STAT 433 Spring 2022
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Examples for 03/22/2022

1. HPS 3.12

12 Consider a birth and death process $X(t)$, $t \geq 0$, such as the branching process, that has state space $\{0, 1, 2, \ldots\}$ and birth and death rates of the form

$$\lambda_x = x\lambda \quad \text{and} \quad \mu_x = x\mu, \quad x \geq 0,$$

where λ and μ are nonnegative constants. Set

$$m_x(t) = E_x(X(t)) = \sum_{y=0}^{\infty} y P_{xy}(t).$$

(a) Write the forward equation for the process.

(b) Use the forward equation to show that $m_x'(t) = (\lambda - \mu)m_x(t)$.

(c) Conclude that

$$m_x(t) = xe^{(\lambda-\mu)t}.$$

a) $$P_{x,y}'(t) = (y-1)\lambda\, P_{x,y-1}(t) - y(\lambda+\mu)\, P_{x,y}(t) + (y+1)\mu\, P_{x,y+1}(t),$$

$$t \geq 0, \quad y \geq 1.$$

$$P_{x,0}'(t) = \mu\, P_{x,1}(t), \qquad t \geq 0.$$

b) $$m_x(t) = \sum_{y=0}^{\infty} y P_{x,y}(t) = \sum_{y=1}^{\infty} y P_{x,y}(t).$$

$$m_x'(t) = \sum_{y=0}^{\infty} y P_{x,y}'(t) = \sum_{y=1}^{\infty} y P_{x,y}'(t)$$

$$= \lambda \sum_{y=1}^{\infty} y(y-1)P_{x,y-1}(t) - (\lambda+\mu) \sum_{y=1}^{\infty} y^2 P_{x,y}(t) + \mu \sum_{y=1}^{\infty} y(y+1)P_{x,y+1}(t)$$

$$= \lambda \sum_{y=2}^{\infty} y(y-1)P_{x,y-1}(t) - (\lambda+\mu) \sum_{y=1}^{\infty} y^2 P_{x,y}(t) + \mu \sum_{y=0}^{\infty} y(y+1)P_{x,y+1}(t)$$

$$= \lambda \sum_{y=1}^{\infty} (y+1) y P_{x,y}(t) - (\lambda + \mu) \sum_{y=1}^{\infty} y^2 P_{x,y}(t) + \mu \sum_{y=1}^{\infty} (y-1) y P_{x,y}(t)$$

$$= \lambda \sum_{y=1}^{\infty} y^2 P_{x,y}(t) + \lambda \sum_{y=1}^{\infty} y P_{x,y}(t) - (\lambda + \mu) \sum_{y=1}^{\infty} y^2 P_{x,y}(t)$$

$$+ \mu \sum_{y=1}^{\infty} y^2 P_{x,y}(t) - \mu \sum_{y=1}^{\infty} y P_{x,y}(t)$$

$$= \lambda \sum_{y=1}^{\infty} y P_{x,y}(t) - \mu \sum_{y=1}^{\infty} y P_{x,y}(t) = (\lambda - \mu) m_x(t), \qquad t \ge 0.$$

c) $\quad m_x'(t) = (\lambda - \mu) m_x(t), \qquad t \ge 0.$

$m_x(0) = x.$

If $f'(t) = \alpha f(t), \quad t \ge 0,$ then $f(t) = f(0) e^{\alpha t}, \quad t \ge 0.$

$$\Rightarrow \qquad m_x(t) = x e^{(\lambda - \mu) t}, \qquad t \ge 0.$$

Consider a branching process in Example 1 (HPS p. 91).

$$\lambda_x = x q p \quad \text{and} \quad \mu_x = x q (1 - p), \qquad x \ge 0.$$

Then $\qquad m_x(t) = x e^{(2p-1) q t}, \qquad t \ge 0.$

If $\lambda > \mu$ or $p > 1/2$, then $m_x(t)$ increases exponentially.

If $\lambda < \mu$ or $p < 1/2$, then $m_x(t)$ decreases exponentially.

If $\lambda = \mu$ or $p = 1/2$, then $m_x(t) = x, \ t \ge 0.$

As t increases, the probability of hitting absorbing state 0 increases.

On the other side, as t increases, the population has a better chance to reach higher states.

The expected population size stays constant.

2. Consider a branching process with immigration in Example 2 (HPS p. 97).
That is, consider a birth and death process with

$$q_x = x\,q + \lambda, \quad \lambda_x = x\,q\,p + \lambda, \quad \mu_x = x\,q\,(1-p), \quad x \ge 0.$$

Find

$$m_x(t) = \mathrm{E}_x(\mathrm{X}(t)) = \sum_{y=0}^{\infty} y\,P_{x,y}(t).$$

Hint: ① Write the forward equation for the process.

② Use the forward equation to obtain a differential equation for $m_x(t)$.

③ If $f'(t) = -\alpha f(t) + g(t), \quad t \ge 0,$ then

$$f(t) = f(0)\,e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}\,g(s)\,ds\,, \quad t \ge 0.$$

$$P'_{x,y}(t) = \{(y-1)\,q\,p + \lambda\}\,P_{x,y-1}(t) - \{y\,q + \lambda\}\,P_{x,y}(t) + (y+1)\,q\,(1-p)\,P_{x,y+1}(t), \qquad t \ge 0, \quad y \ge 1.$$

$$m_x(t) = \sum_{y=0}^{\infty} y\,P_{x,y}(t) = \sum_{y=1}^{\infty} y\,P_{x,y}(t).$$

$$\begin{aligned} m'_x(t) &= \sum_{y=1}^{\infty} y\,P'_{x,y}(t) \\ &= q\,p \sum_{y=1}^{\infty} y(y-1)P_{x,y-1}(t) + \lambda \sum_{y=1}^{\infty} y\,P_{x,y-1}(t) \\ &\qquad - q \sum_{y=1}^{\infty} y^2\,P_{x,y}(t) - \lambda \sum_{y=1}^{\infty} y\,P_{x,y}(t) \\ &\qquad + q\,(1-p) \sum_{y=1}^{\infty} y(y+1)P_{x,y+1}(t) \end{aligned}$$

$$
\begin{aligned}
&= qp \sum_{y=2}^{\infty} y\,(y-1)\,P_{x,y-1}(t) + \lambda \sum_{y=1}^{\infty} y\,P_{x,y-1}(t) \\
&\qquad\qquad - q \sum_{y=1}^{\infty} y^2\,P_{x,y}(t) - \lambda \sum_{y=1}^{\infty} y\,P_{x,y}(t) \\
&\qquad\qquad\qquad + q\,(1-p) \sum_{y=0}^{\infty} y\,(y+1)\,P_{x,y+1}(t) \\
&= qp \sum_{y=1}^{\infty} (y+1)\,y\,P_{x,y}(t) + \lambda \sum_{y=0}^{\infty} (y+1)\,P_{x,y}(t) \\
&\qquad\qquad - q \sum_{y=1}^{\infty} y^2\,P_{x,y}(t) - \lambda \sum_{y=1}^{\infty} y\,P_{x,y}(t) \\
&\qquad\qquad\qquad + q\,(1-p) \sum_{y=1}^{\infty} (y-1)\,y\,P_{x,y}(t) \\
&= qp \sum_{y=1}^{\infty} y^2\,P_{x,y}(t) + qp \sum_{y=1}^{\infty} y\,P_{x,y}(t) + \lambda \sum_{y=0}^{\infty} y\,P_{x,y}(t) + \lambda \sum_{y=0}^{\infty} P_{x,y}(t) \\
&\qquad - q \sum_{y=1}^{\infty} y^2\,P_{x,y}(t) - \lambda \sum_{y=1}^{\infty} y\,P_{x,y}(t) \\
&\qquad\qquad + q\,(1-p) \sum_{y=1}^{\infty} y^2\,P_{x,y}(t) - q\,(1-p) \sum_{y=1}^{\infty} y\,P_{x,y}(t) \\
&= qp \sum_{y=1}^{\infty} y\,P_{x,y}(t) + \lambda \sum_{y=0}^{\infty} P_{x,y}(t) - q\,(1-p) \sum_{y=1}^{\infty} y\,P_{x,y}(t) \\
&= q\,(2p-1)\,m_x(t) + \lambda, \qquad\qquad t \ge 0.
\end{aligned}
$$

$$
\begin{aligned}
m_x(t) &= x\,e^{(2p-1)qt} + \int_0^t e^{q(2p-1)(t-s)}\,\lambda\,ds \\
&= x\,e^{(2p-1)qt} + \lambda\,\frac{e^{q(2p-1)t} - 1}{q\,(2p-1)}, \qquad t \ge 0, \qquad p \ne \frac{1}{2}, \\
&= x + \lambda\,t, \qquad t \ge 0, \qquad p = \frac{1}{2}.
\end{aligned}
$$