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Birth and death Markov chain:

$$\mathrm{P}(x,y) = \begin{cases} q_x & y = x-1 & \quad q_0 = 0 \\ r_x & y = x & \\ p_x & y = x+1 & \quad p_d = 0 \end{cases} \qquad 0 \le x \le d$$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \cdots \\ d-1 \\ d \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & \cdots & d-2 & d-1 & d \end{array} \\ \begin{bmatrix} r_0 & p_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q_1 & r_1 & p_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q_3 & r_3 & p_3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & q_{d-1} & r_{d-1} & p_{d-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & q_d & r_d \end{bmatrix} \end{array}$$

OR

$$\mathrm{P}(x,y) = \begin{cases} q_x & y = x-1 & \quad q_0 = 0 \\ r_x & y = x & \\ p_x & y = x+1 & \end{cases} \qquad x \ge 0$$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \cdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{bmatrix} r_0 & p_0 & 0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots \\ 0 & 0 & q_3 & r_3 & p_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array}$$

$$q_x + r_x + p_x = 1, \qquad x \in S.$$

$a, b \in S \qquad a < b$

$u(x) = \mathrm{P}_x(\mathrm{T}_a < \mathrm{T}_b) \qquad a < x < b$

$u(a) = 1 \qquad u(b) = 0$

$$\Rightarrow \qquad u(y) = q_y \cdot u(y-1) + r_y \cdot u(y) + p_y \cdot u(y+1) \qquad a < y < b$$

Set $\quad \gamma_0 = 1. \qquad \gamma_y = \dfrac{q_1 \cdots q_y}{p_1 \cdots p_y}, \quad y \ge 1.$

$$\Rightarrow \qquad u(x) = \mathrm{P}_x(\mathrm{T}_a < \mathrm{T}_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \qquad \mathrm{P}_x(\mathrm{T}_a > \mathrm{T}_b) = \frac{\sum_{y=a}^{x-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}.$$

Gambler's ruin chain: $\quad S = \{0, 1, 2, \ldots, d\}$ OR $S = \{0, 1, 2, 3, \ldots\}$.

$q_x = q$ (lose a bet), $\qquad r_x = 0, \qquad p_x = p$ (win a bet).

Then $\quad \gamma_y = \dfrac{q_1 \cdots q_y}{p_1 \cdots p_y} = \left(\dfrac{q}{p}\right)^y, \qquad y \in S.$

$p_0 = 0 \qquad$ 0 is an absorbing state.

The probability of winning $d - x$ dollars before losing x dollars $\qquad (0 \le x \le d)$

$$\mathrm{P}_x(\mathrm{T}_0 > \mathrm{T}_d) = \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{d-1} \gamma_y} = \frac{\sum_{y=0}^{x-1} \left(\frac{q}{p}\right)^y}{\sum_{y=0}^{d-1} \left(\frac{q}{p}\right)^y}$$

$$= \frac{x}{d}, \quad \text{if } p = q. \qquad = \frac{\left(\frac{q}{p}\right)^x - 1}{\left(\frac{q}{p}\right)^d - 1}, \quad \text{if } p \ne q.$$

Example 1:

Jack wants to buy his girlfriend Jill a $3'$ World's Softest Bear from VermontTeddyBear.com for Valentine's Day. The teddy bear costs \$100. Unfortunately, Jack only has \$60.

a) Jack will bet \$1 on coin tosses until he either wins \$40 he need to buy the bear, or loses \$60 he has. Find $P_{60}(T_0 > T_{100})$.

b) Craps is a dice game in which the players make wagers on the outcome of the roll, or a series of rolls, of a pair of dice.

Pass Line bet:

- If the come-out roll is 7 or 11, the bet wins.
- If the come-out roll is 2, 3 or 12, the bet loses (known as "crapping out").
- If the roll is any other value, it establishes a *point*.
  - If, with a point established, that point is rolled again before a 7, the bet wins.
  - If, with a point established, a 7 is rolled before the point is rolled again ("seven out"), the bet loses.

$$P(\text{win}) = \left[\frac{6}{36} + \frac{2}{36}\right] + \left(\frac{3}{36}\cdot\frac{3}{9}\right) + \left(\frac{4}{36}\cdot\frac{4}{10}\right) + \left(\frac{5}{36}\cdot\frac{5}{11}\right) + \left(\frac{5}{36}\cdot\frac{5}{11}\right) + \left(\frac{4}{36}\cdot\frac{4}{10}\right) + \left(\frac{3}{36}\cdot\frac{3}{9}\right)$$

$$= \frac{244}{495} = 0.4929\overline{29}.$$

Jack will bet \$1 on Pass Line bets until he either wins \$40 he need to buy the bear, or loses \$60 he has. Find $P_{60}(T_0 > T_{100})$.

c) Jack will bet \$1 on Red in European Roulette until he either wins \$40 he need to buy the bear, or loses \$60 he has. Find $P_{60}(T_0 > T_{100})$.

d) Jack will bet \$1 on Red in American Roulette until he either wins \$40 he need to buy the bear, or loses \$60 he has. Find $P_{60}(T_0 > T_{100})$.

If $S = \{0, 1, 2, \dots, d\}$, if birth and death Markov chain is irreducible, then it is recurrent.

Suppose $S = \{0, 1, 2, 3, \dots\}$.

$$P_1(T_0 < T_n) = 1 - \frac{1}{\sum_{y=0}^{n-1} \gamma_y}. \qquad \Rightarrow \qquad P_1(T_0 < \infty) = 1 - \frac{1}{\sum_{y=0}^{\infty} \gamma_y}.$$

$$P_0(T_0 < \infty) = P(0,0) + P(0,1) \cdot P_1(T_0 < \infty).$$

Irreducible birth and death Markov chain is recurrent if and only if $\sum_{y=0}^{\infty} \gamma_y = \infty$.

Intuition: $\sum_{y=0}^{\infty} \gamma_y < \infty \quad \Leftrightarrow$ there are a lot of "births" and not a lot of "deaths" (we may never return to 0)

$\sum_{y=0}^{\infty} \gamma_y = \infty \quad \Leftrightarrow$ there are enough "deaths" to guarantee return to 0

Example 2:

Determine whether the following birth and death chains are recurrent or transient.

a) $P(0,0) = r_0 = 0.8, \qquad P(0,1) = p_0 = 0.2,$

$$\begin{aligned} P(x,x-1) &= q_x = 0.3, \\ P(x,x) &= r_x = 0.5, \qquad x \geq 1. \\ P(x,x+1) &= p_x = 0.2, \end{aligned}$$

b) $P(0,0) = r_0 = 0.7, \qquad P(0,1) = p_0 = 0.3,$

$$\begin{aligned} P(x,x-1) &= q_x = 0.2, \\ P(x,x) &= r_x = 0.5, \qquad x \geq 1. \\ P(x,x+1) &= p_x = 0.3, \end{aligned}$$

c) $P(0,0) = r_0 = 0.7, \qquad P(0,1) = p_0 = 0.3,$

$$\begin{aligned} P(x,x-1) &= q_x = 0.2, \\ P(x,x) &= r_x = 0.5, \qquad 1 \leq x \leq 99. \\ P(x,x+1) &= p_x = 0.3, \end{aligned}$$

$P(100,100) = r_{100} = 0.8, \qquad P(100,99) = q_{100} = 0.2,$

d) Consider the birth and death chain on $\{0, 1, 2, \dots\}$ defined by $p_x = (x+2)/(2x+2)$ and $q_x = x/(2x+2)$, $x \geq 0$ (i.e., there is a little bias to the right). Any state leads to any other state, so this chain is irreducible. (~ Example 13, p. 33 HPS)

e) Consider the birth and death chain on $\{0, 1, 2, \dots\}$ defined by $p_x = (x+1)/(2x+1)$ and $q_x = x/(2x+1)$, $x \geq 0$ (i.e., there is a little bias to the right). Any state leads to any other state, so this chain is irreducible.