STAT 433
Spring 2022
Version A

Name ANSWERS

NetID ____________________

Exam 2

The exam has 5 problems and 7 pages.

Be sure to show all your work; your partial credit might depend on it.

Please put your final answers at the end of your work and mark them clearly. Box the final answers where appropriate.

No credit will be given without supporting work.

The exam is closed book and closed notes. You are allowed to use a calculator and one 8½" x 11" sheet (both sides) with notes.

Turn in all scratch paper with your exam.

Page	Earned
1	
2	
3	
4	
5	
6	
7	
Total	/80

Academic Integrity

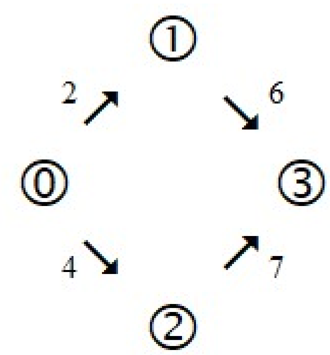
The University statement on your obligation to maintain academic integrity is:

If you engage in an act of academic dishonesty, you become liable to severe disciplinary action. Such acts include cheating; falsification or invention of information or citation in an academic endeavor; helping or attempting to help others commit academic infractions; plagiarism; offering bribes, favors, or threats; academic interference; computer related infractions; and failure to comply with research regulations.

Article 1, Part 4 of the Student Code gives complete details of rules governing academic integrity for all students. You are responsible for knowing and abiding by these rules.

1. Consider a Markov pure jump process on $S = \{ ⓪, ①, ②, ③ \}$ with $q_{01} = 2$, $q_{02} = 4$, $q_{13} = 6$, $q_{23} = 7$.

③ is an absorbing state.



a) (5) Find q_{00}, Q_{00}, Q_{01}, Q_{02}, Q_{03}.

$q_0 = 2 + 4 = 6.$ $\qquad$ $q_{00} = -q_0 = \mathbf{-6}.$

$Q_{00} = \mathbf{0},$ $\qquad Q_{01} = \dfrac{2}{2+4} = \dfrac{2}{6} = \mathbf{\dfrac{1}{3}},$ $\qquad Q_{02} = \dfrac{4}{2+4} = \dfrac{4}{6} = \mathbf{\dfrac{2}{3}},$ $\qquad Q_{03} = \mathbf{0}.$

b) (20) Find $P_{00}(t)$, $P_{01}(t)$, $P_{02}(t)$, $P_{03}(t)$, $t \geq 0$.

$q_{00} = -6$	$q_{01} = 2$	$q_{02} = 4$	$q_{03} = 0$
$q_{10} = 0$	$q_{11} = -6$	$q_{12} = 0$	$q_{13} = 6$
$q_{20} = 0$	$q_{21} = 0$	$q_{22} = -7$	$q_{23} = 7$
$q_{30} = 0$	$q_{31} = 0$	$q_{32} = 0$	$q_{33} = 0$

$P_{10}(t) = 0,\quad P_{20}(t) = 0,\quad P_{30}(t) = 0,$

$P_{21}(t) = 0,\quad P_{31}(t) = 0,$

$P_{12}(t) = 0,\quad P_{32}(t) = 0,$

$P_{33}(t) = 1.$

Starting from 0 and still being at 0 at time t implies that the process has not jumped yet.

$q_0 = 6. \qquad\qquad P_{00}(t) = P_0(\tau_1 > t) = e^{-6t}, \qquad\qquad t \ge 0.$

OR

Forward equation:

$$P'_{00}(t) = P_{00}(t)\,q_{00} + P_{01}(t)\,q_{10} + P_{02}(t)\,q_{20} + P_{03}(t)\,q_{30} = -6\,P_{00}(t).$$

$P_{00}(0) = 1. \qquad \Rightarrow \qquad P_{00}(t) = e^{-6t}, \qquad t \ge 0.$

OR

Backward equation:

$$P'_{00}(t) = q_{00}\,P_{00}(t) + q_{01}\,P_{10}(t) + q_{02}\,P_{20}(t) + q_{03}\,P_{30}(t) = -6\,P_{00}(t),$$

since $P_{10}(t) = P_{20}(t) = P_{30}(t) = 0$ (cannot go from 1 or 2 or 3 to 0).

$P_{00}(0) = 1. \qquad \Rightarrow \qquad P_{00}(t) = e^{-6t}, \qquad t \ge 0.$

Forward equation:

$$\begin{aligned} P'_{01}(t) &= P_{00}(t)\,q_{01} + P_{01}(t)\,q_{11} + P_{02}(t)\,q_{21} + P_{03}(t)\,q_{31} \\ &= 2\,P_{00}(t) - 6\,P_{01}(t) \\ &= 2\,e^{-6t} - 6\,P_{01}(t). \end{aligned}$$

$P_{01}(0) = 0.$

$$\Rightarrow \qquad P_{01}(t) = \int_0^t e^{-6(t-s)}\,2\,e^{-6s}\,ds = 2\,t\,e^{-6t}, \qquad t \ge 0.$$

OR

Backward equation:

$$P'_{01}(t) = q_{00}\,P_{01}(t) + q_{01}\,P_{11}(t) + q_{02}\,P_{21}(t) + q_{03}\,P_{31}(t)$$

$$= -6\,P_{01}(t) + 2\,P_{11}(t),$$

since $P_{21}(t) = P_{31}(t) = 0$ (cannot go from 2 or 3 to 1).

$q_1 = 6.$ $\qquad P_{11}(t) = P_1(\tau_1 > t) = e^{-6t},$ $\qquad t \ge 0.$

$$P'_{01}(t) = -6\,P_{01}(t) + 2\,e^{-2t}.$$

$P_{01}(0) = 0.$

$$\Rightarrow \qquad P_{01}(t) = \int_0^t e^{-6(t-s)}\,2\,e^{-6s}\,ds = 2\,t\,e^{-6t}, \qquad t \ge 0.$$

Forward equation:

$$P'_{02}(t) = P_{00}(t)\,q_{02} + P_{01}(t)\,q_{12} + P_{02}(t)\,q_{22} + P_{03}(t)\,q_{32}$$

$$= 4\,P_{00}(t) - 7\,P_{02}(t)$$

$$= 4\,e^{-6t} - 7\,P_{02}(t).$$

$P_{02}(0) = 0.$

$$\Rightarrow \qquad P_{02}(t) = \int_0^t e^{-7(t-s)}\,4\,e^{-6s}\,ds = 4\,e^{-7t}\,(e^{t} - 1)$$

$$= 4\,(e^{-6t} - e^{-7t}), \qquad t \ge 0.$$

OR

Backward equation:

$$P'_{02}(t) = q_{00}\,P_{02}(t) + q_{01}\,P_{12}(t) + q_{02}\,P_{22}(t) + q_{03}\,P_{32}(t)$$

$$= -6\,P_{02}(t) + 4\,P_{22}(t),$$

since $P_{32}(t) = 0$ (cannot go from 3 to 2)

and $P_{12}(t) = 0$ (cannot go from 1 to 2).

$q_2 = 7.$ $\qquad P_{22}(t) = P_2(\tau_1 > t) = e^{-7t}, \qquad t \geq 0.$

$$P'_{02}(t) = -6\,P_{02}(t) + 4\,e^{-7t}.$$

$$P_{02}(0) = 0.$$

$$\Rightarrow \qquad P_{02}(t) = \int_0^t e^{-6(t-s)}\,4\,e^{-7s}\,ds = 4\,e^{-6t}(1 - e^{-t})$$

$$= 4\,(e^{-6t} - e^{-7t}), \qquad t \geq 0.$$

$$P_{03}(t) = 1 - P_{00}(t) - P_{01}(t) - P_{02}(t)$$

$$= 1 - e^{-6t} - 2\,t\,e^{-6t} - 4\,(e^{-6t} - e^{-7t}), \qquad t \geq 0.$$

OR

Forward equation:

$$P'_{03}(t) = P_{00}(t)\,q_{03} + P_{01}(t)\,q_{13} + P_{02}(t)\,q_{23} + P_{03}(t)\,q_{33}$$

$$= 6\,P_{01}(t) + 7\,P_{02}(t)$$

$$= 12\,t\,e^{-6t} + 28\,(e^{-6t} - e^{-7t}).$$

$$P_{03}(0) = 0.$$

$$\Rightarrow \qquad P_{03}(t) = \int_0^t \left(12\, s\, e^{-6s} + 28\left(e^{-6s} - e^{-7s}\right)\right) ds = \ldots, \qquad t \geq 0.$$

OR

Backward equation:

$$P'_{03}(t) = q_{00}\, P_{03}(t) + q_{01}\, P_{13}(t) + q_{02}\, P_{23}(t) + q_{03}\, P_{33}(t)$$

$$= -6\, P_{03}(t) + 2\, P_{13}(t) + 4\, P_{23}(t)$$

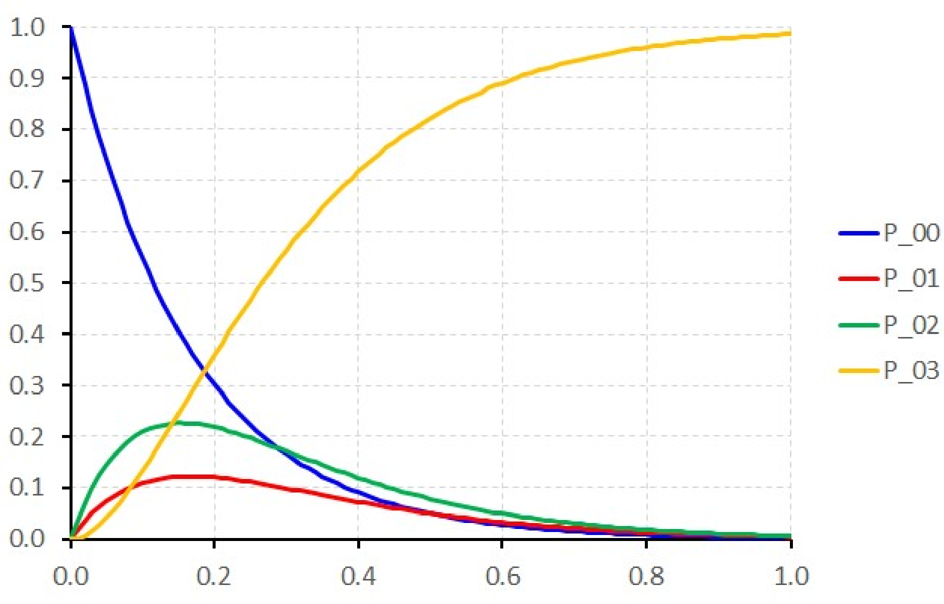
$$= \ldots$$

$$P_{00}(t) = e^{-6t},$$

$$P_{01}(t) = 2\, t\, e^{-6t},$$

$$P_{02}(t) = 4\,(e^{-6t} - e^{-7t}),$$

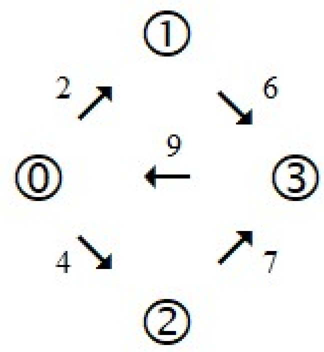
$$P_{03}(t) = 1 + 4\, e^{-7t} - 5\, e^{-6t} - 2\, t\, e^{-6t}, \qquad t \geq 0.$$



2. Consider a Markov pure jump process on

$S = \{ ⓪, ①, ②, ③ \}$ with $q_{01} = 2$,

$q_{02} = 4$, $q_{13} = 6$, $q_{23} = 7$, $q_{30} = 9$.



a) (10) Find the stationary distribution, if it exists.

$q_{00} = -6$	$q_{01} = 2$	$q_{02} = 4$	$q_{03} = 0$
$q_{10} = 0$	$q_{11} = -6$	$q_{12} = 0$	$q_{13} = 6$
$q_{20} = 0$	$q_{21} = 0$	$q_{22} = -7$	$q_{23} = 7$
$q_{30} = 9$	$q_{31} = 0$	$q_{32} = 0$	$q_{33} = -9$

$-6\,\pi(0) + 9\,\pi(3) = 0.$ (1)

$2\,\pi(0) - 6\,\pi(1) = 0.$ (2)

$4\,\pi(0) - 7\,\pi(2) = 0.$ (3)

$6\,\pi(1) + 7\,\pi(2) - 9\,\pi(3) = 0.$ (4)

$\pi(0) + \pi(1) + \pi(2) + \pi(3) = 1.$ (5)

(1) $\Rightarrow$ $\pi(3) = \frac{2}{3}\,\pi(0).$ (6)

(2) $\Rightarrow$ $\pi(1) = \frac{1}{3}\,\pi(0).$ (7)

(3) $\Rightarrow$ $\pi(2) = \frac{4}{7}\,\pi(0).$ (6)

(5), (6), (7), (8) $\Rightarrow$ $\pi(0) + \frac{1}{3}\,\pi(0) + \frac{4}{7}\,\pi(0) + \frac{2}{3}\,\pi(0) = 1.$

$\Rightarrow$ $\frac{21+7+12+14}{21}\,\pi(0) = 1.$

$$\Rightarrow \qquad \pi(0) = \frac{21}{54}.$$

$$\Rightarrow \qquad \pi(1) = \frac{7}{54}, \qquad \pi(2) = \frac{12}{54}, \qquad \pi(3) = \frac{14}{54}.$$

$$\pi(0) = \mathbf{\frac{21}{54}}, \qquad \pi(1) = \mathbf{\frac{7}{54}}, \qquad \pi(2) = \mathbf{\frac{12}{54}}, \qquad \pi(3) = \mathbf{\frac{14}{54}}.$$

OR

$$\begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\pi_{\text{emb}}(0) = \pi_{\text{emb}}(3).$

$\pi_{\text{emb}}(1) = \frac{1}{3}\,\pi_{\text{emb}}(0).$

$\pi_{\text{emb}}(2) = \frac{2}{3}\,\pi_{\text{emb}}(0).$

$\pi_{\text{emb}}(3) = \pi_{\text{emb}}(3)$

$\pi_{\text{emb}}(0) + \pi_{\text{emb}}(1) + \pi_{\text{emb}}(2) + \pi_{\text{emb}}(3) = 1.$

$$\Rightarrow \qquad \pi_{\text{emb}}(0) + \frac{1}{3}\,\pi_{\text{emb}}(0) + \frac{2}{3}\,\pi_{\text{emb}}(0) + \pi_{\text{emb}}(0) = 1.$$

$$\Rightarrow \qquad \frac{9}{3}\,\pi_{\text{emb}}(0) = 1. \qquad \Rightarrow \qquad \pi_{\text{emb}}(0) = \frac{1}{3}.$$

$\Rightarrow \qquad \pi_{\text{emb}}(1) = \frac{1}{9}, \qquad \pi_{\text{emb}}(2) = \frac{2}{9}, \qquad \pi_{\text{emb}}(3) = \frac{1}{3}.$

$\pi_{\text{emb}}(0) = \frac{1}{3}. \qquad \pi_{\text{emb}}(1) = \frac{1}{9}, \qquad \pi_{\text{emb}}(2) = \frac{2}{9}, \qquad \pi_{\text{emb}}(3) = \frac{1}{3}.$

$$\pi_{\text{emb}}(0)/q_0 = \frac{1}{18} = \frac{21}{378}, \qquad \pi_{\text{emb}}(1)/q_1 = \frac{1}{54} = \frac{7}{378},$$
$$\pi_{\text{emb}}(2)/q_2 = \frac{2}{63} = \frac{12}{378}, \qquad \pi_{\text{emb}}(3)/q_3 = \frac{1}{27} = \frac{14}{378}.$$

$21 + 7 + 12 + 14 = 54.$

$\pi(0) = \mathbf{\frac{21}{54}}, \qquad \pi(1) = \mathbf{\frac{7}{54}}, \qquad \pi(2) = \mathbf{\frac{12}{54}}, \qquad \pi(3) = \mathbf{\frac{14}{54}}.$

b) (3) Find the mean return time to state ①, $m_1 = \mathrm{E}_1(\mathrm{T}_1)$.

$\pi(x) = \frac{1}{q_x\, m_x}. \qquad \pi(1) = \frac{7}{54} = \frac{1}{q_1\, m_1} = \frac{1}{6\, m_1}. \qquad m_0 = \mathbf{\frac{9}{7}}.$

3. (5) Jack and Jill got married and lived happily until their kitchen sink started leaking. Suppose that the time it would take Jack to become motivated to fix the sink is exponentially distributed with mean 6 hours. As soon as Jack becomes motivated to fix the sink, the time it would take for him to fix it follows an independent exponential distribution with mean 3 hours. Independently of anything Jack does, the time it would take Jill to give up on Jack and call a professional plumber is exponentially distributed with mean 9 hours. What is the probability that Jack would fix the sink before Jill calls a professional plumber?

rates: $\lambda_{\text{M(otivated)}} = \frac{1}{6}$, $\lambda_{\text{F(ix sink)}} = \frac{1}{3}$,

$\lambda_{\text{C(all plumber)}} = \frac{1}{9}$.

Jack becomes motivated to fix the sink BEFORE Jill calls a professional plumber

AND

Jack fixes the sink BEFORE Jill calls a professional plumber

(do not forget the memoryless property of the exponential distribution)

$$\frac{\lambda_M}{\lambda_M + \lambda_C} \times \frac{\lambda_F}{\lambda_F + \lambda_C} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{9}} \times \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{9}} = 0.60 \times 0.75 = \mathbf{0.45}.$$

OR

Jill calls a professional plumber BEFORE Jack becomes motivated to fix the sink

OR

Jack becomes motivated to fix the sink BEFORE Jill calls a professional plumber

AND

Jill calls a professional plumber BEFORE Jack fixes the sink

$$\frac{\frac{1}{9}}{\frac{1}{6}+\frac{1}{9}} \;+\; \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{9}} \times \frac{\frac{1}{9}}{\frac{1}{3}+\frac{1}{9}} \;=\; 0.40 \;+\; 0.60 \times 0.25 \;=\; 0.55.$$

$1 - 0.55 \;=\; \mathbf{0.45}.$

For fun:

```
> S = 1000000
> count = 0
> for (i in 1:S) {
+     M = rexp(1, rate=1/6)
+     F = rexp(1, rate=1/3)
+     C = rexp(1, rate=1/9)
+     if (C>M+F) { count = count+1 }
+ }
> count/S
[1] 0.450156
```

Yay!!! It is really close! 1,000,000 may have been an overkill, but it is really close!

4. In addition to many other wonderful services it provides, BookwormAlliance.com helps interested "straight A students" maintain their artificially inflated high GPA by helping them cheat on their homework. After a new homework assignment is posted on Canvas at 5:00 pm on Friday, STAT 433 "straight A students" flock to BookwormAlliance.com to post the homework for a stranger to do it for them (because they care so much about their grades, obviously) according to a Poisson process with rate 3 per hour, while STAT 410 "straight A students" converge to BookwormAlliance.com according to a Poisson process with rate 2 per hour. Assume that the two processes are independent.

a) (4) Find the probability that (exactly) 4 STAT 433 "straight A students" show up at BookwormAlliance.com during the first hour.

1 hour $\Rightarrow$ expect 3 STAT 433 "straight A students"

$$P(N_{433}(1)=4) = \frac{3^4 e^{-3}}{4!} \approx \mathbf{0.16803}.$$

b) (5) Given that 4 STAT 433 "straight A students" show up at BookwormAlliance.com during the first hour, what is the probability that (exactly) 14 STAT 433 "straight A students" show up at BookwormAlliance.com during the first five hours?

For a Poisson process $X(t)$ with rate λ, $0 \le s \le t$,

$X(t)-X(s)$ has a Poisson distribution with mean $\lambda(t-s)$,

$X(t)-X(s)$ and $X(s)$ are independent.

$$P(N_{433}(5)=14 \mid N_{433}(1)=4) = P(N_{433}(5)-N_{433}(1)=14-4=10)$$

4 hours $\Rightarrow$ expect 12 STAT 433 "straight A students"

$$= \frac{12^{10} e^{-12}}{10!} \approx \mathbf{0.10484}.$$

4. (continued) 433 – 3 per hour, 410 – 2 per hour.

c) (5) Given that 14 STAT 433 "straight A students" show up at BookwormAlliance.com during the first five hours, what is the probability that (exactly) 4 STAT 433 "straight A students" show up at BookwormAlliance.com during the first hour?

For a Poisson process $X(t)$ with rate λ, $0 \le s \le t$,

$X(s) \mid X(t) = n$ has a Binomial$(n,\ p = \frac{s}{t})$ distribution.

$$P(N_{433}(1) = 4 \mid N_{433}(5) = 14) = \binom{14}{4} \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^{10} \approx \mathbf{0.17197}.$$

d) (5) Find the probability that (exactly) 20 "straight A students" (from either class) show up at BookwormAlliance.com during the first five hours.

$N(t) = N_{433}(t) + N_{410}(t)$ is a Poisson process

with rate $\lambda = \lambda_{433} + \lambda_{410} = 5$ "straight A students" per hour.

5 hours $\Rightarrow$ expect 25 "straight A students"

$$P(N(5) = 20) = \frac{25^{20}\, e^{-25}}{20\,!} \approx \mathbf{0.05192}.$$

4. (continued) 433 – 3 per hour, 410 – 2 per hour.

e) (5) Given that 20 "straight A students" (from either class) show up at BookwormAlliance.com during the first five hours, what is the probability that (exactly) 14 of them are STAT 433 "straight A students"?

$\mathrm{N}_{433}(t) \,|\, \mathrm{N}(t) = \mathrm{N}_{433}(t) + \mathrm{N}_{410}(t) = n$ has a Binomial distribution,

$$p = \frac{\lambda_{433}}{\lambda_{433} + \lambda_{410}} = \frac{3}{3+2} = \frac{3}{5}.$$

$$\mathrm{P}(\mathrm{N}_{433}(5) = 14 \,|\, \mathrm{N}(5) = 20) = \binom{20}{14} \left(\frac{3}{5}\right)^{14} \left(\frac{2}{5}\right)^{6} \approx \mathbf{0.12441}.$$

f) (5) Each STAT 433 "straight A student" who shows up at BookwormAlliance.com claims that they are there "just to check their answers" with probability 0.70, independently of others. What is the probability that (exactly) 11 STAT 433 "straight A students" show up at BookwormAlliance.com and claim that they are there "just to check their answers" during the first five hours?

$\mathrm{N}_{433}(5) = \mathrm{N}_{\mathrm{JCA}}(t) + \mathrm{N}_{\mathrm{SHC}}(t)$.

$\mathrm{N}_{\mathrm{JCA}}(t)$ is a Poisson process with rate $\lambda_{433}\, p = 3 \cdot 0.70 = 2.1$ per hour.

(JCA = Just Checking Answer, SHC = Somewhat Honest Cheater)

5 hours $\Rightarrow$ expect 10.5 STAT 433 "straight A students" "just checking answers"

$$\mathrm{P}(\mathrm{N}_{\mathrm{JCA}}(5) = 11) = \frac{10.5^{11}\, e^{-10.5}}{11\,!} \approx \mathbf{0.11799}.$$

5. (8) In addition to many other wonderful services it provides, BookwormAlliance.com helps interested "straight A students" maintain their artificially inflated high GPA by helping them cheat on their homework. After a new homework assignment is posted on Canvas at 5:00 pm on Friday, STAT 433 "straight A students" flock to BookwormAlliance.com and immediately post the homework for a stranger to do it for them (because they care so much about their grades, obviously) according to a Poisson process with rate 3 per hour. Each post is then answered by someone who benefits financially from the unwillingness and / or inability of "straight A students" to do their own homework after an exponentially distributed time with mean 6 hours, independently of all other posts. Find the long-term distribution of the number of unanswered STAT 433 homework posts, if it exists.

Infinite server queue. $\lambda_x = \lambda = 3, \quad x \ge 0, \qquad \mu_x = x\,\mu = \frac{x}{6}, \quad x \ge 1.$

The long-term (stationary) probability distribution is a **Poisson** distribution with mean $\frac{\lambda}{\mu}$.

$\frac{\lambda}{\mu} = \mathbf{18}. \qquad \pi(x) = \frac{18^x e^{-18}}{x!}, \qquad x \ge 0.$

OR

$$⓪ \underset{\frac{1}{6}}{\overset{3}{\rightleftarrows}} ① \underset{\frac{2}{6}}{\overset{3}{\rightleftarrows}} ② \underset{\frac{3}{6}}{\overset{3}{\rightleftarrows}} ③ \underset{\frac{4}{6}}{\overset{3}{\rightleftarrows}} ④ \underset{\frac{5}{6}}{\overset{3}{\rightleftarrows}} \bullet\bullet\bullet$$

$\pi_0 = 1, \qquad \pi_x = \frac{\lambda_0 \ldots \lambda_{x-1}}{\mu_1 \ldots \mu_x} = \frac{3 \cdot 3 \cdot 3 \cdot \ldots \cdot 3}{\frac{1}{6} \cdot \frac{2}{6} \cdot \frac{3}{6} \cdot \ldots \cdot \frac{x}{6}} = \frac{18^x}{x!}, \qquad x \ge 1.$

$$\sum_{x=0}^{\infty} \pi_x = \sum_{x=0}^{\infty} \frac{18^x}{x!} = e^{18}.$$

$\pi(0) = e^{-18}. \qquad \pi(x) = \pi_x\,\pi(0) = \frac{18^x}{x!} e^{-18}, \qquad x \ge 0.$