STAT 433
Spring 2022

Version **B**

Name ANSWERS

NetID ______________________

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Final Exam

The exam has 12 problems and 14 pages.

Please put your final answers at the end of your work and mark them clearly.

Be sure to show all your work; your partial credit might depend on it.

No credit will be given without supporting work.

The exam is closed book and closed notes.
You are allowed to use a calculator and
two 8½" x 11" sheets with notes.

Turn in all scratch paper with your exam.
(*I would be happy to provide scratch paper.*)

Academic Integrity

The University statement on your obligation to maintain academic integrity is:

If you engage in an act of academic dishonesty, you become liable to severe disciplinary action. Such acts include cheating; falsification or invention of information or citation in an academic endeavor; helping or attempting to help others commit academic infractions; plagiarism; offering bribes, favors, or threats; academic interference; computer related infractions; and failure to comply with research regulations.

Article 1, Part 4 of the Student Code gives complete details of rules governing academic integrity for all students. You are responsible for knowing and abiding by these rules.

1. Consider a Markov chain on $\{1, 2, 3, 4, 5\}$ with the following transition matrix.

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

a) (6) Determine which states are

Recurrent: **1, 4, 5**.

Transient: **2, 3**.

Absorbing: **1**.

$$\begin{tikzcd}
& \boxed{1} & & \\
2 \arrow[ur] \arrow[rr, shift left] & & 3 \arrow[ul] \arrow[ll, shift left] \arrow[dr] & \\
& & & \boxed{4 \rightleftarrows 5}
\end{tikzcd}$$

(transitions $x \to x$ are not shown on the diagram)

$S_R = \{1\} \cup \{4, 5\} = C_1 \cup C_2$.

1. (continued)

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

b) (6) Find $\rho_{21} = P_2(T_1 < \infty)$.

$\rho_{21} = \rho_{C_1}(2)$.

$$\rho_{C_1}(2) = \frac{1}{2} + \frac{1}{2}\rho_{C_1}(3).$$

$$\rho_{C_1}(3) = \frac{1}{3} + \frac{1}{3}\rho_{C_1}(2) + \frac{1}{6}\rho_{C_1}(3).$$

$$\Rightarrow \qquad \rho_{C_1}(3) = \frac{1}{3} + \frac{1}{6} + \frac{1}{6}\rho_{C_1}(3) + \frac{1}{6}\rho_{C_1}(3).$$

$$\Rightarrow \qquad \rho_{C_1}(3) = \frac{3}{4}.$$

$$\Rightarrow \qquad \rho_{C_1}(2) = \frac{1}{2} + \frac{1}{2}\cdot\frac{3}{4} = \mathbf{\frac{7}{8}}.$$

OR

$$\mathbf{Q} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}, \qquad \mathbf{I} - \mathbf{Q} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{3} & \frac{5}{6} \end{bmatrix}, \qquad \det(\mathbf{I} - \mathbf{Q}) = \frac{5}{6} - \frac{1}{6} = \frac{4}{6},$$

$$\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{2}{4} & \frac{6}{4} \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}, \qquad \mathbf{F}\,\mathbf{R} = \begin{bmatrix} \mathbf{\frac{7}{8}} & \frac{1}{8} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

1. (continued)

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

c) (10) Does a stationary distribution exist?

Circle one: **Yes** No

IF a stationary distribution does exist, is it unique?

Circle one: Yes **No**

IF a stationary distribution does exists, and it is unique, find the stationary distribution.

IF a stationary distribution does exist, but it is not unique, find two distinct stationary distributions.

$S_R = \{1\} \cup \{4, 5\} = C_1 \cup C_2.$

The set of positive recurrent states is nonempty, but not irreducible.

$\Rightarrow$ The chain has an infinite number of distinct stationary distributions.

$$\pi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\pi_2(4) = \frac{2}{3}\,\pi_2(4) + \frac{1}{4}\,\pi_2(5)$$

$$\pi_2(5) = \frac{1}{3}\,\pi_2(4) + \frac{3}{4}\,\pi_2(5)$$

$$\pi_2(4) + \pi_2(5) = 1$$

$$\Rightarrow \qquad \pi_2(4) = \frac{3}{7}, \qquad \pi_2(5) = \frac{4}{7}.$$

$$\pi_2 = \begin{bmatrix} 0 & 0 & 0 & \frac{3}{7} & \frac{4}{7} \end{bmatrix}$$

$$\begin{bmatrix} 1-a & 0 & 0 & \frac{3}{7}\,a & \frac{4}{7}\,a \end{bmatrix}, \qquad 0 \le a \le 1.$$

2. Let $X(t)$ be a Poisson process with parameter $\lambda = 1.3$.

a) (5) Find $P(X(9) = 15 \mid X(3) = 6)$.

For a Poisson process $X(t)$ with rate λ, $0 \le s \le t$,

$X(t) - X(s)$ has a Poisson distribution with mean $\lambda(t - s)$,

$X(t) - X(s)$ and $X(s)$ are independent.

$P(X(9) = 15 \mid X(3) = 6) = P(X(9) - X(3) = 9)$

$9 - 3 = 6 \qquad \Rightarrow \qquad$ expect $6 \times 1.3 = 7.8$.

$$\ldots = \frac{7.8^9 \, e^{-7.8}}{9!} \approx \mathbf{0.120668}.$$

b) (5) Find $P(X(3) = 6 \mid X(9) = 15)$.

For a Poisson process $X(t)$ with rate λ, $0 \le s \le t$,

$X(s) \mid X(t) = n$ has a Binomial$(n,\ p = \frac{s}{t})$ distribution.

$$P(X(3) = 6 \mid X(9) = 15) = \binom{15}{6} \left(\frac{3}{9}\right)^6 \left(\frac{6}{9}\right)^9 \approx \mathbf{0.178589}.$$

3. Let $B(t)$ be a Brownian motion with parameter $\sigma = 5$.

a) (4) Find $P(B(12) < 3 \mid B(3) = -6)$.

$(B(t) \mid B(s) = B)$ has a Normal distribution with mean B

and variance $\sigma^2 (t - s)$, $\qquad 0 \le s \le t$.

$$P(B(12) < 3 \mid B(3) = -6) = P\left(Z < \frac{3 - (-6)}{\sqrt{5^2 \cdot (12 - 3)}}\right) = P(Z < 0.60) = \mathbf{0.7257}.$$

b) (6) Find $P(B(3) < -6 \mid B(12) = 3)$.

$(B(s) \mid B(t) = B)$ has a Normal distribution with mean $\frac{s}{t}B$

and variance $\frac{s(t-s)}{t}\sigma^2$, $\qquad 0 \le s \le t$.

$$P(B(3) < -6 \mid B(12) = 3) = P\left(Z < \frac{-6 - \frac{3}{12} \cdot 3}{\sqrt{\frac{3 \cdot (12 - 3)}{12} \cdot 5^2}}\right) = P\left(Z < \frac{-6 - 0.75}{7.5}\right)$$

$$= P(Z < -0.90) = \mathbf{0.1841}.$$

c) (2) Find the probability $P(T_{-6} < T_3)$.

$$P(T_{-6} < T_3) = 1 - P(T_3 < T_{-6}) = 1 - P(T_A < T_{-B}) = 1 - \frac{B}{A+B} = 1 - \frac{6}{3+6} = \mathbf{\frac{1}{3}}.$$

OR

by symmetry $\qquad P(T_{-6} < T_3) = P(T_6 < T_{-3}) = P(T_A < T_{-B}) = \frac{B}{A+B} = \frac{3}{6+3} = \mathbf{\frac{1}{3}}.$

4. Find $P(X(7) < 80 \mid X(3) = 50)$ if $X(t)$ is …

a) (5) … an Arithmetic Brownian motion with drift parameter $\alpha = 5$ and diffusion parameter (volatility) $\sigma = 4$.

$A(t) - A(s)$ has a Normal distribution with mean $\alpha(t-s)$ and variance $\sigma^2(t-s)$, $0 \le s \le t$.

$$P(X(7) < 80 \mid X(3) = 50) = P\left(Z < \frac{(80-50) - 5(7-3)}{\sqrt{4^2(7-3)}}\right) = P\left(Z < \frac{30 - 20}{8}\right)$$

$$= P(Z < 1.25) = \mathbf{0.8944}.$$

b) (7) … a Geometric Brownian motion with percentage drift $\alpha = 0.08$ and percentage volatility $\sigma = 0.15$.

$\ln \dfrac{G(t)}{G(s)}$ has a Normal distribution with mean $\left(\alpha - \dfrac{\sigma^2}{2}\right)(t-s)$ and variance $\sigma^2(t-s)$, $0 \le s \le t$.

$$P(X(7) < 80 \mid X(3) = 50) = P\left(Z < \frac{\ln\left(\frac{80}{50}\right) - \left(0.08 - \frac{0.15^2}{2}\right)(7-3)}{\sqrt{0.15^2(7-3)}}\right)$$

$$\approx P\left(Z < \frac{0.47 - 0.06875 \cdot 4}{0.30}\right) = P(Z < 0.65) = \mathbf{0.7422}.$$

5. (4) Consider a Markov chain on $\{1, 2, 3, 4, 5, 6\}$ with the following transition matrix.

$$\mathbf{P} = \begin{bmatrix} 0 & 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.7 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Show that this Markov chain is irreducible, but not aperiodic. Find the period.

A possible path:

$1 \to 2 \to 4 \to 1 \to 3 \to 5 \to 1 \to 3 \to 6 \to 1$

The chain is indeed irreducible.

$\ldots \to \{1\} \to \{2, 3\} \to \{4, 5, 6\} \to \{1\} \to \ldots$

Period = **3**.

6. Wraps'Я'Us offers their customers a choice of Beef wrap, Chicken wrap, or Veggie wrap. Bob has a wrap for lunch every day at Wraps'Я'Us. His daily preference can be modeled by a Markov chain with the following transition probability matrix:

$$\begin{array}{c} \\ B \\ C \\ V \end{array} \begin{array}{c} \begin{array}{ccc} \;\;B\;\; & \;\;C\;\; & \;\;V\;\; \end{array} \\ \begin{bmatrix} 0.60 & 0.30 & 0.10 \\ 0 & 0.70 & 0.30 \\ 0.30 & 0.50 & 0.20 \end{bmatrix} \end{array}$$

a) (3) Suppose Bob had a Beef wrap for lunch on Wednesday. Find the probability that he will have a Veggie wrap on Friday.

$$P(X_{Fr} = V \mid X_{We} = B) = P^2(B, V)$$

$$= P(B, B) \cdot P(B, V) + P(B, C) \cdot P(C, V) + P(B, V) \cdot P(V, V)$$

$$= 0.60 \cdot 0.10 + 0.30 \cdot 0.30 + 0.10 \cdot 0.20 = \mathbf{0.17}.$$

$$\begin{bmatrix} 0.60 & 0.30 & 0.10 \\ 0 & 0.70 & 0.30 \\ 0.30 & 0.50 & 0.20 \end{bmatrix} \cdot \begin{bmatrix} 0.60 & 0.30 & 0.10 \\ 0 & 0.70 & 0.30 \\ 0.30 & 0.50 & 0.20 \end{bmatrix} = \begin{bmatrix} 0.39 & 0.44 & \mathbf{0.17} \\ 0.09 & 0.64 & 0.27 \\ 0.24 & 0.54 & 0.22 \end{bmatrix}$$

b) (6) Suppose Bob had a Beef wrap for lunch on Wednesday. Find the expected number of days until the next time he has a Veggie wrap for lunch.

$$m_{BV} = 1 + 0.60\, m_{BV} + 0.30\, m_{CV}. \qquad \Rightarrow \qquad m_{BV} = 2.5 + 0.75\, m_{CV}.$$

$$m_{CV} = 1 \qquad\qquad + 0.70\, m_{CV}. \qquad \Rightarrow \qquad m_{CV} = \frac{10}{3}.$$

$$\Rightarrow \qquad m_{CV} = 2.5 + 0.75 \cdot \frac{10}{3} = \mathbf{5}.$$

6. (continued)

$$\begin{array}{c} \\ B \\ C \\ V \end{array}\begin{array}{c} \begin{array}{ccc} B & C & V \end{array} \\ \begin{bmatrix} 0.60 & 0.30 & 0.10 \\ 0 & 0.70 & 0.30 \\ 0.30 & 0.50 & 0.20 \end{bmatrix} \end{array}$$

c) (8) Find the long-term proportion of days when Bob has a

(i) Beef wrap (ii) Chicken wrap (iii) Veggie wrap

if they exist.

$\pi(B) = 0.60\,\pi(B) + 0.30\,\pi(V).$ (1)

$\pi(C) = 0.30\,\pi(B) + 0.70\,\pi(C) + 0.50\,\pi(V).$ (2)

$\pi(V) = 0.10\,\pi(B) + 0.30\,\pi(C) + 0.20\,\pi(V).$ (3)

$\pi(B) + \pi(C) + \pi(V) = 1.$ (4)

(1) $\Rightarrow$ $\pi(B) = 0.75\,\pi(V).$ (5)

(2), (5) $\Rightarrow$ $0.30\,\pi(C) = 0.225\,\pi(V) + 0.50\,\pi(V).$

$\Rightarrow$ $\pi(C) = \dfrac{725}{300}\,\pi(V).$ (6)

(4), (5), (6) $\Rightarrow$ $\dfrac{225}{300}\,\pi(V) + \dfrac{725}{300}\,\pi(V) + \pi(V) = 1.$

$\Rightarrow$ $\pi(V) = 0.24.$ (7)

(5), (7) $\Rightarrow$ $\pi(B) = 0.18.$

(6), (7) $\Rightarrow$ $\pi(C) = 0.58.$

$\pi(B) = \mathbf{0.18}$, $\pi(C) = \mathbf{0.58}$, $\pi(V) = \mathbf{0.24}$.

7. (10) In Anytown, the biggest tourist attraction is Cloud-Buster Tower, the tallest building in Anytown. At the top of Cloud-Buster Tower, there is an observation deck with a telescope for viewing distant features. Tourists ascend to the observation deck to use the telescope (the town could only afford one, but it swivels in all directions) according to a Poisson process with the rate of 10 per hour. The time a tourist would spend looking through the telescope is exponentially distributed with mean 5 minutes, independently of all others, at which point the tourist would immediately leave the observation deck. Find the long-term probability distribution for the number of tourists on the observation deck, if it exists.

Birth and death process. (1 server queue) M/M/1 or M/M/1/∞.

$\lambda_x = \lambda, \quad x \ge 0,$ $\qquad \mu_x = \mu, \quad x \ge 1.$

only one telescope

$\lambda = 10.$ mean 5 minutes $\Rightarrow$ rate $\frac{60}{5} = 12$ tourists per hour. $\mu = 12.$

$$⓪ \underset{12}{\overset{10}{\rightleftarrows}} ① \underset{12}{\overset{10}{\rightleftarrows}} ② \underset{12}{\overset{10}{\rightleftarrows}} ③ \underset{12}{\overset{10}{\rightleftarrows}} ④ \cdots$$

$\pi_0 = 1,$ $\qquad \pi_x = \dfrac{\lambda_0 \ldots \lambda_{x-1}}{\mu_1 \ldots \mu_x} = \left(\dfrac{10}{12}\right)^x = \left(\dfrac{5}{6}\right)^x, \qquad x \ge 1.$

$$\sum_{x=0}^{\infty} \pi_x = \sum_{x=0}^{\infty} \left(\frac{5}{6}\right)^x = \frac{1}{1-\frac{5}{6}} = 6 < \infty. \quad \Rightarrow \quad \text{stationary distribution does exist.}$$

$\pi(0) = \dfrac{1}{6}.$ $\qquad \pi(x) = \pi_x \, \pi(0) = \left(\dfrac{5}{6}\right)^x \left(\dfrac{1}{6}\right), \qquad x \ge 0.$

8. (14) Consider a Markov pure jump process on $\{1, 2, 3, 4\}$ with $Q_{12} = Q_{23} = Q_{34} = 1$, and $q_1 = 9$, $q_2 = 6$, $q_3 = 3$, and $q_4 = 0$ (i.e., 4 is an absorbing state).

Find $P_{23}(t)$, $t \geq 0$. $1 \to 2 \to 3 \to 4$

$q_{11} = -9$	$q_{12} = 9$	$q_{13} = 0$	$q_{14} = 0$
$q_{21} = 0$	$q_{22} = -6$	$q_{23} = 6$	$q_{24} = 0$
$q_{31} = 0$	$q_{32} = 0$	$q_{33} = -3$	$q_{34} = 3$
$q_{41} = 0$	$q_{42} = 0$	$q_{43} = 0$	$q_{44} = 0$

Forward equation:

$$P'_{23}(t) = P_{21}(t)\,q_{13} + P_{22}(t)\,q_{23} + P_{23}(t)\,q_{33} + P_{24}(t)\,q_{43}$$

$$= 6\,P_{22}(t) - 3\,P_{23}(t).$$

Need $P_{22}(t)$.

$$P'_{22}(t) = P_{21}(t)\,q_{12} + P_{22}(t)\,q_{22} + P_{23}(t)\,q_{32} + P_{24}(t)\,q_{42}$$

$$= 9\,P_{21}(t) - 6\,P_{22}(t) = -6\,P_{22}(t),$$

since $P_{21}(t) = 0$ (cannot go from 2 to 1).

$$P_{22}(0) = 1. \qquad \Rightarrow \qquad P_{22}(t) = e^{-6t}, \qquad t \geq 0.$$

OR

$$P_{22}(t) = P_2(\tau_1 > t) = e^{-6t}, \qquad t \geq 0, \qquad \text{since } q_2 = 6.$$

$$P'_{23}(t) = 6\,e^{-6t} - 3\,P_{23}(t).$$

$P_{23}(0) = 0.$

$$\Rightarrow \qquad P_{23}(t) = \int_0^t e^{-3(t-s)}\, 6e^{-6s}\, ds = 6e^{-3t} \int_0^t e^{-3s}\, ds$$

$$= 6e^{-3t} \cdot \frac{1}{3}\left(1 - e^{-3t}\right) = 2\left(e^{-3t} - e^{-6t}\right), \qquad t \ge 0.$$

Backward equation:

$$P'_{23}(t) = q_{21} P_{13}(t) + q_{22} P_{23}(t) + q_{23} P_{33}(t) + q_{24} P_{43}(t)$$

$$= -6 P_{23}(t) + 6 P_{33}(t).$$

Need $P_{33}(t)$.

$$P'_{33}(t) = q_{31} P_{13}(t) + q_{32} P_{23}(t) + q_{33} P_{33}(t) + q_{34} P_{43}(t)$$

$$= -3 P_{33}(t) - 3 P_{43}(t) = -3 P_{33}(t),$$

since $P_{43}(t) = 0$ (cannot go from 4 to 3).

$P_{33}(0) = 1. \qquad \Rightarrow \qquad P_{33}(t) = e^{-3t}, \qquad t \ge 0.$

OR

$P_{33}(t) = P_3(\tau_1 > t) = e^{-3t}, \qquad t \ge 0, \qquad$ since $q_3 = 3.$

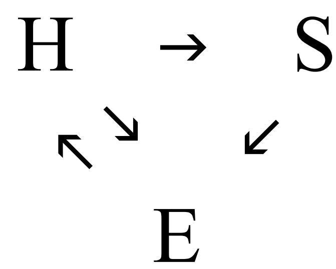
$$P'_{23}(t) = -6 P_{23}(t) + 6e^{-3t}.$$

$P_{23}(0) = 0.$

$$\Rightarrow \qquad P_{23}(t) = \int_0^t e^{-6(t-s)}\, 6e^{-3s}\, ds = 6e^{-6t} \int_0^t e^{3s}\, ds$$

$$= 6e^{-6t} \cdot \frac{1}{3}\left(e^{3t} - 1\right) = 2\left(e^{-3t} - e^{-6t}\right), \qquad t \ge 0.$$

9. Bob is a typical ☺+CS major student, he does only three things during a semester – **H**omework, **S**leep, and **E**at. (Like so many other students, Bob chooses to miss his classes.) Suppose that when Bob does homework, the duration of his homework session follows an exponential distribution with mean 240 minutes. When Bob sleeps, the duration of his sleep follows an exponential distribution with mean 300 minutes. When Bob eats, the duration of his meal (including preparation) follows an exponential distribution with mean 24 minutes. After Bob does homework, there is 0.60 probability that he would eat next and 0.40 probability that he would immediately go to sleep. After Bob sleeps, he always eats next (one cannot do homework on an empty stomach). After Bob eats, he always dives into homework (with all this new energy). Assume that all times are independent. Consider a Markov pure jump process with three states { **H**omework, **S**leep, and **E**at }.



"Hint": Rates (per hour): $q_H = \frac{60}{240} = 0.25$, $q_S = \frac{60}{300} = 0.20$, $q_E = \frac{60}{24} = 2.5$.

a) (12) This has been going on for a while now, so the stationary distribution has already been reached. Find the stationary distribution.

$Q_{HE} = 0.60$, $Q_{HS} = 0.40$. $Q_{SE} = 1$. $Q_{EH} = 1$.

$q_{HH} = -0.25$, $q_{HE} = 0.15$, $q_{HS} = 0.10$,

$q_{EH} = 2.50$, $q_{EE} = -2.50$, $q_{ES} = 0$,

$q_{SH} = 0$, $q_{SE} = 0.20$, $q_{SS} = -0.20$.

$$-0.25\,\pi(H) + 2.50\,\pi(E) = 0. \qquad (1)$$

$$0.15\,\pi(H) - 2.50\,\pi(E) + 0.20\,\pi(S) = 0. \qquad (2)$$

$$0.10\,\pi(H) - 0.20\,\pi(S) = 0. \qquad (3)$$

$$\pi(H) + \pi(E) + \pi(S) = 1. \qquad (4)$$

(1) $\Rightarrow$ $\pi(\mathrm{H}) = 10\,\pi(\mathrm{E}).$ (5)

(3) $\Rightarrow$ $\pi(\mathrm{H}) = 2\,\pi(\mathrm{S}).$ (6)

(5), (6) $\Rightarrow$ $\pi(\mathrm{S}) = 5\,\pi(\mathrm{E}).$ (7)

(4), (5), (7) $\Rightarrow$ $10\,\pi(\mathrm{E}) + \pi(\mathrm{E}) + 5\,\pi(\mathrm{E}) = 1.$

$$\pi(\mathrm{E}) = \frac{1}{16} = 0.0625. \quad (8)$$

(5), (8) $\pi(\mathrm{H}) = \frac{10}{16} = 0.6250.$

(7), (8) $\pi(\mathrm{S}) = \frac{5}{16} = 0.3125.$

$$\pi(\mathrm{H}) = \mathbf{\frac{10}{16}} = \mathbf{0.6250}, \qquad \pi(\mathrm{E}) = \mathbf{\frac{1}{16}} = \mathbf{0.0625}, \qquad \pi(\mathrm{S}) = \mathbf{\frac{5}{16}} = \mathbf{0.3125}.$$

b) (3) Bob's Mom is very concerned with his eating habits. Find the mean return time to state Find the mean return time to state **E**, $m_{\mathrm{E}} = \mathrm{E}_{\mathrm{E}}(\mathrm{T}_{\mathrm{E}})$.

$$\pi(x) = \frac{1}{q_x\, m_x}. \qquad \pi(\mathrm{E}) = \frac{1}{16} = \frac{1}{q_{\mathrm{E}}\, m_{\mathrm{E}}} = \frac{1}{2.5\, m_{\mathrm{E}}}.$$

$$m_{\mathrm{E}} = \frac{16}{2.5} = \mathbf{6.4} \text{ hours.}$$

10. (8) Let $B(t)$ denote a (standard) Brownian motion. Let $\lambda \in \mathbf{R}$ and let

$$X(t) = e^{\lambda^2 t/2} \sin(\lambda B(t)), \quad t \geq 0.$$ Use Itô's Lemma to find $dX(t)$.

$$X(t) = e^{\lambda^2 t/2} \sin(\lambda B(t)) = f(t, B(t)).$$

$$f(t, b) = e^{\lambda^2 t/2} \sin(\lambda b).$$

Ito's Lemma: $$dX = f_t\, dt + f_b\, dB + \frac{1}{2} f_{bb}\, (dB)^2 = (f_t + \frac{1}{2} f_{bb})\, dt + f_b\, dB.$$

$$f_t(t, b) = \frac{\lambda^2}{2} e^{\lambda^2 t/2} \sin(\lambda b), \qquad f_b(t, b) = \lambda e^{\lambda^2 t/2} \cos(\lambda b),$$

$$f_{bb}(t, b) = -\lambda^2 e^{\lambda^2 t/2} \sin(\lambda b).$$

$$f_t(t, b) + \frac{1}{2} f_{bb}(t, b) = 0.$$

$$dX(t) = 0\, dt + \lambda e^{\lambda^2 t/2} \cos(\lambda B(t))\, dB(t)$$

$$= \lambda e^{\lambda^2 t/2} \cos(\lambda B(t))\, dB(t).$$

For fun:

$X(t)$ has zero drift. $\Rightarrow$ $X(t)$ is a martingale.

11. (10) Let $\mathrm{B}(t)$ denote a (standard) Brownian motion. Find the mean function $m_{\mathrm{X}}(t)$ and the covariance function $r_{\mathrm{X}}(s,t)$, $0 \le s \le t$, of $\quad \mathrm{X}(t) = \mathrm{B}^3(t)$.

$\mathrm{B}(t)$ has a Normal distribution with mean 0 and variance $\sigma^2 t$, $\qquad t \ge 0$.

$$\mathrm{E}(\mathrm{B}^{2n}(t)) = \frac{(2n)!\,\sigma^{2n}\,t^n}{2^n\,(n)!}, \qquad t \ge 0.$$

$$\mathrm{E}(\mathrm{B}^2(t)) = t, \qquad \mathrm{E}(\mathrm{B}^4(t)) = 3t^2, \qquad \mathrm{E}(\mathrm{B}^6(t)) = 15t^3, \qquad t \ge 0.$$

$$m_{\mathrm{X}}(t) = \mathrm{E}(\mathrm{B}^3(t)) = \mathbf{0}, \qquad t \ge 0.$$

Let $0 \le s \le t$. $\qquad$ Then $\mathrm{B}(s)$ and $\{\mathrm{B}(t) - \mathrm{B}(s)\}$ are independent.

$$\begin{aligned}
r_{\mathrm{X}}(s,t) &= \mathrm{Cov}(\mathrm{B}^3(s), \mathrm{B}^3(t)) \\
&= \mathrm{E}(\mathrm{B}^3(s)\,\mathrm{B}^3(t)) - \mathrm{E}(\mathrm{B}^3(s)) \cdot \mathrm{E}(\mathrm{B}^3(t)) \\
&= \mathrm{E}(\mathrm{B}^3(s) \cdot \{\mathrm{B}(t) - \mathrm{B}(s) + \mathrm{B}(s)\}^3) - 0 \cdot 0 \\
&= \mathrm{E}(\mathrm{B}^3(s) \cdot \{\mathrm{B}(t) - \mathrm{B}(s)\}^3) + 3\,\mathrm{E}(\mathrm{B}^4(s) \cdot \{\mathrm{B}(t) - \mathrm{B}(s)\}^2) \\
&\qquad + 3\,\mathrm{E}(\mathrm{B}^5(s) \cdot \{\mathrm{B}(t) - \mathrm{B}(s)\}) + \mathrm{E}(\mathrm{B}^6(s)) \\
&= 3\,\mathrm{E}(\mathrm{B}^4(s)) \cdot \mathrm{E}(\{\mathrm{B}(t) - \mathrm{B}(s)\}^2) + \mathrm{E}(\mathrm{B}^6(s)) \\
&= 3 \times 3s^2 \cdot (t - s) + 15s^3 \\
&= \mathbf{9s^2 t + 6s^3} = \mathbf{3s^2(3t + 2s)}, \qquad 0 \le s \le t.
\end{aligned}$$

If $\mathrm{B}(t)$ is a Brownian motion with parameter σ,

$$r_{\mathrm{X}}(s,t) = 9\sigma^6 s^2 t + 6\sigma^6 s^3 = 3\sigma^6 s^2(3t + 2s), \qquad 0 \le s \le t.$$

12. (6) An American roulette wheel has 38 (equally likely) slots: 18 red, 18 black, and 2 green. Bob comes to a casino with \$13 and the intention to place repeated \$1 bets on red. Find the probability that Bob would have \$20 before he runs out of money.

American Roulette – 38 slots – 18 red, 18 black, 2 green.

$$p = \frac{18}{38}, \qquad q = \frac{20}{38}, \qquad \gamma_y = \left(\frac{20}{18}\right)^y .$$

$$\mathrm{P}_{13}(\mathrm{T}_0 > \mathrm{T}_{20}) = \frac{\sum_{y=0}^{13-1} \gamma_y}{\sum_{y=0}^{20-1} \gamma_y} = \frac{\sum_{y=0}^{13-1} \left(\frac{20}{18}\right)^y}{\sum_{y=0}^{20-1} \left(\frac{20}{18}\right)^y} = \frac{\left(\frac{20}{18}\right)^{13} - 1}{\left(\frac{20}{18}\right)^{20} - 1} \approx \mathbf{0.4060915}.$$