Examples for 02/10/2022

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If $\pi(x)$, $x \in S$, are nonnegative numbers summing to one, and if

$$\sum_{x \in S} \pi(x) \times \mathrm{P}(x, y) = \pi(y), \qquad y \in \mathrm{S},$$

then π is called a **stationary distribution**. That is, a stationary distribution is a probability vector $\boldsymbol{\pi}$ such that

$$\boldsymbol{\pi}\,\mathbf{P} = \boldsymbol{\pi}.$$

1. Consider a Markov chain on $\{0, 1, 2, 3, 4, \ldots\}$

with $\mathrm{P}(x, y) = \dfrac{1}{x+2}$ for $0 \le y \le x+1$

and $\mathrm{P}(x, y) = 0$ for $y > x+1$.

Find a stationary distribution if it exists.

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & \cdots \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Hint: Use mathematical induction to prove that $\pi(n) = \dfrac{1}{n!}\,\pi(0), \quad n \ge 0.$

2. Consider a birth and death Markov chain on $\{1, 2, \ldots, \mathrm{N}\}$ having a transition function

$$\mathrm{P}(x, x-1) = 1/x, \qquad 1 < x \le \mathrm{N},$$
$$\mathrm{P}(x, x+1) = 1/x, \qquad 1 \le x < \mathrm{N},$$
$$\mathrm{P}(x, x) = 1 - 2/x, \qquad 1 < x < \mathrm{N},$$
$$\mathrm{P}(\mathrm{N}, \mathrm{N}) = 1 - 1/\mathrm{N}.$$

Find the stationary distribution.

Birth and death Markov chains:

$$P(x,y) = \begin{cases} q_x & y = x-1 \qquad\qquad q_0 = 0 \\ r_x & y = x \\ p_x & y = x+1 \qquad\qquad p_d = 0 \end{cases} \qquad\qquad 0 \le x \le d$$

OR

$$P(x,y) = \begin{cases} q_x & y = x-1 \qquad\qquad q_0 = 0 \\ r_x & y = x \\ p_x & y = x+1 \end{cases} \qquad\qquad x \ge 0$$

$$q_x + r_x + p_x = 1, \qquad\qquad x \in S.$$

$$\pi(0)\, r_0 \quad + \quad \pi(1)\, q_1 \quad = \quad \pi(0)$$

$$\pi(y-1)\, p_{y-1} \quad + \quad \pi(y)\, r_y \quad + \quad \pi(y+1)\, q_{y+1} \quad = \quad \pi(y) \qquad\qquad y \ge 1$$

$$\Rightarrow \qquad \pi(x) = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x}\, \pi(0), \qquad\qquad x \in S.$$

Set $\qquad \pi_0 = 1, \qquad\qquad \pi_x = \dfrac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x}, \qquad\qquad x \ge 1.$

$$\pi(x) = \pi_x\, \pi(0), \qquad\qquad x \in S.$$

Must have $\qquad \displaystyle\sum_{x \in S} \pi(x) = 1.$

If $S = \{0, 1, 2, \dots, d\}$, then $\qquad \pi(x) = \dfrac{\pi_x}{\displaystyle\sum_{z=0}^{d} \pi_z}, \qquad\qquad x \in S.$

Suppose $S = \{0, 1, 2, 3, \dots\}$.

$\Rightarrow$ If $\sum\limits_{x=0}^{\infty} \pi_x = \infty$, there is no stationary distribution.

Intuition: $\sum\limits_{x=0}^{\infty} \pi_x = \infty \quad \Leftrightarrow$ there are a lot of "births" and not a lot of "deaths"

the Markov chain gravitates towards infinity

If $\sum\limits_{x=0}^{\infty} \pi_x < \infty$, then $\pi(x) = \dfrac{\pi_x}{\sum\limits_{z=0}^{\infty} \pi_z}$, $\quad x \in S$.

3. Example 2 from 02/03/2022.

Find the stationary distribution if it exists.

a) $P(0,0) = r_0 = 0.8$, $\quad P(0,1) = p_0 = 0.2$,

$$\left.\begin{aligned} P(x, x-1) &= q_x = 0.3, \\ P(x, x) &= r_x = 0.5, \\ P(x, x+1) &= p_x = 0.2, \end{aligned}\right\} \quad x \ge 1.$$

b) $P(0,0) = r_0 = 0.7$, $\quad P(0,1) = p_0 = 0.3$,

$$\left.\begin{aligned} P(x, x-1) &= q_x = 0.2, \\ P(x, x) &= r_x = 0.5, \\ P(x, x+1) &= p_x = 0.3, \end{aligned}\right\} \quad x \ge 1.$$

c) $P(0,0) = r_0 = 0.7,$ $P(0,1) = p_0 = 0.3,$

$P(x, x-1) = q_x = 0.2,$

$P(x, x) = r_x = 0.5,$ $1 \le x \le 99.$

$P(x, x+1) = p_x = 0.3,$

$P(100, 100) = r_{100} = 0.8,$ $P(100, 99) = q_{100} = 0.2,$

d) Consider the birth and death chain on $\{0, 1, 2, \dots\}$ defined by $p_x = (x+2)/(2x+2)$ and $q_x = x/(2x+2)$, $x \ge 0$ (i.e., there is a little bias to the right). Any state leads to any other state, so this chain is irreducible. (~ Example 13, p. 33 HPS)

e) Consider the birth and death chain on $\{0, 1, 2, \dots\}$ defined by $p_x = (x+1)/(2x+1)$ and $q_x = x/(2x+1)$, $x \ge 0$ (i.e., there is a little bias to the right). Any state leads to any other state, so this chain is irreducible.