STAT 433
Spring 2022
Version A

Name ANSWERS

NetID ____________________

Page	Earned
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

/ 80

Exam 1

The exam has 6 problems and 10 pages.

Be sure to show all your work; your partial credit might depend on it.

Please put your final answers at the end of your work and mark them clearly. Box the final answers where appropriate.

No credit will be given without supporting work.

The exam is closed book and closed notes. You are allowed to use a calculator and one 8½" x 11" sheet (both sides) with notes.

Turn in all scratch paper with your exam.

Academic Integrity

The University statement on your obligation to maintain academic integrity is:

If you engage in an act of academic dishonesty, you become liable to severe disciplinary action. Such acts include cheating; falsification or invention of information or citation in an academic endeavor; helping or attempting to help others commit academic infractions; plagiarism; offering bribes, favors, or threats; academic interference; computer related infractions; and failure to comply with research regulations.

Article 1, Part 4 of the Student Code gives complete details of rules governing academic integrity for all students. You are responsible for knowing and abiding by these rules.

1. An auto insurance company classifies its customers into three categories based on recent driving history: **L**ow risk, **M**edium risk, and **P**oor risk. At the end of each year, a customer could either be upgraded into a higher category or downgraded into a lower one. (One cannot move from Poor risk to Low risk in one year, that is not how accident forgiveness works.) Consider a Markov chain with the following transition probability matrix:

$$\begin{array}{c|ccc} & L & M & P \\ \hline L & 0.6 & 0.3 & 0.1 \\ M & 0.1 & 0.7 & 0.2 \\ P & 0 & 0.3 & 0.7 \end{array}$$

a) (2) Given that a customer was a Low risk at the end of 2020 and a Medium risk at the end of 2021, what is the probability that this customer will be a Poor risk at the end of 2022?

Since this is a Markov chain,

$$P(X_{22} = P \mid X_{20} = L, X_{21} = M) = P(X_{22} = P \mid X_{21} = M) = P(M, P) = \mathbf{0.20}.$$

b) (2) Given that a customer was a Low risk at the end of 2021, what is the probability that this customer will be a Medium risk at the end of 2022 and a Poor risk at the end of 2023?

$$P(X_{22} = M, X_{23} = P \mid X_{21} = L) = P(L, M) \cdot P(M, P) = 0.30 \cdot 0.20 = \mathbf{0.06}.$$

c) (3) Given that a customer was a Low risk at the end of 2021, what is the probability that this customer will be a Poor risk at the end of 2023?

$$P(X_{23} = P \mid X_{21} = L) = P(L, L) \cdot P(L, P) + P(L, M) \cdot P(M, P) + P(L, P) \cdot P(P, P)$$

$$= 0.60 \cdot 0.10 + 0.30 \cdot 0.20 + 0.10 \cdot 0.70 = \mathbf{0.19}.$$

$$\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.3 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.39 & 0.42 & \mathbf{0.19} \\ 0.13 & 0.58 & 0.29 \\ 0.03 & 0.42 & 0.55 \end{bmatrix}$$

1. (continued)

$$\begin{array}{c} \\ L \\ M \\ P \end{array}\begin{array}{c} \begin{array}{ccc} L & M & P \end{array} \\ \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.3 & 0.7 \end{bmatrix} \end{array}$$

d) (5) Given that a customer is a Low risk now, what is the expected number of years until this customer is a Poor risk for the first time?

$$m_{\text{LP}} = 1 + 0.60\, m_{\text{LP}} + 0.30\, m_{\text{MP}}. \qquad (1)$$

$$m_{\text{MP}} = 1 + 0.10\, m_{\text{LP}} + 0.70\, m_{\text{MP}}. \qquad (2)$$

(1) $\Rightarrow$ $m_{\text{LP}} = 2.5 + 0.75\, m_{\text{MP}}.$ (3)

(2), (3) $\Rightarrow$ $m_{\text{MP}} = 1 + 0.25 + 0.075\, m_{\text{MP}} + 0.70\, m_{\text{MP}}.$

$\Rightarrow$ $0.225\, m_{\text{MP}} = 1.25.$

$\Rightarrow$ $m_{\text{MP}} = \frac{50}{9}.$ (4)

(3), (4) $\Rightarrow$ $m_{\text{LP}} = \frac{5}{2} + \frac{3}{4} \cdot \frac{50}{9} = \frac{5}{2} + \frac{25}{6} = \frac{40}{6} = \boldsymbol{\frac{20}{3}} \approx 6.666667.$

e) (5) Suppose that 20% of the customers were Low risk, 70% were Medium risk, and 10% were Poor risk at the end of 2021. Find the proportions for the three categories at the end of 2022.

$$\pi_{22} = \pi_{21} \times \mathbf{P} = \begin{bmatrix} 0.2 & 0.7 & 0.1 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} \mathbf{0.19} & \mathbf{0.58} & \mathbf{0.23} \end{bmatrix}.$$

19% Low risk, 58% Medium risk, 23% Poor risk.

1. (continued)

$$\begin{array}{c} \\ L \\ M \\ P \end{array} \begin{array}{c} \begin{array}{ccc} L & M & P \end{array} \\ \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.3 & 0.7 \end{bmatrix} \end{array}$$

f) (8) Find the long-term probability distribution if it exists.

$\pi(L) = 0.60\,\pi(L) + 0.10\,\pi(M).$ (1)

$\pi(M) = 0.30\,\pi(L) + 0.70\,\pi(M) + 0.30\,\pi(P).$ (2)

$\pi(P) = 0.10\,\pi(L) + 0.20\,\pi(M) + 0.70\,\pi(P).$ (3)

$\pi(L) + \pi(M) + \pi(P) = 1.$ (4)

(1) $\Rightarrow$ $4\,\pi(L) = \pi(M).$ (5)

(2) $\Rightarrow$ $\pi(M) = \pi(L) + \pi(P).$ (6)

(4), (6) $\Rightarrow$ $\pi(M) = \pi(L) + \pi(P) = 0.50.$ (7)

(5), (7) $\Rightarrow$ $\pi(L) = 0.125.$ (8)

(6), (8) $\Rightarrow$ $\pi(P) = 0.375.$

$\pi(L) = \mathbf{0.125}.$ $\pi(M) = \mathbf{0.50}.$ $\pi(P) = \mathbf{0.375}.$

2. (8) Consider a Markov Chain with $S = \mathbb{Z} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$ (all integers) and transition probability function

$$P(0,0) = 0, \qquad P(0,k) = \left(\frac{1}{3}\right)^{|k|}, \qquad k \neq 0,$$

$$P(k,k) = \frac{2}{3}, \qquad P(k,k-2) = \frac{1}{3}, \qquad k > 0,$$

$$P(k,k) = \frac{2}{3}, \qquad P(k,k+2) = \frac{1}{3}, \qquad k < 0.$$

For each state, determine whether it is recurrent or transient. **Justify your answer.**

From a non-zero integer, when we move, we move in the direction of 0 by 2.
That is, from an even (non-zero) state, when we move, we move to an even state,
from an odd state, when we move, we move to an odd state.

From an odd positive integer, we would eventually visit 1.
From an odd negative integer, we would eventually visit -1.

From an even (non-zero) integer, we would eventually visit 0.

From 0, if we go to an even (non-zero) integer, we would eventually return to 0.
However, from 0, we would eventually go to an odd integer.
If we go from 0 to an odd positive integer, we would eventually visit 1.
If we go from 0 to an odd negative integer, we would eventually visit -1.

That is, we would eventually visit $\{-1, 1\}$.

$\{-1, 1\}$ is a closed irreducible set of states. From $\{-1, 1\}$, we would never leave, we either stay in the same state or switch to the other one, visiting both infinitely many times.

Therefore, $\mathbf{-1}$ and $\mathbf{1}$ are **recurrent**,

all other states are transient.

3. Suppose the number of members at the University of Illinois at Urbana-Champaign University Introverts Unite Club can be modeled as a birth and death chain with

$$p_x = 0.3, \qquad x \ge 0,$$

$$r_0 = 0.7, \qquad r_x = 0.3, \qquad x \ge 1,$$

$$q_x = 0.4, \qquad x \ge 1.$$

a) (7) Find the long-term probability distribution for the number of members at the University of Illinois at Urbana-Champaign University Introverts Unite Club, if it exists.

$$\pi_0 = 1, \qquad \pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \left(\frac{3}{4}\right)^x, \qquad x \ge 1.$$

$$\sum_{x=0}^{\infty} \pi_x = \sum_{x=0}^{\infty} \left(\frac{3}{4}\right)^x = \frac{1}{1-\frac{3}{4}} = 4 < \infty.$$

Stationary distribution DOES exist.

$$\pi(x) = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^x, \qquad x \ge 0.$$

b) (6) If the club membership reaches 14, the club would organize a pizza-and-wings party. Find the probability that there will be a club pizza-and-wings party before the club membership hits 0, given that the club has 10 members now.

$$\gamma_0 = 1, \qquad \gamma_y = \frac{q_1 \cdots q_y}{p_1 \cdots p_y} = \left(\frac{4}{3}\right)^y, \qquad y \ge 1.$$

$x = 10, \qquad a = 0, \qquad b = 14.$

$$\mathrm{P}_{10}(\mathrm{T}_0 > \mathrm{T}_{14}) = \mathrm{P}_x(\mathrm{T}_a > \mathrm{T}_b) = \frac{\sum_{y=a}^{x-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y} = \frac{\sum_{y=0}^{9} \left(\frac{4}{3}\right)^y}{\sum_{y=0}^{13} \left(\frac{4}{3}\right)^y} = \frac{\left(\frac{4}{3}\right)^{10} - 1}{\left(\frac{4}{3}\right)^{14} - 1} \approx \mathbf{0.304}.$$

4. Consider a Markov chain having the transition matrix

$$\begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.3 & 0.1 & 0 & 0.2 & 0.4 \\ 0 & 0 & 0.7 & 0 & 0.3 \end{bmatrix} \end{array}$$

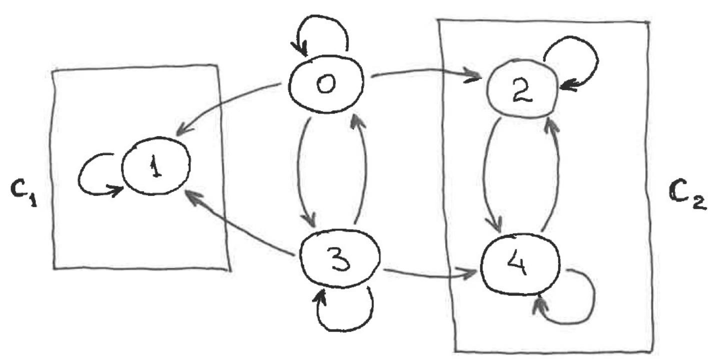
a) (7) Determine which states are

$$\begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.3 & 0.1 & 0 & 0.2 & 0.4 \\ 0 & 0 & 0.7 & 0 & 0.3 \end{bmatrix} \end{array}$$

Recurrent: **1**, **2**, **4**. (An absorbing state is recurrent.)

Transient: **0**, **3**.

Absorbing: **1**.



b) (2) Write the set of the recurrent states as the union of disjoint irreducible closed set of recurrent states.

$\{\mathbf{1}\} \cup \{\mathbf{2}, \mathbf{4}\} = C_1 \cup C_2$.

4. (continued)

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.3 & 0.1 & 0 & 0.2 & 0.4 \\ 0 & 0 & 0.7 & 0 & 0.3 \end{bmatrix} \end{array}$$

c) (12) Find …

(i) $\rho_{30} = P_3(T_0 < \infty)$;

(ii) $\rho_{04} = P_0(T_4 < \infty)$;

(iii) $\rho_{41} = P_4(T_1 < \infty)$.

(i) $$\rho_{xy} = P(x,y) + \sum_{\substack{z \in S_T \\ z \neq y}} P(x,z) \cdot \rho_{zy}, \qquad x, y \in S_T.$$

$$\rho_{30} = 0.3 + 0.2\,\rho_{30} \qquad \Rightarrow \qquad \rho_{30} = \mathbf{\frac{3}{8}} = \mathbf{0.375}.$$

(ii) $\rho_{04} = \rho_{C_2}(0)$.

$$\rho_C(x) = \sum_{y \in C} P(x,y) + \sum_{y \in S_T} P(x,y) \cdot \rho_C(y), \qquad x \in S_T.$$

$$\rho_{C_2}(0) = 0.3 + 0.1\,\rho_{C_2}(0) + 0.4\,\rho_{C_2}(3) \qquad (1)$$

$$\rho_{C_2}(3) = 0.4 + 0.3\,\rho_{C_2}(0) + 0.2\,\rho_{C_2}(3) \qquad (2)$$

(1) $\Rightarrow$ $0.4\,\rho_{C_2}(3) = 0.9\,\rho_{C_2}(0) - 0.3$ (3)

(2), (3) $\Rightarrow$ $0.3\,\rho_{C_2}(0) = 0.8\,\rho_{C_2}(3) - 0.4 = 1.8\,\rho_{C_2}(0) - 1$

$1 = 1.5\,\rho_{C_2}(0)$

$$\Rightarrow \qquad \rho_{C_2}(0) = \mathbf{\frac{2}{3}}, \qquad \rho_{C_2}(3) = \frac{3}{4}.$$

OR

$$
\begin{array}{c} \\ C_1 \\ C_2 \\ 0 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccc} C_1 & C_2 & 0 & 3 \end{array} \\
\left[\begin{array}{cc|cc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\ \hline
0.2 & 0.3 & 0.1 & 0.4 \\
0.1 & 0.4 & 0.3 & 0.2
\end{array}\right]
\end{array}
$$

$$\mathbf{Q} = \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \qquad \mathbf{I} - \mathbf{Q} = \begin{bmatrix} 0.9 & -0.4 \\ -0.3 & 0.8 \end{bmatrix}$$

$$\det(\mathbf{I} - \mathbf{Q}) = 0.60$$

$$\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} \frac{0.8}{0.6} & \frac{0.4}{0.6} \\ \frac{0.3}{0.6} & \frac{0.9}{0.6} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$\mathbf{F}\,\mathbf{R} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \mathbf{\frac{2}{3}} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

(iii) $\rho_{41} = \mathbf{0}$ since C_2 is a closed set.

5. (6) A life insurance company wants to find out how much money to charge its clients. The following 3-state Markov chain model is proposed for the changes in health for the 40+ age group: H – healthy, S – sick, D – dead. The matrix of one-step (one-year) transition probabilities is given below:

$$\begin{array}{c} \\ H \\ S \\ D \end{array} \begin{array}{c} \begin{array}{ccc} H & S & D \end{array} \\ \begin{bmatrix} 0.95 & 0.04 & 0.01 \\ 0.50 & 0.20 & 0.30 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

According to this model, what is the expected number of years until death for an individual from the 40+ age group who is currently healthy?

$$a_H = 1 + 0.95\,a_H + 0.04\,a_S \qquad (1)$$

$$a_S = 1 + 0.50\,a_H + 0.20\,a_S \qquad (2)$$

$$(2) \quad \Rightarrow \quad a_S = 1.25 + 0.625\,a_H \qquad (3)$$

$$(1), (3) \quad \Rightarrow \quad a_H = 1 + 0.95\,a_H + 0.05 + 0.025\,a_H$$

$$\Rightarrow \quad 0.025\,a_H = 1.05 \qquad a_H = \mathbf{42}.$$

OR

$$\mathbf{Q} = \begin{bmatrix} 0.95 & 0.04 \\ 0.50 & 0.20 \end{bmatrix} \qquad \mathbf{I} - \mathbf{Q} = \begin{bmatrix} 0.05 & -0.04 \\ -0.50 & 0.80 \end{bmatrix}$$

$$\det(\mathbf{I} - \mathbf{Q}) = 0.02$$

$$\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} \frac{0.80}{0.02} & \frac{0.04}{0.02} \\ \frac{0.50}{0.02} & \frac{0.05}{0.02} \end{bmatrix} = \begin{bmatrix} 40 & 2 \\ 25 & 2.5 \end{bmatrix}$$

$$\begin{bmatrix} 40 & 2 \\ 25 & 2.5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{42} \\ 27.5 \end{bmatrix}.$$

Answer: **42**.

6. (7) Some species of Whiptail Lizard exist only as females and reproduce asexually. Suppose that the number of offspring a Bluebelly Whiptail Lizard would have over its lifetime has the following probability distribution:

$$f(0) = 0.40, \quad f(1) = 0.20, \quad f(2) = 0.30, \quad f(3) = 0.10.$$

Determine whether Bluebelly Whiptail Lizard extinction is certain.

If the extinction is not certain, find the probability of extinction $\rho = \rho_{10}$.

$\mu = 0 \times 0.40 + 1 \times 0.20 + 2 \times 0.30 + 3 \times 0.10 = 1.1 > 1.$

Extinction is NOT certain.

$$\rho = 0.40 + 0.20\,\rho + 0.30\,\rho^2 + 0.10\,\rho^3$$

$$0 = 0.40 - 0.80\,\rho + 0.30\,\rho^2 + 0.10\,\rho^3$$

$$\rho^3 + 3\,\rho^2 - 8\,\rho + 4 = 0$$

$$(\rho - 1)(\rho^2 + 4\,\rho - 4) = 0$$

$$\rho = 1 \qquad \text{or} \qquad \rho = \frac{-4 \pm \sqrt{16+16}}{2} = -2 \pm \sqrt{8}\,.$$

Need $0 \le \rho < 1$. $\qquad \rho = \sqrt{8} - 2 \approx 0.828427.$