Examples for 03/03/2022

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Time until the next jump τ:

Exponential with rate q_x

$$f_x(t) = q_x e^{-q_x t}, \quad t > 0.$$

Transition probabilities:

$$\mathrm{Q}_{xy} = \mathrm{P}_x(\mathrm{X}(\tau) = y), \quad x, y \in S, \quad x \neq y.$$

$$\mathrm{Q}_{xx} = 0. \qquad \sum_{y \in S} \mathrm{Q}_{xy} = 1, \quad x \in S.$$

Transition functions: $\mathrm{P}_{xy}(t) = \mathrm{P}_x(\mathrm{X}(t) = y), \quad t \geq 0, \quad x, y \in S.$

$$\sum_{y \in S} \mathrm{P}_{xy}(t) = 1, \quad t \geq 0, \quad x \in S. \qquad \mathrm{P}_{xy}(0) = \delta_{xy}, \quad x, y \in S.$$

Chapman-Kolmogorov equation:

$$\mathrm{P}_{xy}(t+s) = \sum_{z \in S} \mathrm{P}_{xz}(t)\,\mathrm{P}_{zy}(s), \quad s, t \geq 0, \quad x, y \in S.$$

Infinitesimal parameters: $q_{xy} = \mathrm{P}'_{xy}(0).$

$$q_{xx} = -q_x, \qquad q_{xy} = q_x \mathrm{Q}_{xy}, \quad x \neq y. \qquad \sum_{y \in S} q_{xy} = 0, \quad x \in S.$$

Forward equation:

$$\mathrm{P}'_{xy}(t) = \sum_{z \in S} \mathrm{P}_{xz}(t)\, q_{zy}$$

Backward equation:

$$\mathrm{P}'_{xy}(t) = \sum_{z \in S} q_{xz}\, \mathrm{P}_{zy}(t)$$

Some special cases of differential equations:

If $f'(t) = -\alpha f(t), \quad t \ge 0,$ then

$$f(t) = f(0)\, e^{-\alpha t}, \quad t \ge 0.$$

If $f'(t) = g(t), \quad t \ge 0,$ then

$$f(t) = f(0) + \int_0^t g(s)\,ds, \quad t \ge 0.$$

If $f'(t) = -\alpha f(t) + g(t), \quad t \ge 0,$ then

$$f(t) = f(0)\, e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}\, g(s)\,ds, \quad t \ge 0.$$

1. Suppose the time spent in Anytown Emergency Room is exponentially distributed with mean 4 hours (rate $q_{\mathrm{E}} = 6$). Suppose also that $1/3$ of the ER patients are admitted to the Anytown Hospital, and $2/3$ are discharged. The time spent in the Hospital is exponentially distributed with mean 2 days (rate $q_{\mathrm{H}} = 0.5$). Consider a Markov pure jump process $\mathrm{X}(t)$ with three states $\{$ E(mergency Room), H(ospital), D(ischarged) $\}$. D(ischarged) is an absorbing state. Suppose a person has just arrived to the Emergency Room. Find the transition functions

$$\mathrm{P}_{\mathrm{EE}}(t), \qquad \mathrm{P}_{\mathrm{EH}}(t), \qquad \mathrm{P}_{\mathrm{ED}}(t), \qquad t \ge 0.$$

$$\begin{tikzcd} \mathrm{E} \arrow[rr] \arrow[dr] & & \mathrm{D} \\ & \mathrm{H} \arrow[ur] & \end{tikzcd}$$

a) Identify all infinitesimal parameters of $\mathrm{X}(t)$.

$\mathrm{Q}_{\mathrm{EH}} = \frac{1}{3}, \quad \mathrm{Q}_{\mathrm{ED}} = \frac{2}{3} \qquad\qquad \mathrm{Q}_{\mathrm{HD}} = 1$

$q_{\mathrm{EE}} = -q_{\mathrm{E}} = -6$ $\quad q_{\mathrm{EH}} = q_{\mathrm{E}}\,\mathrm{Q}_{\mathrm{EH}} = 2$ $\quad q_{\mathrm{ED}} = q_{\mathrm{E}}\,\mathrm{Q}_{\mathrm{ED}} = 4$

$q_{\mathrm{HE}} = q_{\mathrm{H}}\,\mathrm{Q}_{\mathrm{HE}} = 0$ $\quad q_{\mathrm{HH}} = -q_{\mathrm{H}} = -0.5$ $\quad q_{\mathrm{HD}} = q_{\mathrm{H}}\,\mathrm{Q}_{\mathrm{HD}} = 0.5$

$q_{\mathrm{DE}} = 0$ $\quad q_{\mathrm{DH}} = 0$ $\quad q_{\mathrm{DD}} = 0$

b) Use forward equations to find $\mathrm{P}_{\mathrm{EE}}(t)$, $\mathrm{P}_{\mathrm{EH}}(t)$, and $\mathrm{P}_{\mathrm{ED}}(t)$, $t \ge 0$.

$$\mathrm{P}'_{\mathrm{EE}}(t) = \mathrm{P}_{\mathrm{EE}}(t)\,q_{\mathrm{EE}} + \mathrm{P}_{\mathrm{EH}}(t)\,q_{\mathrm{HE}} + \mathrm{P}_{\mathrm{ED}}(t)\,q_{\mathrm{DE}} = -6\,\mathrm{P}_{\mathrm{EE}}(t).$$

$$\mathrm{P}_{\mathrm{EE}}(0) = 1. \qquad \Rightarrow \qquad \mathrm{P}_{\mathrm{EE}}(t) = e^{-6t}, \qquad t \ge 0.$$

$$\mathrm{P}'_{\mathrm{EH}}(t) = \mathrm{P}_{\mathrm{EE}}(t)\,q_{\mathrm{EH}} + \mathrm{P}_{\mathrm{EH}}(t)\,q_{\mathrm{HH}} + \mathrm{P}_{\mathrm{ED}}(t)\,q_{\mathrm{DH}} = 2\,e^{-6t} - 0.5\,\mathrm{P}_{\mathrm{EH}}(t).$$

$$\mathrm{P}_{\mathrm{EH}}(0) = 0.$$

$$\Rightarrow \qquad \mathrm{P}_{\mathrm{EH}}(t) = \int_0^t e^{-0.5\,(t-s)}\,2\,e^{-6s}\,ds = 2\,e^{-0.5\,t}\int_0^t e^{-5.5\,s}\,ds$$

$$= \frac{4}{11}\left(e^{-0.5\,t} - e^{-6\,t}\right), \qquad t \ge 0.$$

$$\mathrm{P}_{\mathrm{ED}}(t) = 1 - \mathrm{P}_{\mathrm{EE}}(t) - \mathrm{P}_{\mathrm{EH}}(t) = 1 - \frac{4}{11}\,e^{-0.5\,t} - \frac{7}{11}\,e^{-6\,t}, \qquad t \ge 0.$$

OR

$$\mathrm{P}'_{\mathrm{ED}}(t) = \mathrm{P}_{\mathrm{EE}}(t)\,q_{\mathrm{ED}} + \mathrm{P}_{\mathrm{EH}}(t)\,q_{\mathrm{HD}} + \mathrm{P}_{\mathrm{ED}}(t)\,q_{\mathrm{DD}}$$

$$= 4\,e^{-6t} + \frac{2}{11}\left(e^{-0.5\,t} - e^{-6\,t}\right) = \frac{2}{11}\,e^{-0.5\,t} + \frac{42}{11}\,e^{-6\,t}.$$

$P_{ED}(0) = 0.$

$$\Rightarrow \qquad P_{EH}(t) = \int_0^t \left(\frac{2}{11} e^{-0.5s} + \frac{42}{11} e^{-6s} \right) ds$$

$$= 1 - \frac{4}{11} e^{-0.5t} - \frac{7}{11} e^{-6t}, \qquad t \ge 0.$$

c) Use backward equations to find $P_{EE}(t)$, $P_{EH}(t)$, and $P_{ED}(t)$, $t \ge 0.$

$$P'_{EE}(t) = q_{EE} P_{EE}(t) + q_{EH} P_{HE}(t) + q_{ED} P_{DE}(t) = -6 P_{EE}(t).$$

$$P_{EE}(0) = 1. \qquad \Rightarrow \qquad P_{EE}(t) = e^{-6t}, \qquad t \ge 0.$$

$$P'_{HH}(t) = q_{HE} P_{EH}(t) + q_{HH} P_{HH}(t) + q_{HD} P_{DH}(t) = -0.5 P_{HH}(t).$$

$$P_{HH}(0) = 1. \qquad \Rightarrow \qquad P_{HH}(t) = e^{-0.5t}, \qquad t \ge 0.$$

$$P'_{EH}(t) = q_{EE} P_{EH}(t) + q_{EH} P_{HH}(t) + q_{ED} P_{DH}(t) = -6 P_{EH}(t) + 2 e^{-0.5t}.$$

$P_{EH}(0) = 0.$

$$\Rightarrow \qquad P_{EH}(t) = \int_0^t e^{-6(t-s)} 2 e^{-0.5s} \, ds = 2 e^{-6t} \int_0^t e^{5.5s} \, ds$$

$$= \frac{4}{11} \left(e^{-0.5t} - e^{-6t} \right), \qquad t \ge 0.$$

$$P_{ED}(t) = 1 - P_{EE}(t) - P_{EH}(t) = 1 - \frac{4}{11} e^{-0.5t} - \frac{7}{11} e^{-6t}, \qquad t \ge 0.$$

OR

$$P'_{ED}(t) = q_{EE}\,P_{ED}(t) + q_{EH}\,P_{HD}(t) + q_{ED}\,P_{DD}(t)$$
$$= -6\,P_{ED}(t) + 2\,P_{HD}(t) + 4\,P_{DD}(t).$$

Need $P_{HD}(t)$ and $P_{DD}(t)$.

$P_{DD}(t) = 1.$

$$P_{HD}(t) = 1 - P_{HE}(t) - P_{HH}(t) = 1 - 0 - e^{-0.5\,t} = 1 - e^{-0.5\,t}, \qquad t \ge 0.$$

or

$$P'_{HD}(t) = q_{HE}\,P_{ED}(t) + q_{HH}\,P_{HD}(t) + q_{HD}\,P_{DD}(t) = -0.5\,P_{HD}(t) + 0.5.$$

$P_{HD}(0) = 0.$

$$\Rightarrow \qquad P_{EH}(t) = \int_0^t e^{-0.5\,(t-s)}\,0.5\,ds = e^{-0.5\,t}\int_0^t 0.5\,e^{0.5\,s}\,ds$$
$$= 1 - e^{-0.5\,t}, \qquad t \ge 0.$$

Then $$P'_{ED}(t) = -6\,P_{ED}(t) + 2\,P_{HD}(t) + 4\,P_{DD}(t)$$
$$= -6\,P_{ED}(t) + 2 - 2\,e^{-0.5\,t} + 4 = -6\,P_{ED}(t) + 6 - 2\,e^{-0.5\,t}.$$

$P_{ED}(0) = 0.$

$$\Rightarrow \qquad P_{ED}(t) = \int_0^t e^{-6\,(t-s)}\left(6 - 2\,e^{-0.5\,s}\right)ds$$
$$= e^{-6\,t}\int_0^t \left(6\,e^{6\,s} - 2\,e^{5.5\,s}\right)ds$$
$$= 1 - e^{-6\,t} - \frac{4}{11}\,e^{-0.5\,t} + \frac{4}{11}\,e^{-6\,t}$$
$$= 1 - \frac{4}{11}\,e^{-0.5\,t} - \frac{7}{11}\,e^{-6\,t}, \qquad t \ge 0.$$

2. Sue's sewing machine is very old, and it malfunctions often. When a machine fails, it needs either a small repair (which happens with probability 0.75) or a large repair (probability 0.25). If the machine needs a small repair, the time of the repair is exponentially distributed with mean 3 minutes (rate = 20). If the machine needs a large repair, the time of the repair is exponential with mean 6 minutes (rate = 10). After a repair, the machine works for an exponentially distributed time with mean 15 minutes (rate = 4). Assume that all times are independent. Consider a Markov pure jump process $X(t)$ with three states $\{$W(orking), S(mall repair), L(arge repair)$\}$.

a) Identify all infinitesimal parameters of $X(t)$.

$q_W = 4$ $\qquad$ $q_S = 20$ $\qquad$ $q_L = 10$

$Q_{WS} = 0.75,\quad Q_{WL} = 0.25$ $\qquad$ $Q_{SW} = 1$ $\qquad$ $Q_{LW} = 1$

$q_{WW} = -4$ $\qquad$ $q_{WS} = 3$ $\qquad$ $q_{WL} = 1$

$q_{SW} = 20$ $\qquad$ $q_{SS} = -20$ $\qquad$ $q_{SL} = 0$

$q_{LW} = 10$ $\qquad$ $q_{LS} = 0$ $\qquad$ $q_{LL} = -10$

b) Write the forward equation for the transition functions $P_{W\bullet}(t)$.
Include the initial condition.

$$P'_{xy}(t) = \sum_{z \in S} P_{xz}(t)\, q_{zy}$$

$$P'_{WW}(t) = P_{WW}(t)\, q_{WW} + P_{WS}(t)\, q_{SW} + P_{WL}(t)\, q_{LW}$$

$$= -4\, P_{WW}(t) + 20\, P_{WS}(t) + 10\, P_{WL}(t).$$

$$P'_{WS}(t) = P_{WW}(t)\,q_{WS} + P_{WS}(t)\,q_{SS} + P_{WL}(t)\,q_{LS}$$

$$= 3\,P_{WW}(t) - 20\,P_{WS}(t).$$

$$P'_{WL}(t) = P_{WW}(t)\,q_{WL} + P_{WS}(t)\,q_{SL} + P_{WL}(t)\,q_{LL}$$

$$= P_{WW}(t) - 10\,P_{WL}(t).$$

$P_{WW}(0) = 1,$ $\qquad$ $P_{WS}(0) = 0,$ $\qquad$ $P_{WL}(0) = 0.$

c) Write the backward equation for the transition functions $P_{\bullet W}(t)$.
Include the initial condition.

$$P'_{xy}(t) = \sum_{z \in S} q_{xz}\,P_{zy}(t)$$

$$P'_{WW}(t) = q_{WW}\,P_{WW}(t) + q_{WS}\,P_{SW}(t) + q_{WL}\,P_{LW}(t)$$

$$= -4\,P_{WW}(t) + 3\,P_{SW}(t) + P_{LW}(t).$$

$$P'_{SW}(t) = q_{SW}\,P_{WW}(t) + q_{SS}\,P_{SW}(t) + q_{SL}\,P_{LW}(t)$$

$$= 20\,P_{WW}(t) - 20\,P_{SW}(t).$$

$$P'_{LW}(t) = q_{LW}\,P_{WW}(t) + q_{LS}\,P_{SW}(t) + q_{LL}\,P_{LW}(t)$$

$$= 10\,P_{WW}(t) - 10\,P_{LW}(t).$$

$P_{WW}(0) = 1,$ $\qquad$ $P_{SW}(0) = 0,$ $\qquad$ $P_{LW}(0) = 0.$