STAT 433 | Spring 2022
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# Examples for 03/24/2022

Stationary distribution: $\displaystyle\sum_x \pi(x) q_{xy} = 0, \quad y \in S.$

**1.** Suppose the time spent in Anytown Emergency Room is exponentially distributed with mean 4 hours ( rate $q_{\text{E}} = 6$ ). Suppose also that $1/3$ of the ER patients are admitted to the Anytown Hospital, and $2/3$ are discharged. The time spent in the Hospital is exponentially distributed with mean 2 days ( rate $q_{\text{H}} = 0.5$ ). People in Anytown are extremely accident-prone; the time until a person goes to the Emergency Room is exponentially distributed with mean 20 days ( rate $q_{\text{D}} = 0.05$ ). Consider a Markov pure jump process $X(t)$ with three states $\{$ E(mergency Room), H(ospital), D(ischarged) $\}$. Find the long-term distribution of $X(t)$.

$$\begin{array}{ccc} \text{E} & \leftrightarrows & \text{D} \\ & \searrow \quad \nearrow & \\ & \text{H} & \end{array}$$

$Q_{\text{EH}} = \dfrac{1}{3}, \quad Q_{\text{ED}} = \dfrac{2}{3}$ $\qquad Q_{\text{HD}} = 1$ $\qquad Q_{\text{DE}} = 1$

$q_{\text{EE}} = -q_{\text{E}} = -6$ $\qquad q_{\text{EH}} = q_{\text{E}}\, Q_{\text{EH}} = 2$ $\qquad q_{\text{ED}} = q_{\text{E}}\, Q_{\text{ED}} = 4$

$q_{\text{HE}} = q_{\text{H}}\, Q_{\text{HE}} = 0$ $\qquad q_{\text{HH}} = -q_{\text{H}} = -0.5$ $\qquad q_{\text{HD}} = q_{\text{H}}\, Q_{\text{HD}} = 0.5$

$q_{\text{DE}} = q_{\text{D}}\, Q_{\text{DE}} = 0.05$ $\qquad q_{\text{DH}} = q_{\text{D}}\, Q_{\text{DH}} = 0$ $\qquad q_{\text{DD}} = -q_{\text{D}} = -0.05$

$$\sum_x \pi(x)\, q_{xy} = 0, \quad y \in S.$$

$-6\,\pi(\text{E}) + 0.05\,\pi(\text{D}) = 0.$ $\qquad \Rightarrow \qquad \pi(\text{D}) = 120\,\pi(\text{E}).$

$2\,\pi(\text{E}) - 0.5\,\pi(\text{H}) = 0.$ $\qquad \Rightarrow \qquad \pi(\text{H}) = 4\,\pi(\text{E}).$

$4\,\pi(\text{E}) + 0.5\,\pi(\text{H}) - 0.4\,\pi(\text{D}) = 0.$

$\pi(E) + \pi(H) + \pi(D) = 1. \qquad \Rightarrow \qquad \pi(E) + 4\,\pi(E) + 120\,\pi(E) = 1.$

$\Rightarrow \quad \pi(E) = \mathbf{\dfrac{1}{125}} = \mathbf{0.008}. \quad \pi(H) = \mathbf{\dfrac{4}{125}} = \mathbf{0.032}. \quad \pi(E) = \mathbf{\dfrac{120}{125}} = \mathbf{0.96}.$

```
> q = rbind( c(-6,2,4), c(0,-0.5,0.5), c(0.05,0,-0.05) )
> eigen( t(q) )
$values
[1] -6.031923e+00 -5.180769e-01 -1.239310e-17

$vectors
           [,1]         [,2]        [,3]
[1,] -0.8062567 -0.006419964 0.008328419
[2,]  0.2914924  0.710294905 0.033313675
[3,]  0.5147643 -0.703874941 0.999410244

> eigen(t(q))$vectors[,3]/sum(eigen(t(q))$vectors[,3])
[1] 0.008 0.032 0.960
```

**2.** Sue's sewing machine is very old, and it malfunctions often. When a machine fails, it needs either a small repair ( which happens with probability 0.75 ) or a large repair ( probability 0.25 ). If the machine needs a small repair, the time of the repair is exponentially distributed with mean 3 minutes ( rate = 20 ). If the machine needs a large repair, the time of the repair is exponential with mean 6 minutes ( rate = 10 ). After a repair, the machine works for an exponentially distributed time with mean 15 minutes ( rate = 4 ). Assume that all times are independent. Consider a Markov pure jump process $X(t)$ with three states { W(orking), S(mall repair), L(arge repair) }. Find the long-term distribution of $X(t)$.

$q_W = 4 \qquad\qquad q_S = 20 \qquad\qquad q_L = 10$

$Q_{WS} = 0.75, \quad Q_{WL} = 0.25 \qquad\qquad Q_{SW} = 1 \qquad\qquad Q_{LW} = 1$

$$S \underset{20}{\overset{3}{\leftrightarrows}} W \underset{1}{\overset{10}{\leftrightarrows}} L$$

| | | |
|---|---|---|
| $q_{WW} = -4$ | $q_{WS} = 3$ | $q_{WL} = 1$ |
| $q_{SW} = 20$ | $q_{SS} = -20$ | $q_{SL} = 0$ |
| $q_{LW} = 10$ | $q_{LS} = 0$ | $q_{LL} = -10$ |

$-4\,\pi(W) + 20\,\pi(S) + 10\,\pi(L) = 0.$

$3\,\pi(W) - 20\,\pi(S) = 0.$

$\pi(W) - 10\,\pi(L) = 0. \quad \Rightarrow \quad \pi(W) = 10\,\pi(L).$

$\Rightarrow \quad -40\,\pi(L) + 20\,\pi(S) + 10\,\pi(L) = 0.$

$\Rightarrow \quad \pi(S) = 1.5\,\pi(L).$

$\pi(W) + \pi(S) + \pi(L) = 1. \quad \Rightarrow \quad 10\,\pi(L) + 1.5\,\pi(L) + \pi(L) = 1.$

$\Rightarrow \quad \pi(W) = \mathbf{0.80}. \quad \pi(S) = \mathbf{0.12}. \quad \pi(L) = \mathbf{0.08}.$

```
> q = rbind( c(-4,3,1), c(20,-20,0), c(10,0,-10) )
> eigen( t(q) )
$values
[1] -2.324500e+01 -1.075500e+01 -1.363493e-15

$vectors
            [,1]       [,2]        [,3]
[1,]  0.73315671 -0.5913538 -0.98413566
[2,] -0.67780324 -0.1918942 -0.14762035
[3,] -0.05535348  0.7832479 -0.09841357

> eigen(t(q))$vectors[,3]/sum(eigen(t(q))$vectors[,3])
[1] 0.80 0.12 0.08
```

**3.** The Department of Statistics has two photocopy machines. The time to breakdown for each machine has an exponential distribution with parameter $\lambda$. The time to repair for each machine has an exponential distribution with parameter $\mu$. (The two machines could be undergoing repairs at the same time.) Assume that all times to breakdown and all times to repair are independent. For each machine, let 1 denote the *working* condition, and 0 denote the *broken* condition. Then the status of both machines can be described using 4 states, i.e.,

$$0 = (0, 0) \qquad 1 = (1, 0) \qquad 2 = (0, 1) \qquad 3 = (1, 1).$$

Let $X(t)$ denote the conditions of both machines at time $t$.

a) Identify all infinitesimal parameters of $X(t)$.

| | | | |
|---|---|---|---|
| $q_{00} = -2\mu$ | $q_{01} = \mu$ | $q_{02} = \mu$ | $q_{03} = 0$ |
| $q_{10} = \lambda$ | $q_{11} = -\lambda - \mu$ | $q_{12} = 0$ | $q_{13} = \mu$ |
| $q_{20} = \lambda$ | $q_{21} = 0$ | $q_{22} = -\lambda - \mu$ | $q_{23} = \mu$ |
| $q_{30} = 0$ | $q_{31} = \lambda$ | $q_{32} = \lambda$ | $q_{33} = -2\lambda$ |

b) Find the stationary distribution for $X(t)$.

$$\sum_x \pi(x) q_{xy} = 0, \qquad y \in S.$$

$$\pi(0) + \pi(1) + \pi(2) + \pi(3) = 1.$$

$$-2\mu\,\pi(0) + \lambda\,\pi(1) + \lambda\,\pi(2) = 0.$$

$$\mu\,\pi(0) - (\lambda + \mu)\,\pi(1) + \lambda\,\pi(3) = 0.$$

$$\mu\,\pi(0) - (\lambda + \mu)\,\pi(2) + \lambda\,\pi(3) = 0.$$

$$\mu\,\pi(1) + \mu\,\pi(2) - 2\lambda\,\pi(3) = 0.$$

$$\Rightarrow \qquad \pi(1) = \pi(2). \qquad \Rightarrow \qquad \pi(0) = \frac{\lambda}{\mu}\,\pi(1), \qquad \pi(3) = \frac{\mu}{\lambda}\,\pi(1).$$

$$\Rightarrow \qquad \frac{\lambda}{\mu}\,\pi(1) + \pi(1) + \pi(1) + \frac{\mu}{\lambda}\,\pi(1) = 1.$$

$$\Rightarrow \qquad \pi(1) = \pi(2) = \frac{\lambda\mu}{\lambda^2 + 2\lambda\mu + \mu^2} = \frac{\lambda\mu}{(\lambda+\mu)^2}.$$

$$\Rightarrow \qquad \pi(0) = \frac{\lambda^2}{(\lambda+\mu)^2}, \qquad \pi(3) = \frac{\mu^2}{(\lambda+\mu)^2}.$$

**4.** The Department of Statistics has three printers. Each printer breaks down independently at rate $\mu$, then it is sent to the repair shop. The repair shop can only repair one printer at a time and each printer takes an exponential amount of time with parameter $\lambda$ to repair. Let $X(t)$ denote the number of working printers.

a) Identify all infinitesimal parameters of $X(t)$.

| | | | |
|---|---|---|---|
| $q_{00} = -\lambda$ | $q_{01} = \lambda$ | $q_{02} = 0$ | $q_{03} = 0$ |
| $q_{10} = \mu$ | $q_{11} = -\lambda - \mu$ | $q_{12} = \lambda$ | $q_{13} = 0$ |
| $q_{20} = 0$ | $q_{21} = 2\mu$ | $q_{22} = -\lambda - 2\mu$ | $q_{23} = \lambda$ |
| $q_{30} = 0$ | $q_{31} = 0$ | $q_{32} = 3\mu$ | $q_{33} = -3\mu$ |

b) Find the stationary distribution for $X(t)$.

$$\sum_x \pi(x) q_{xy} = 0, \qquad y \in S.$$

$$\pi(0) + \pi(1) + \pi(2) + \pi(3) = 1.$$

$$-\lambda\,\pi(0) + \mu\,\pi(1) = 0.$$

$$\lambda\,\pi(0) - (\lambda + \mu)\,\pi(1) + 2\mu\,\pi(2) = 0.$$

$$\lambda\,\pi(1) - (\lambda + 2\mu)\,\pi(2) + 3\mu\,\pi(3) = 0.$$

$$\lambda\,\pi(2) - 3\mu\,\pi(3) = 0.$$

$$\Rightarrow \qquad \pi(0) = \frac{\mu}{\lambda}\,\pi(1), \qquad \pi(1) = \frac{2\mu}{\lambda}\,\pi(2), \qquad \pi(2) = \frac{3\mu}{\lambda}\,\pi(3).$$

$$\Rightarrow \qquad \pi(1) = \frac{6\mu^2}{\lambda^2}\,\pi(3), \qquad \pi(0) = \frac{6\mu^3}{\lambda^3}\,\pi(3).$$

$$\Rightarrow \qquad \frac{6\mu^3}{\lambda^3}\,\pi(3) + \frac{6\mu^2}{\lambda^2}\,\pi(3) + \frac{3\mu}{\lambda}\,\pi(3) + \pi(3) = 1.$$

$$\Rightarrow \qquad \pi(3) = \frac{\lambda^3}{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3}.$$

$$\Rightarrow \qquad \pi(0) = \frac{6\mu^3}{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3}, \qquad \pi(1) = \frac{6\mu^2\lambda}{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3},$$

$$\pi(2) = \frac{3\mu\lambda^2}{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3}.$$

OR

This is a birth and death process.

$$\pi_0 = 1, \qquad \pi_x = \frac{\lambda_0 \ldots \lambda_{x-1}}{\mu_1 \ldots \mu_x}, \qquad 1 \le x \le 3.$$

Then $$\pi(x) = \frac{\pi_x}{\pi_0 + \pi_1 + \pi_2 + \pi_3}, \qquad 0 \le x \le 3.$$

$$\pi_1 = \frac{\lambda}{\mu}. \qquad \pi_2 = \frac{\lambda \cdot \lambda}{\mu \cdot 2\mu} = \frac{\lambda^2}{2\mu^2}. \qquad \pi_3 = \frac{\lambda \cdot \lambda \cdot \lambda}{\mu \cdot 2\mu \cdot 3\mu} = \frac{\lambda^3}{6\mu^3}.$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{6\mu^3} = \frac{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3}{6\mu^3}.$$

$$\pi(0) = \frac{1}{\dfrac{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3}{6\mu^3}} = \frac{6\mu^3}{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3},$$

$$\pi(1) = \frac{\dfrac{\lambda}{\mu}}{\dfrac{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3}{6\mu^3}} = \frac{6\mu^2\lambda}{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3},$$

$$\pi(2) = \frac{\dfrac{\lambda^2}{2\mu^2}}{\dfrac{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3}{6\mu^3}} = \frac{3\mu\lambda^2}{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3},$$

$$\pi(3) = \frac{\dfrac{\lambda^3}{6\mu^3}}{\dfrac{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3}{6\mu^3}} = \frac{\lambda^3}{6\mu^3 + 6\mu^2\lambda + 3\mu\lambda^2 + \lambda^3}.$$

**Birth and death process:**

$$\pi_0 = 1, \qquad \pi_x = \frac{\lambda_0 \ldots \lambda_{x-1}}{\mu_1 \ldots \mu_x}, \quad x \ge 1. \qquad\qquad \pi(x) = \frac{\pi_x}{\sum_y \pi_y}, \quad x \ge 0.$$

$$\gamma_0 = 1, \qquad \gamma_x = \frac{\mu_1 \ldots \mu_x}{\lambda_1 \ldots \lambda_x}, \quad x \ge 1.$$

$$\sum_{x=0}^{\infty} \gamma_x = \infty. \qquad \sum_{x=0}^{\infty} \pi_x < \infty. \qquad \Rightarrow \qquad \text{Positive recurrent.}$$

$$\sum_{x=0}^{\infty} \gamma_x = \infty. \qquad \sum_{x=0}^{\infty} \pi_x = \infty. \qquad \Rightarrow \qquad \text{Null recurrent.}$$

$$\sum_{x=0}^{\infty} \gamma_x < \infty. \qquad \sum_{x=0}^{\infty} \pi_x = \infty. \qquad \Rightarrow \qquad \text{Transient.}$$

Infinite server queue: $\lambda_x = \lambda, \quad x \ge 0, \qquad \mu_x = x\,\mu, \quad x \ge 1.$

$$\pi_0 = 1,$$

$$\pi_x = \frac{\lambda_0 \ldots \lambda_{x-1}}{\mu_1 \ldots \mu_x} = \frac{\lambda \cdot \lambda \cdot \lambda \cdot \ldots \cdot \lambda}{\mu \cdot 2\,\mu \cdot 3\,\mu \cdot \ldots \cdot x\,\mu} = \frac{(\lambda/\mu)^x}{x!}, \qquad x \ge 1.$$

$$\sum_{x=0}^{\infty} \pi_x = \sum_{x=0}^{\infty} \frac{(\lambda/\mu)^x}{x!} = e^{\lambda/\mu}.$$

$$\pi(0) = e^{-\lambda/\mu}. \qquad\qquad \pi(x) = \pi_x\,\pi(0) = \frac{(\lambda/\mu)^x}{x!}\,e^{-\lambda/\mu}, \qquad x \ge 0.$$

**5.** HPS 3.16

**16** **Consider a birth and death process on the nonnegative integers whose death rates are given by $\mu_x = x$, $x \geq 0$. Determine whether the process is transient, null recurrent, or positive recurrent if the birth rates are**

**(a) $\lambda_x = x + 1$, $x \geq 0$;**
**(b) $\lambda_x = x + 2$, $x \geq 0$.**

a) $\mu_x = x, \qquad \lambda_x = x + 1, \qquad x \geq 0.$

$$\gamma_x = \frac{\mu_1 \dots \mu_x}{\lambda_1 \dots \lambda_x} = \frac{1 \cdot 2 \cdot \dots \cdot x}{2 \cdot 3 \cdot \dots \cdot (x+1)} = \frac{1}{x+1}, \qquad x \geq 1.$$

$$\sum_{x=1}^{\infty} \gamma_x = \sum_{x=1}^{\infty} \frac{1}{x+1} = \infty. \qquad \Rightarrow \qquad \text{Recurrent.}$$

$$\pi_x = \frac{\lambda_0 \dots \lambda_{x-1}}{\mu_1 \dots \mu_x} = \frac{1 \cdot 2 \cdot \dots \cdot x}{1 \cdot 2 \cdot \dots \cdot x} = 1, \qquad x \geq 1.$$

$$\sum_{x=1}^{\infty} \pi_x = \sum_{x=1}^{\infty} 1 = \infty. \qquad \Rightarrow \qquad \textbf{Null recurrent}.$$

b) $\mu_x = x, \qquad \lambda_x = x + 2, \qquad x \geq 0.$

$$\gamma_x = \frac{\mu_1 \dots \mu_x}{\lambda_1 \dots \lambda_x} = \frac{1 \cdot 2 \cdot \dots \cdot x}{3 \cdot 4 \cdot \dots \cdot (x+2)} = \frac{1 \cdot 2}{(x+1) \cdot (x+2)}, \qquad x \geq 1.$$

$$\sum_{x=1}^{\infty} \gamma_x = \sum_{x=1}^{\infty} \frac{1 \cdot 2}{(x+1) \cdot (x+2)} = 2 \cdot \sum_{x=1}^{\infty} \left[ \frac{1}{(x+1)} - \frac{1}{(x+2)} \right] = 2 \cdot \frac{1}{2} = 1 < \infty.$$

$$\Rightarrow \qquad \textbf{Transient}.$$