STAT 433 Spring 2022

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Examples for 01/25/2022

$\rho_{xy} = P_x(T_y < \infty)$.

(The probability that the Markov chain eventually visits y starting from x.)

State y is **recurrent** if $\rho_{yy} = 1$.

($\rho_{yy} = P_y(T_y < \infty) = 1 \;\Rightarrow\;$ the Markov chain is guaranteed to return to y starting from y.)

State y is **transient** if $\rho_{yy} < 1$.

($\rho_{yy} < 1 \;\Rightarrow\;$ the Markov chain is NOT guaranteed to return to y starting from y.)

State y is **absorbing** if $P(y, y) = 1$. ($\Rightarrow\;$ $P(y, z) = 0$ for all $z \neq y$.)

State y is absorbing $\Rightarrow$ y is recurrent.

$$N(y) = \sum_{n=1}^{\infty} 1_y(X_n)$$ – the number of times the Markov chain visits state y.

$P_x(N(y) \geq 1) = P_x(T_y < \infty) = \rho_{xy}$.

(To visit y at least once starting from x, the Markov chain needs to eventually visit y starting from x.)

$P_x(N(y) \geq 2) = \rho_{xy} \times \rho_{yy}$.

(To visit y at least twice starting from x, the Markov chain needs to eventually visit y starting from x, and then return to y starting from y.)

$P_x(N(y) \geq m) = \rho_{xy} \times \rho_{yy}^{m-1}$.

(To visit y at least m times starting from x, the Markov chain needs to eventually visit y starting from x, and then return to y ($m - 1$) times starting from y.)

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Fact: Let X be a random variable of the discrete type with pmf $p(x)$ that is positive on the nonnegative integers and is equal to zero elsewhere. Then

$$\mathrm{E}(\mathrm{X}) = \sum_{m=1}^{\infty} \mathrm{P}(\mathrm{X} \ge m).$$

Proof:

$$\mathrm{P}(\mathrm{X} \ge m) = p(m) + p(m+1) + p(m+2) + p(m+3) + p(m+4) + \dots$$

$$\begin{array}{lllllllll}
\mathrm{P}(\mathrm{X} \ge 1) & p(1) + & p(2) + & p(3) + & p(4) + & p(5) + & p(6) + & p(7) + & \dots \\
\mathrm{P}(\mathrm{X} \ge 2) & & p(2) + & p(3) + & p(4) + & p(5) + & p(6) + & p(7) + & \dots \\
\mathrm{P}(\mathrm{X} \ge 3) & & & p(3) + & p(4) + & p(5) + & p(6) + & p(7) + & \dots \\
\mathrm{P}(\mathrm{X} \ge 4) & & & & p(4) + & p(5) + & p(6) + & p(7) + & \dots \\
\mathrm{P}(\mathrm{X} \ge 5) & & & & & p(5) + & p(6) + & p(7) + & \dots \\
\quad \dots & & & \dots & & & & &
\end{array}$$

$$\Rightarrow \quad \sum_{m=1}^{\infty} \mathrm{P}(\mathrm{X} \ge m) = 1 \times p(1) + 2 \times p(2) + 3 \times p(3) + 4 \times p(4) + \dots = \mathrm{E}(\mathrm{X}).$$

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Let X be a random variable with a Geometric distribution with probability of "success" p.

Then

$$\mathrm{P}(\mathrm{X} > a) = \sum_{k=a+1}^{\infty} (1-p)^{k-1} p = \frac{(1-p)^a p}{1-(1-p)} = (1-p)^a, \qquad a = 0, 1, 2, 3, \dots .$$

OR

X = number of independent attempts needed to get the first "success".

$\mathrm{P}(\mathrm{X} > a) = \mathrm{P}($the first a attempts are "failures"$) = (1-p)^a, \qquad a = 0, 1, 2, 3, \dots .$

$$\mathrm{E}(\mathrm{X}) = \sum_{m=1}^{\infty} \mathrm{P}(\mathrm{X} \ge m) = \sum_{x=0}^{\infty} \mathrm{P}(\mathrm{X} > x) = \sum_{x=0}^{\infty} [1-p]^x = \frac{1}{1-[1-p]} = \frac{1}{p}.$$

$$\mathrm{P}(\mathrm{X}=\infty) = \lim_{m\to\infty} \mathrm{P}(\mathrm{X}>m) = \lim_{m\to\infty} (1-p)^m = 0. \qquad \mathrm{P}(\mathrm{X}<\infty) = 1.$$

No matter how small the probability of "success" p is, if it is positive (if it is possible to get a "success"), then eventually a "success" will occur.

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$$\mathrm{E}_x(1_y(\mathrm{X}_n)) = \mathrm{P}_x(\mathrm{X}_n=y) = \mathrm{P}^n(x,y).$$

$$\mathrm{G}(x,y) = \mathrm{E}_x(\mathrm{N}(y)) = \sum_{n=1}^{\infty} \mathrm{P}^n(x,y)$$ – expected number of visits to y starting at x

Theorem (i) If y is transient,

$$\mathrm{P}_x(\mathrm{N}(y)<\infty) = 1, \qquad \mathrm{G}(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}}, \qquad x\in S.$$

(ii) If y is recurrent,

$$\mathrm{P}_y(\mathrm{N}(y)=\infty) = 1, \qquad \mathrm{G}(y,y) = \infty,$$

$$\mathrm{P}_x(\mathrm{N}(y)=\infty) = \mathrm{P}_x(\mathrm{T}_y<\infty) = \rho_{xy}, \qquad x\in S.$$

If $\rho_{xy}=0$, then $\mathrm{G}(x,y) = 0$.

If $\rho_{xy}>0$, then $\mathrm{G}(x,y) = \infty$.

Since $\mathrm{P}_x(\mathrm{N}(y)\ge m) = \rho_{xy}\times\rho_{yy}^{m-1}$,

$$\mathrm{G}(x,y) = \mathrm{E}_x(\mathrm{N}(y)) = \sum_{m=1}^{\infty} \rho_{xy}\times\rho_{yy}^{m-1} = \ldots$$

y is transient $\quad \rho_{yy}<1$

$$\ldots = \frac{\rho_{xy}}{1-\rho_{yy}}.$$

y is recurrent $\quad \rho_{yy}=1$

$\ldots = 0, \quad$ if $\rho_{xy}=0$,

$\ldots = \infty, \quad$ if $\rho_{xy}>0$.

If y is transient, $\quad \mathrm{G}(x,y) = \sum_{n=1}^{\infty} \mathrm{P}^n(x,y) < \infty, \quad \Rightarrow \quad \lim_{n\to\infty} \mathrm{P}^n(x,y) = 0.$

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x leads to y if $\rho_{xy} > 0 \;\Leftrightarrow\; \mathrm{P}^n(x, y) > 0$ for some n.

($\Leftrightarrow$ it is possible to go from x to y, not necessarily in one step)

Theorem x is recurrent, x leads to y

$\Rightarrow$ y is recurrent, and $\rho_{xy} = \rho_{yx} = 1$.

$\mathrm{C} \subseteq S$ is closed if $\rho_{xy} = 0$ for $x \in \mathrm{C}$, $y \notin \mathrm{C}$.

($\Leftrightarrow$ it is NOT possible to go outside C from C)

Closed C is irreducible if x leads to y $\forall\, x, y \in \mathrm{C}$.

($\Leftrightarrow$ it is possible to go from every state in C to every state in C, not necessarily in one step)

Theorem C is a finite irreducible closed set of states

$\Rightarrow$ every state in C is recurrent.

$S = S_{\mathrm{R}} \cup S_{\mathrm{T}}$

S_{R} = set of recurrent states

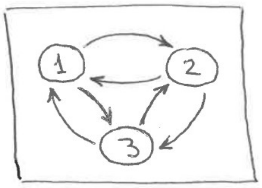
S_{T} = set of transient states

Theorem $S_{\mathrm{R}} \neq \varnothing$

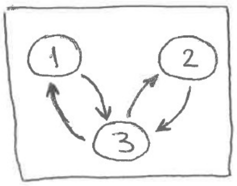
$\Rightarrow$ S_{R} is the union of a finite or countably infinite number of disjoint irreducible closed sets $\mathrm{C}_1, \mathrm{C}_2, \ldots$.

If you start in a recurrent state, you will never visit any of the transient states. Whichever "box" (closed irreducible set of recurrent states) contains the starting state, the chain will remain in that "box" forever.

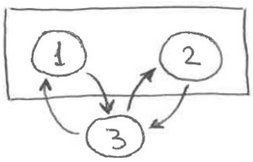
If you start in a transient state, you may visit transient states for a while, but eventually you would arrive to a recurrent state. Once you arrive to a recurrent state, you will never visit any of the transient states again. Whichever "box" (closed irreducible set of recurrent states) contains that recurrent state, the chain will remain in that "box" forever.



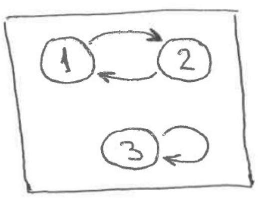
closed irreducible



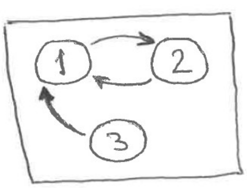
closed irreducible



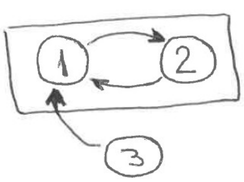
NOT closed



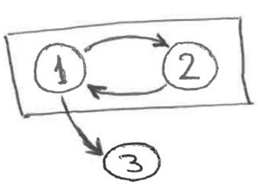
closed NOT irreducible can be divided into two closed irreducible sets



closed NOT irreducible



closed irreducible



NOT closed

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In words: Probability that …

$P^n(x,y) = P_x(X_n = y) = P(X_n = y \mid X_0 = x)$

… starting from state x, after n steps, you are in state y.
Nothing is assumed about what happened in the meantime
(including possibly visiting y before step n).

$P_x(T_y = n) = P_x(X_1 \neq y, X_2 \neq y, \dots, X_{n-1} \neq y, X_n = y)$

… starting from x, after n steps, you are in y for the first time
(not including the initial "visit" on step 0 if $x = y$).

$P_x(T_y \leq n)$

… starting from x, you have visited y on step n or before that
(not including the initial "visit" on step 0 if $x = y$).
(You may or may not be at y on step n, as long as you have been there.)

$P_x(T_y < \infty)$

… starting from x, you have visited y at some point between now and the end of the world (not including the initial "visit" on step 0 if $x = y$).

$P_x(N(y) = n)$

… starting from x, you visit y exactly n times between now and the end of the world (not including the initial "visit" on step 0 if $x = y$).

$P_x(N(y) < \infty)$

… starting from x, you visit y finitely many times between now and the end of the world.

$P_x(N(y) = \infty)$

… starting from x, you visit y infinitely many times between now and the end of the world.