STAT 433 | Spring 2022
A. Stepanov

Examples for 03/31/2022

Example 1: Let $X(t)$ be a Poisson process with rate λ, $t \ge 0$.

Find the mean function $m_X(t) = E(X(t))$

and the covariance function $r_X(s,t) = \text{Cov}(X(s), X(t))$.

Hint: $X(t) = \{X(t) - X(s)\} + X(s)$.

$$m_X(t) = E(X(t)) = \lambda t, \qquad t \ge 0.$$

$$r_X(s,t) = \text{Cov}(X(s), X(t)) = \text{Cov}(X(s), \{X(t) - X(s)\} + X(s))$$

$$= \text{Cov}(X(s), X(t) - X(s)) + \text{Cov}(X(s), X(s))$$

$$= 0 + \text{Var}[X(s)] = \lambda s, \qquad 0 \le s \le t.$$

OR

$$E[X(s)X(t)] = E[X(s)\{X(t) - X(s)\}] + E[X(s)X(s)]$$

$X(s)$ and $X(t) - X(s)$ are independent

$$= E[X(s)] \times E[X(t) - X(s)] + E[(X(s))^2]$$

$E[\text{Poisson}(\mu)] = \mu$

$\text{Var}[\text{Poisson}(\mu)] = \mu$

$E[(\text{Poisson}(\mu))^2] = \mu + \mu^2$

$$= \lambda s \times \lambda(t - s) + [\lambda s + (\lambda s)^2]$$

$$= \lambda^2 t s + \lambda s, \qquad 0 \le s \le t.$$

$$\text{Cov}(X(s), X(t)) = E[X(s)X(t)] - E[X(s)] \times E[X(t)]$$

$$= \lambda^2 t s + \lambda s - \lambda s \times \lambda t = \lambda s, \qquad 0 \le s \le t.$$

$$r_X(s,t) = \lambda \min(s,t), \qquad s,t \ge 0.$$

<u>Example 2</u>: Let $X(t)$ be a Poisson process with rate λ, $\quad t \ge 0$.

Let $Y(t) = X(t+1) - X(t), \quad t \ge 0$.

Find the mean function $m_Y(t) = E(Y(t))$

and the covariance function $r_Y(s,t) = \mathrm{Cov}(Y(s), Y(t))$.

$$m_Y(t) = E(Y(t)) = E[X(t+1) - X(t)] = \lambda, \qquad t \ge 0.$$

$$r_Y(s,t) = \mathrm{Cov}(Y(s), Y(t)) = \mathrm{Cov}[X(s+1) - X(s), X(t+1) - X(t)].$$

$$r_Y(s,t) = 0 \qquad \text{if} \qquad s+1 \le t.$$

Suppose $0 \le s \le t \le s+1$.

$$r_Y(s,t) = \mathrm{Cov}[X(s+1) - X(s), X(t+1) - X(t)]$$

$$= \mathrm{Cov}[X(s+1) - X(t) + X(t) - X(s), X(t+1) - X(s+1) + X(s+1) - X(t)]$$

$$\begin{aligned} = \mathrm{Cov}[X(s+1) - X(t), X(t+1) - X(s+1)] \\ + \mathrm{Cov}[X(s+1) - X(t), X(s+1) - X(t)] \\ + \mathrm{Cov}[X(t) - X(s), X(t+1) - X(s+1)] \\ + \mathrm{Cov}[X(t) - X(s), X(s+1) - X(t)] \end{aligned}$$

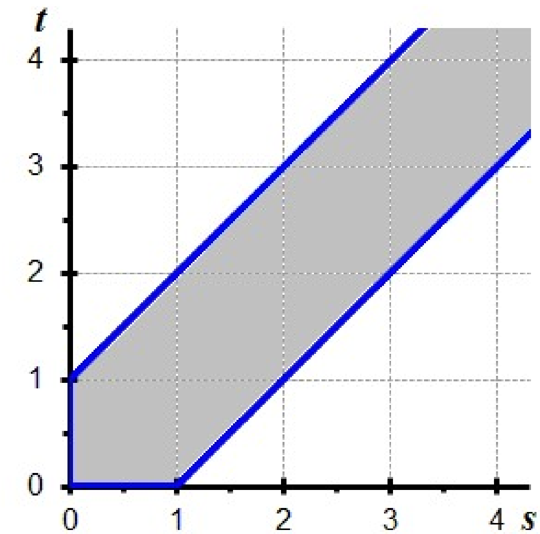
$$= 0 + \mathrm{Var}[X(s+1) - X(t)] + 0 + 0$$

$$= \lambda(s+1-t) = \lambda(1-(t-s)), \qquad 0 \le s \le t \le s+1.$$

$$r_Y(s,t) = \lambda(1 - |t-s|), \qquad |t-s| \le 1,$$

$$r_Y(s,t) = 0, \qquad |t-s| > 1.$$

The region where
$r_Y(s,t) = \text{Cov}(Y(s), Y(t)) > 0.$



$X(t)$ is NOT a second order stationary process.

($X(t)$ is a second order process, but it is not stationary.)

$Y(t)$ is a second order stationary process.

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To simulate Brownian Motion:

```
SimBrMot = function(t,N,sigma,K) {
time = t*(1:N)/N
for (i in 1:K) {
   incr = rnorm(N, 0, sigma*sqrt(t/N))
   X = cumsum(incr)
   plot(time,X,type="l",ylim=c(-2*sigma*sqrt(t),2*sigma*sqrt(t)))
   abline(h=0,lty=2)
   Sys.sleep(2)
}
}
```

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