STAT 433

Examples for 01/18/2022

Spring 2022
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Multiplication Rule:

$$P(A \cap B) = P(A) \times P(B \mid A) \qquad \text{since} \qquad P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Similarly,

$$P(A \cap B \cap C) = P(A) \times P(B \mid A) \times P(C \mid A \cap B)$$

$$\text{since} \qquad P(A \cap B \cap C) = P(A) \times \frac{P(A \cap B)}{P(A)} \times \frac{P(A \cap B \cap C)}{P(A \cap B)}$$

$$P(A \cap B \cap C \cap D) = P(A) \times P(B \mid A) \times P(C \mid A \cap B) \times P(D \mid A \cap B \cap C)$$

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Furthermore,

$$P(A \cap B \mid C) = P(A \mid C) \times P(B \mid A \cap C)$$

$$\text{since} \qquad \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap C)}{P(C)} \times \frac{P(A \cap B \cap C)}{P(A \cap C)}$$

Law of Total Probability:

$$P(A) = P(A \cap B) + P(A \cap B') = P(B) \times P(A \mid B) + P(B') \times P(A \mid B')$$

If $B_1 \cup B_2 \cup \ldots \cup B_m = S$ and $B_i \cap B_j = \varnothing$ for $i \neq j$,

$$P(A) = P(B_1) \times P(A \mid B_1) + P(B_2) \times P(A \mid B_2) + \ldots + P(B_m) \times P(A \mid B_m)$$

Similarly,

$$P(A \mid C) = P(B \mid C) \times P(A \mid B \cap C) + P(B' \mid C) \times P(A \mid B' \cap C)$$

If $B_1 \cup B_2 \cup \ldots \cup B_m = S$ and $B_i \cap B_j = \varnothing$ for $i \neq j$,

$$P(A \mid C) = P(B_1 \mid C) \times P(A \mid B_1 \cap C) + P(B_2 \mid C) \times P(A \mid B_2 \cap C) + \ldots + P(B_m \mid C) \times P(A \mid B_m \cap C)$$

Markov Chain: $\mathrm{X}_0,\ \mathrm{X}_1,\ \mathrm{X}_2,\ \dots,\ \mathrm{X}_{n-1},\ \mathrm{X}_n,\ \mathrm{X}_{n+1},\ \dots$

$$\mathrm{P}(\mathrm{X}_{n+1}=x_{n+1} \mid \mathrm{X}_n=x_n, \mathrm{X}_{n-1}=x_{n-1}, \dots, \mathrm{X}_0=x_0) = \mathrm{P}(\mathrm{X}_{n+1}=x_{n+1} \mid \mathrm{X}_n=x_n)$$

$x_0, \dots, x_{n-1}, x_n, x_{n+1} \in S$ – state space

$\mathrm{P}(\mathrm{X}_{n+1}=x_{n+1} \mid \mathrm{X}_n=x_n)$ – transition probabilities

stationary transition probabilities – $\mathrm{P}(\mathrm{X}_{n+1}=x_{n+1} \mid \mathrm{X}_n=x_n)$ does not depend on n

$$\mathrm{P}(x,y) = \mathrm{P}^1(x,y) = \mathrm{P}(\mathrm{X}_1=y \mid \mathrm{X}_0=x) = \mathrm{P}(\mathrm{X}_{n+1}=y \mid \mathrm{X}_n=x)$$

$$\forall\, x \in S \qquad \sum_{y\in S} \mathrm{P}(x,y) = 1$$

$$\mathrm{P}^n(x,y) = \mathrm{P}(\mathrm{X}_n=y \mid \mathrm{X}_0=x) = \mathrm{P}(\mathrm{X}_{n+m}=y \mid \mathrm{X}_m=x)$$

$$\mathrm{P}^0(x,y) = \mathrm{P}(\mathrm{X}_0=y \mid \mathrm{X}_0=x) = \begin{cases} 1 & x=y \\ 0 & \text{otherwise} \end{cases}$$

initial distribution

$$\pi_0(x) = \mathrm{P}(\mathrm{X}_0=x) \qquad x \in S \qquad \sum_{x\in S} \pi_0(x) = 1$$

$$\pi_n(x) = \mathrm{P}(\mathrm{X}_n=x) \qquad x \in S \qquad \sum_{x\in S} \pi_n(x) = 1$$

$$\mathrm{P}(\mathrm{X}_0=x_0, \mathrm{X}_1=x_1) = \pi_0(x_0) \times \mathrm{P}(x_0,x_1)$$

$$\mathrm{P}(\mathrm{X}_0=x_0, \mathrm{X}_1=x_1, \dots, \mathrm{X}_{n-1}=x_{n-1}, \mathrm{X}_n=x_n)$$
$$= \pi_0(x_0) \times \mathrm{P}(x_0,x_1) \times \dots \times \mathrm{P}(x_{n-1},x_n)$$

$$\mathrm{P}(\mathrm{X}_{n+m}=x_{n+m}, \dots, \mathrm{X}_{n+1}=x_{n+1} \mid \mathrm{X}_n=x_n, \mathrm{X}_{n-1}=x_{n-1}, \dots, \mathrm{X}_0=x_0)$$
$$= \mathrm{P}(x_n,x_{n+1}) \times \dots \times \mathrm{P}(x_{n+m-1},x_{n+m})$$

Fact: $$\mathrm{P}^2(x,y) = \mathrm{P}(\mathrm{X}_2 = y \mid \mathrm{X}_0 = x) = \sum_{z \in S} \mathrm{P}(x,z) \times \mathrm{P}(z,y)$$

Proof: Since $\{\mathrm{X}_2 = y\} = \bigcup_{z \in S} \{\mathrm{X}_1 = z,\ \mathrm{X}_2 = y\}$,

$$\mathrm{P}^2(x,y) = \mathrm{P}(\mathrm{X}_2 = y \mid \mathrm{X}_0 = x) = \sum_{z \in S} \mathrm{P}\left(\mathrm{X}_1 = z,\ \mathrm{X}_2 = y \,\middle|\, \mathrm{X}_0 = x\right)$$

$$= \sum_{z \in S} \mathrm{P}\left(\mathrm{X}_1 = z \,\middle|\, \mathrm{X}_0 = x\right) \times \mathrm{P}\left(\mathrm{X}_2 = y \,\middle|\, \mathrm{X}_1 = z,\ \mathrm{X}_0 = x\right)$$

$$= \sum_{z \in S} \mathrm{P}\left(\mathrm{X}_1 = z \,\middle|\, \mathrm{X}_0 = x\right) \times \mathrm{P}\left(\mathrm{X}_2 = y \,\middle|\, \mathrm{X}_1 = z\right) = \sum_{z \in S} \mathrm{P}(x,z) \times \mathrm{P}(z,y).$$

$$\mathrm{P}^{n+1}(x,y) = \mathrm{P}(\mathrm{X}_{n+1} = y \mid \mathrm{X}_0 = x) = \sum_{z \in S} \mathrm{P}(x,z) \times \mathrm{P}^n(z,y)$$

$$\mathrm{P}^{n+1}(x,y) = \mathrm{P}(\mathrm{X}_{n+1} = y \mid \mathrm{X}_0 = x) = \sum_{z \in S} \mathrm{P}^n(x,z) \times \mathrm{P}(z,y)$$

$$\mathrm{P}^{n+m}(x,y) = \mathrm{P}(\mathrm{X}_{n+m} = y \mid \mathrm{X}_0 = x) = \sum_{z \in S} \mathrm{P}^n(x,z) \times \mathrm{P}^m(z,y)$$

$$\mathrm{P}^3(x,y) = \mathrm{P}(\mathrm{X}_3 = y \mid \mathrm{X}_0 = x) = \sum_{z_1 \in S} \sum_{z_2 \in S} \mathrm{P}(x,z_1) \times \mathrm{P}(z_1,z_2) \times \mathrm{P}(z_2,y)$$

$$\pi_n(x) = \mathrm{P}(\mathrm{X}_n = x) = \sum_{y \in S} \pi_0(y) \times \mathrm{P}^n(y,x)$$

$$\pi_{n+1}(x) = \mathrm{P}(\mathrm{X}_{n+1} = x) = \sum_{y \in S} \pi_n(y) \times \mathrm{P}(y,x)$$

1. The winter weather in Central Illinois is not kind – there are never two nice days in a row. If there is a nice day, the next day would either have rain or snow with equal chance. If it rains on a given day, there is a 50% chance it would rain again the next day, a 25% chance it would snow, and a 25% chance the next day would be nice. If it snows on a given day, there is equal chance that it would rain, it would snow, or the day would be nice.

$S = \{ \mathrm{N}, \mathrm{R}, \mathrm{S} \}$

$\mathrm{P}(\mathrm{N}, \mathrm{N}) = 0$ $\qquad \mathrm{P}(\mathrm{N}, \mathrm{R}) = \frac{1}{2}$ $\qquad \mathrm{P}(\mathrm{N}, \mathrm{S}) = \frac{1}{2}$

$\mathrm{P}(\mathrm{R}, \mathrm{N}) = \frac{1}{4}$ $\qquad \mathrm{P}(\mathrm{R}, \mathrm{R}) = \frac{1}{2}$ $\qquad \mathrm{P}(\mathrm{R}, \mathrm{S}) = \frac{1}{4}$

$\mathrm{P}(\mathrm{S}, \mathrm{N}) = \frac{1}{3}$ $\qquad \mathrm{P}(\mathrm{S}, \mathrm{R}) = \frac{1}{3}$ $\qquad \mathrm{P}(\mathrm{S}, \mathrm{S}) = \frac{1}{3}$

The weather forecast for Monday claims that there is a 50% chance it would be nice, 20% or rain, and 30% of snow.

$\pi_0(\mathrm{N}) = 0.50$ $\qquad \pi_0(\mathrm{R}) = 0.20$ $\qquad \pi_0(\mathrm{S}) = 0.30$

a) Find the probability that Monday would be nice, and it would rain on Tuesday.

$$\mathrm{P}(\mathrm{X}_0 = \mathrm{N},\ \mathrm{X}_1 = \mathrm{R}) = \mathrm{P}(\mathrm{X}_0 = \mathrm{N}) \times \mathrm{P}(\mathrm{X}_1 = \mathrm{R} \mid \mathrm{X}_0 = \mathrm{N}) = 0.50 \times \frac{1}{2} = \mathbf{0.25}.$$

b) Find the probability that it would rain on Monday, nice on Tuesday, snow on Wednesday, snow again on Thursday, and nice on Friday.

$$\mathrm{P}(\mathrm{X}_0 = \mathrm{R},\ \mathrm{X}_1 = \mathrm{N},\ \mathrm{X}_2 = \mathrm{S},\ \mathrm{X}_3 = \mathrm{S},\ \mathrm{X}_4 = \mathrm{N}) = 0.20 \times \frac{1}{4} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} = \frac{\mathbf{1}}{\mathbf{360}}$$

c) Find the probability that it would be nice on Tuesday.

$$\mathrm{P}(\mathrm{X}_1 = \mathrm{N}) = \pi_0(\mathrm{N}) \times \mathrm{P}(\mathrm{N}, \mathrm{N}) + \pi_0(\mathrm{R}) \times \mathrm{P}(\mathrm{R}, \mathrm{N}) + \pi_0(\mathrm{S}) \times \mathrm{P}(\mathrm{S}, \mathrm{N})$$

$$= 0.50 \times 0 + 0.20 \times \frac{1}{4} + 0.30 \times \frac{1}{3} = \mathbf{0.15}.$$

d) Find the probability that it would rain on Tuesday.

$$P(X_1 = R) = 0.50 \times \frac{1}{2} + 0.20 \times \frac{1}{2} + 0.30 \times \frac{1}{3} = \mathbf{0.45}.$$

d ½) Find the probability that it would snow on Tuesday.

$$P(X_1 = S) = 0.50 \times \frac{1}{2} + 0.20 \times \frac{1}{4} + 0.30 \times \frac{1}{3} = \mathbf{0.40}. \quad \text{OR} \quad 1 - 0.15 - 0.45 = \mathbf{0.40}.$$

e) Find the probability that it would rain on Wednesday.

$$\begin{aligned} P(X_2 = R) = {} & 0.50 \times 0 \times \frac{1}{2} + 0.50 \times \frac{1}{2} \times \frac{1}{2} + 0.50 \times \frac{1}{2} \times \frac{1}{3} \\ & + 0.20 \times \frac{1}{4} \times \frac{1}{2} + 0.20 \times \frac{1}{2} \times \frac{1}{2} + 0.20 \times \frac{1}{4} \times \frac{1}{3} \\ & + 0.30 \times \frac{1}{3} \times \frac{1}{2} + 0.30 \times \frac{1}{3} \times \frac{1}{2} + 0.30 \times \frac{1}{3} \times \frac{1}{3} = \mathbf{\frac{104}{240}} = \frac{13}{30}. \end{aligned}$$

e ¼) Find the probability that it would be nice on Wednesday.

$$\begin{aligned} P(X_2 = N) = {} & 0.50 \times 0 \times 0 + 0.50 \times \frac{1}{2} \times \frac{1}{4} + 0.50 \times \frac{1}{2} \times \frac{1}{3} \\ & + 0.20 \times \frac{1}{4} \times 0 + 0.20 \times \frac{1}{2} \times \frac{1}{4} + 0.20 \times \frac{1}{4} \times \frac{1}{3} \\ & + 0.30 \times \frac{1}{3} \times 0 + 0.30 \times \frac{1}{3} \times \frac{1}{4} + 0.30 \times \frac{1}{3} \times \frac{1}{3} = \mathbf{\frac{59}{240}}. \end{aligned}$$

e ½) Find the probability that it would snow on Wednesday.

$$\begin{aligned} P(X_2 = S) = {} & 0.50 \times 0 \times \frac{1}{2} + 0.50 \times \frac{1}{2} \times \frac{1}{4} + 0.50 \times \frac{1}{2} \times \frac{1}{3} \\ & + 0.20 \times \frac{1}{4} \times \frac{1}{2} + 0.20 \times \frac{1}{2} \times \frac{1}{4} + 0.20 \times \frac{1}{4} \times \frac{1}{3} \\ & + 0.30 \times \frac{1}{3} \times \frac{1}{2} + 0.30 \times \frac{1}{3} \times \frac{1}{4} + 0.30 \times \frac{1}{3} \times \frac{1}{3} = \mathbf{\frac{77}{240}}. \end{aligned}$$

e¾) Given that it is nice on Monday, what is the probability it would be nice on Wednesday?

$$P(X_2 = N \mid X_0 = N) = P(X_2 = N, X_1 = N \mid X_0 = N) + P(X_2 = N, X_1 = R \mid X_0 = N) + P(X_2 = N, X_1 = S \mid X_0 = N)$$

$$= P(X_1 = N \mid X_0 = N) \times P(X_2 = N \mid X_1 = N, X_0 = N) + P(X_1 = R \mid X_0 = N) \times P(X_2 = N \mid X_1 = R, X_0 = N) + P(X_1 = S \mid X_0 = N) \times P(X_2 = N \mid X_1 = S, X_0 = N)$$

$$= P(X_1 = N \mid X_0 = N) \times P(X_2 = N \mid X_1 = N) + P(X_1 = R \mid X_0 = N) \times P(X_2 = N \mid X_1 = R) + P(X_1 = S \mid X_0 = N) \times P(X_2 = N \mid X_1 = S)$$

$$= P(N, N) \times P(N, N) + P(N, R) \times P(R, N) + P(N, S) \times P(S, N)$$

$$= 0 \times 0 + \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{3} = \mathbf{\frac{21}{72}} = \frac{7}{24}.$$

f) Given that it is nice on Monday, what is the probability it would snow on Wednesday?

$$P(X_1 = N \mid X_0 = N) \times P(X_2 = S \mid X_1 = N) + P(X_1 = R \mid X_0 = N) \times P(X_2 = S \mid X_1 = R) + P(X_1 = S \mid X_0 = N) \times P(X_2 = S \mid X_1 = S)$$

$$= P(N, N) \times P(N, S) + P(N, R) \times P(R, S) + P(N, S) \times P(S, S)$$

$$= 0 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{3} = \mathbf{\frac{21}{72}} = \frac{7}{24}.$$

f¼) Given that it is nice on Monday, what is the probability it would rain on Wednesday?

$$P(X_1 = N \mid X_0 = N) \times P(X_2 = R \mid X_1 = N) + P(X_1 = R \mid X_0 = N) \times P(X_2 = R \mid X_1 = R) + P(X_1 = S \mid X_0 = N) \times P(X_2 = R \mid X_1 = S)$$

$$= P(N, N) \times P(N, R) + P(N, R) \times P(R, R) + P(N, S) \times P(S, R)$$

$$= 0 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \mathbf{\frac{30}{72}} = \frac{10}{24}.$$

g) Given that it rains on Monday, what is the probability it would be nice on Wednesday?

$$P(R, N) \times P(N, N) + P(R, R) \times P(R, N) + P(R, S) \times P(S, N)$$

$$= \frac{1}{4} \times 0 + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{3} = \mathbf{\frac{15}{72}} = \frac{5}{24}.$$

g¼) Given that it rains on Monday, what is the probability it would rain on Wednesday?

$$P(R, N) \times P(N, R) + P(R, R) \times P(R, R) + P(R, S) \times P(S, R)$$

$$= \frac{1}{4} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{3} = \mathbf{\frac{33}{72}} = \frac{11}{24}.$$

g½) Given that it rains on Monday, what is the probability it would snow on Wednesday?

$$P(R, N) \times P(N, S) + P(R, R) \times P(R, S) + P(R, S) \times P(S, S)$$

$$= \frac{1}{4} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{3} = \mathbf{\frac{24}{72}} = \frac{8}{24}.$$

h) Given that it snows on Monday, what is the probability it would rain on Wednesday?

$$\frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{3} = \mathbf{\frac{32}{72}} = \frac{16}{36}.$$

h¼) Given that it snows on Monday, what is the probability it would be nice on Wednesday?

$$\frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{3} = \mathbf{\frac{14}{72}} = \frac{7}{36}.$$

h½) Given that it snows on Monday, what is the probability it would snow on Wednesday?

$$\frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{3} = \mathbf{\frac{26}{72}} = \frac{13}{36}.$$

Suppose $S = \{ 0, 1, \ldots, d \}$.

The transition matrix

$$\begin{array}{c} \\ 0 \\ \vdots \\ i \\ \vdots \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & \cdots & j & \cdots & d \end{array} \\ \begin{bmatrix} P(0,0) & \cdots & P(0,j) & \cdots & P(0,d) \\ \vdots & & \vdots & & \vdots \\ P(i,0) & \cdots & P(i,j) & \cdots & P(i,d) \\ \vdots & & \vdots & & \vdots \\ P(d,0) & \cdots & P(d,j) & \cdots & P(d,d) \end{bmatrix} \end{array} = \mathbf{P}$$

$$\mathbf{P}_{ij} = P(i,j) = P^1(i,j) = P(X_1 = j \mid X_0 = i)$$

$$[\mathbf{P}^n]_{ij} = P^n(i,j) = P(X_n = j \mid X_0 = i)$$

$$P^2(x,y) = P(X_2 = y \mid X_0 = x) = \sum_{z \in S} P(x,z) \times P(z,y)$$

$$P^{n+m}(x,y) = P(X_{n+m} = y \mid X_0 = x) = \sum_{z \in S} P^n(x,z) \times P^m(z,y)$$

The initial distribution

$$\boldsymbol{\pi_0} = [\ \pi_0(0)\ \ldots\ \pi_0(i)\ \ldots\ \pi_0(d)\ ] = [\ P(X_0 = 0)\ \ldots\ P(X_0 = i)\ \ldots\ P(X_0 = d)\ ]$$

$$\boldsymbol{\pi_n} = [\ \pi_n(0)\ \ldots\ \pi_n(i)\ \ldots\ \pi_n(d)\ ] = [\ P(X_n = 0)\ \ldots\ P(X_n = i)\ \ldots\ P(X_n = d)\ ]$$

$$\boldsymbol{\pi_n} = \boldsymbol{\pi_0} \times \mathbf{P}^n \qquad \pi_n(x) = P(X_n = x) = \sum_{y \in S} \pi_0(y) \times P^n(y,x)$$

$$\boldsymbol{\pi_{n+1}} = \boldsymbol{\pi_n} \times \mathbf{P} \qquad \pi_{n+1}(x) = P(X_{n+1} = x) = \sum_{y \in S} \pi_n(y) \times P(y,x)$$

Consider the transition matrix

$$\begin{array}{c} \\ N \\ R \\ S \end{array}\begin{array}{c} \begin{array}{ccc} N & R & S \end{array} \\ \begin{bmatrix} P(N,N) & P(N,R) & P(N,S) \\ P(R,N) & P(R,R) & P(R,S) \\ P(S,N) & P(S,R) & P(S,S) \end{bmatrix} \end{array} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \mathbf{P}$$

$$\mathbf{P}^2 = \mathbf{P} \times \mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \times \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{21}{72} & \frac{30}{72} & \frac{21}{72} \\ \frac{15}{72} & \frac{33}{72} & \frac{24}{72} \\ \frac{14}{72} & \frac{32}{72} & \frac{26}{72} \end{bmatrix}.$$

$$\boldsymbol{\pi}_{\mathbf{0}} = [\ \pi_0(N)\ \ \pi_0(R)\ \ \pi_0(S)\ ] = [\ P(X_0 = N)\ \ P(X_0 = R)\ \ P(X_0 = S)\ ]$$
$$= [\ 0.50 \quad 0.20 \quad 0.30\ ]$$

$$\boldsymbol{\pi}_{\boldsymbol{n}} = [\ \pi_n(N)\ \ \pi_n(R)\ \ \pi_n(S)\ ] = [\ P(X_n = N)\ \ P(X_n = R)\ \ P(X_n = S)\ ]$$

$$\boldsymbol{\pi}_{\mathbf{1}} = \boldsymbol{\pi}_{\mathbf{0}} \times \mathbf{P} = [0.50 \quad 0.20 \quad 0.30] \times \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = [\ 0.15 \quad 0.45 \quad 0.40\ ]$$

$$\boldsymbol{\pi}_{\mathbf{2}} = \boldsymbol{\pi}_{\mathbf{0}} \times \mathbf{P}^2 = [\ 0.50 \quad 0.20 \quad 0.30\ ] \times \begin{bmatrix} \frac{21}{72} & \frac{30}{72} & \frac{21}{72} \\ \frac{15}{72} & \frac{33}{72} & \frac{24}{72} \\ \frac{14}{72} & \frac{32}{72} & \frac{26}{72} \end{bmatrix} = \begin{bmatrix} \frac{59}{240} & \frac{104}{240} & \frac{77}{240} \end{bmatrix}$$

```
> P = rbind( c(0,1/2,1/2), c(1/4,1/2,1/4), c(1/3,1/3,1/3) )
>
> P
          [,1]      [,2]      [,3]
[1,] 0.0000000 0.5000000 0.5000000
[2,] 0.2500000 0.5000000 0.2500000
[3,] 0.3333333 0.3333333 0.3333333
>
>
> P %*% P
          [,1]      [,2]      [,3]
[1,] 0.2916667 0.4166667 0.2916667
[2,] 0.2083333 0.4583333 0.3333333
[3,] 0.1944444 0.4444444 0.3611111
>
> 72 * (P %*% P)
     [,1] [,2] [,3]
[1,]   21   30   21
[2,]   15   33   24
[3,]   14   32   26
>
>
> pi0 = c(0.5, 0.2, 0.3)         ## weather on Monday
>
> t(pi0) %*% P                   ## weather on Tuesday
     [,1] [,2] [,3]
[1,] 0.15 0.45  0.4
>
> t(pi0) %*% P %*% P             ## weather on Wednesday
          [,1]      [,2]      [,3]
[1,] 0.2458333 0.4333333 0.3208333
>
>
> P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P
          [,1]      [,2]      [,3]
[1,] 0.2222222 0.4444444 0.3333333
[2,] 0.2222222 0.4444444 0.3333333
[3,] 0.2222222 0.4444444 0.3333333
>
> pi00 = c( 2/9, 4/9, 3/9 )
> t(pi00) %*% P
          [,1]      [,2]      [,3]
[1,] 0.2222222 0.4444444 0.3333333
```