STAT 433

Spring 2022
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Examples for 01/18/2022

Multiplication Rule:

$$P(A \cap B) = P(A) \times P(B \mid A) \qquad \text{since} \qquad P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Similarly,

$$P(A \cap B \cap C) = P(A) \times P(B \mid A) \times P(C \mid A \cap B)$$

$$\text{since} \qquad P(A \cap B \cap C) = P(A) \times \frac{P(A \cap B)}{P(A)} \times \frac{P(A \cap B \cap C)}{P(A \cap B)}$$

$$P(A \cap B \cap C \cap D) = P(A) \times P(B \mid A) \times P(C \mid A \cap B) \times P(D \mid A \cap B \cap C)$$

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Furthermore,

$$P(A \cap B \mid C) = P(A \mid C) \times P(B \mid A \cap C)$$

$$\text{since} \qquad \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap C)}{P(C)} \times \frac{P(A \cap B \cap C)}{P(A \cap C)}$$

Law of Total Probability:

$$P(A) = P(A \cap B) + P(A \cap B') = P(B) \times P(A \mid B) + P(B') \times P(A \mid B')$$

If $B_1 \cup B_2 \cup \ldots \cup B_m = S$ and $B_i \cap B_j = \varnothing$ for $i \neq j$,

$$P(A) = P(B_1) \times P(A \mid B_1) + P(B_2) \times P(A \mid B_2) + \ldots + P(B_m) \times P(A \mid B_m)$$

Similarly,

$$P(A \mid C) = P(B \mid C) \times P(A \mid B \cap C) + P(B' \mid C) \times P(A \mid B' \cap C)$$

If $B_1 \cup B_2 \cup \ldots \cup B_m = S$ and $B_i \cap B_j = \varnothing$ for $i \neq j$,

$$P(A \mid C) = P(B_1 \mid C) \times P(A \mid B_1 \cap C) + P(B_2 \mid C) \times P(A \mid B_2 \cap C) + \ldots + P(B_m \mid C) \times P(A \mid B_m \cap C)$$

Markov Chain: $X_0, X_1, X_2, \dots, X_{n-1}, X_n, X_{n+1}, \dots$

$$P(X_{n+1}=x_{n+1} \mid X_n=x_n, X_{n-1}=x_{n-1}, \dots, X_0=x_0) = P(X_{n+1}=x_{n+1} \mid X_n=x_n)$$

$x_0, \dots, x_{n-1}, x_n, x_{n+1} \in S$ – state space

$P(X_{n+1}=x_{n+1} \mid X_n=x_n)$ – transition probabilities

stationary transition probabilities – $P(X_{n+1}=x_{n+1} \mid X_n=x_n)$ does not depend on n

$$P(x,y) = P^1(x,y) = P(X_1=y \mid X_0=x) = P(X_{n+1}=y \mid X_n=x)$$

$$\forall x \in S \qquad \sum_{y \in S} P(x,y) = 1$$

$$P^n(x,y) = P(X_n=y \mid X_0=x) = P(X_{n+m}=y \mid X_m=x)$$

$$P^0(x,y) = P(X_0=y \mid X_0=x) = \begin{cases} 1 & x=y \\ 0 & \text{otherwise} \end{cases}$$

initial distribution

$\pi_0(x) = P(X_0=x) \qquad x \in S \qquad \sum_{x \in S} \pi_0(x) = 1$

$\pi_n(x) = P(X_n=x) \qquad x \in S \qquad \sum_{x \in S} \pi_n(x) = 1$

$$P(X_0=x_0, X_1=x_1) = \pi_0(x_0) \times P(x_0,x_1)$$

$$P(X_0=x_0, X_1=x_1, \dots, X_{n-1}=x_{n-1}, X_n=x_n)$$
$$= \pi_0(x_0) \times P(x_0,x_1) \times \dots \times P(x_{n-1},x_n)$$

$$P(X_{n+m}=x_{n+m}, \dots, X_{n+1}=x_{n+1} \mid X_n=x_n, X_{n-1}=x_{n-1}, \dots, X_0=x_0)$$
$$= P(x_n,x_{n+1}) \times \dots \times P(x_{n+m-1},x_{n+m})$$

Fact: $$\mathrm{P}^2(x,y) = \mathrm{P}(\mathrm{X}_2 = y \mid \mathrm{X}_0 = x) = \sum_{z\in S} \mathrm{P}(x,z)\times \mathrm{P}(z,y)$$

Proof: Since $\{\mathrm{X}_2 = y\} = \bigcup_{z\in S} \{\mathrm{X}_1 = z,\ \mathrm{X}_2 = y\}$,

$$\begin{aligned}
\mathrm{P}^2(x,y) = \mathrm{P}(\mathrm{X}_2 = y \mid \mathrm{X}_0 = x) &= \sum_{z\in S} \mathrm{P}\left(\mathrm{X}_1 = z,\ \mathrm{X}_2 = y \,\middle|\, \mathrm{X}_0 = x\right) \\
&= \sum_{z\in S} \mathrm{P}\left(\mathrm{X}_1 = z \,\middle|\, \mathrm{X}_0 = x\right)\times \mathrm{P}\left(\mathrm{X}_2 = y \,\middle|\, \mathrm{X}_1 = z,\ \mathrm{X}_0 = x\right) \\
&= \sum_{z\in S} \mathrm{P}\left(\mathrm{X}_1 = z \,\middle|\, \mathrm{X}_0 = x\right)\times \mathrm{P}\left(\mathrm{X}_2 = y \,\middle|\, \mathrm{X}_1 = z\right) = \sum_{z\in S} \mathrm{P}(x,z)\times \mathrm{P}(z,y).
\end{aligned}$$

$$\mathrm{P}^{n+1}(x,y) = \mathrm{P}(\mathrm{X}_{n+1} = y \mid \mathrm{X}_0 = x) = \sum_{z\in S} \mathrm{P}(x,z)\times \mathrm{P}^n(z,y)$$

$$\mathrm{P}^{n+1}(x,y) = \mathrm{P}(\mathrm{X}_{n+1} = y \mid \mathrm{X}_0 = x) = \sum_{z\in S} \mathrm{P}^n(x,z)\times \mathrm{P}(z,y)$$

$$\mathrm{P}^{n+m}(x,y) = \mathrm{P}(\mathrm{X}_{n+m} = y \mid \mathrm{X}_0 = x) = \sum_{z\in S} \mathrm{P}^n(x,z)\times \mathrm{P}^m(z,y)$$

$$\mathrm{P}^3(x,y) = \mathrm{P}(\mathrm{X}_3 = y \mid \mathrm{X}_0 = x) = \sum_{z_1\in S}\sum_{z_2\in S} \mathrm{P}(x,z_1)\times \mathrm{P}(z_1,z_2)\times \mathrm{P}(z_2,y)$$

$$\pi_n(x) = \mathrm{P}(\mathrm{X}_n = x) = \sum_{y\in S} \pi_0(y)\times \mathrm{P}^n(y,x)$$

$$\pi_{n+1}(x) = \mathrm{P}(\mathrm{X}_{n+1} = x) = \sum_{y\in S} \pi_n(y)\times \mathrm{P}(y,x)$$

1. The winter weather in Central Illinois is not kind – there are never two nice days in a row. If there is a nice day, the next day would either have rain or snow with equal chance. If it rains on a given day, there is a 50% chance it would rain again the next day, a 25% chance it would snow, and a 25% chance the next day would be nice. If it snows on a given day, there is equal chance that it would rain, it would snow, or the day would be nice.

$S = \{N, R, S\}$

$P(N, N) =$	$P(N, R) =$	$P(N, S) =$
$P(R, N) =$	$P(R, R) =$	$P(R, S) =$
$P(S, N) =$	$P(S, R) =$	$P(S, S) =$

The weather forecast for Monday claims that there is a 50% chance it would be nice, 20% or rain, and 30% of snow.

$\pi_0(N) =$ $\qquad$ $\pi_0(R) =$ $\qquad$ $\pi_0(S) =$

a) Find the probability that Monday would be nice, and it would rain on Tuesday.

$P(X_0 = N,\ X_1 = R) =$

b) Find the probability that it would rain on Monday, nice on Tuesday, snow on Wednesday, snow again on Thursday, and nice on Friday.

$P(X_0 = R,\ X_1 = N,\ X_2 = S,\ X_3 = S,\ X_4 = N) =$

c) Find the probability that it would be nice on Tuesday.

$P(X_1 = N) = \pi_0(N) \times P(N, N) + \pi_0(R) \times P(R, N) + \pi_0(S) \times P(S, N)$

d) Find the probability that it would rain on Tuesday.

$P(X_1 = R) =$

e) Find the probability that it would rain on Wednesday.

$P(X_2 = R) =$

f) Given that it is nice on Monday, what is the probability it would snow on Wednesday?

$$P(X_1 = N \mid X_0 = N) \times P(X_2 = S \mid X_1 = N) + P(X_1 = R \mid X_0 = N) \times P(X_2 = S \mid X_1 = R)$$
$$+ P(X_1 = S \mid X_0 = N) \times P(X_2 = S \mid X_1 = S)$$
$$= P(N, N) \times P(N, S) + P(N, R) \times P(R, S) + P(N, S) \times P(S, S)$$

g) Given that it rains on Monday, what is the probability it would be nice on Wednesday?

$$P(R, N) \times P(N, N) + P(R, R) \times P(R, N) + P(R, S) \times P(S, N)$$

h) Given that it snows on Monday, what is the probability it would rain on Wednesday?

Suppose $S = \{ 0, 1, \dots , d \}$.

The transition matrix

$$\begin{array}{c} \\ 0 \\ \vdots \\ i \\ \vdots \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & \cdots & j & \cdots & d \end{array} \\ \left[\begin{array}{ccccc} P(0,0) & \cdots & P(0,j) & \cdots & P(0,d) \\ \vdots & & \vdots & & \vdots \\ P(i,0) & \cdots & P(i,j) & \cdots & P(i,d) \\ \vdots & & \vdots & & \vdots \\ P(d,0) & \cdots & P(d,j) & \cdots & P(d,d) \end{array} \right] \end{array} = \mathbf{P}$$

$$\mathbf{P}_{ij} = P(i,j) = P^1(i,j) = P(X_1 = j \mid X_0 = i)$$

$$[\mathbf{P}^n]_{ij} = P^n(i,j) = P(X_n = j \mid X_0 = i)$$

$$P^2(x,y) = P(X_2 = y \mid X_0 = x) = \sum_{z \in S} P(x,z) \times P(z,y)$$

$$P^{n+m}(x,y) = P(X_{n+m} = y \mid X_0 = x) = \sum_{z \in S} P^n(x,z) \times P^m(z,y)$$

The initial distribution

$$\boldsymbol{\pi}_{\mathbf{0}} = [\ \pi_0(0)\ \dots\ \pi_0(i)\ \dots\ \pi_0(d)\ ] = [\ P(X_0 = 0)\ \dots\ P(X_0 = i)\ \dots\ P(X_0 = d)\ ]$$

$$\boldsymbol{\pi}_{\boldsymbol{n}} = [\ \pi_n(0)\ \dots\ \pi_n(i)\ \dots\ \pi_n(d)\ ] = [\ P(X_n = 0)\ \dots\ P(X_n = i)\ \dots\ P(X_n = d)\ ]$$

$$\boldsymbol{\pi}_{\boldsymbol{n}} = \boldsymbol{\pi}_{\mathbf{0}} \times \mathbf{P}^n \qquad \pi_n(x) = P(X_n = x) = \sum_{y \in S} \pi_0(y) \times P^n(y,x)$$

$$\boldsymbol{\pi}_{\boldsymbol{n}+\mathbf{1}} = \boldsymbol{\pi}_{\boldsymbol{n}} \times \mathbf{P} \qquad \pi_{n+1}(x) = P(X_{n+1} = x) = \sum_{y \in S} \pi_n(y) \times P(y,x)$$