STAT 433

Examples for 01/20/2022 (3)

Spring 2022
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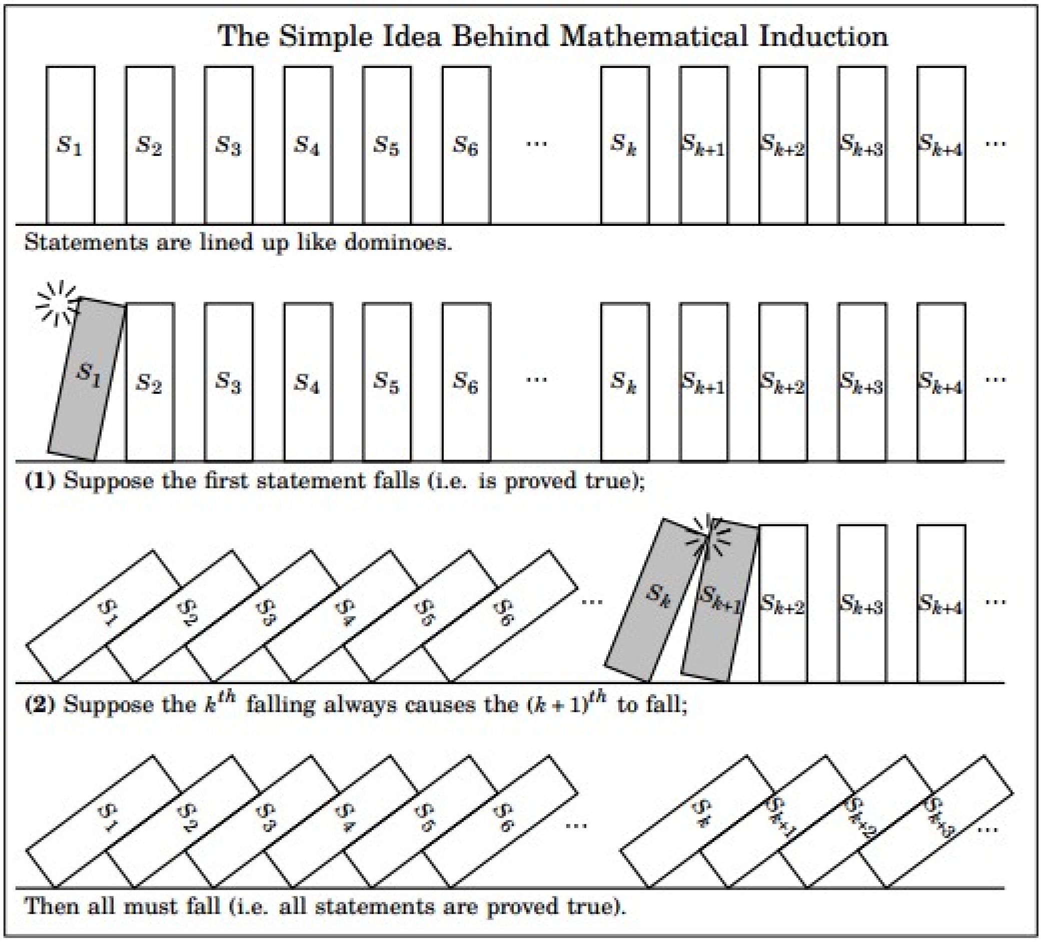


1. Use mathematical induction to prove that for all positive integers n, …

a) … $1+2+3+\ldots n = \dfrac{n(n+1)}{2}$, $n \ge 1$.

Base. $n=1$. $1 = \dfrac{1(1+1)}{2}$. ✓

Step. Suppose $1+2+3+\ldots k = \dfrac{k(k+1)}{2}$.

$$1+2+3+\ldots k+(k+1) = \frac{k(k+1)}{2}+(k+1)$$

$$= (k+1)\left[\frac{k}{2}+1\right] = \frac{(k+1)(k+2)}{2}. \quad ✓$$

b) … $1^2 + 2^2 + 3^2 + \ldots n^2 = \dfrac{n(n+1)(2n+1)}{6}, \qquad n \geq 1.$

Base. $n = 1.$ $\quad 1^2 = \dfrac{1(1+1)(2 \cdot 1+1)}{6}.$ ✓

Step. Suppose $\quad 1^2 + 2^2 + 3^2 + \ldots k^2 = \dfrac{k(k+1)(2k+1)}{6}.$

$$
\begin{aligned}
1^2 + 2^2 + 3^2 + \ldots k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left[\frac{k(2k+1)}{6} + k + 1\right] \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}. \quad ✓
\end{aligned}
$$

c) … $n^3 + 2n$ is divisible by 3, $\quad n \geq 1.$

Base. $n = 1.$ $\quad 1^3 + 2 \cdot 1 = 3$ is divisible by 3. ✓

Step. Suppose $\quad k^3 + 2k$ is divisible by 3.

$$
\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
&= [k^3 + 2k] + 3[k^2 + k + 1] \quad \text{is divisible by 3.} \quad ✓
\end{aligned}
$$

d) … $1 + 3 + 5 + \ldots (2n - 1) = n^2, \qquad n \geq 1.$

Base. $\quad n = 1. \quad$ LHS = 1 $\quad$ RHS = $1^2 = 1$ ✓

Step. $\quad$ Suppose $\quad 1 + 3 + 5 + \ldots (2k - 1) = k^2.$

$$1 + 3 + 5 + \ldots (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2. \quad ✓$$

e) … $1^3 + 2^3 + 3^3 + \ldots n^3 = \dfrac{n^2(n+1)^2}{4}, \qquad n \geq 1.$

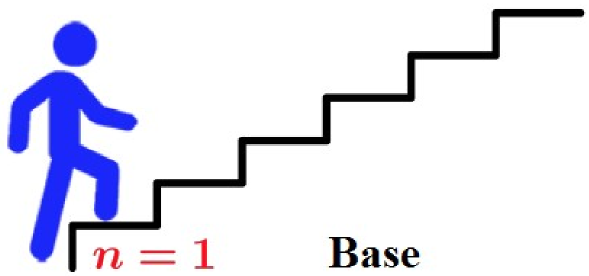
Base. $\quad n = 1. \quad 1^3 = \dfrac{1^2(1+1)^2}{4}$ ✓

Step. $\quad$ Suppose $\quad 1^3 + 2^3 + 3^3 + \ldots k^3 = \dfrac{k^2(k+1)^2}{4}.$

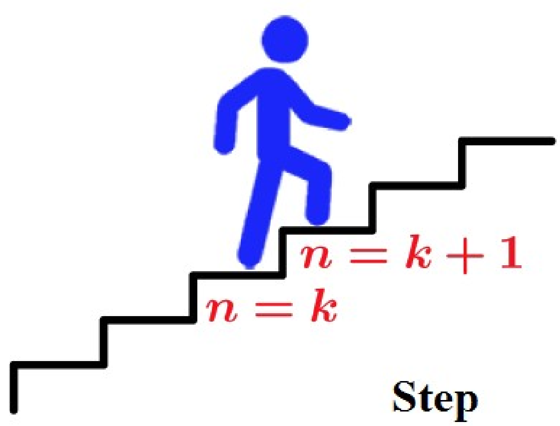
$$1^3 + 2^3 + 3^3 + \ldots k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2\left[\frac{k^2}{4} + k + 1\right] = (k+1)^2\left[\frac{k^2 + 4k + 4}{4}\right]$$

$$= \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2\left((k+1)+1\right)^2}{4}. \quad ✓$$



Base



Step

f) $\dots$ $$\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right), \qquad n \geq 1,$$

where there are n 2s in the expression on the left.

Hint: $\cos(2x) = 2(\cos x)^2 - 1$

Base. $n = 1$. $\sqrt{2} = 2\cos\left(\frac{\pi}{4}\right)$. ✓

Step. Suppose $$\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}} = 2\cos\left(\frac{\pi}{2^{k+1}}\right),$$

where there are k 2s in the expression on the left.

Consider $\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}$ where there are $k+1$ 2s.

$$= \sqrt{2+2\cos\left(\frac{\pi}{2^{k+1}}\right)}$$ by induction assumption.

$$\cos(2x) = 2(\cos x)^2 - 1 \quad \Rightarrow \quad 1+\cos(2x) = 2(\cos x)^2$$

$$\Rightarrow \quad 2+2\cos(2x) = 4(\cos x)^2$$

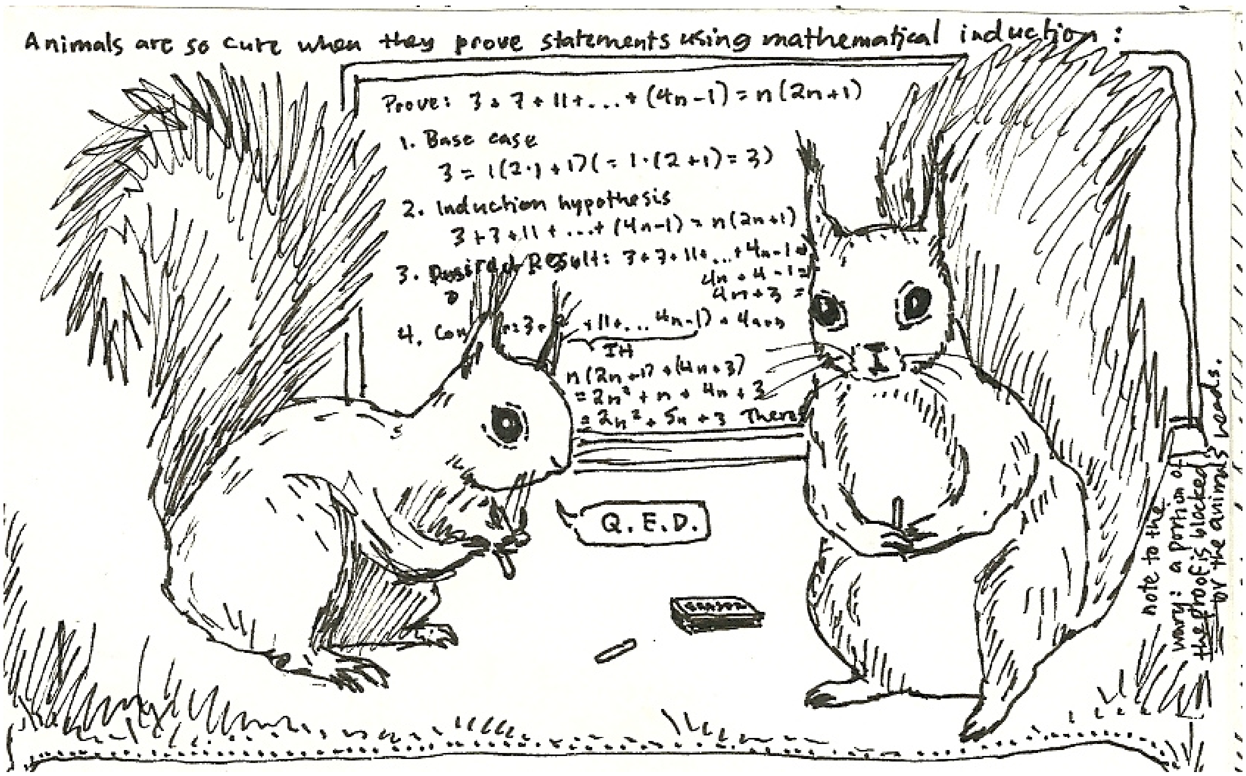
$$\Rightarrow \quad \sqrt{2+2\cos(2x)} = 2\cos x$$

$(\text{if } \cos x \geq 0)$

$\Rightarrow$ $\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}$ where there are $k+1$ 2s.

$$= \sqrt{2+2\cos\left(\frac{\pi}{2^{k+1}}\right)}$$

$$= 2\cos\left(\frac{\pi}{2^{k+2}}\right).$$ ✓


Animals are so cute when they prove statements using mathematical induction:
Prove: 3 + 7 + 11 + . . . + (4n − 1) = n(2n + 1)
1. Base case
3 = 1(2·1 + 1) (= 1·(2 + 1) = 3)
2. Induction hypothesis
3 + 7 + 11 + . . . + (4n − 1) = n(2n + 1)
3. ~~Desired~~ Result: 3 + 7 + 11 + . . . + 4n − 1 +
4n + 4 − 1 =
4n + 3 =
+ 11 + . . . 4n − 1) + 4n + 3
IH
n(2n + 1) + (4n + 3)
= 2n² + n + 4n + 3
= 2n² + 5n + 3 There
Q.E.D.
note to the wary: a portion of the proof is blocked by the animals' heads.