Examples for 03/29/2022 (2)

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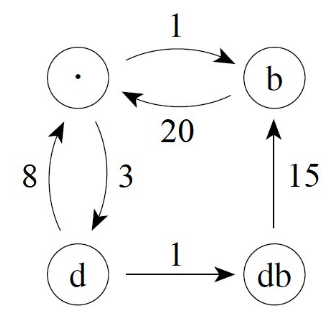
1. Consider a machine that is subject to failures (with rate 1) and repairs (with rate 20 if the repairman is not drunk). If the machine is working properly, the repairman gets bored and starts drinking after an exponentially distributed time with rate 3. The time it takes him to stop drinking is also exponentially distributed with rate 8 if the machine is working properly and rate 15 if the machine is broken (more urgency). Consider a Markov pure jump process on $\{\,\cdot, \text{b}, \text{d}, \text{db}\,\}$.
(In the state's name the letter "b" means broken, and "d" means drunk.)



[The machine breaks after an exponentially distributed random time with mean 120 days; if the repairman is not drunk, the time it takes to fix the machine is exponentially distributed with mean 6 days; the repairman starts drinking after an exponentially distributed random time with mean 40 days; the time it takes the repairman to sober up is exponentially distributed with mean 15 days if the machine is working properly and with mean 8 days if the machine is broken.]

The whole system operates for a long time, so the stationary distribution is already acquired.

"Hint": Obviously, 0) Identify all infinitesimal parameters of $X(t)$. Find the stationary distribution for $X(t)$.

a) What is the probability that the machine is broken?

b) What is the probability that the repairman is drunk?

c) Suppose the repairman is drunk, but the machine is working properly. What is the probability that the machine breaks before the repairman sobers up?

2. Customers arrive to a store to purchase a thingamabob (one per customer) according to a Poisson process with rate λ. As soon as the store is sold out of thingamabobs, the inventory software submits a restocking order. The time it takes for a new batch of thingamabobs to arrive has an Exponential distribution with rate μ. The number of thingamabobs in a new batch, Y, has he following probability distribution:

$$p(y) = \mathrm{P}(\mathrm{Y} = y), \quad y \geq 1, \;\; y - \text{integer.}$$

Assume that the size of each new batch is independent of the past, and that $\mathrm{E}(\mathrm{Y}) < \infty$. Assume that all times are independent.

Find the long-term distribution of $\mathrm{X}(t)$, the number of thingamabobs at the store.

The state space is $\{0, 1, 2, \ldots, d\}$ if Y has finitely many values and d is the largest possible value of Y. The state space is $\{0, 1, 2, 3, \ldots\}$ (all nonnegative integers) if Y has infinitely many possible values. It would be easier to assume that the state space is $\{0, 1, 2, 3, \ldots\}$ (all nonnegative integers), and if Y has finitely many values and d is the largest possible value of Y, then we will simply end up with $\pi(y) = 0$, $y > d$ (all these states are transient).

Once the process leaves zero, it moves back one thingamabob at a time, and will get back to zero eventually. The process is irreducible. The assumption $\mathrm{E}(\mathrm{Y}) < \infty$ makes the process positive recurrent. (The process is null recurrent if $\mathrm{E}(\mathrm{Y}) = \infty$.)

Hint 1: $\mathrm{Q}_{0y} = p(y), \quad y \geq 1.$

Obviously, $q_0 = \mu$. $\mathrm{Q}_{y,y-1} = 1$, $q_y = \lambda$, $y \geq 1$.
(That was not even worth including in the hint.)

Hint 2: Consider considering $\pi(y) = \heartsuit(y)\,\pi(0), \quad y \geq 1.$

This is NOT a birth and death process. Nevertheless, consider considering … .

Hint 3: $1 = \mathrm{P}(\mathrm{Y} \geq 1), \qquad p(y) - \mathrm{P}(\mathrm{Y} \geq y) + \mathrm{P}(\mathrm{Y} \geq y+1) = 0, \quad y \geq 1.$

Hint 4: $$\mathrm{E}(\mathrm{Y}) = \sum_{y=1}^{\infty} \mathrm{P}(\mathrm{Y} \geq y).$$

3. Bob makes his living fixing thingamabobs. Customers bring thingamabobs to Bob's workshop (one at a time) according to a Poisson process with rate of one in 4 weeks. The time to fix a thingamabob follows an Exponential distribution with mean 3 weeks. Bob repairs thingamabobs one at a time; and if a customer brings a thingamabob to be repaired while Bob is already working on a different thingamabob, the customer would leave it at Bob's workshop to be fixed on a first-come, first-served basis.

a) Find the long-term distribution of $X(t)$, the number of thingamabobs at Bob's workshop, if it exists.

b) Find the long-term expected number of thingamabobs at Bob's workshop.

c) When (if) Bob will fix all thingamabobs at his workshop, he will go on a well-deserved vacation. However, if the number of thingamabobs reaches 4, Bob will give up on his vacation plan. Given that there are 2 thingamabobs at Bob's workshop right now, find the probability that that number of thingamabobs will reach zero before it reaches 4.

Hint: Consider the embedded chain which is a birth and death chain. For the embedded chain,

$$p_x = Q_{x,x+1} = \frac{\lambda_x}{\lambda_x + \mu_x}, \qquad q_x = Q_{x,x-1} = \frac{\mu_x}{\lambda_x + \mu_x}.$$

Find γ_y, $y = 0, 1, 2, 3$. Then find $P_2(T_0 < T_4)$.

3. (continued)

Bob hires an apprentice to help him fix thingamabobs. Suppose the time is takes the apprentice to fix a thingamabob follows an Exponential distribution with mean 6 weeks. If there is only one thingamabob at Bob's workshop, then Bob is the one fixing it. However, if there is more than one thingamabob at Bob's workshop, then Bob and the apprentice each work on fixing a different thingamabob (one at a time).

d) Find the long-term distribution of $X(t)$, the number of thingamabobs at Bob's workshop, if it exists.

e) Find the long-term expected number of thingamabobs at Bob's workshop.

f) Find $P_2(T_0 < T_4)$.

4. Potential customers arrive to a single-server queue according to a Poisson process with rate λ. However, if a newcomer finds that n customers are already in the system, then this newcomer will join the queue with probability of only $1/(n+1)$. That is, with probability $n/(n+1)$ such a newcomer will not join the queue. The service times are independent and exponentially distributed with rate μ.

That is,

if there are no customers in the system, and a new person arrives after an exponentially distributed time with rate λ, that person would join the system with probability 1.

However,

if there is 1 customer currently in the system, this customer is receiving service with exponentially distributed service time with rate μ, and a new person arrives after an exponentially distributed time with rate λ, that person would join the system with probability 0.50 OR would simply leave with probability 0.50, in which case we would never even know that that person existed.

...

if there are 4 customers currently in the system, one of them receiving service with exponentially distributed service time with rate μ, and a new person arrives after an exponentially distributed time with rate λ, that person would join the system with probability 0.20 OR would simply leave with probability 0.80, in which case we would never even know that that person existed.

...

a) Find the long-term distribution of the number of customers in the system, if it exists.

b) Suppose that $\lambda = 4$, $\mu = 2$. If there are 3 customers in the system now, what is the probability that the number of customers in the system would reach 5 before it reaches 0 ?

Hint 1: Consider the embedded chain. For the embedded chain, find $P_3(T_0 > T_5)$.

Hint 2: Recall: $\gamma_0 = 1.$ $\gamma_y = \dfrac{q_1 \cdots q_y}{p_1 \cdots p_y}, \quad y \ge 1.$

$$P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \qquad P_x(T_a > T_b) = \frac{\sum_{y=a}^{x-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \qquad a < x < b.$$

Since $\quad p_x = Q_{x,x+1} = \dfrac{\lambda_x}{\lambda_x + \mu_x}, \qquad q_x = Q_{x,x-1} = \dfrac{\mu_x}{\lambda_x + \mu_x}, \qquad x \geq 1,$

$$\gamma_y = \frac{q_1 \dots q_y}{p_1 \dots p_y} = \frac{\mu_1 \dots \mu_y}{\lambda_1 \dots \lambda_y}, \qquad y \geq 1.$$

6. Consider a Markov pure jump process on $S = \{ ①, ②, ③ \}$ with

$q_1 = 4, \quad Q_{12} = 0.75, \quad Q_{13} = 0.25,$

$q_2 = 8, \quad Q_{21} = 0.50, \quad Q_{23} = 0.50,$

$q_3 = 0, \quad$ ③ is an absorbing state.

$$\begin{tikzcd} ① \arrow[rr, shift left] \arrow[dr] & & ② \arrow[ll, shift left] \arrow[dl] \\ & ③ & \end{tikzcd}$$

Obviously, $\quad P_{31}(t) = 0, \quad P_{32}(t) = 0,$

$P_{33}(t) = 1, \quad t \geq 0.$

Possible fairy tales:

	①	②	③
a machine	adjusted properly	out of alignment	broken
a person	healthy	ill	dead

a) (11) Use the forward equation to obtain $P_{11}(t)$, $P_{12}(t)$, and $P_{13}(t)$, $t \geq 0$.

(i) Set up the system of three differential equations for $P_{11}(t)$, $P_{12}(t)$, and $P_{13}(t)$.

Spoiler: Two of the three equations form a system of two differential equations for $P_{11}(t)$ and $P_{12}(t)$. The third equation expresses $P'_{13}(t)$ in terms of $P_{11}(t)$ and $P_{12}(t)$. Obviously, first …

(ii) Solve the system for $P_{11}(t)$ and $P_{12}(t)$, $t \geq 0$.

Recall: Given a square matrix $\mathbf{A}$, suppose there are a constant λ and a nonzero vector $\vec{\mathbf{v}}$ such that $\mathbf{A}\,\vec{\mathbf{v}} = \lambda\,\vec{\mathbf{v}}$, then λ is called an eigenvalue of $\mathbf{A}$, and $\vec{\mathbf{v}}$ is an eigenvector of $\mathbf{A}$ corresponding to λ.

Consider a homogeneous system of linear first order differential equations: $\vec{\mathbf{x}}\,' = \mathbf{A}\,\vec{\mathbf{x}}$.

That is,

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

IF $\mathbf{A}$ has distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (this is indeed the case here), and their respective eigenvectors are $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n$, then the system $\vec{\mathbf{x}}\,' = \mathbf{A}\,\vec{\mathbf{x}}$ has a general solution

$$\vec{\mathbf{x}}(t) = C_1\,\vec{\mathbf{v}}_1\,e^{\lambda_1 t} + C_2\,\vec{\mathbf{v}}_2\,e^{\lambda_2 t} + \ldots + C_n\,\vec{\mathbf{v}}_n\,e^{\lambda_n t}.$$

$C_1, C_2, \ldots, C_n$ are then determined from the initial condition:

$$\vec{\mathbf{x}}(0) = C_1\,\vec{\mathbf{v}}_1 + C_2\,\vec{\mathbf{v}}_2 + \ldots + C_n\,\vec{\mathbf{v}}_n.$$

(iii) Use the third equation to obtain $\mathrm{P}_{13}(t)$, $t \geq 0$.

You are welcome to use $\mathrm{P}_{11}(t) + \mathrm{P}_{12}(t) + \mathrm{P}_{13}(t) = 1$ to *double-check* your answer. However, to obtain $\mathrm{P}_{13}(t)$, you have to use the differential equation you obtained in (a) (i).

b) (7) Use the backward equation to obtain $\mathrm{P}_{11}(t)$, $\mathrm{P}_{21}(t)$, and $\mathrm{P}_{31}(t)$, $t \geq 0$.

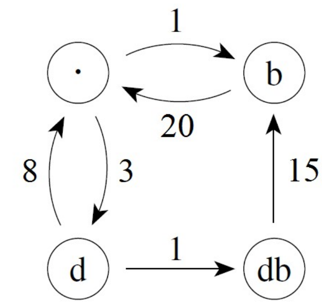
(i) Set up the system of three differential equations for $\mathrm{P}_{11}(t)$, $\mathrm{P}_{21}(t)$, and $\mathrm{P}_{31}(t)$.

Spoiler: Two of the three equations form a system of two differential equations for $\mathrm{P}_{11}(t)$ and $\mathrm{P}_{21}(t)$. The third equation is $\mathrm{P}_{31}'(t) = 0$, which yields $\mathrm{P}_{31}(t) = 0$, $t \geq 0$, something we already know.

(ii) Solve the system for $\mathrm{P}_{11}(t)$ and $\mathrm{P}_{21}(t)$, $t \geq 0$.

To *double-check* your answer, you already have $\mathrm{P}_{11}(t)$ from part (a).

1. Consider a machine that is subject to failures (with rate 1) and repairs (with rate 20 if the repairman is not drunk). If the machine is working properly, the repairman gets bored and starts drinking after an exponentially distributed time with rate 3. The time it takes him to stop drinking is also exponentially distributed with rate 8 if the machine is working properly and rate 15 if the machine is broken (more urgency). Consider a Markov pure jump process on $\{\cdot, \mathrm{b}, \mathrm{d}, \mathrm{db}\}$. (In the state's name the letter "b" means broken, and "d" means drunk.)



[The machine breaks after an exponentially distributed random time with mean 120 days; if the repairman is not drunk, the time it takes to fix the machine is exponentially distributed with mean 6 days; the repairman starts drinking after an exponentially distributed random time with mean 40 days; the time it takes the repairman to sober up is exponentially distributed with mean 15 days if the machine is working properly and with mean 8 days if the machine is broken.]

The whole system operates for a long time, so the stationary distribution is already acquired.

$q_{\bullet\bullet} = -4$	$q_{\bullet \mathrm{b}} = 1$	$q_{\bullet \mathrm{d}} = 3$	$q_{\bullet \mathrm{db}} = 0$
$q_{\mathrm{b}\bullet} = 20$	$q_{\mathrm{b}\,\mathrm{b}} = -20$	$q_{\mathrm{b}\,\mathrm{d}} = 0$	$q_{\mathrm{b}\,\mathrm{db}} = 0$
$q_{\mathrm{d}\bullet} = 8$	$q_{\mathrm{d}\,\mathrm{b}} = 0$	$q_{\mathrm{d}\,\mathrm{d}} = -9$	$q_{\mathrm{d}\,\mathrm{db}} = 1$
$q_{\mathrm{db}\bullet} = 0$	$q_{\mathrm{db}\,\mathrm{b}} = 15$	$q_{\mathrm{db}\,\mathrm{d}} = 0$	$q_{\mathrm{db}\,\mathrm{db}} = -15$

$$\sum_x \pi(x) q_{xy} = 0, \qquad y \in S.$$

$$-4\,\pi(\bullet) + 20\,\pi(\mathrm{b}) + 8\,\pi(\mathrm{d}) = 0. \tag{1}$$

$$\pi(\bullet) - 20\,\pi(\mathrm{b}) + 15\,\pi(\mathrm{db}) = 0. \tag{2}$$

$$3\,\pi(\bullet) - 9\,\pi(\mathrm{d}) = 0. \tag{3}$$

$$\pi(\mathrm{d}) - 15\,\pi(\mathrm{db}) = 0. \tag{4}$$

$$\pi(\bullet) + \pi(\mathrm{b}) + \pi(\mathrm{d}) + \pi(\mathrm{db}) = 1. \tag{5}$$

(3)	$\Rightarrow$	$\pi(\bullet) = 3\,\pi(\mathrm{d})$.	(6)
(4)	$\Rightarrow$	$\pi(\mathrm{d}) = 15\,\pi(\mathrm{db})$.	(7)
(2), (6), (7)	$\Rightarrow$	$3\,\pi(\mathrm{d}) - 20\,\pi(\mathrm{b}) + \pi(\mathrm{d}) = 0$.	
	$\Rightarrow$	$\pi(\mathrm{d}) = 5\,\pi(\mathrm{b})$.	(8)

(5), (6), (7), (8) $\Rightarrow$ $3\,\pi(\mathrm{d}) + \frac{1}{5}\,\pi(\mathrm{d}) + \pi(\mathrm{d}) + \frac{1}{15}\,\pi(\mathrm{d}) = 1$.

$\Rightarrow$ $\pi(\mathrm{d}) = \frac{15}{64}$.

$$\Rightarrow \qquad \pi(\bullet) = \frac{45}{64}, \qquad \pi(\mathrm{b}) = \frac{3}{64}, \qquad \pi(\mathrm{d}) = \frac{15}{64}, \qquad \pi(\mathrm{db}) = \frac{1}{64}.$$

To double-check (just don't tell Alex I did this):

```
> q = matrix( rep(0,16), nrow=4 )
> q[1,1] = -4;  q[1,2] = 1;  q[1,3] = 3
> q[2,1] = 20;  q[2,2] = -20
> q[3,1] = 8;  q[3,3] = -9;  q[3,4] = 1
> q[4,2] = 15;  q[4,4] = -15
> q
     [,1] [,2] [,3] [,4]
[1,]   -4    1    3    0
[2,]   20  -20    0    0
[3,]    8    0   -9    1
[4,]    0   15    0  -15
> eigen( t(q) )
$values
[1] -2.188221e+01+0.00000e+00i -1.305890e+01+2.22492e+00i
[3] -1.305890e+01-2.22492e+00i  3.295975e-16+0.00000e+00i

$vectors
                     [,1]                     [,2]
[1,]  0.76766244+0i  0.7968733+0.0000000i
[2,] -0.61486577+0i -0.1797808+0.1879526i
[3,] -0.17877275+0i -0.4528973-0.2482593i
[4,]  0.02597608+0i -0.1641953+0.0603067i
```

```
                        [,3]            [,4]
[1,]  0.7968733+0.0000000i -0.94658211+0i
[2,] -0.1797808-0.1879526i -0.06310547+0i
[3,] -0.4528973+0.2482593i -0.31552737+0i
[4,] -0.1641953-0.0603067i -0.02103516+0i

> eigen(t(q))$vectors[,4] / sum(eigen(t(q))$vectors[,4])
[1] 0.703125+0i 0.046875+0i 0.234375+0i 0.015625+0i
> 64 * eigen(t(q))$vectors[,4] / sum(eigen(t(q))$vectors[,4])
[1] 45+0i  3+0i 15+0i  1+0i
```

a) What is the probability that the machine is broken?

$$\pi(\mathrm{b}) + \pi(\mathrm{db}) = \frac{3}{64} + \frac{1}{64} = \mathbf{\frac{1}{16}}.$$

b) What is the probability that the repairman is drunk?

$$\pi(\mathrm{d}) + \pi(\mathrm{db}) = \frac{15}{64} + \frac{1}{64} = \mathbf{\frac{1}{4}}.$$

c) Suppose the repairman is drunk, but the machine is working properly. What is the probability that the machine breaks before the repairman sobers up?

$Q_{\mathrm{d\,db}} = ?$ $\qquad q_{\mathrm{d}\bullet} = 8, \quad q_{\mathrm{d\,b}} = 0, \quad q_{\mathrm{d\,d}} = -9, \quad q_{\mathrm{d\,db}} = 1.$

$$\Rightarrow \quad q_{\mathrm{d}} = 9. \qquad Q_{\mathrm{d}\bullet} = \frac{8}{9}, \qquad Q_{\mathrm{d\,db}} = \mathbf{\frac{1}{9}}.$$

2. Customers arrive to a store to purchase a thingamabob (one per customer) according to a Poisson process with rate λ. As soon as the store is sold out of thingamabobs, the inventory software submits a restocking order. The time it takes for a new batch of thingamabobs to arrive has an Exponential distribution with rate μ. The number of thingamabobs in a new batch, Y, has he following probability distribution:

$$p(y) = \mathrm{P}(\mathrm{Y} = y), \quad y \ge 1, \ y - \text{integer}.$$

Assume that the size of each new batch is independent of the past, and that $\mathrm{E}(\mathrm{Y}) < \infty$. Assume that all times are independent.

Find the long-term distribution of $\mathrm{X}(t)$, the number of thingamabobs at the store.

The state space is $\{0, 1, 2, \ldots, d\}$ if Y has finitely many values and d is the largest possible value of Y. The state space is $\{0, 1, 2, 3, \ldots\}$ (all nonnegative integers) if Y has infinitely many possible values. It would be easier to assume that the state space is $\{0, 1, 2, 3, \ldots\}$ (all nonnegative integers), and if Y has finitely many values and d is the largest possible value of Y, then we will simply end up with $\pi(y) = 0, \ y > d$ (all these states are transient).

Once the process leaves zero, it moves back one thingamabob at a time, and will get back to zero eventually. The process is irreducible. The assumption $\mathrm{E}(\mathrm{Y}) < \infty$ makes the process positive recurrent. (The process is null recurrent if $\mathrm{E}(\mathrm{Y}) = \infty$.)

Hint 1: $\mathrm{Q}_{0y} = p(y), \quad y \ge 1.$

Obviously, $q_0 = \mu. \quad \mathrm{Q}_{y,y-1} = 1, \ q_y = \lambda, \quad y \ge 1.$

(That was not even worth including in the hint.)

Hint 2: Consider considering $\pi(y) = \heartsuit(y)\,\pi(0), \quad y \ge 1.$

This is NOT a birth and death process. Nevertheless, consider considering … .

Hint 3: $1 = \mathrm{P}(\mathrm{Y} \ge 1), \qquad p(y) - \mathrm{P}(\mathrm{Y} \ge y) + \mathrm{P}(\mathrm{Y} \ge y+1) = 0, \quad y \ge 1.$

Hint 4: $\mathrm{E}(\mathrm{Y}) = \sum_{y=1}^{\infty} \mathrm{P}(\mathrm{Y} \ge y).$

$$
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & \cdots \\
\hline
0 & -\mu & \mu p(1) & \mu p(2) & \mu p(3) & \mu p(4) & \cdots \\
1 & \lambda & -\lambda & 0 & 0 & 0 & \cdots \\
2 & 0 & \lambda & -\lambda & 0 & 0 & \cdots \\
3 & 0 & 0 & \lambda & -\lambda & 0 & \cdots \\
4 & 0 & 0 & 0 & \lambda & -\lambda & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

$$-\mu\,\pi(0) + \lambda\,\pi(1) = 0. \qquad \Rightarrow \qquad \pi(1) = \frac{\mu}{\lambda}\,\pi(0) = \frac{\mu}{\lambda}\,\mathrm{P}(\mathrm{Y} \ge 1)\,\pi(0).$$

$$\mu\,p(1)\,\pi(0) - \lambda\,\pi(1) + \lambda\,\pi(2) = 0.$$

$$\mu\,p(1)\,\pi(0) - \mu\,\mathrm{P}(\mathrm{Y} \ge 1)\,\pi(0) + \lambda\,\pi(2) = 0.$$

$$-\mu\,\mathrm{P}(\mathrm{Y} \ge 2)\,\pi(0) + \lambda\,\pi(2) = 0.$$

$$\Rightarrow \qquad \pi(2) = \frac{\mu}{\lambda}\,\mathrm{P}(\mathrm{Y} \ge 2)\,\pi(0).$$

$$\mu\,p(2)\,\pi(0) - \lambda\,\pi(2) + \lambda\,\pi(3) = 0.$$

$$\mu\,p(2)\,\pi(0) - \mu\,\mathrm{P}(\mathrm{Y} \ge 2)\,\pi(0) + \lambda\,\pi(3) = 0.$$

$$-\mu\,\mathrm{P}(\mathrm{Y} \ge 3)\,\pi(0) + \lambda\,\pi(3) = 0.$$

$$\Rightarrow \qquad \pi(3) = \frac{\mu}{\lambda}\,\mathrm{P}(\mathrm{Y} \ge 3)\,\pi(0).$$

Assume $\pi(y) = \dfrac{\mu}{\lambda}\,\mathrm{P}(\mathrm{Y} \ge y)\,\pi(0),$ $\qquad y$ – integer, $y \ge 1$.

$$\mu\,p(y)\,\pi(0) - \lambda\,\pi(y) + \lambda\,\pi(y+1) = 0.$$

$$\mu\,p(y)\,\pi(0) - \mu\,\mathrm{P}(\mathrm{Y} \ge y)\,\pi(0) + \lambda\,\pi(y+1) = 0.$$

$$-\mu\,\mathrm{P}(\mathrm{Y} \ge y+1)\,\pi(0) + \lambda\,\pi(y+1) = 0.$$

$$\Rightarrow \qquad \pi(y+1) = \frac{\mu}{\lambda}\,\mathrm{P}(\mathrm{Y} \ge y+1)\,\pi(0).$$

By induction, $\pi(y) = \frac{\mu}{\lambda} \mathrm{P}(\mathrm{Y} \ge y)\, \pi(0), \qquad y \ge 1. \qquad\qquad \heartsuit(y) = \frac{\mu}{\lambda} \mathrm{P}(\mathrm{Y} \ge y).$

$$1 = \sum_{y=0}^{\infty} \pi(y) = \pi(0) \left[1 + \frac{\mu}{\lambda} \sum_{y=1}^{\infty} \mathrm{P}(\mathrm{Y} \ge y) \right] = \pi(0)\, \frac{\lambda + \mu\, \mathrm{E}(\mathrm{Y})}{\lambda}.$$

$$\pi(0) = \frac{\lambda}{\lambda + \mu\, \mathrm{E}(\mathrm{Y})}, \qquad\qquad \pi(y) = \frac{\mu\, \mathrm{P}(\mathrm{Y} \ge y)}{\lambda + \mu\, \mathrm{E}(\mathrm{Y})}, \qquad y \ge 1.$$

OR

$$\pi(0) = \frac{1}{q_0\, m_0}. \qquad\qquad q_0 = \mu.$$

$m_0 = \ldots$ first must leave zero, receive (on average) $\mathrm{E}(\mathrm{Y})$ thingamabobs, need (on average) $\mathrm{E}(\mathrm{Y})$ customers to show up

$$\ldots = \frac{1}{\mu} + \frac{1}{\lambda} \mathrm{E}(\mathrm{Y}).$$

$$\pi(0) = \frac{1}{q_0\, m_0} = \frac{1}{\mu \left(\frac{1}{\mu} + \frac{1}{\lambda} \mathrm{E}(\mathrm{Y}) \right)} = \frac{\lambda}{\lambda + \mu\, \mathrm{E}(\mathrm{Y})}.$$

Intuition:

If μ is large (new batches tend to arrive fast), $\pi(0)$ is small.

If μ is small (takes a long time for a new batch to arrive), $\pi(0)$ is close to 1.

If λ is large (customers arrive often), $\pi(0)$ is close to 1.

If λ is small (long time between customers), $\pi(0)$ is small.

If $\mathrm{E}(\mathrm{Y})$ is large (large new batches), $\pi(0)$ is small.

If $\mathrm{E}(\mathrm{Y}) = \infty$, stationary distribution does not exist, the process is null recurrent.

If $\mathrm{P}(\mathrm{Y} = 1) = 1$, this is a two-state birth and death process (λ and μ are switched around compared to the textbook).

OR

Embedded Markov chain for this process:

$$
\begin{array}{c|ccccccc|}
 & 0 & 1 & 2 & 3 & 4 & \cdots \\
\hline
0 & 0 & p(1) & p(2) & p(3) & p(4) & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 0 & 1 & 0 & 0 & 0 & \cdots \\
3 & 0 & 0 & 1 & 0 & 0 & \cdots \\
4 & 0 & 0 & 0 & 1 & 0 & \cdots \\
5 & 0 & 0 & 0 & 0 & 1 & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

$\pi_e(0) = \pi_e(1),$ $\qquad$ $\pi_e(y) = p(y)\,\pi_e(0) + \pi_e(y+1), \qquad y \ge 1.$

$1 = P(Y \ge 1).$ $\qquad \Rightarrow \qquad \pi_e(0) = P(Y \ge 1)\,\pi_e(0),$

$P(Y \ge y) = p(y) + P(Y \ge y+1), \quad y \ge 1.$

$$\Rightarrow \qquad P(Y \ge y)\,\pi_e(0) = p(y)\,\pi_e(0) + P(Y \ge y+1)\,\pi_e(0), \qquad y \ge 1.$$

$\Rightarrow \qquad \pi_e(y) = P(Y \ge y)\,\pi_e(0), \qquad y \ge 1.$

(or use mathematical induction to show this)

$$1 = \sum_{y=0}^{\infty} \pi(y) = \pi_e(0) + \sum_{y=1}^{\infty} \pi_e(y)$$

$$= \pi_e(0) + \sum_{y=1}^{\infty} P(Y \ge y)\,\pi_e(0) = \pi_e(0)\left[1 + E(Y)\right].$$

$\pi_e(0) = \dfrac{1}{1 + E(Y)},$ $\qquad$ $\pi_e(y) = \dfrac{P(Y \ge y)}{1 + E(Y)}, \qquad y \ge 1.$

$\pi(y)$ is proportional to $\pi_{\text{emb}}(y)/q_y$.

$$\pi_e(0) = \frac{1}{1 + E(Y)}, \qquad \pi_e(y) = \frac{P(Y \ge y)}{1 + E(Y)}, \quad y \ge 1.$$

$$q_0 = \mu, \qquad q_y = \lambda, \quad y \ge 1.$$

$$\pi(0) \sim \frac{1}{\mu}, \qquad \pi(y) \sim \frac{P(Y \ge y)}{\lambda}, \quad y \ge 1.$$

$$\frac{1}{\mu} + \sum_{y=1}^{\infty} \frac{P(Y \ge y)}{\lambda} = \frac{1}{\mu} + \frac{E(Y)}{\lambda} = \frac{\lambda + \mu E(Y)}{\mu\lambda}.$$

$$\pi(0) = \frac{\frac{1}{\mu}}{\frac{\lambda + \mu E(Y)}{\mu\lambda}} = \frac{\lambda}{\lambda + \mu E(Y)}, \qquad \pi(y) = \frac{\frac{P(Y \ge y)}{\lambda}}{\frac{\lambda + \mu E(Y)}{\mu\lambda}} = \frac{\mu P(Y \ge y)}{\lambda + \mu E(Y)},$$

$$y \ge 1.$$

The embedded chain for this problem is the same as the Markov chain in example 1 from Examples for 02/10/2022 (2) (Alex eats an apple a day), with one exception: $p(0) = 0$.

3. Bob makes his living fixing thingamabobs. Customers bring thingamabobs to Bob's workshop (one at a time) according to a Poisson process with rate of one in 4 weeks. The time to fix a thingamabob follows an Exponential distribution with mean 3 weeks. Bob repairs thingamabobs one at a time; and if a customer brings a thingamabob to be repaired while Bob is already working on a different thingamabob, the customer would leave it at Bob's workshop to be fixed on a first-come, first-served basis.

a) Find the long-term distribution of $X(t)$, the number of thingamabobs at Bob's workshop, if it exists.

Birth and death process.

$$\lambda_x = \lambda = \frac{1}{4}, \quad x \geq 0, \qquad \mu_x = \mu = \frac{1}{3}, \quad x \geq 1.$$

$$0 \underset{\frac{1}{3}}{\overset{\frac{1}{4}}{\rightleftarrows}} 1 \underset{\frac{1}{3}}{\overset{\frac{1}{4}}{\rightleftarrows}} 2 \underset{\frac{1}{3}}{\overset{\frac{1}{4}}{\rightleftarrows}} 3 \underset{\frac{1}{3}}{\overset{\frac{1}{4}}{\rightleftarrows}} 4 \cdots$$

$$\pi_0 = 1, \qquad \pi_x = \frac{\frac{1}{4} \cdot \frac{1}{4} \cdot \ldots \cdot \frac{1}{4}}{\frac{1}{3} \cdot \frac{1}{3} \cdot \ldots \cdot \frac{1}{3}} = \left(\frac{3}{4}\right)^x, \qquad x \geq 1.$$

$$\sum_{x=0}^{\infty} \pi_x = \sum_{x=0}^{\infty} \left(\frac{3}{4}\right)^x = \frac{1}{1-\frac{3}{4}} = 4.$$

$$\pi(0) = \frac{1}{4}. \qquad \pi(x) = \pi_x \, \pi(0) = \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right), \qquad x \geq 0.$$

b) Find the long-term expected number of thingamabobs at Bob's workshop.

$$\sum_{x=0}^{\infty} x \cdot \left(\frac{3}{4}\right)^{x} \left(\frac{1}{4}\right) = \left(\frac{3}{4}\right) \cdot \sum_{x=0}^{\infty} x \cdot \left(\frac{3}{4}\right)^{x-1} \left(\frac{1}{4}\right) = \frac{3}{4} \cdot \mathrm{E}(\mathrm{Y}) = \ldots$$

where Y has a Geometric$\left(p = \frac{1}{4}\right)$ distribution, $\mathrm{E}(\mathrm{Y}) = \frac{1}{p} = 4$

$\ldots = \mathbf{3}$.

OR

Fun fact: $$\sum_{k=0}^{\infty} k\, z^{k} = \frac{z}{(1-z)^{2}} \qquad \text{for } -1 < z < 1. \qquad z = \frac{3}{4}.$$

$$\sum_{x=0}^{\infty} x \cdot \left(\frac{3}{4}\right)^{x} \left(\frac{1}{4}\right) = \left(\frac{1}{4}\right) \cdot \sum_{x=0}^{\infty} x \cdot \left(\frac{3}{4}\right)^{x} = \frac{1}{4} \cdot \frac{\frac{3}{4}}{\left(1-\frac{3}{4}\right)^{2}} = \mathbf{3}.$$

c) When (if) Bob will fix all thingamabobs at his workshop, he will go on a well-deserved vacation. However, if the number of thingamabobs reaches 4, Bob will give up on his vacation plan. Given that there are 2 thingamabobs at Bob's workshop right now, find the probability that that number of thingamabobs will reach zero before it reaches 4.

Hint: Consider the embedded chain which is a birth and death chain. For the embedded chain,

$$p_x = \mathrm{Q}_{x,\,x+1} = \frac{\lambda_x}{\lambda_x + \mu_x}, \qquad q_x = \mathrm{Q}_{x,\,x-1} = \frac{\mu_x}{\lambda_x + \mu_x}.$$

Find γ_y, $y = 0, 1, 2, 3$. Then find $\mathrm{P}_2(\mathrm{T}_0 < \mathrm{T}_4)$.

$p_0 = 1$; $\qquad p_x = \frac{3}{7}, \qquad q_x = \frac{4}{7}, \qquad x \geq 1.$

$$\gamma_0 = 1. \qquad \gamma_y = \frac{q_1 \cdots q_y}{p_1 \cdots p_y} = \left(\frac{4}{3}\right)^y, \qquad y \ge 1.$$

$$P_2(T_0 < T_4) = 1 - P_2(T_0 > T_4) = 1 - \frac{\sum_{y=0}^{2-1} \gamma_y}{\sum_{y=0}^{4-1} \gamma_y} = 1 - \frac{\left(\frac{4}{3}\right)^2 - 1}{\left(\frac{4}{3}\right)^4 - 1} = 1 - \frac{63}{175} = \mathbf{0.64}.$$

OR

$$P_2(T_0 < T_4) = \frac{\sum_{y=2}^{4-1} \gamma_y}{\sum_{y=0}^{4-1} \gamma_y} = \frac{\left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3}{1 + \left(\frac{4}{3}\right)^1 + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3} = \frac{48+64}{27+36+48+64} = \mathbf{0.64}.$$

3. (continued)

Bob hires an apprentice to help him fix thingamabobs. Suppose the time is takes the apprentice to fix a thingamabob follows an Exponential distribution with mean 6 weeks. If there is only one thingamabob at Bob's workshop, then Bob is the one fixing it. However, if there is more than one thingamabob at Bob's workshop, then Bob and the apprentice each work on fixing a different thingamabob (one at a time).

d) Find the long-term distribution of $X(t)$, the number of thingamabobs at Bob's workshop, if it exists.

$$\textcircled{0} \underset{\frac{1}{3}}{\overset{\frac{1}{4}}{\rightleftarrows}} \textcircled{1} \underset{\frac{1}{3}+\frac{1}{6}}{\overset{\frac{1}{4}}{\rightleftarrows}} \textcircled{2} \underset{\frac{1}{3}+\frac{1}{6}}{\overset{\frac{1}{4}}{\rightleftarrows}} \textcircled{3} \underset{\frac{1}{3}+\frac{1}{6}}{\overset{\frac{1}{4}}{\rightleftarrows}} \textcircled{4} \bullet\bullet\bullet$$

$$\pi_0 = 1, \qquad \pi_1 = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4},$$

$$\pi_x = \frac{\frac{1}{4}\cdot\frac{1}{4}\cdot\ \ldots\ \cdot\frac{1}{4}}{\frac{1}{3}\cdot\frac{1}{2}\cdot\ \ldots\ \cdot\frac{1}{2}} = \left(\frac{3}{4}\right)\cdot\left(\frac{1}{2}\right)^{x-1} = \left(\frac{3}{2}\right)\cdot\left(\frac{1}{2}\right)^{x} = 3\cdot\left(\frac{1}{2}\right)^{x+1}, \qquad x \ge 2.$$

$$\sum_{x=0}^{\infty} \pi_x = 1 + \sum_{x=1}^{\infty} \left(\frac{3}{4}\right)\cdot\left(\frac{1}{2}\right)^{x-1} = 1 + \frac{3}{4}\cdot\frac{1}{1-\frac{1}{2}} = 2.5.$$

$$\pi(0) = \frac{1}{2.5} = 0.40. \qquad \pi(x) = \pi_x\,\pi(0) = 0.30\cdot\left(\frac{1}{2}\right)^{x-1} = 0.60\cdot\left(\frac{1}{2}\right)^{x}, \qquad x \ge 1.$$

e) Find the long-term expected number of thingamabobs at Bob's workshop.

$$0\cdot 0.40 + \sum_{x=1}^{\infty} x\cdot 0.30\cdot\left(\frac{1}{2}\right)^{x-1} = 0.60\cdot\sum_{x=0}^{\infty} x\cdot\left(\frac{1}{2}\right)^{x} = \mathbf{1.2}.$$

f) Find $P_2(T_0 < T_4)$.

$$\gamma_0 = 1. \qquad \gamma_y = \frac{q_1 \cdots q_y}{p_1 \cdots p_y} = \frac{\frac{1}{3}\cdot\frac{1}{2}\cdot\ \ldots\ \cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{4}\cdot\ \ldots\ \cdot\frac{1}{4}} = \frac{4}{3}\cdot 2^{y-1}, \qquad y \ge 1.$$

$$P_2(T_0 < T_4) = \frac{\sum_{y=2}^{4-1}\gamma_y}{\sum_{y=0}^{4-1}\gamma_y} = \frac{\frac{8}{3}+\frac{16}{3}}{1+\frac{4}{3}+\frac{8}{3}+\frac{16}{3}} = \mathbf{\frac{24}{31}} \approx \mathbf{0.7742}.$$

4. Potential customers arrive to a single-server queue according to a Poisson process with rate λ. However, if a newcomer finds that n customers are already in the system, then this newcomer will join the queue with probability of only $1/(n+1)$. That is, with probability $n/(n+1)$ such a newcomer will not join the queue. The service times are independent and exponentially distributed with rate μ.

That is,

if there are no customers in the system, and a new person arrives after an exponentially distributed time with rate λ, that person would join the system with probability 1.

However,

if there is 1 customer currently in the system, this customer is receiving service with exponentially distributed service time with rate μ, and a new person arrives after an exponentially distributed time with rate λ, that person would join the system with probability 0.50 OR would simply leave with probability 0.50, in which case we would never even know that that person existed.

...

if there are 4 customers currently in the system, one of them receiving service with exponentially distributed service time with rate μ, and a new person arrives after an exponentially distributed time with rate λ, that person would join the system with probability 0.20 OR would simply leave with probability 0.80, in which case we would never even know that that person existed.

...

a) Find the long-term distribution of the number of customers in the system, if it exists.

Birth and death process.

$$\lambda_x = \lambda \times \frac{1}{x+1}, \qquad x \ge 0, \qquad\qquad \mu_x = \mu, \qquad x \ge 1.$$

$$\pi_0 = 1, \qquad \pi_x = \frac{\lambda_0 \ldots \lambda_{x-1}}{\mu_1 \ldots \mu_x} = \frac{\lambda \cdot \frac{\lambda}{2} \cdot \frac{\lambda}{3} \cdot \ldots \cdot \frac{\lambda}{x}}{\mu \cdot \mu \cdot \mu \cdot \ldots \cdot \mu} = \frac{(\lambda/\mu)^x}{x!}, \qquad x \ge 1.$$

$$\sum_{x=0}^{\infty} \pi_x = \sum_{x=0}^{\infty} \frac{(\lambda/\mu)^x}{x!} = e^{\lambda/\mu}.$$

$$\pi(0) = e^{-\lambda/\mu}.$$

$$\pi(x) = \pi_x \, \pi(0) = \frac{(\lambda/\mu)^x}{x!} e^{-\lambda/\mu}, \qquad x \ge 0.$$

The long-term distribution of the number of customers in the system is a Poisson distribution with mean $\dfrac{\lambda}{\mu}$.

b) Suppose that $\lambda = 4$, $\mu = 2$. If there are 3 customers in the system now, what is the probability that the number of customers in the system would reach 5 before it reaches 0 ?

Hint 1: Consider the embedded chain. For the embedded chain, find $\mathrm{P}_3(\mathrm{T}_0 > \mathrm{T}_5)$.

Hint 2: Recall: $\gamma_0 = 1.$ $\qquad \gamma_y = \dfrac{q_1 \cdots q_y}{p_1 \cdots p_y}, \quad y \ge 1.$

$$\mathrm{P}_x(\mathrm{T}_a < \mathrm{T}_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \qquad \mathrm{P}_x(\mathrm{T}_a > \mathrm{T}_b) = \frac{\sum_{y=a}^{x-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \qquad a < x < b.$$

Since $\quad p_x = \mathrm{Q}_{x,x+1} = \dfrac{\lambda_x}{\lambda_x + \mu_x}, \qquad q_x = \mathrm{Q}_{x,x-1} = \dfrac{\mu_x}{\lambda_x + \mu_x}, \qquad x \ge 1,$

$$\gamma_y = \frac{q_1 \cdots q_y}{p_1 \cdots p_y} = \frac{\mu_1 \cdots \mu_y}{\lambda_1 \cdots \lambda_y}, \qquad y \ge 1.$$

$$\lambda_1 = \frac{\lambda}{2}, \qquad \lambda_2 = \frac{\lambda}{3}, \qquad \lambda_3 = \frac{\lambda}{4}, \qquad \lambda_4 = \frac{\lambda}{5}, \qquad \ldots$$

$$\gamma_y = \frac{\mu_1 \cdots \mu_y}{\lambda_1 \cdots \lambda_y} = \frac{\mu \;\; \mu \;\; \mu \; \cdots \; \mu \;\; \mu}{\frac{\lambda}{2} \; \frac{\lambda}{3} \; \frac{\lambda}{4} \cdots \frac{\lambda}{y} \; \frac{\lambda}{y+1}} = (y+1)! \cdot \left(\frac{\mu}{\lambda}\right)^y, \qquad y \ge 1.$$

$$\gamma_0 = 1, \qquad \gamma_1 = \frac{2!}{2^1} = 1, \qquad \gamma_2 = \frac{3!}{2^2} = 1.5,$$

$$\gamma_3 = \frac{4!}{2^3} = 3, \qquad \gamma_4 = \frac{5!}{2^4} = 7.5, \qquad \ldots$$

$$P_3(T_0 > T_5) = \frac{\sum_{y=0}^{3-1} \gamma_y}{\sum_{y=0}^{5-1} \gamma_y} = \frac{\gamma_0 + \gamma_1 + \gamma_2}{\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4}$$

$$= \frac{1+1+1.5}{1+1+1.5+3+7.5} = \frac{3.5}{14} = \mathbf{\frac{1}{4}} = \mathbf{0.25}.$$

```
> sim = 1000000
> lambda = 4
> mu = 2
> count = 0
> for (i in 1:sim) {
+    xt = 3
+    while ((xt>0)&&(xt<5)) {
+       p = (lambda/(xt+1))/(lambda/(xt+1)+mu)
+       r = runif(1,0,1)
+       xt = xt-1
+       if (r<p) { xt = xt+2 }
+    }
+    if (xt==5) { count = count+1 }
+ }
> count / sim
[1] 0.250121
```

Yay!!! It is really close! 1,000,000 may have been an overkill, but it is really close!

5. Consider a Markov pure jump process on $S = \{ ①, ②, ③ \}$ with

$q_1 = 4, \quad Q_{12} = 0.75, \quad Q_{13} = 0.25,$

$q_2 = 8, \quad Q_{21} = 0.50, \quad Q_{23} = 0.50,$

$q_3 = 0,$ ③ is an absorbing state.

$$\begin{tikzcd} \textcircled{1} \arrow[rr, shift right] \arrow[dr] & & \textcircled{2} \arrow[ll, shift right] \arrow[dl] \\ & \textcircled{3} & \end{tikzcd}$$

Obviously, $P_{31}(t) = 0, \quad P_{32}(t) = 0,$
$P_{33}(t) = 1, \quad t \geq 0.$

Possible fairy tales:

	①	②	③
a machine	adjusted properly	out of alignment	broken
a person	healthy	ill	dead

$q_{11} = -4, \qquad q_{12} = 4 \cdot 0.75 = 3, \qquad q_{13} = 4 \cdot 0.25 = 1,$

$q_{21} = 8 \cdot 0.50 = 4, \qquad q_{22} = -8, \qquad q_{23} = 8 \cdot 0.50 = 4,$

$q_{31} = 0, \qquad q_{32} = 0, \qquad q_{33} = 0.$

a) Use the **forward** equation to obtain $P_{11}(t)$, $P_{12}(t)$, and $P_{13}(t)$, $t \geq 0$.

(i) Set up the system of three differential equations for $P_{11}(t)$, $P_{12}(t)$, and $P_{13}(t)$.

Spoiler: Two of the three equations form a system of two differential equations for $P_{11}(t)$ and $P_{12}(t)$. The third equation expresses $P'_{13}(t)$ in terms of $P_{11}(t)$ and $P_{12}(t)$. Obviously, first …

$$P'_{xy}(t) = \sum_{z \in S} P_{xz}(t)\, q_{zy}$$

$$\mathrm{P}'_{11}(t) = \mathrm{P}_{11}(t)\,q_{11} + \mathrm{P}_{12}(t)\,q_{21} + \mathrm{P}_{13}(t)\,q_{31} = -4\,\mathrm{P}_{11}(t) + 4\,\mathrm{P}_{12}(t),$$

$$\mathrm{P}'_{12}(t) = \mathrm{P}_{11}(t)\,q_{12} + \mathrm{P}_{12}(t)\,q_{22} + \mathrm{P}_{13}(t)\,q_{32} = 3\,\mathrm{P}_{11}(t) - 8\,\mathrm{P}_{12}(t),$$

$$\mathrm{P}'_{13}(t) = \mathrm{P}_{11}(t)\,q_{13} + \mathrm{P}_{12}(t)\,q_{23} + \mathrm{P}_{13}(t)\,q_{33} = \mathrm{P}_{11}(t) + 4\,\mathrm{P}_{12}(t),$$

$$\mathrm{P}_{11}(0) = 1, \qquad \mathrm{P}_{12}(0) = 0, \qquad \mathrm{P}_{13}(0) = 0.$$

(ii) Solve the system for $\mathrm{P}_{11}(t)$ and $\mathrm{P}_{12}(t)$, $t \ge 0$.

Recall: Given a square matrix $\mathbf{A}$, suppose there are a constant λ and a nonzero vector $\vec{\mathbf{v}}$ such that $\mathbf{A}\,\vec{\mathbf{v}} = \lambda\,\vec{\mathbf{v}}$, then λ is called an eigenvalue of $\mathbf{A}$, and $\vec{\mathbf{v}}$ is an eigenvector of $\mathbf{A}$ corresponding to λ.

Consider a homogeneous system of linear first order differential equations: $\vec{\mathbf{x}}\,' = \mathbf{A}\,\vec{\mathbf{x}}$.

That is,

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

IF $\mathbf{A}$ has distinct real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (this is indeed the case here), and their respective eigenvectors are $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \dots, \vec{\mathbf{v}}_n$, then the system $\vec{\mathbf{x}}\,' = \mathbf{A}\,\vec{\mathbf{x}}$ has a general solution

$$\vec{\mathbf{x}}(t) = C_1\,\vec{\mathbf{v}}_1\,e^{\lambda_1 t} + C_2\,\vec{\mathbf{v}}_2\,e^{\lambda_2 t} + \dots + C_n\,\vec{\mathbf{v}}_n\,e^{\lambda_n t}.$$

$C_1, C_2, \dots, C_n$ are then determined from the initial condition:

$$\vec{\mathbf{x}}(0) = C_1\,\vec{\mathbf{v}}_1 + C_2\,\vec{\mathbf{v}}_2 + \dots + C_n\,\vec{\mathbf{v}}_n.$$

$$\begin{pmatrix} \mathrm{P}'_{11}(t) \\ \mathrm{P}'_{12}(t) \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 3 & -8 \end{pmatrix} \begin{pmatrix} \mathrm{P}_{11}(t) \\ \mathrm{P}_{12}(t) \end{pmatrix}. \qquad \begin{pmatrix} \mathrm{P}_{11}(0) \\ \mathrm{P}_{12}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues:

$$\det\begin{pmatrix} -4-\lambda & 4 \\ 3 & -8-\lambda \end{pmatrix} = (-4-\lambda)(-8-\lambda) - 12 = 20 + 12\lambda + \lambda^2$$

$$= (\lambda+2)(\lambda+10) = 0.$$

$$\Rightarrow \qquad \lambda_1 = -2, \quad \lambda_2 = -10.$$

Eigenvectors:

$$\begin{pmatrix} -2 & 4 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad \vec{\mathbf{v}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad \vec{\mathbf{v}}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

```
> A = rbind( c(-4,4), c(3,-8) )
> A
     [,1] [,2]
[1,]   -4    4
[2,]    3   -8
> eigen(A)
$values
[1] -10  -2

$vectors
           [,1]      [,2]
[1,] -0.5547002 0.8944272
[2,]  0.8320503 0.4472136

> 2*eigen(A)$vectors[,2]/eigen(A)$vectors[1,2]
[1] 2 1
> 2*eigen(A)$vectors[,1]/eigen(A)$vectors[1,1]
[1]  2 -3
```

$$C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \qquad \begin{pmatrix} 2 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$2C_1 + 2C_2 = 1. \qquad (1)$$

$$C_1 - 3C_2 = 0. \qquad (2)$$

$$(2) \quad \Rightarrow \quad C_1 = 3C_2. \qquad (3)$$

$$(1), (3) \quad \Rightarrow \quad C_2 = \frac{1}{8}. \qquad (4)$$

$$(3), (4) \quad \Rightarrow \quad C_1 = \frac{3}{8}.$$

```
> solve( rbind( c(2,2), c(1,-3) ), c(1,0) )
[1] 0.375 0.125
```

Then $\begin{pmatrix} P_{11}(t) \\ P_{12}(t) \end{pmatrix} = C_1 \vec{\mathbf{v}}_1 e^{\lambda_1 t} + C_2 \vec{\mathbf{v}}_2 e^{\lambda_2 t} = \frac{3}{8}\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + \frac{1}{8}\begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-10t}.$

$$P_{11}(t) = \frac{3}{4} e^{-2t} + \frac{1}{4} e^{-10t},$$

$$P_{12}(t) = \frac{3}{8} e^{-2t} - \frac{3}{8} e^{-10t}, \qquad t \ge 0.$$

(iii) Use the third equation to obtain $P_{13}(t)$, $t \ge 0$.

You are welcome to use $P_{11}(t) + P_{12}(t) + P_{13}(t) = 1$ to *double-check* your answer. However, to obtain $P_{13}(t)$, you have to use the differential equation you obtained in (a) (i).

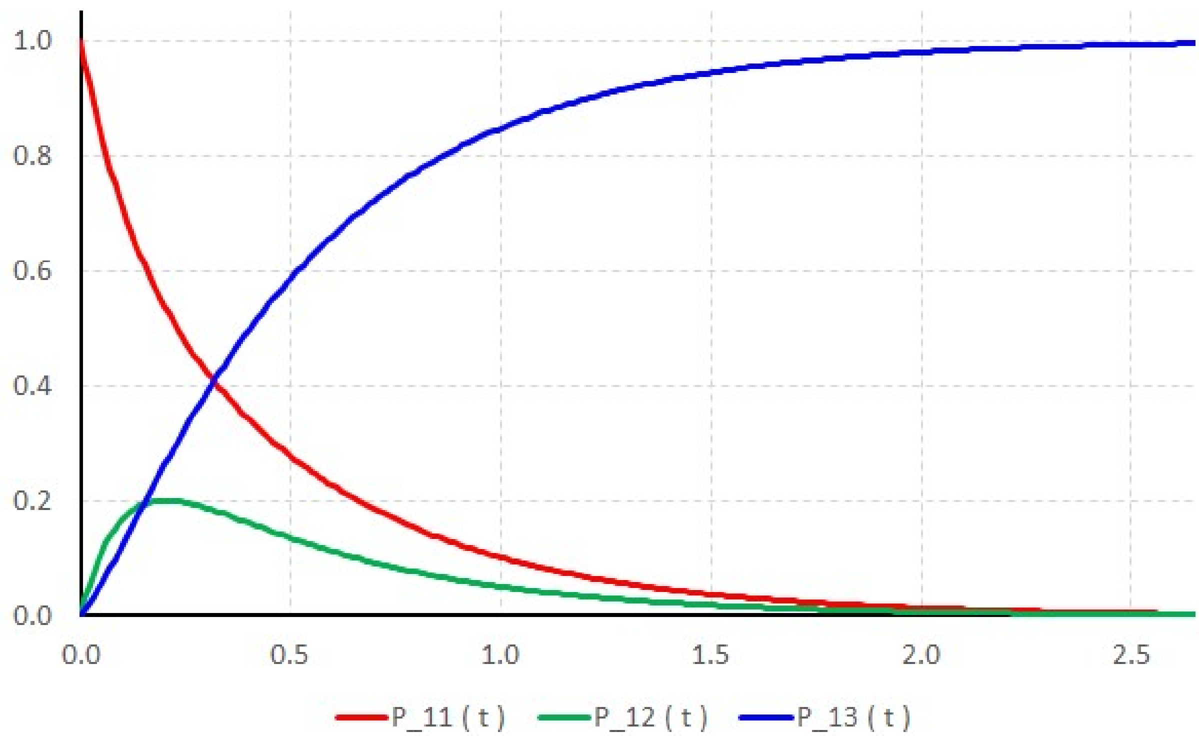
$$P'_{13}(t) = P_{11}(t) + 4\,P_{12}(t) = \frac{9}{4} e^{-2t} - \frac{5}{4} e^{-10t}.$$

$$P_{13}(0) = 0.$$

$$\Rightarrow \qquad P_{13}(t) = \int_0^t \left(\frac{9}{4} e^{-2s} - \frac{5}{4} e^{-10s} \right) ds = \left(-\frac{9}{8} e^{-2s} + \frac{1}{8} e^{-10s} \right) \Big|_0^t$$

$$= 1 - \frac{9}{8} e^{-2t} + \frac{1}{8} e^{-10t}, \qquad t \geq 0.$$

Indeed, $\quad P_{11}(t) + P_{12}(t) + P_{13}(t) = 1.$ ☺



For fun:

$P_{13}(t) = P_1(X(t) = 3) = P_1(T_3 \leq t)$, since ③ is an absorbing state.

$$E_1(T_3) = \int_0^\infty \left(\frac{9}{8} e^{-2t} - \frac{1}{8} e^{-10t} \right) dt = 0.55.$$

For fun:

a½) Use the **forward** equation to obtain $P_{21}(t)$, $P_{22}(t)$, and $P_{23}(t)$, $t \geq 0$.

$$P'_{21}(t) = P_{21}(t)\,q_{11} + P_{22}(t)\,q_{21} + P_{23}(t)\,q_{31} = -4\,P_{21}(t) + 4\,P_{22}(t),$$

$$P'_{22}(t) = P_{21}(t)\,q_{12} + P_{22}(t)\,q_{22} + P_{23}(t)\,q_{32} = 3\,P_{21}(t) - 8\,P_{22}(t),$$

$$P'_{23}(t) = P_{21}(t)\,q_{13} + P_{22}(t)\,q_{23} + P_{23}(t)\,q_{33} = P_{21}(t) + 4\,P_{22}(t),$$

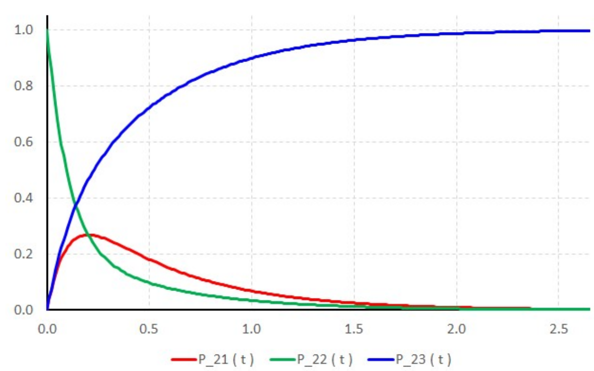
$$P_{21}(0) = 0, \qquad P_{22}(0) = 1, \qquad P_{23}(0) = 0.$$

$$\begin{pmatrix} P'_{21}(t) \\ P'_{22}(t) \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 3 & -8 \end{pmatrix} \begin{pmatrix} P_{21}(t) \\ P_{22}(t) \end{pmatrix}. \qquad \begin{pmatrix} P_{21}(0) \\ P_{22}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Same eigenvalues and eigenvectors.

$$C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad \begin{pmatrix} 2 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\Rightarrow \qquad C_1 = \frac{1}{4}, \qquad C_2 = -\frac{1}{4}.$$



$$P_{21}(t) = \frac{1}{2}\,e^{-2t} - \frac{1}{2}\,e^{-10t},$$

$$P_{22}(t) = \frac{1}{4}\,e^{-2t} + \frac{3}{4}\,e^{-10t},$$

$$P_{23}(t) = 1 - P_{21}(t) - P_{22}(t) = 1 - \frac{3}{4}\,e^{-2t} - \frac{1}{4}\,e^{-10t}, \qquad t \geq 0.$$

$$E_2(T_3) = \int_0^\infty \left(\frac{3}{4}\,e^{-2t} + \frac{1}{4}\,e^{-10t} \right) dt = 0.40.$$

b) Use the backward equation to obtain $P_{11}(t)$, $P_{21}(t)$, and $P_{31}(t)$, $t \geq 0$.

(i) Set up the system of three differential equations for $P_{11}(t)$, $P_{21}(t)$, and $P_{31}(t)$.

Spoiler: Two of the three equations form a system of two differential equations for $P_{11}(t)$ and $P_{21}(t)$. The third equation is $P'_{31}(t) = 0$, which yields $P_{31}(t) = 0$, $t \geq 0$, something we already know.

$$P'_{xy}(t) = \sum_{z \in S} q_{xz} P_{zy}(t)$$

$$P'_{11}(t) = q_{11} P_{11}(t) + q_{12} P_{21}(t) + q_{13} P_{31}(t) = -4 P_{11}(t) + 3 P_{21}(t),$$

$$P'_{21}(t) = q_{21} P_{11}(t) + q_{22} P_{21}(t) + q_{23} P_{31}(t) = 4 P_{11}(t) - 8 P_{21}(t),$$

$$P'_{31}(t) = q_{31} P_{11}(t) + q_{32} P_{21}(t) + q_{33} P_{31}(t) = 0,$$

$$P_{11}(0) = 1, \qquad P_{21}(0) = 0, \qquad P_{31}(0) = 0.$$

(ii) Solve the system for $P_{11}(t)$ and $P_{21}(t)$, $t \geq 0$.

To *double-check* your answer, you already have $P_{11}(t)$ from part (a).

$$\begin{pmatrix} P'_{11}(t) \\ P'_{21}(t) \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} P_{11}(t) \\ P_{21}(t) \end{pmatrix}.$$

$$\begin{pmatrix} P_{11}(0) \\ P_{21}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Eigenvalues:

$$\det\begin{pmatrix} -4-\lambda & 3 \\ 4 & -8-\lambda \end{pmatrix} = (-4-\lambda)(-8-\lambda) - 12 = 20 + 12\lambda + \lambda^2$$

$$= (\lambda+2)(\lambda+10) = 0.$$

$$\Rightarrow \qquad \lambda_1 = -2, \quad \lambda_2 = -10.$$

Eigenvectors:

$$\begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad \vec{\mathbf{v}}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

$$\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad \vec{\mathbf{v}}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

```
> A = rbind( c(-4,3), c(4,-8) )
> A
     [,1] [,2]
[1,]   -4    3
[2,]    4   -8
> eigen(A)
$values
[1] -10  -2

$vectors
           [,1]      [,2]
[1,] -0.4472136 0.8320503
[2,]  0.8944272 0.5547002

> 2*eigen(A)$vectors[,2]/eigen(A)$vectors[2,2]
[1] 3 2
> 2*eigen(A)$vectors[,1]/eigen(A)$vectors[2,1]
[1] -1  2
```

$$C_1\begin{pmatrix} 3 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \qquad \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$3\,C_1 - C_2 = 1. \qquad (1)$

$2\,C_1 + 2\,C_2 = 0. \qquad (2)$

(2) $\Rightarrow$ $C_2 = -C_1. \qquad (3)$

(1), (3) $\Rightarrow$ $C_1 = \dfrac{1}{4}. \qquad (4)$

(3), (4) $\Rightarrow$ $C_2 = -\dfrac{1}{4}.$

```
> solve( rbind( c(3,-1), c(2,2) ), c(1,0) )
[1]  0.25 -0.25
```

Then $\begin{pmatrix} P_{11}(t) \\ P_{12}(t) \end{pmatrix} = C_1\,\vec{\mathbf{v}}_1\,e^{\lambda_1 t} + C_2\,\vec{\mathbf{v}}_2\,e^{\lambda_2 t} = \dfrac{1}{4}\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-2t} - \dfrac{1}{4}\begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-10t}.$

$P_{11}(t) = \dfrac{3}{4}\,e^{-2t} + \dfrac{1}{4}\,e^{-10t},$

$P_{21}(t) = \dfrac{1}{2}\,e^{-2t} - \dfrac{1}{2}\,e^{-10t}, \qquad t \geq 0.$