Cov}(X_1, X_3) + 6\operatorname{Cov}(X_2, X_3)$$

$$= 36 + 36 + 16 - 24 + 12 - 12 = 64.$$

Multivariate Normal Distribution:

$\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2} (\mathbf{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu}) \right\}$$

$\mathbf{Z} \sim N_n(\mathbf{0}, \mathbf{I}_n)$

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}} \exp\left\{ -\frac{1}{2} \mathbf{z}' \mathbf{z} \right\} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} z_i^2 \right\}$$

$\mathbf{X} = \boldsymbol{\Sigma}^{1/2} \mathbf{Z} + \boldsymbol{\mu}$

$$\mathrm{M}_{\mathbf{X}}(\mathbf{t}) = \exp\left\{ \mathbf{t}' \boldsymbol{\mu} + \frac{1}{2} \mathbf{t}' \boldsymbol{\Sigma} \mathbf{t} \right\} \qquad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ \cdots \\ t_n \end{pmatrix} \in \mathbf{R}^n$$

$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ $\qquad$ $\mathbf{A} - m \times n$ $\qquad$ $\mathbf{b} \in \mathbf{R}^m$

$\Rightarrow \qquad \mathbf{Y} \sim N_m(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$

$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}$ $\qquad$ $\mathbf{X}_1$ is of dimension $m < n$; $\mathbf{X}_2$ is of dimension $n - m$ $\qquad$ $\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}$ $\qquad$ $\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$

$\Rightarrow \qquad \mathbf{X}_1 | \mathbf{X}_2 \sim N_m\left(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} (\mathbf{X}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21} \right)$

2.

$$\mathbf{X} \sim N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \qquad \boldsymbol{\mu} = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix}$$

a) Find $P(X_1 > 6)$.

$$X_1 \sim N(5, 4)$$

$$P(X_1 > 6) = P\left(Z > \frac{6-5}{\sqrt{4}}\right) = P(Z > 0.5) = \mathbf{0.3085}.$$

b) Find $P(5X_2 + 4X_3 > 70)$.

$$E(5X_2 + 4X_3) = 5 \cdot 3 + 4 \cdot 7 = 43.$$

$$\begin{aligned} \operatorname{Var}(5X_2 + 4X_3) &= 25 \operatorname{Var}(X_2) + 40 \operatorname{Cov}(X_2, X_3) + 16 \operatorname{Var}(X_3) \\ &= 25 \cdot 4 + 40 \cdot 2 + 16 \cdot 9 = 324. \end{aligned}$$

OR

$$\operatorname{Var}(5X_2 + 4X_3) = \begin{pmatrix} 0 & 5 & 4 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 & 28 & 46 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 4 \end{pmatrix} = 324.$$

$$P(5X_2 + 4X_3 > 70) = P\left(Z > \frac{70-43}{\sqrt{324}}\right) = P(Z > 1.50) = \mathbf{0.0668}.$$

c) Find $P(3X_3 - 5X_2 < 17)$.

$$E(3X_3 - 5X_2) = 3 \cdot 7 - 5 \cdot 3 = 6.$$

$$\begin{aligned} \operatorname{Var}(3X_3 - 5X_2) &= 9 \operatorname{Var}(X_3) - 30 \operatorname{Cov}(X_2, X_3) + 25 \operatorname{Var}(X_2) \\ &= 9 \cdot 9 - 30 \cdot 2 + 25 \cdot 4 = 121. \end{aligned}$$

OR

$$\operatorname{Var}(3X_3 - 5X_2) = \begin{pmatrix} 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 & -14 & 17 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 3 \end{pmatrix} = 121.$$

$$P(3X_3 - 5X_2 < 17) = P\left(Z < \frac{17-6}{\sqrt{121}}\right) = P(Z < 1.00) = \mathbf{0.8413}.$$

d) Find $P(4X_1 - 3X_2 + 5X_3 < 80)$.

$$E(4X_1 - 3X_2 + 5X_3) = 4\mu_1 - 3\mu_2 + 5\mu_3 = 4 \cdot 5 - 3 \cdot 3 + 5 \cdot 7 = 46.$$

$$\operatorname{Var}(4X_1 - 3X_2 + 5X_3) = \begin{pmatrix} 4 & -3 & 5 \end{pmatrix} \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 19 & -6 & 39 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$$

$$= 289.$$

$$4X_1 - 3X_2 + 5X_3 \sim N(46, 289)$$

$$P(4X_1 - 3X_2 + 5X_3 < 80) = P\left(Z < \frac{80-46}{\sqrt{289}}\right) = P(Z < 2.00) = \mathbf{0.9772}.$$

e)* Find $P(X_1 > 8 \mid X_2 = 1, X_3 = 10)$.

$$\boldsymbol{\Sigma}_{22} = \begin{pmatrix} 4 & 2 \\ 2 & 9 \end{pmatrix} \qquad \boldsymbol{\Sigma}_{22}^{-1} = \frac{1}{32}\begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} = \frac{1}{32}\begin{pmatrix} -1 & 0 \end{pmatrix}\begin{pmatrix} 9 & -2 \\ -2 & 4 \end{pmatrix} = \frac{1}{32}\begin{pmatrix} -9 & 2 \end{pmatrix}$$

$$\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_2 - \boldsymbol{\mu}_2) = 5 + \frac{1}{32}\begin{pmatrix} -9 & 2 \end{pmatrix}\begin{pmatrix} 1-3 \\ 10-7 \end{pmatrix} = 5.75.$$

$$\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} = 4 - \frac{1}{32}\begin{pmatrix} -9 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = 3.71875.$$

$$X_1 \mid X_2 = 1, X_3 = 10 \sim N(5.75, 3.71875)$$

$$P(X_1 > 8 \mid X_2 = 1, X_3 = 10) = P\left(Z > \frac{8-5.75}{\sqrt{3.71875}}\right) = P(Z > 1.17) = \mathbf{0.1210}.$$