Examples for 01/20/2022 (2)

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Let $A \subset S$. The *hitting time* T_A of A is defined by $T_A = \min\{ n > 0 : X_n \in A \}$.

T_A is the first (positive) time the Markov chain hits A.

$T_A = \infty$ if $X_n \notin A$ for all $n > 0$ (if the Markov chain never hits A).

Let $a \in S$. Let $T_a = T_{\{a\}}$.

Notation: $P_x(\ldots) = P(\ldots \mid X_0 = x)$.

$$P_x(X_n = y) = P^n(x, y) = \sum_{m=1}^{n} P_x\left(T_y = m\right) \times P^{n-m}\left(y, y\right), \qquad n \geq 1.$$

(To go from x to y in n steps, it takes m steps to hit y for the first time, and then $n - m$ steps to go from y to y.)

$\Rightarrow$ IF y is absorbing, then $P_x(X_n = y) = P_x(T_y \leq n)$.

$$P_x(T_y = 1) = P(X_1 = y \mid X_0 = x) = P(x, y).$$

$$P_x(T_y = 2) = \sum_{z \neq y} P\left(x, z\right) \times P\left(z, y\right).$$

$$P_x(T_y = n+1) = \sum_{z \neq y} P\left(x, z\right) \times P_z\left(T_y = n\right), \qquad n \geq 1.$$

(To hit y from x for the first time in n +1 steps, move from x to $z \neq y$, and then hit y from z for the first time in n steps.)

Example: Winter weather in Central Illinois.

$$\begin{array}{c c} & \begin{array}{ccc} N & R & S \end{array} \\ \begin{array}{c} N \\ R \\ S \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

$$P_N(T_N=1) = 0 \qquad P_R(T_N=1) = \frac{1}{4} \qquad P_S(T_N=1) = \frac{1}{3}$$

$$P_N(T_N=2) = P(N,R)\times P_R(T_N=1) + P(N,S)\times P_S(T_N=1) = \frac{1}{2}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{3} = \frac{7}{24}$$

$$P_R(T_N=2) = P(R,R)\times P_R(T_N=1) + P(R,S)\times P_S(T_N=1) = \frac{1}{2}\times\frac{1}{4} + \frac{1}{4}\times\frac{1}{3} = \frac{5}{24}$$

$$P_S(T_N=2) = P(S,R)\times P_R(T_N=1) + P(S,S)\times P_S(T_N=1) = \frac{1}{3}\times\frac{1}{4} + \frac{1}{3}\times\frac{1}{3} = \frac{7}{36}$$

$$\begin{aligned} P_N(T_N=3) &= P(N,R)\times P_R(T_N=2) + P(N,S)\times P_S(T_N=2) \\ &= \frac{1}{2}\times\frac{5}{24} + \frac{1}{2}\times\frac{7}{36} = \frac{29}{144} \end{aligned}$$

$$\begin{aligned} P_R(T_N=3) &= P(R,R)\times P_R(T_N=2) + P(R,S)\times P_S(T_N=2) \\ &= \frac{1}{2}\times\frac{5}{24} + \frac{1}{4}\times\frac{7}{36} = \frac{11}{72} \end{aligned}$$

$$\begin{aligned} P_S(T_N=3) &= P(S,R)\times P_R(T_N=2) + P(S,S)\times P_S(T_N=2) \\ &= \frac{1}{3}\times\frac{5}{24} + \frac{1}{3}\times\frac{7}{36} = \frac{29}{216} \end{aligned}$$

$$\begin{aligned} P_N(T_N=4) &= P(N,R)\times P_R(T_N=3) + P(N,S)\times P_S(T_N=3) \\ &= \frac{1}{2}\times\frac{11}{72} + \frac{1}{2}\times\frac{29}{216} = \frac{31}{216} \end{aligned}$$

$$\begin{aligned} P_R(T_N=4) &= P(R,R)\times P_R(T_N=3) + P(R,S)\times P_S(T_N=3) \\ &= \frac{1}{2}\times\frac{11}{72} + \frac{1}{4}\times\frac{29}{216} = \frac{95}{864} \end{aligned}$$

$$\begin{aligned} P_S(T_N=4) &= P(S,R)\times P_R(T_N=3) + P(S,S)\times P_S(T_N=3) \\ &= \frac{1}{3}\times\frac{11}{72} + \frac{1}{3}\times\frac{29}{216} = \frac{31}{324} \end{aligned}$$

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```
> hitting = function(k,N,P) {
+ n = length(P[1,])                    or          n = nrow(P)
+ result = matrix(rep(0,n*N), nrow=N, ncol=n)
+ for (j in 1:n) {result[1,j] = P[j,k]}
+ for (i in 2:N) {
+    for (j in 1:n) {
+       result[i,j] = t(P[j,-k]) %*% result[i-1,-k]
+    }
+ }
+ result
+ }
```

Computes $\mathrm{P}_x(\mathrm{T}_k = i)$, $1 \le i \le \mathrm{N}$, $x \in S$.

```
>
> P = rbind( c(0,1/2,1/2), c(1/4,1/2,1/4), c(1/3,1/3,1/3) )
>
> hitting(1,10,P)   ## Nice
            [,1]       [,2]       [,3]
 [1,] 0.00000000 0.25000000 0.33333333
 [2,] 0.29166667 0.20833333 0.19444444
 [3,] 0.20138889 0.15277778 0.13425926
 [4,] 0.14351852 0.10995370 0.09567901
 [5,] 0.10281636 0.07889660 0.06854424
 [6,] 0.07372042 0.05658436 0.04914695
 [7,] 0.05286566 0.04057892 0.03524377
 [8,] 0.03791134 0.02910040 0.02527423
 [9,] 0.02718732 0.02086876 0.01812488
[10,] 0.01949682 0.01496560 0.01299788
>
> hitting(2,10,P)   ## Rain
             [,1]        [,2]        [,3]
 [1,] 0.500000000 0.500000000 0.333333333
 [2,] 0.166666667 0.208333333 0.277777778
 [3,] 0.138888889 0.111111111 0.148148148
 [4,] 0.074074074 0.071759259 0.095679012
 [5,] 0.047839506 0.042438272 0.056584362
 [6,] 0.028292181 0.026105967 0.034807956
 [7,] 0.017403978 0.015775034 0.021033379
 [8,] 0.010516690 0.009609339 0.012812452
 [9,] 0.006406226 0.005832285 0.007776381
[10,] 0.003888190 0.003545652 0.004727536
```

```
> hitting(3,10,P)   ## Snow
             [,1]       [,2]        [,3]
 [1,] 0.500000000 0.25000000 0.333333333
 [2,] 0.125000000 0.25000000 0.250000000
 [3,] 0.125000000 0.15625000 0.125000000
 [4,] 0.078125000 0.10937500 0.093750000
 [5,] 0.054687500 0.07421875 0.062500000
 [6,] 0.037109375 0.05078125 0.042968750
 [7,] 0.025390625 0.03466797 0.029296875
 [8,] 0.017333984 0.02368164 0.020019531
 [9,] 0.011840820 0.01617432 0.013671875
[10,] 0.008087158 0.01104736 0.009338379
```

That is,

$P_N(T_S = 1) = 0.5000000$	$P_R(T_S = 1) = 0.25000000$	$P_S(T_S = 1) = 0.3333333$
$P_N(T_S = 2) = 0.1250000$	$P_R(T_S = 2) = 0.25000000$	$P_S(T_S = 2) = 0.2500000$
$P_N(T_S = 3) = 0.1250000$	$P_R(T_S = 3) = 0.15625000$	$P_S(T_S = 3) = 0.1250000$
$P_N(T_S = 4) = 0.0781250$	$P_R(T_S = 4) = 0.10937500$	$P_S(T_S = 4) = 0.0937500$
$P_N(T_S = 5) = 0.0546875$	$P_R(T_S = 5) = 0.07421875$	$P_S(T_S = 5) = 0.0625000$
$\cdots$	$\cdots$	$\cdots$

= = = = = = = = = = = = =

For states i and j, let $m_{ij} = E_i(T_j)$ denote the expected number of steps it takes a Markov chain starting in state i to visit state j for the first time. Then

$$m_{ij} = P(i,j)\cdot 1 + \sum_{k \neq j} P(i,k)\cdot\left(1 + m_{kj}\right) \underset{\uparrow}{=} 1 + \sum_{k \neq j} P(i,k)\cdot m_{kj},$$

since $P(i,j) + \sum_{k \neq j} P(i,k) = 1.$

(To go from i to j, we go to state j in one step OR we go to a different state k in one step and then we will go to state j from state k in m_{kj} steps (on average).)

Example: Winter weather in Central Illinois.

$$\begin{array}{c} \\ N \\ R \\ S \end{array} \begin{array}{c} \begin{array}{ccc} N & R & S \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

$$m_{NN} = 1 + \frac{1}{2} \times m_{RN} + \frac{1}{2} \times m_{SN}$$

$$m_{RN} = 1 + \frac{1}{2} \times m_{RN} + \frac{1}{4} \times m_{SN} \qquad \Rightarrow \qquad m_{RN} = 2 + \frac{1}{2} \times m_{SN}$$

$$m_{SN} = 1 + \frac{1}{3} \times m_{RN} + \frac{1}{3} \times m_{SN} \qquad \Rightarrow \qquad m_{RN} = 2 \times m_{SN} - 3$$

$$\Rightarrow \qquad \frac{3}{2} \times m_{SN} = 5$$

$$\Rightarrow \qquad m_{SN} = \frac{10}{3} \qquad \Rightarrow \qquad m_{RN} = \frac{11}{3}$$

$$\Rightarrow \qquad m_{NN} = \frac{27}{6} = \frac{9}{2}$$

That is, $\quad E_N(T_N) = 4.5 = \frac{9}{2}, \quad E_R(T_N) = \frac{11}{3}, \quad E_S(T_N) = \frac{10}{3}.$

If it is a nice day today, the expected (average) number of days until the next (first) nice day is 4.5.

$$m_{NR} = \frac{1}{2} \times 1 + 0 \times (1 + m_{NR}) + \frac{1}{2} \times (1 + m_{SR}) = 1 + \frac{1}{2} \times m_{SR}$$

$$m_{SR} = \frac{1}{3} \times 1 + \frac{1}{3} \times (1 + m_{NR}) + \frac{1}{3} \times (1 + m_{SR}) = 1 + \frac{1}{3} \times m_{NR} + \frac{1}{3} \times m_{SR}$$

$$\Rightarrow \qquad m_{SR} = 1 + \frac{1}{3} + \frac{1}{6} \times m_{SR} + \frac{1}{3} \times m_{SR}$$

$$\Rightarrow \qquad m_{SR} = \frac{8}{3} \qquad \Rightarrow \qquad m_{NR} = \frac{7}{3}$$

$$m_{RR} = \frac{1}{2} \times 1 + \frac{1}{4} \times (1 + m_{NR}) + \frac{1}{4} \times (1 + m_{SR}) = 1 + \frac{1}{4} \times m_{NR} + \frac{1}{4} \times m_{SR}$$

$$= 1 + \frac{1}{4} \times \frac{8}{3} + \frac{1}{4} \times \frac{7}{3} = \frac{9}{4}$$

That is, $\mathrm{E}_N(T_R) = \frac{7}{3}$, $\mathrm{E}_R(T_R) = \frac{9}{4}$, $\mathrm{E}_S(T_R) = \frac{8}{3}$.

```
> ExpHitTime = function(j,P) {
+ n = length(P[1,])                    or         n = nrow(P)
+ newP = P
+ for (i in 1:n) { newP[i,j] = 0 }
+ result = solve( diag(n) - newP ) %*% rep(1,n)
+ result
+ }
```

Computes m_{ij}, $i \in S$.

```
>
> P = rbind( c(0,1/2,1/2), c(1/4,1/2,1/4), c(1/3,1/3,1/3) )
>
> ## expected number of days until next Nice
> ExpHitTime(1,P)
         [,1]
[1,] 4.500000
[2,] 3.666667
[3,] 3.333333
> ## expected number of days until next Rain
> ExpHitTime(2,P)
         [,1]
[1,] 2.333333
[2,] 2.250000
[3,] 2.666667
> ## expected number of days until next Snow
> ExpHitTime(3,P)
         [,1]
[1,] 2.666667
[2,] 3.333333
[3,] 3.000000
```

“Explanations” for `hitting`:

```
> hitting = function(k,N,P) {
```

The function computes $P_x(T_k = i), \quad 1 \le i \le N, \quad x \in S.$

P is the transition probability matrix, **k** is the state we want to hit.

```
+ n = length(P[1,])                    or          n = nrow(P)
```

n is the number of states.

```
+ result = matrix(rep(0,n*N), nrow=N, ncol=n)
```

Reserving space for **n*N** probabilities to be stored after they are computed.

```
+ for (j in 1:n) {result[1,j] = P[j,k]}
```

$P_j(T_k = 1) = P(j,k). \qquad i = 1.$

```
+ for (i in 2:N) {
+     for (j in 1:n) {
+         result[i,j] = t(P[j,-k]) %*% result[i-1,-k]
```

$$P_j(T_k = i) = \sum_{z \neq k} P(j,z) \times P_z\left(T_k = i-1\right), \qquad 2 \le i \le N.$$

```
+     }
+ }
+ result
+ }
```

“Explanations” for `ExpHitTime`:

```
> ExpHitTime = function(j,P) {
```

The function computes $m_{ij}, \quad i \in S.$

`P` is the transition probability matrix, `j` is the state we want to hit.

```
+ n = length(P[1,])                    or          n = nrow(P)
```

`n` is the number of states.

```
+ newP = P
+ for (i in 1:n) { newP[i,j] = 0 }
```

$$m_{ij} = 1 + \sum_{k \neq j} \mathrm{P}(i,k) \cdot m_{kj}, \qquad i \in S.$$

$$\Rightarrow \qquad m_{ij} - \sum_{k \neq j} \mathrm{P}(i,k) \cdot m_{kj} = 1, \qquad i \in S.$$

Since we need $k \neq j$, `newP` is the transition probability matrix `P` with zeroes in column j.

```
+ result = solve( diag(n) - newP ) %*% rep(1,n)
```

$$[\mathbf{I}_n - \mathbf{newP}] \cdot \vec{\mathbf{m}}_{\bullet j} = \vec{\mathbf{1}}. \qquad \Rightarrow \qquad \vec{\mathbf{m}}_{\bullet j} = [\mathbf{I}_n - \mathbf{newP}]^{-1} \cdot \vec{\mathbf{1}}.$$

```
+ result
+ }
```

Notations: $\mathbf{I}_n = \underset{n \times n}{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}}, \qquad \vec{\mathbf{1}} = \underset{n \times 1}{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}.$

= = = = = = = = = = = = =

Expected hitting time (barbaric way): Do NOT even bother with this, I was just bored.

```
> result = hitting(1,100,P)
>
> sum(result[,1]); sum(result[,2]); sum(result[,3])
[1] 1
[1] 1
[1] 1
```

These are actually slightly less than 1. However, $P_N(T_N > 100)$, $P_R(T_N > 100)$, and $P_N(T_N > 100)$ are very small.

```
> result[100,1]; result[100,2]; result[100,3]
[1] 1.966683e-15
[1] 1.50961e-15
[1] 1.311122e-15
```

These are $P_N(T_N = 100)$, $P_R(T_N = 100)$, and $P_N(T_N = 100)$.

```
> ave = rep(0,3)
> for (j in 1:3) {
+     for (i in 1:100) {
+         ave[j] = ave[j] + i * result[i,j]
+     }
+ }
> ave  ## expected number of days until next Nice
[1] 4.500000 3.666667 3.333333

> result = hitting(2,100,P)
> ave = rep(0,3)
> for (j in 1:3) {
+     for (i in 1:100) {
+         ave[j] = ave[j] + i * result[i,j]
+     }
+ }
> ave  ## expected number of days until next Rain
[1] 2.333333 2.250000 2.666667
```

```
> result = hitting(3,100,P)
> ave = rep(0,3)
> for (j in 1:3) {
+    for (i in 1:100) {
+       ave[j] = ave[j] + i * result[i,j]
+    }
+ }
> ave  ## expected number of days until next Snow
[1] 2.666667 3.333333 3.000000
```