STAT 433 Spring 2022

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Examples for 02/03/2022

Birth and death Markov chain:

$$P(x,y) = \begin{cases} q_x & y = x-1 \qquad\qquad q_0 = 0 \\ r_x & y = x \\ p_x & y = x+1 \qquad\qquad p_d = 0 \end{cases} \qquad\qquad 0 \le x \le d$$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \cdots \\ d-1 \\ d \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & \cdots & d-2 & d-1 & d \end{array} \\ \begin{bmatrix} r_0 & p_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q_1 & r_1 & p_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q_3 & r_3 & p_3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & q_{d-1} & r_{d-1} & p_{d-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & q_d & r_d \end{bmatrix} \end{array}$$

OR

$$P(x,y) = \begin{cases} q_x & y = x-1 \qquad\qquad q_0 = 0 \\ r_x & y = x \\ p_x & y = x+1 \end{cases} \qquad\qquad x \ge 0$$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \cdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{bmatrix} r_0 & p_0 & 0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots \\ 0 & 0 & q_3 & r_3 & p_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array}$$

$q_x + r_x + p_x = 1, \qquad\qquad x \in S.$

$a, b \in S \qquad a < b$

$u(x) = P_x(T_a < T_b) \qquad a < x < b$

$u(a) = 1 \qquad u(b) = 0$

$\Rightarrow \qquad u(y) = q_y \cdot u(y-1) + r_y \cdot u(y) + p_y \cdot u(y+1) \qquad a < y < b$

Set $\qquad \gamma_0 = 1. \qquad \gamma_y = \dfrac{q_1 \cdots q_y}{p_1 \cdots p_y}, \quad y \geq 1.$

$$\Rightarrow \qquad u(x) = P_x(T_a < T_b) = \frac{\sum_{y=x}^{b-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}, \qquad P_x(T_a > T_b) = \frac{\sum_{y=a}^{x-1} \gamma_y}{\sum_{y=a}^{b-1} \gamma_y}.$$

Gambler's ruin chain: $\qquad S = \{0, 1, 2, \ldots, d\}$ OR $S = \{0, 1, 2, 3, \ldots\}$.

$q_x = q$ (lose a bet), $\qquad r_x = 0, \qquad p_x = p$ (win a bet).

Then $\qquad \gamma_y = \dfrac{q_1 \cdots q_y}{p_1 \cdots p_y} = \left(\dfrac{q}{p}\right)^y, \qquad y \in S.$

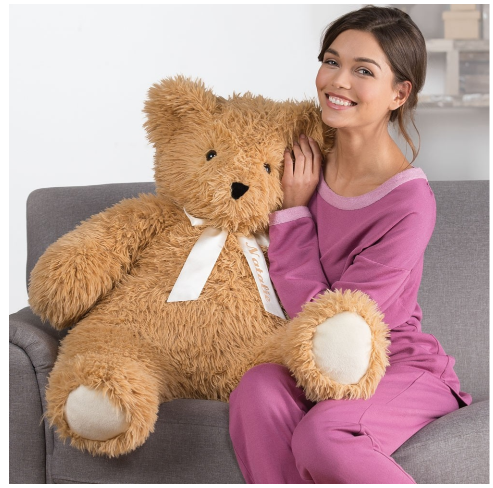
$p_0 = 0 \qquad$ 0 is an absorbing state.

The probability of winning $d - x$ dollars before losing x dollars $\qquad (0 \leq x \leq d)$

$$P_x(T_0 > T_d) = \frac{\sum_{y=0}^{x-1} \gamma_y}{\sum_{y=0}^{d-1} \gamma_y} = \frac{\sum_{y=0}^{x-1} \left(\frac{q}{p}\right)^y}{\sum_{y=0}^{d-1} \left(\frac{q}{p}\right)^y}$$

$$= \frac{x}{d}, \quad \text{if } p = q. \qquad = \frac{\left(\frac{q}{p}\right)^x - 1}{\left(\frac{q}{p}\right)^d - 1}, \quad \text{if } p \neq q.$$

Example 1:



Jack wants to buy his girlfriend Jill a $3'$ World's Softest Bear from VermontTeddyBear.com for Valentine's Day. The teddy bear costs \$100. Unfortunately, Jack only has \$60.

a) Jack will bet \$1 on coin tosses until he either wins \$40 he need to buy the bear, or loses \$60 he has. Find $P_{60}(T_0 > T_{100})$.

$$p = \frac{1}{2}, \qquad q = \frac{1}{2}, \qquad \gamma_y = 1.$$

$$P_{60}(T_0 > T_{100}) = \frac{\sum_{y=0}^{60-1} \gamma_y}{\sum_{y=0}^{100-1} \gamma_y} = \frac{60}{100} = 0.60.$$

Jack has a 60% chance to get to \$100 before losing all his money.

b) Craps is a dice game in which the players make wagers on the outcome of the roll, or a series of rolls, of a pair of dice.

Pass Line bet:

- If the come-out roll is 7 or 11, the bet wins.
- If the come-out roll is 2, 3 or 12, the bet loses (known as "crapping out").
- If the roll is any other value, it establishes a *point*.
  - If, with a point established, that point is rolled again before a 7, the bet wins.
  - If, with a point established, a 7 is rolled before the point is rolled again ("seven out"), the bet loses.

$$P(\text{win}) = \left[\frac{6}{36}+\frac{2}{36}\right]+\left(\frac{3}{36}\cdot\frac{3}{9}\right)+\left(\frac{4}{36}\cdot\frac{4}{10}\right)+\left(\frac{5}{36}\cdot\frac{5}{11}\right)+\left(\frac{5}{36}\cdot\frac{5}{11}\right)+\left(\frac{4}{36}\cdot\frac{4}{10}\right)+\left(\frac{3}{36}\cdot\frac{3}{9}\right)$$

$$= \frac{244}{495} = 0.492\overline{929}.$$

Jack will bet \$1 on Pass Line bets until he either wins \$40 he need to buy the bear, or loses \$60 he has. Find $P_{60}(T_0 > T_{100})$.

$$p = \frac{244}{495}, \qquad q = \frac{251}{495}, \qquad \gamma_y = \left(\frac{251}{244}\right)^y.$$

$$P_{60}(T_0 > T_{100}) = \frac{\sum_{y=0}^{60-1}\gamma_y}{\sum_{y=0}^{100-1}\gamma_y} = \frac{\left(\frac{251}{244}\right)^{60}-1}{\left(\frac{251}{244}\right)^{100}-1} \approx 0.28.$$

Jack has a 28% chance to get to \$100 before losing all his money.

Suppose that instead of the timid strategy of betting \$1 each time, Jack uses the bold strategy of betting as much as possible but not more than necessary to bring his fortune up to \$100.

$$\begin{array}{ccccccccccc}
 & & 100 & & & & & & & & \\
 & \nearrow & & & & & 80 & \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & \begin{matrix}100\\ \\ 60\end{matrix} & & \text{try again} \\
60 & & & & & \nearrow & & & & & \\
 & \searrow & & & 40 & & & & & & \\
 & & 20 & \begin{smallmatrix}\nearrow\\ \searrow\end{smallmatrix} & & \searrow & & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

To get \$100, Jack could WIN or LOSE WIN WIN WIN or LOSE WIN WIN LOSE try again.

$$P(☺) = \frac{244}{495} + \frac{251}{495}\cdot\frac{244}{495}\cdot\frac{244}{495}\cdot\frac{244}{495} + \frac{251}{495}\cdot\frac{244}{495}\cdot\frac{244}{495}\cdot\frac{251}{495}\cdot P(☺).$$

$\Rightarrow \quad P(☺) \approx 0.590557.$!!!

c) Jack will bet \$1 on Red in European Roulette until he either wins \$40 he need to buy the bear, or loses \$60 he has. Find $P_{60}(T_0 > T_{100})$.

European Roulette – 37 slots – 18 red, 18 black, 1 green.

$$p = \frac{18}{37}, \qquad q = \frac{19}{37}, \qquad \gamma_y = \left(\frac{19}{18}\right)^y.$$

$$P_{60}(T_0 > T_{100}) = \frac{\left(\frac{19}{18}\right)^{60} - 1}{\left(\frac{19}{18}\right)^{100} - 1} \approx 0.111.$$

Jack has a 11.1% chance to get to \$100 before losing all his money.

0.581928 with the bold strategy

d) Jack will bet \$1 on Red in American Roulette until he either wins \$40 he need to buy the bear, or loses \$60 he has. Find $P_{60}(T_0 > T_{100})$.

American Roulette – 38 slots – 18 red, 18 black, 2 green.

$$p = \frac{18}{38}, \qquad q = \frac{20}{38}, \qquad \gamma_y = \left(\frac{20}{18}\right)^y.$$

$$P_{60}(T_0 > T_{100}) = \frac{\left(\frac{20}{18}\right)^{60} - 1}{\left(\frac{20}{18}\right)^{100} - 1} \approx 0.014755.$$

Jack has a 1.5% chance to get to \$100 before losing all his money.

0.564723 with the bold strategy

If $S = \{0, 1, 2, \dots, d\}$, if birth and death Markov chain is irreducible,

then it is recurrent.

Suppose $S = \{0, 1, 2, 3, \dots\}$.

$$P_1(T_0 < T_n) = 1 - \frac{1}{\sum_{y=0}^{n-1} \gamma_y}. \qquad \Rightarrow \qquad P_1(T_0 < \infty) = 1 - \frac{1}{\sum_{y=0}^{\infty} \gamma_y}.$$

$$P_0(T_0 < \infty) = P(0,0) + P(0,1) \cdot P_1(T_0 < \infty).$$

Irreducible birth and death Markov chain is recurrent if and only if $\sum_{y=0}^{\infty} \gamma_y = \infty$.

Intuition: $\sum_{y=0}^{\infty} \gamma_y < \infty \quad \Leftrightarrow$ there are a lot of "births" and not a lot of "deaths"

(we may never return to 0)

$\sum_{y=0}^{\infty} \gamma_y = \infty \quad \Leftrightarrow$ there are enough "deaths" to guarantee return to 0

Example 2:

Determine whether the following birth and death chains are recurrent or transient.

a) $P(0,0) = r_0 = 0.8, \qquad P(0,1) = p_0 = 0.2,$

$$\begin{aligned} P(x, x-1) &= q_x = 0.3, \\ P(x, x) &= r_x = 0.5, \qquad\qquad x \ge 1. \\ P(x, x+1) &= p_x = 0.2, \end{aligned}$$

$$\gamma_y = \left(\frac{3}{2}\right)^y, \qquad y \ge 1.$$

$$\sum_{y=0}^{\infty} \gamma_y = \infty.$$ The chain is recurrent.

b) $P(0,0) = r_0 = 0.7, \qquad P(0,1) = p_0 = 0.3,$

$$\begin{aligned} P(x, x-1) &= q_x = 0.2, \\ P(x, x) &= r_x = 0.5, \qquad\qquad x \ge 1. \\ P(x, x+1) &= p_x = 0.3, \end{aligned}$$

$$\gamma_y = \left(\frac{2}{3}\right)^y, \qquad y \ge 1.$$

$$\sum_{y=0}^{\infty} \gamma_y = 3 < \infty.$$ The chain is transient.

c) $P(0,0) = r_0 = 0.7, \qquad P(0,1) = p_0 = 0.3,$

$$\begin{aligned} P(x, x-1) &= q_x = 0.2, \\ P(x, x) &= r_x = 0.5, \qquad\qquad 1 \le x \le 99. \\ P(x, x+1) &= p_x = 0.3, \end{aligned}$$

$$P(100, 100) = r_{100} = 0.8, \qquad P(100, 99) = q_{100} = 0.2,$$

$S = \{0, 1, 2, \dots, 100\}.$ The chain is recurrent.

d) Consider the birth and death chain on $\{0, 1, 2, \ldots\}$ defined by $p_x = (x+2)/(2x+2)$ and $q_x = x/(2x+2)$, $x \geq 0$ (i.e., there is a little bias to the right). Any state leads to any other state, so this chain is irreducible. (~ Example 13, p. 33 HPS)

$$\gamma_0 = 1. \qquad \frac{q_x}{p_x} = \frac{x}{x+2}, \qquad x \geq 1.$$

$$\gamma_y = \frac{q_1 \cdots q_y}{p_1 \cdots p_y} = \frac{1 \cdot 2 \cdot \ldots \cdot y}{3 \cdot \ldots \cdot (y+1) \cdot (y+2)} = \frac{2}{(y+1) \cdot (y+2)}, \qquad y \geq 1.$$

$$\sum_{y=0}^{\infty} \gamma_y = \sum_{y=0}^{\infty} \frac{2}{(y+1) \cdot (y+2)} = 2 \sum_{y=0}^{\infty} \left(\frac{1}{y+1} - \frac{1}{y+2} \right) = 2 < \infty.$$

$\Rightarrow$ the Markov chain is **transient**.

(i.e., everything will eventually move to the right without return)

e) Consider the birth and death chain on $\{0, 1, 2, \ldots\}$ defined by $p_x = (x+1)/(2x+1)$ and $q_x = x/(2x+1)$, $x \geq 0$ (i.e., there is a little bias to the right). Any state leads to any other state, so this chain is irreducible.

$$\gamma_0 = 1. \qquad \frac{q_x}{p_x} = \frac{x}{x+1}, \qquad x \geq 1.$$

$$\gamma_y = \frac{q_1 \cdots q_y}{p_1 \cdots p_y} = \frac{1 \cdot 2 \cdot \ldots \cdot y}{2 \cdot 3 \cdot \ldots \cdot (y+1)} = \frac{1}{y+1}, \qquad y \geq 1.$$

$$\sum_{y=0}^{\infty} \gamma_y = \sum_{y=0}^{\infty} \frac{1}{y+1} = \infty.$$

$\Rightarrow$ the Markov chain is **recurrent**.

(i.e., any state will be visited infinitely many times)