STAT 410

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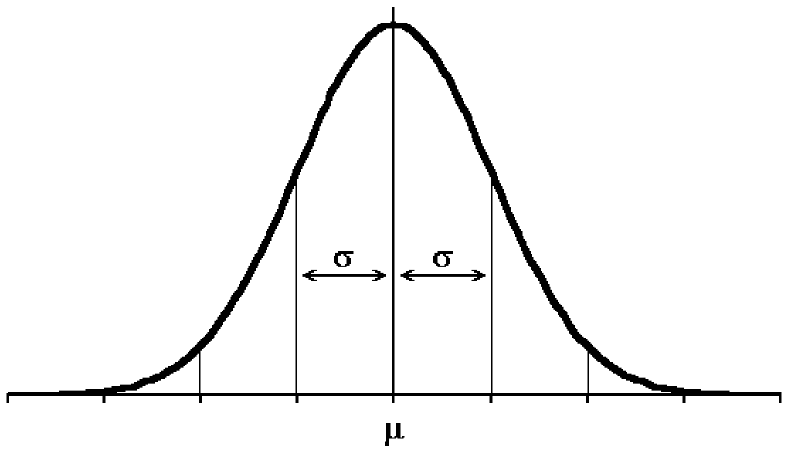
# Normal (Gaussian) Distribution.




**μ** – mean

**σ** – standard deviation

$$\mathbf{N}\left(\mu,\sigma^2\right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty.$$

Standard Normal Distribution – $\mathrm{N}(0, 1)$: $\mu = 0$, $\sigma^2 = 1$.

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EXCEL: ( Z – Standard Normal $\mathrm{N}(0,1)$ )

| | | |
|---|---|---|
| = NORMSDIST( $z$ ) | gives | $\Phi(z) = \mathrm{P}(\mathrm{Z} \le z)$ |
| = NORMSINV( $p$ ) | gives | $z$ such that $\mathrm{P}(\mathrm{Z} \le z) = p$ |
| = NORMDIST( $x, \mu, \sigma, 1$ ) | gives | $\mathrm{P}(\mathrm{X} \le x)$, where X is $\mathrm{N}(\mu, \sigma^2)$ |
| = NORMDIST( $x, \mu, \sigma, 0$ ) | gives | $f(x)$, p.d.f. of $\mathrm{N}(\mu, \sigma^2)$ |
| = NORMINV( $p, \mu, \sigma$ ) | gives | $x$ such that $\mathrm{P}(\mathrm{X} \le x) = p$, where X is $\mathrm{N}(\mu, \sigma^2)$ |

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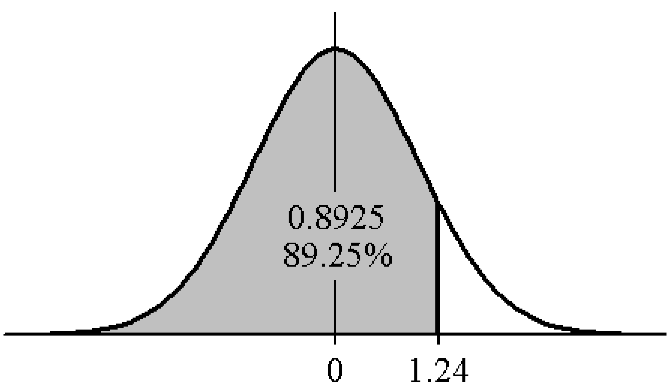
Example:

For the standard normal distribution, find the area to the left of

$z$ = 1.24

1.24

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.53 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.57 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.61 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.65 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.68 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.72 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.75 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.78 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.81 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.83 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.86 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.88 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.90 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.91 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.93 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.94 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.95 |




Area to the left

of $z$ = 1.24

is **0.8925**.

$$Z \sim N(0,1)$$

$$X \sim N(\mu,\sigma^2)$$

$$Z = \frac{X-\mu}{\sigma}$$

$$X = \mu + \sigma Z$$

**1.** Models of the pricing of stock options often make the assumption of a normal distribution. An analyst believes that the price of an *Initech* stock option varies from day to day according to normal distribution with mean \$9.22 and unknown standard deviation.

a) The analyst also believes that 77% of the time the price of the option is greater than \$7.00. Find the standard deviation of the price of the option.

$\mu = 9.22, \qquad \sigma = ? \qquad$ Know $\mathrm{P}(X > 7.00) = 0.77.$

Find $z$ such that $\mathrm{P}(Z > z) = 0.77.$

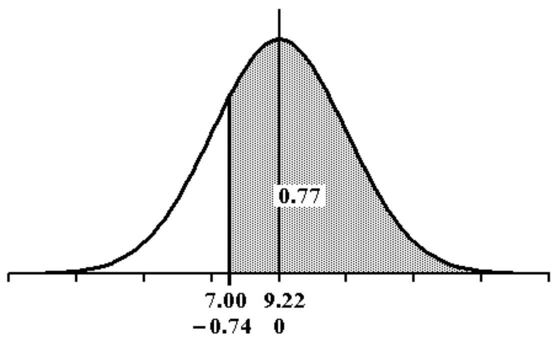
$\Phi(z) = 1 - 0.77 = \mathbf{0.23}.$

$z = \mathbf{-0.74}.$

$x = \mu + \sigma \cdot z.$

$7.00 = 9.22 + \sigma \cdot (-0.74).$

$\sigma = \mathbf{\$3.00}.$




b) Find the proportion of days when the price of the option is greater than \$10.00?

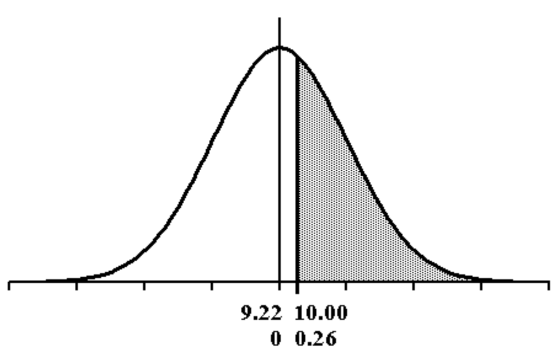
$$\mathrm{P}(X > 10.00) = \mathrm{P}\left(Z > \frac{10.00 - 9.22}{3.00}\right)$$

$= \mathrm{P}(Z > 0.26)$

$= 1 - \Phi(0.26)$

$= 1 - 0.6026$

$= \mathbf{0.3974}.$



c) Following the famous “buy low, sell high” principle, the analyst recommends buying *Initech* stock option if the price falls into the lowest 14% of the price distribution, and selling if the price rises into the highest 9% of the distribution. Mr. Statman doesn’t know much about history, doesn’t know much about biology, doesn’t know much about statistics, but he does want to be rich someday. Help Mr. Statman find the price below which he should buy *Initech* stock option and the price above which he should sell.

Need $x = ?$ such that $P(X < x) = 0.14$.

Find $z$ such that $P(Z < z) = 0.14$.

The area to the left is **0.14** $= \Phi(z)$.

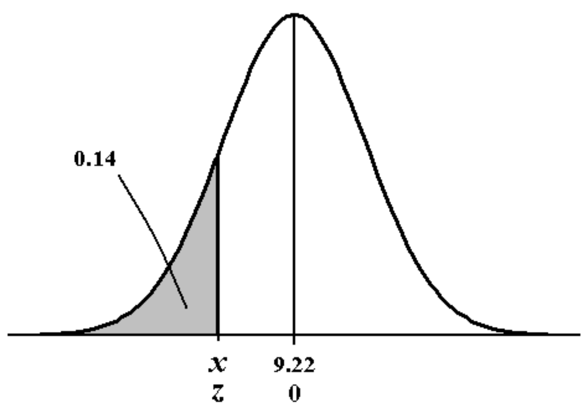
$z = \mathbf{-1.08}$.

$x = \mu + \sigma \cdot z$.

$x = 9.22 + 3 \cdot (-1.08)$

$= \mathbf{\$5.98}$.

Buy if the price is below $5.98.




Need $x = ?$ such that $P(X < x) = 0.09$.

① Find $z$ such that $P(Z < z) = 0.09$.

The area to the left is **0.91** $= \Phi(z)$.

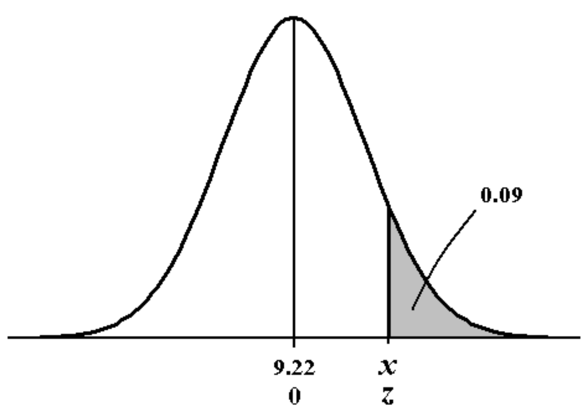
$z = \mathbf{1.34}$.

② $x = \mu + \sigma \cdot z$.

$x = 9.22 + 3 \cdot (1.34)$

$= \mathbf{\$13.24}$.

Sell if the price is above $13.24.



$$X \sim N(\mu, \sigma^2) \qquad \Leftrightarrow \qquad M_X(t) = e^{\mu t + \sigma^2 t^2/2}.$$

Proof: $\quad M_X(t) = E(e^{tX}) = \displaystyle\int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$

$$= \int_{-\infty}^{\infty} e^{t(\mu + \sigma z)} \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= e^{\mu t + \sigma^2 t^2/2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z - \sigma t)^2/2} dz$$

$$= e^{\mu t + \sigma^2 t^2/2},$$

since $\dfrac{1}{\sqrt{2\pi}} e^{-(z-\sigma t)^2/2}$ is the probability density function of a $N(\sigma t, 1)$ random variable.

Let $Y = aX + b$. Then $M_Y(t) = e^{bt} M_X(at)$.

If $X$ is normally distributed with mean $\mu$ and standard deviation $\sigma$, $Y$ is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$ (standard deviation $|a|\sigma$).

**2.** Suppose the average daily temperature [in degrees Fahrenheit] in July in Anytown is a random variable $T$ with mean $\mu_T = 85$ and standard deviation $\sigma_T = 7$. The daily air conditioning cost $Q$, in dollars, for Anytown State University, is related to $T$ by

$$Q = 120\,T + 750.$$

Suppose that $T$ is a normal random variable. Compute the probability that the daily air conditioning cost on a typical July day for the factory will exceed \$12,210.

$Q$ has Normal distribution.

$\mu_Q = 120\,\mu_T + 750 = 120 \cdot 85 + 750 = \$10{,}950.$

$\sigma_Q^2 = (120)^2 \cdot \sigma_T^2 = (120)^2 \cdot 7^2 = 840^2. \qquad \sigma_Q = \$840.$

$$P(Q > 12{,}210) = P\left(Z > \frac{12{,}210 - 10{,}950}{840}\right)$$

$$= P(Z > 1.50) = 1 - \Phi(1.50)$$

$$= 1 - 0.9332 = \mathbf{0.0668}.$$

OR

$12{,}210 = 120\,T + 750. \qquad \Leftrightarrow \qquad T = 95.5.$

$$P(Q > 12{,}210) = P(T > 95.5)$$

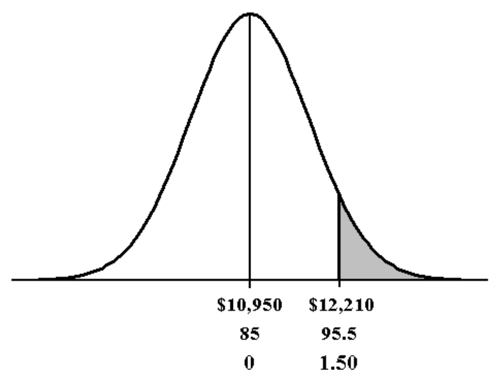
$$= P\left(Z > \frac{95.5 - 85}{7}\right)$$

$$= P(Z > 1.50)$$

$$= 1 - \Phi(1.50)$$

$$= 1 - 0.9332$$

$$= \mathbf{0.0668}.$$




Recall: $\mathrm{E}(a\,X + b\,Y) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y),$

$\mathrm{Var}(a\,X + b\,Y) = a^2\,\mathrm{Var}(X) + 2\,a\,b\,\mathrm{Cov}(X, Y) + b^2\,\mathrm{Var}(Y).$

If $X_1, X_2, \ldots, X_n$ are $n$ random variables and $a_0, a_1, a_2, \ldots, a_n$ are $n + 1$ constants, then the random variable $U = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$ has mean

$$\mathrm{E}(U) = a_0 + a_1\,\mathrm{E}(X_1) + a_2\,\mathrm{E}(X_2) + \ldots + a_n\,\mathrm{E}(X_n)$$

and variance

$$\mathrm{Var}(U) = \sum_{i=1}^{n} a_i^2\,\mathrm{Var}(X_i) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} a_i\,a_j\,\mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} a_i^2\,\mathrm{Var}(X_i) + 2\sum_{i<j}\sum a_i\,a_j\,\mathrm{Cov}(X_i, X_j)$$

If $X_1, X_2, \dots, X_n$ are $n$ independent random variables and $a_0, a_1, a_2, \dots, a_n$ are $n+1$ constants, then the random variable $U = a_0 + a_1 X_1 + a_2 X_2 + \dots + a_n X_n$ has variance

$$\mathrm{Var}(U) = a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2) + \dots + a_n^2\,\mathrm{Var}(X_n)$$

Also

$$M_U(t) = e^{a_0 t} \cdot M_{X_1}(a_1 t) \cdot M_{X_2}(a_2 t) \cdot \dots \cdot M_{X_n}(a_n t).$$

If $X_1, X_2, \dots, X_n$ are normally distributed random variables, then $U$ is also normally distributed.

**3.** Models of the pricing of stock options often make the assumption of a normal distribution. An investor believes that the price of an *Burger Queen* stock option is a normally distributed random variable with mean \$18 and standard deviation \$3. He also believes that the price of an *Dairy King* stock option is a normally distributed random variable with mean \$14 and standard deviation \$2. Assume the stock options of these two companies are independent. The investor buys 8 shares of *Burger Queen* stock option and 9 shares of *Dairy King* stock option. What is the probability that the value of this portfolio will exceed \$300?

BQ has Normal distribution, $\mu_{BQ} = \$18$, $\sigma_{BQ} = \$3$.

DK has Normal distribution, $\mu_{DK} = \$14$, $\sigma_{DK} = \$2$.

Value of the portfolio $VP = 8 \times BQ + 9 \times DK$.

Then VP has Normal distribution.

$$\mu_{VP} = 8 \times \mu_{BQ} + 9 \times \mu_{DK} = 8 \times 18 + 9 \times 14 = \$270.$$

$$\sigma_{VP}^2 = 8^2 \times \sigma_{BQ}^2 + 9^2 \times \sigma_{DK}^2 = 64 \times 9 + 81 \times 4 = 900. \qquad \sigma_{VP} = \$30.$$

$$P(VP > 300) = P\left(Z > \frac{300 - 270}{30}\right) = P(Z > 1.00) = 1 - 0.8413 = \mathbf{0.1587}.$$

**4.** In Neverland, the weights of adult men are normally distributed with mean of 170 pounds and standard deviation of 10 pounds, and the weights of adult women are normally distributed with mean of 125 pounds and standard deviation of 8 pounds. Six women and four men got on an elevator. Assume that all their weights are independent. What is the probability that their total weight exceeds 1500 pounds?

$$\text{Total} = W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + M_1 + M_2 + M_3 + M_4.$$

$$\begin{aligned} E(\text{Total}) &= E(W_1) + E(W_2) + E(W_3) + E(W_4) + E(W_5) + E(W_6) \\ &\quad + E(M_1) + E(M_2) + E(M_3) + E(M_4) \\ &= 125 + 125 + 125 + 125 + 125 + 125 + 170 + 170 + 170 + 170 = 1430. \end{aligned}$$

$$\begin{aligned} \text{Var}(\text{Total}) &= \text{Var}(W_1) + \text{Var}(W_2) + \text{Var}(W_3) + \text{Var}(W_4) + \text{VarE}(W_5) + \text{Var}(W_6) \\ &\quad + \text{Var}(M_1) + \text{Var}(M_2) + \text{Var}(M_3) + \text{Var}(M_4) \\ &= 64 + 64 + 64 + 64 + 64 + 64 + 100 + 100 + 100 + 100 = 784. \end{aligned}$$

$\text{SD}(\text{Total}) = \sqrt{784} = 28.$ Total has Normal distribution.

$$P(\text{ Total} > 1500 \text{ }) = P\left( Z > \frac{1500 - 1430}{28} \right) = P(\text{ } Z > 2.50 \text{ }) = 1 - 0.9938 = \mathbf{0.0062}.$$

<u>Note</u>: It is tempting to set Total = 6 W + 4 M, but that would imply that the six women who got on the elevator all have the same weight, and so do the four men, which is most likely not the case here.

**5.** A machine fastens plastic screw-on caps onto containers of motor oil. If the machine applies more torque than the cap can withstand, the cap will break. Both the torque applied and the strength of the caps vary. The capping machine torque has the normal distribution with mean 7.9 inch-pounds and standard deviation 0.9 inch-pounds. The cap strength (the torque that would break the cap) has the normal distribution with mean 10 inch-pounds and standard deviation 1.2 inch-pounds. The cap strength and the torque applied by the machine are independent. What is the probability that a cap will break while being fastened by the capping machine? That is, what is the probability $P(\text{Strength} < \text{Torque})$?

Need $P(\text{Strength} < \text{Torque}) = P(\text{Strength} - \text{Torque} < 0) = ?$

$E(\text{Strength} - \text{Torque}) = 10 - 7.9 = 2.1.$

$Var(\text{Strength} - \text{Torque}) = 1.2^2 + 0.9^2 = 2.25.$ $SD(\text{Strength} - \text{Torque}) = 1.5.$

$(\text{Strength} - \text{Torque})$ is normally distributed.

$$P(\text{Strength} - \text{Torque} < 0) = P\left(Z < \frac{0 - 2.1}{1.5}\right) = P(Z < -1.40) = \Phi(-1.40) = \mathbf{0.0808}.$$

**6.** One piece of PVC pipe is to be inserted inside another piece. The length of the first piece is normally distributed with mean value 20 in. and standard deviation 0.7 in. The length of the second piece is a normal random variable with mean and standard deviation 15 in. and 0.6 in., respectively. The amount of overlap is normally distributed with mean value 1 in. and standard deviation 0.2 in. Assuming that the lengths and amount of overlap are independent of one another, what is the probability that the total length after insertion is between 32.65 in. and 35.35 in.?

Total = First + Second – Overlap.

E(Total) = E(First) + E(Second) – E(Overlap) = 20 + 15 – 1 = 34.

$$\begin{aligned} Var(\text{Total}) &= Var(\text{First}) + Var(\text{Second}) + (-1)^2\, Var(\text{Overlap}) \\ &= 0.7^2 + 0.6^2 + 0.2^2 = 0.49 + 0.36 + 0.04 = 0.89. \end{aligned}$$

$SD(\text{Total}) = \sqrt{0.89} = 0.9434.$ Total has Normal distribution.

$$P(32.65 < \text{Total} < 35.35) = P\left(\frac{32.65 - 34}{0.9434} < Z < \frac{35.35 - 34}{0.9434}\right) = P(-1.43 < Z < 1.43)$$

$$= 0.9236 - 0.0764 = \mathbf{0.8472}.$$

**Bivariate Normal Distribution:**

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 -2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\},$$

$-\infty < x < \infty,\ \ -\infty < y < \infty.$

(a) the marginal distributions of X and Y are $\mathbf{N}\left(\mu_1, \sigma_1^2\right)$ and $\mathbf{N}\left(\mu_2, \sigma_2^2\right)$, respectively;

(b) the correlation coefficient of X and Y is $\rho_{XY} = \rho$, and X and Y are independent if and only if $\rho = 0$;

(c) the conditional distribution of Y, given X = $x$, is

$$\mathbf{N}\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \left(1-\rho^2\right)\sigma_2^2\right);$$

(d) the conditional distribution of X, given Y = $y$, is

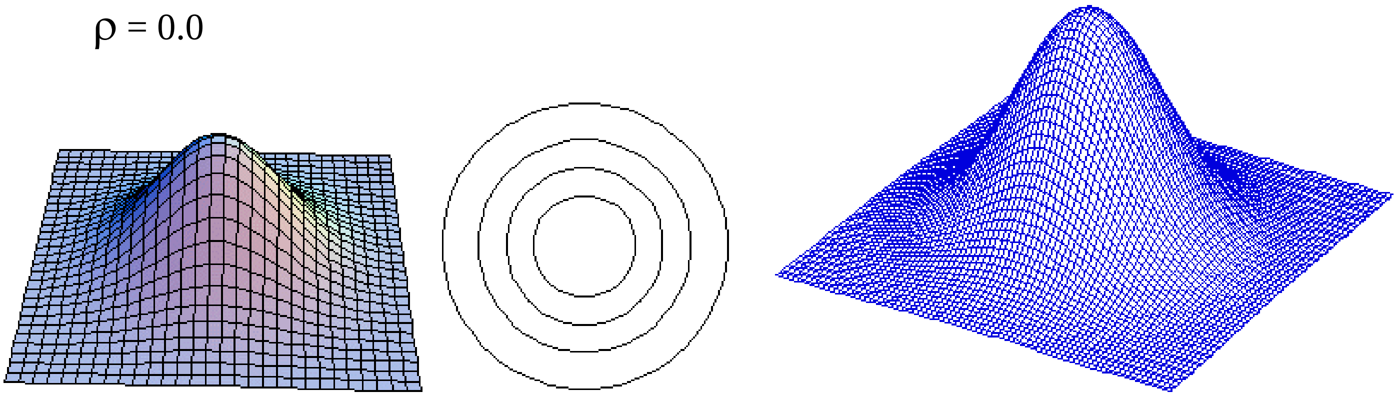
$$\mathbf{N}\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2),\ \left(1-\rho^2\right)\sigma_1^2\right).$$

(e) $a\,\mathrm{X} + b\,\mathrm{Y}$ is normally distributed with

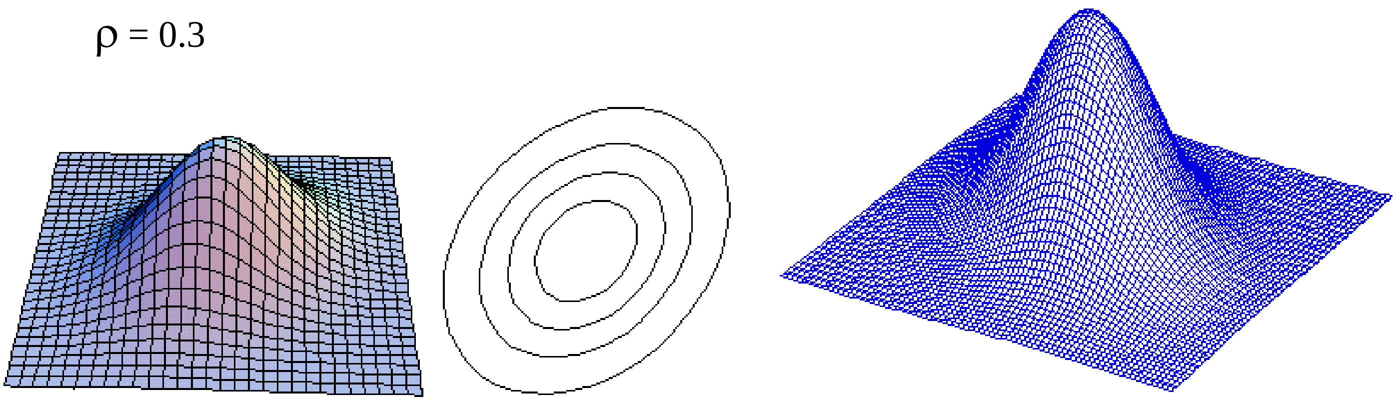
mean $\mathrm{E}(a\,\mathrm{X} + b\,\mathrm{Y}) = a\,\mu_1 + b\,\mu_2$ and

variance $\mathrm{Var}(a\,\mathrm{X} + b\,\mathrm{Y}) = a^2\sigma_1^2 + 2\,a\,b\,\rho\,\sigma_1\sigma_2 + b^2\sigma_2^2.$

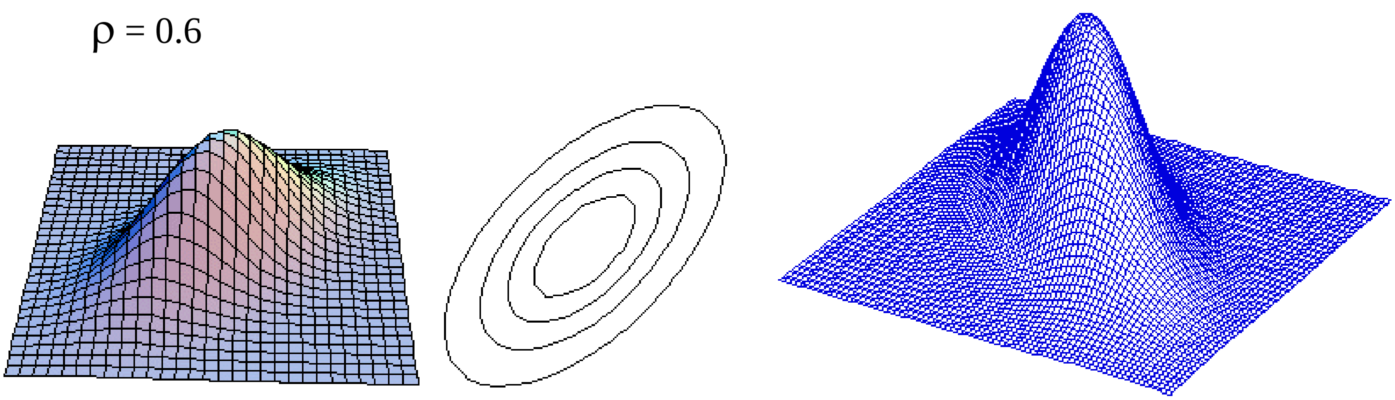
$\rho = 0.0$



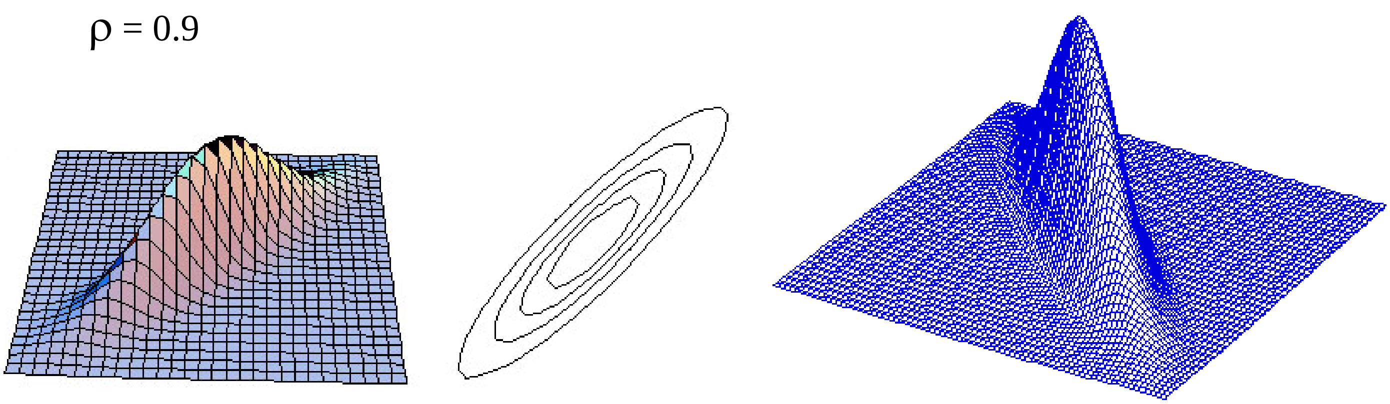
$\rho = 0.3$



$\rho = 0.6$



$\rho = 0.9$

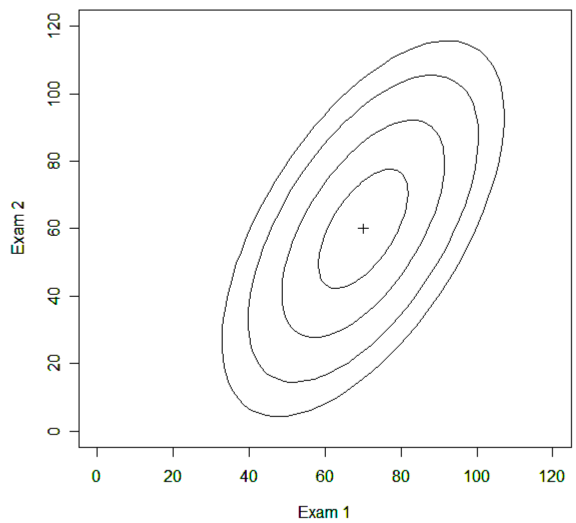


**1.** A large class took two exams. Suppose the exam scores X ( Exam 1 ) and Y ( Exam 2 ) follow a bivariate normal distribution with

$\mu_1 = 70, \qquad \sigma_1 = 10,$

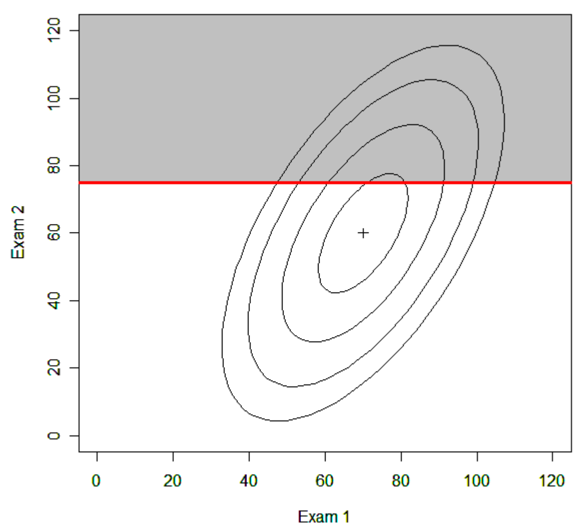
$\mu_2 = 60, \qquad \sigma_2 = 15,$

$\rho = 0.6.$




a) A students is selected at random. What is the probability that his/her score on Exam 2 is over 75?

$$P(Y > 75) = P\left(Z > \frac{75-60}{15}\right) = P(Z > 1.00) = \mathbf{0.1587}.$$



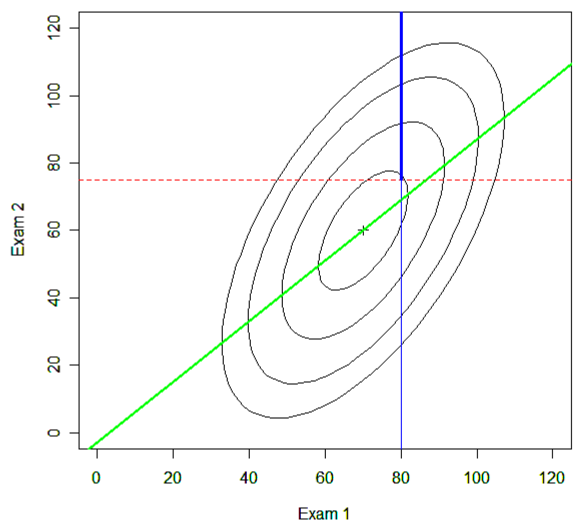
b) Suppose you're told that a student got a 80 on Exam 1. What is the probability that his/her score on Exam 2 is over 75?

Given X = 80, Y has Normal distribution

with mean $60 + 0.6 \cdot \frac{15}{10} \cdot (80 - 70) = 69$

and variance $\left(1 - 0.6^2\right) \cdot 15^2 = 144$ ( standard deviation 12 ).

$P(Y > 75 \mid X = 80) = P(Z > \frac{75 - 69}{12}) = P(Z > 0.50) =$ **0.3085**.



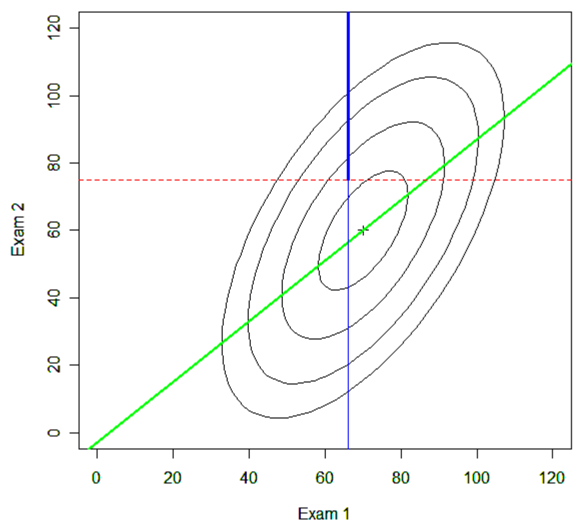
c) Suppose you're told that a student got a 66 on Exam 1. What is the probability that his/her score on Exam 2 is over 75?

Given X = 66, Y has Normal distribution

with mean $60 + 0.6 \cdot \frac{15}{10} \cdot (66 - 70) = 56.4$

and variance $\left(1 - 0.6^2\right) \cdot 15^2 = 144$ ( standard deviation 12 ).

$P(Y > 75 \mid X = 66) = P(Z > \frac{75 - 56.4}{12}) = P(Z > 1.55) = \mathbf{0.0606}$.



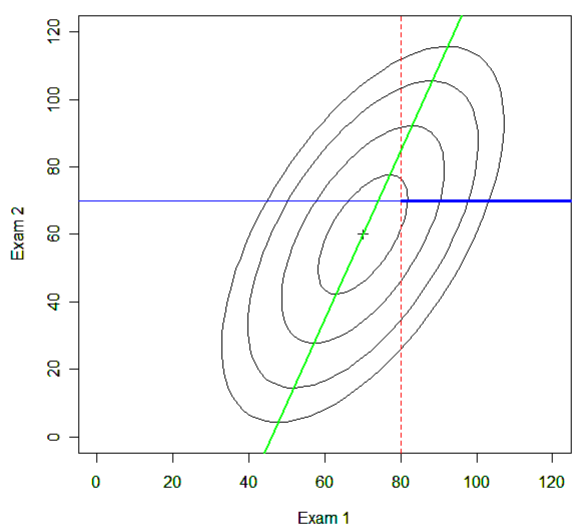
d) Suppose you're told that a student got a 70 on Exam 2. What is the probability that his/her score on Exam 1 is over 80?

Given Y = 70, X has Normal distribution

with mean $70 + 0.6 \cdot \frac{10}{15} \cdot (70 - 60) = 74$

and variance $\left(1 - 0.6^2\right) \cdot 10^2 = 64$ ( standard deviation 8 ).

$P(X > 80 \mid Y = 70) = P(Z > \frac{80 - 74}{8}) = P(Z > 0.75) = \mathbf{0.2266}$.



e) A students is selected at random. What is the probability that the sum of his/her Exam 1 and Exam 2 scores is over 150?

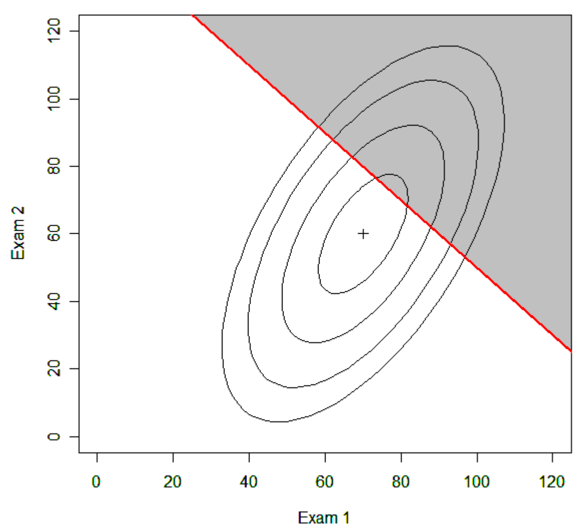
$X + Y$ has Normal distribution,

$$E(X+Y) = \mu_X + \mu_Y = 70 + 60 = 130,$$

$$\mathrm{Var}(X+Y) = \sigma_X^2 + 2\,\sigma_{XY} + \sigma_Y^2 = \sigma_X^2 + 2\,\rho\,\sigma_X\,\sigma_Y + \sigma_Y^2$$

$$= 10^2 + 2 \cdot 0.6 \cdot 10 \cdot 15 + 15^2 = 505 \quad (\text{ standard deviation} \approx 22.4722 ).$$

$$P(X+Y > 150) = P\left(Z > \frac{150-130}{22.4722}\right) = P(Z > 0.89) = \mathbf{0.1867}.$$



f) What proportion of students did better on Exam 1 than on Exam 2?

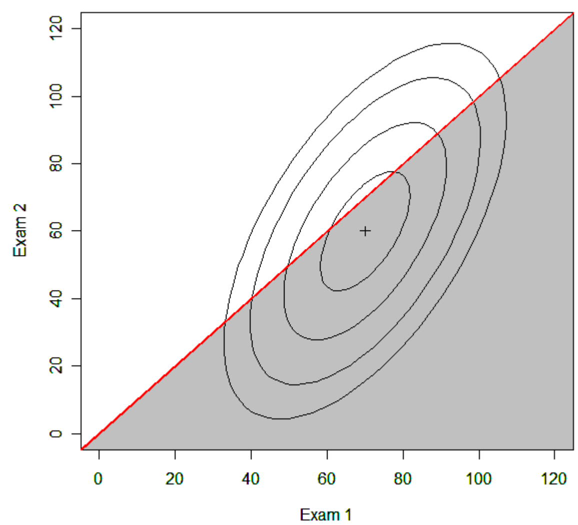
Want $P(X > Y) = P(X - Y > 0) = ?$

$X - Y$ has Normal distribution,

$E(X - Y) = \mu_X - \mu_Y = 70 - 60 = 10,$

$$\text{Var}(X - Y) = \sigma_X^2 - 2\sigma_{XY} + \sigma_Y^2 = \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2$$
$$= 10^2 - 2 \cdot 0.6 \cdot 10 \cdot 15 + 15^2 = 145 \quad (\text{standard deviation} \approx 12.0416).$$

$$P(X - Y > 0) = P\left(Z > \frac{0 - 10}{12.0416}\right) = P(Z > -0.83) = \mathbf{0.7967}.$$



g) Find $P(2X + 3Y > 350)$.

$2X + 3Y$ has Normal distribution,

$$E(2X + 3Y) = 2\mu_X + 3\mu_Y = 2 \times 70 + 3 \times 60 = 320,$$

$$Var(2X + 3Y) = 4\sigma_X^2 + 12\sigma_{XY} + 9\sigma_Y^2 = 4\sigma_X^2 + 12\rho\sigma_X\sigma_Y + 9\sigma_Y^2$$

$$= 4 \times 10^2 + 12 \cdot 0.6 \cdot 10 \cdot 15 + 9 \times 15^2 = 3505$$

( standard deviation $\approx 59.203$ ).

$$P(2X + 3Y > 350) = P(Z > \frac{350 - 320}{59.203}) = P(Z > 0.5067) \approx \mathbf{0.3050}.$$

h) Find $P(5X + 3Y < 570)$.

$5X + 3Y$ has Normal distribution,

$$E(5X + 3Y) = 5\mu_X + 3\mu_Y = 5 \times 70 + 3 \times 60 = 530,$$

$$Var(5X + 3Y) = 25\sigma_X^2 + 30\sigma_{XY} + 9\sigma_Y^2 = 25\sigma_X^2 + 30\rho\sigma_X\sigma_Y + 9\sigma_Y^2$$

$$= 25 \times 10^2 + 30 \cdot 0.6 \cdot 10 \cdot 15 + 9 \times 15^2 = 7225$$

( standard deviation = 85 ).

$$P(5X + 3Y < 570) = P(Z < \frac{570 - 530}{85}) = P(Z < 0.47) = \mathbf{0.6808}.$$

i) Find $P(5X - 4Y > 150)$.

$5X - 4Y$ has Normal distribution,

$$E(5X - 4Y) = 5\mu_X - 4\mu_Y = 5 \times 70 - 4 \times 60 = 110,$$

$$Var(5X - 4Y) = 25\sigma_X^2 - 40\sigma_{XY} + 16\sigma_Y^2 = 25\sigma_X^2 - 40\rho\sigma_X\sigma_Y + 16\sigma_Y^2$$

$$= 25 \times 10^2 - 40 \cdot 0.6 \cdot 10 \cdot 15 + 16 \times 15^2 = 2500$$

( standard deviation = 50 ).

$$P(5X - 4Y > 150) = P(Z > \frac{150 - 110}{50}) = P(Z > 0.80) = \mathbf{0.2119}.$$

**2.** Suppose that company A and company B are in the same industry sector, and the prices of their stocks, \$X per share for company A and \$Y per share for company B, vary from day to day randomly according to a bivariate normal distribution with parameters $\mu_X = 45$, $\sigma_X = 5.6$, $\mu_Y = 25$, $\sigma_Y = 5$, $\rho = 0.8$.

a) What is the probability that on a given day the price of stock for company B $(Y)$ exceeds \$33?

Y has Normal distribution with mean $\mu_Y = 25$

and standard deviation $\sigma_Y = 5$.

$$P(Y > 33) = P\left(Z > \frac{33-25}{5}\right) = P(Z > 1.60)$$

$$= 1 - \Phi(1.60) = 1 - 0.9452 = \mathbf{0.0548}.$$

b) Suppose that on a given day the price of stock for company A $(X)$ is \$52. What is the probability that the price of stock for company B $(Y)$ exceeds \$33?

Given $X = 52$, Y has Normal distribution

$$\text{with mean} \quad \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 25 + 0.8\cdot\frac{5}{5.6}\cdot(52-45) = 30$$

$$\text{and variance} \quad \left(1-\rho^2\right)\cdot\sigma_Y^2 = \left(1-0.8^2\right)\cdot 5^2 = 9$$

( standard deviation = 3 ).

$$P(Y > 33 \mid X = 52) = P\left(Z > \frac{33-30}{3}\right) = P(Z > 1.00) = 1 - \Phi(1.00)$$

$$= 1 - \Phi(1.00) = 1 - 0.8413 = \mathbf{0.1587}.$$

c) Alex bought 5 shares of company A stock and 3 shares of company B stock. What is the probability that on a given day the value of his portfolio $(5X + 3Y)$ is below \$250?

Portfolio = $5X + 3Y$.

Portfolio has Normal distribution

with mean $5\mu_X + 3\mu_Y = 5 \cdot 45 + 3 \cdot 25 = 300$

and variance

$$\text{Var}(5X + 3Y) = \text{Cov}(5X + 3Y, 5X + 3Y)$$

$$= \text{Cov}(5X, 5X) + \text{Cov}(5X, 3Y) + \text{Cov}(3Y, 5X) + \text{Cov}(3Y, 3Y)$$

$$= 25\,\sigma_X^2 + 30\,\sigma_{XY} + 9\,\sigma_Y^2 = 25\,\sigma_X^2 + 30\,\rho\,\sigma_X\sigma_Y + 9\,\sigma_Y^2$$

$$= 25 \cdot 5.6^2 + 30 \cdot 0.8 \cdot 5.6 \cdot 5 + 9 \cdot 5^2 = 1681$$

( standard deviation = $\sqrt{1681} = 41$ ).

$$P(\text{Portfolio} < 250) = P\left(Z < \frac{250 - 300}{41}\right) = P(Z < -1.22) = \Phi(-1.22)$$

$$= \mathbf{0.1112}.$$

d) What is the probability that 1 share of company A stock is worth more than 2 shares of company B stock?

Want $P(X > 2Y) = P(X - 2Y > 0) = ?$

$X - 2Y$ has Normal distribution,

$E(X - 2Y) = \mu_X - 2\mu_Y = 45 - 2 \cdot 25 = -5$,

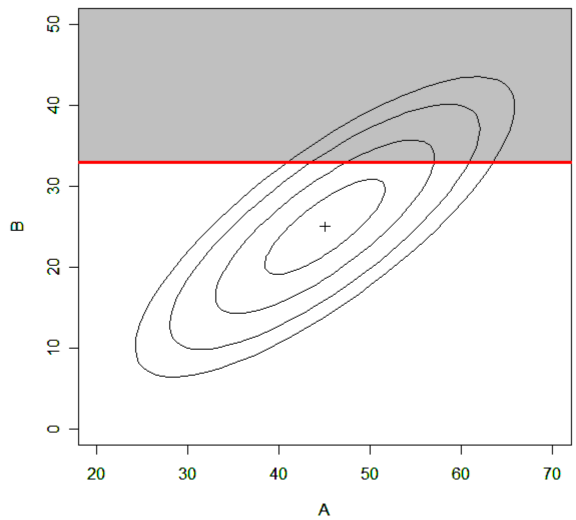
$$\text{Var}(X - 2Y) = \sigma_X^2 - 4\,\sigma_{XY} + 4\sigma_Y^2 = \sigma_X^2 - 4\,\rho\,\sigma_X\,\sigma_Y + 4\sigma_Y^2$$

$$= 5.6^2 - 4 \cdot 0.8 \cdot 5.6 \cdot 5 + 4 \cdot 5^2 = 41.76$$ ( standard deviation $\approx 6.462$ ).

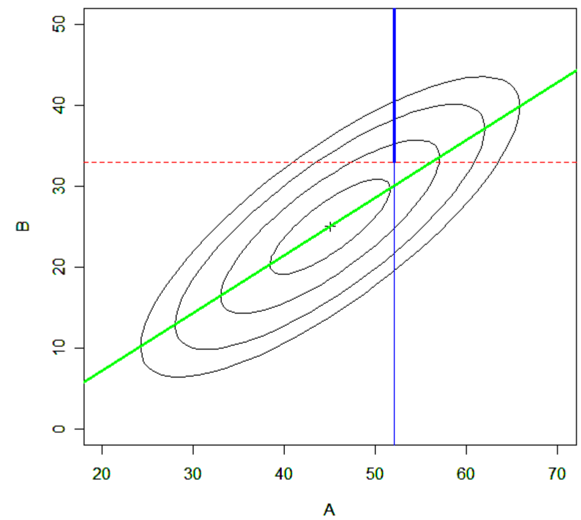
$$P(X - 2Y > 0) = P\left(Z > \frac{0 + 5}{6.462}\right) = P(Z > 0.77) = 1 - \Phi(0.77)$$

$$= 1 - 0.7794 = \mathbf{0.2206}.$$

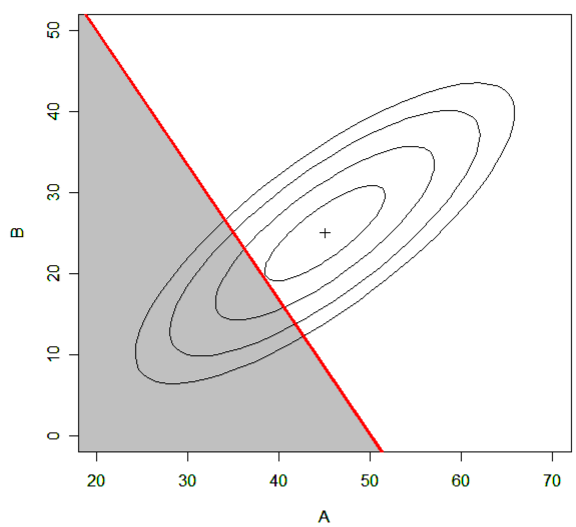
a)


B
0 10 20 30 40 50
20 30 40 50 60 70
A

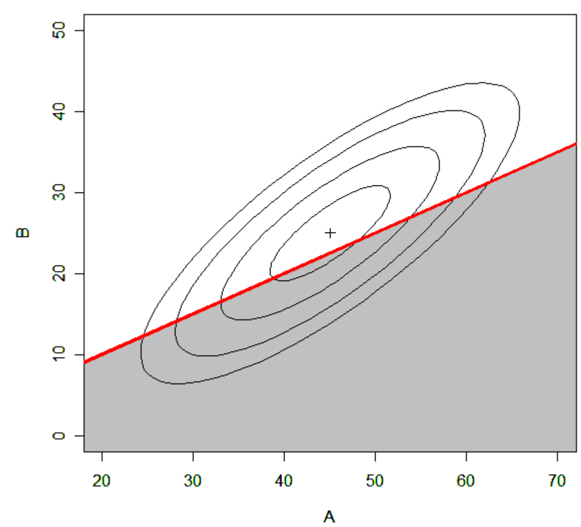
b)


B
0 10 20 30 40 50
20 30 40 50 60 70
A

c)


B
0 10 20 30 40 50
20 30 40 50 60 70
A

d)


B
0 10 20 30 40 50
20 30 40 50 60 70
A

**3.** In a college health fitness program, let X denote the weight in kilograms of a male freshman at the beginning of the program and let Y denote his weight change during a semester. Assume that X and Y have a bivariate normal distribution with $\mu_X = 75$, $\sigma_X = 9$, $\mu_Y = 2.5$, $\sigma_Y = 1.5$, $\rho = -0.6$. (The lighter students tend to gain weight, while the heavier students tend to lose weight.)

$\mu_X = 75$, $\sigma_X = 9$, $\mu_Y = 2.5$, $\sigma_Y = 1.5$, $\rho = -0.6$.

a) What proportion of the students that weigh 85 kg end up losing weight during the semester? That is, find $P(Y < 0 \mid X = 85)$.

Given X = 85, Y has Normal distribution

with mean $\mu_Y + \rho \dfrac{\sigma_Y}{\sigma_X}(x - \mu_X) = 2.5 + (-0.6) \cdot \dfrac{1.5}{9} \cdot (85 - 75) = 1.5$

and variance $\left(1 - \rho^2\right) \cdot \sigma_Y^2 = \left(1 - (-0.6)^2\right) \cdot 1.5^2 = 1.44$

( standard deviation = 1.2 ).

$$P(Y < 0 \mid X = 85) = P\left(Z < \frac{0 - 1.5}{1.2}\right) = P(Z < -1.25) = \Phi(-1.25) = \mathbf{0.1056}.$$

b) What proportion of the students that weigh over 87 kg at the end of the semester? That is, find $P(X + Y > 87)$.

X + Y has Normal distribution,

$E(X + Y) = \mu_X + \mu_Y = 75 + 2.5 = 77.5$,

$\operatorname{Var}(X + Y) = \sigma_X^2 + 2\sigma_{XY} + \sigma_Y^2 = \sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2$

$= 9^2 + 2 \cdot (-0.6) \cdot 9 \cdot 1.5 + 1.5^2 = 67.05.$

$SD(X + Y) \approx 8.1884.$

$$P(X + Y > 87) = P\left(Z > \frac{87 - 77.5}{8.1884}\right) = P(Z > 1.16) = \mathbf{0.1230}.$$