STAT 433 | Spring 2022
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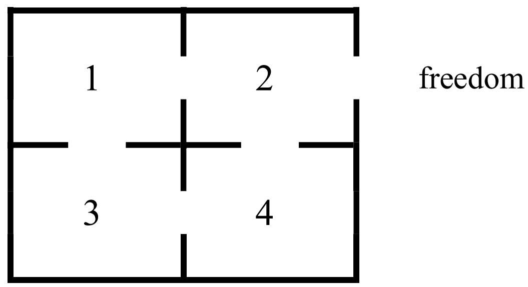
Examples for 01/20/2022 (1)

<u>Example 1</u>:

A rat runs through the following maze:



The rat starts in a given cell, and at each step, it moves to a neighboring cell (chosen with equal probability from those available, independently of the past). It continues moving between cells in this way until it escapes to the outside. Assume that the outside (freedom) is denoted by state 0 (and it can only be reached from cell 2). Assume also that once the rat has escaped, it remains escaped forever.

$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \end{array}$$

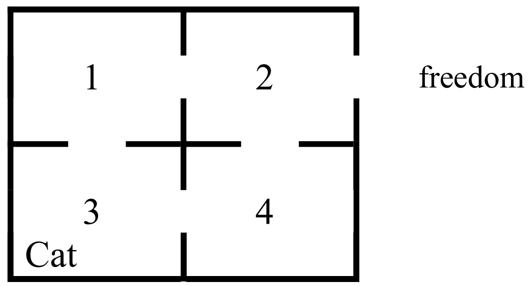
State y is **absorbing** if $P(y, y) = 1$. $\quad (\Rightarrow \; P(y, z) = 0$ for all $z \neq y$.)

State 0 is absorbing.

Find the expected number of steps until the rat escapes, if it starts in cell $i = 1, 2, 3, 4$.

Example 2:

A rat runs through the following maze:



The rat starts in a given cell, and at each step, it moves to a neighboring cell (chosen with equal probability from those available, independently of the past). It continues moving between cells in this way until it escapes to the outside. Assume that the outside (freedom) is denoted by state 0 (and it can only be reached from cell 2). Assume also that once the rat has escaped, it remains escaped forever. However, suppose there is a cat in cell 3, and the cat would eat the rat if the rat enters cell 3.

$$\begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

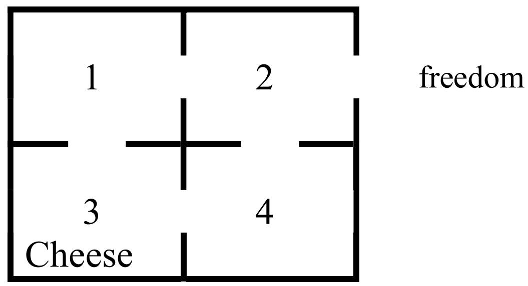
States 0 and 3 are absorbing.

a) Find the probability that the rat escapes (instead of being eaten) if it starts in cell $i = 1, 2, 4$.

b) Find the expected number of steps until the rat either escapes or is eaten, if it starts in cell $i = 1, 2, 4$.

Example 3:

A rat runs through the following maze:



The rat starts in a given cell, and at each step, it moves to a neighboring cell (chosen with equal probability from those available, independently of the past). It continues moving between cells in this way until it escapes to the outside. Assume that the outside (freedom) is denoted by state 0 (and it can only be reached from cell 2). Assume also that once the rat has escaped, it remains escaped forever.

a) A piece of cheese is placed in cell 3 (once), and the rat would eat the cheese if the rat enters cell 3. Find the probability that the rat would eat the cheese before (eventually) escaping, if it starts in cell $i = 1, 2, 4$.

State 0 is absorbing. State 3 is NOT absorbing.

However, this is equivalent to Example 2 (a):

If all we care about is whether the rat enters cell 3 or not, then pretend that state 3 is absorbing, then the probability that the rat eats the cheese is the same as the probability that the cat eats the rat is Example 2 (a).

b) The cheese is replaced each time after the rat leaves cell 3. Find the expected number of times the rat would eat a piece of cheese before the rat (eventually) escapes. That is, find the expected number of visits to cell 3 before the rat (eventually) escapes, if it starts in cell $i = 1, 2, 4$.

As exciting as rats running through mazes are, here is a (still very simplified) more meaningful similar example:

Example 4:

A life insurance company wants to find out how much money to charge its clients. The following 4-state Markov chain model is proposed for the changes in health for the 50+ age group: H – healthy, M – mildly ill, S – seriously ill, D – dead. The matrix of one-step (one-year) transition probabilities is given below:

$$\begin{array}{c} \\ H \\ M \\ S \\ D \end{array} \begin{array}{c} \begin{array}{cccc} H & M & S & D \end{array} \\ \begin{bmatrix} 0.90 & 0.05 & 0.03 & 0.02 \\ 0.80 & 0.10 & 0.05 & 0.05 \\ 0.08 & 0.16 & 0.50 & 0.26 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

a) According to this model, what is the expected number of years until death for an individual from the 50+ age group who is currently …

i) … healthy? ii) … seriously ill?

b) What is the probability that an individual from the 50+ age group who is currently healthy will not become seriously ill before death?

c) For an individual who is currently healthy, what is the expected number of years this individual will spend being seriously ill before death? That is, starting from state H, what is the expected number of visits to state S before entering the absorbing state D?

Example 5:

Consider the game of tennis when *deuce* is reached. If a player wins the next point, he has *advantage*. On the following point, he either wins the game or the game returns to deuce. Assume that for any point, player A has probability p of winning the point and player B has probability $(1-p)$ of winning the point.

$$\begin{array}{c} \\ \text{A wins} \\ \text{adv A} \\ \text{Deuce} \\ \text{adv B} \\ \text{B wins} \end{array}
\begin{array}{c} \begin{array}{ccccc} \text{A wins} & \text{adv A} & \text{Deuce} & \text{adv B} & \text{B wins} \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Find the expected number of points needed to determine the winner.

Find the probability that player A wins.

While this example is similar to Example 2, it is also an example of a *gambler's ruin chain*, a special case of *birth and death chains*.

Example 6:

Suppose the number of Padawan Apprentices that a Jedi Knight would train over lifetime has the following probability distribution:

$$f(0) = 0.30, \quad f(1) = 0.25, \quad f(2) = 0.20, \quad f(3) = 0.15, \quad f(4) = 0.10.$$

(A Padawan Apprentice becomes a Jedi Knight when the training is complete.)

Suppose also that Jedi Knights train their Padawan Apprentices independently.

How many Jedi Knights are needed initially for the probability of the Jedi extinction to be less than 0.1%?

This is an example of a *branching chain*.

Example 7:

Suppose that whether or not a bidder is successful on a bid depends on the successes and failures of his previous two bids. If his last two bids were successful, his next bid will be successful with probability 0.50. If only one of his last two bids was successful, the probability is 0.60 that the next bid will be successful. Finally, if none of the last two bids were successful, the probability is 0.7 that the next one will be successful.

If, at step n, the state of the process is defined based on whether bid n is a success or a failure, then the process is NOT a Markov chain.

Define X_n to be *sf*, if at step n the last bid was a failure and the bid before that was a success. Define *ss*, *fs*, and *ff* similarly. Let state 0 = *ff*, state 1 = *fs*, state 2 = *sf*, and state 3 = *ss*. Then $\{X_n : n \geq 2\}$ is a Markov chain. Find the transition probability matrix.

$$\begin{array}{lcll} & & ff & 0.30 \\ & \nearrow & & \\ ff & & & \\ & \searrow & & \\ & & fs & 0.70 \end{array} \qquad \begin{array}{lcll} & & sf & 0.40 \\ & \nearrow & & \\ fs & & & \\ & \searrow & & \\ & & ss & 0.60 \end{array}$$

$$\begin{array}{lcll} & & ff & 0.40 \\ & \nearrow & & \\ sf & & & \\ & \searrow & & \\ & & fs & 0.60 \end{array} \qquad \begin{array}{lcll} & & sf & 0.50 \\ & \nearrow & & \\ ss & & & \\ & \searrow & & \\ & & ss & 0.50 \end{array}$$

$$\begin{array}{c} \\ 0 = ff \\ 1 = fs \\ 2 = sf \\ 3 = ss \end{array} \begin{array}{c} \begin{array}{cccc} 0 = ff & 1 = fs & 2 = sf & 3 = ss \end{array} \\ \begin{bmatrix} 0.30 & 0.70 & 0 & 0 \\ 0 & 0 & 0.40 & 0.60 \\ 0.40 & 0.60 & 0 & 0 \\ 0 & 0 & 0.50 & 0.50 \end{bmatrix} \end{array}$$