STAT 433

Examples for 02/10/2022 (2)

Spring 2022
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If $\pi(x)$, $x \in S$, are nonnegative numbers summing to one, and if

$$\sum_{x \in S} \pi(x) \times \mathrm{P}(x, y) = \pi(y), \qquad y \in \mathrm{S},$$

then π is called a **stationary distribution**. That is, a stationary distribution is a probability vector π such that

$$\pi \mathbf{P} = \pi.$$

1. To keep the doctor away, Alex eats an apple a day (if he has an apple to eat). If/when Alex runs out of apples, he goes to a store and buys Y apples, where Y is a discrete random variable with the following probability distribution:

$$p(y) = \mathrm{P}(\mathrm{Y} = y), \quad y \geq 0, \; y - \text{integer}. \qquad \left(\sum_{y=0}^{\infty} p(y) = 1. \right)$$

Consider a Markov chain that keeps track of how many apples Alex has each day.

The state is $\{0, 1, 2, 3, \ldots\}$ (all nonnegative integers).
[OR $\{0, 1, 2, \ldots, d\}$ if Y has finitely many possible values and d is the largest possible value of Y, but we will assume that Y countably infinitely many possible values of Y, for generality.]

The transition probability function is

$$\mathrm{P}(0, y) = p(y), \quad y \geq 0, \qquad\qquad \mathrm{P}(y, y-1) = 1, \quad y \geq 1.$$

Then the chain is irreducible. Once the chain leaves 0, it moves back one step at a time, and will get back to 0 eventually.

The assumption $\mathrm{E}(\mathrm{Y}) < \infty$ makes the chain positive recurrent (we will discuss this later), and then this chain has a unique stationary distribution.

a) Find the stationary distribution of the number of apples Alex has.

b) Suppose $p(0) = 0.10$, $p(1) = 0.30$, $p(2) = 0.60$.

Find the stationary distribution of the number of apples Alex has.

2. An absent-minded professor has two umbrellas that she uses when commuting from home to office and back. If it rains and an umbrella is available in her location, she takes it. If it is not raining, she always forgets to take an umbrella. Suppose that it rains with probability p each time she commutes, independently of other times.

a) Consider a Markov chain with three states – 0, 1, 2 – the number of umbrellas available at the current location. Write a transition probability matrix.

b) Find a stationary distribution π.

c) What is the steady-state probability she gets wet on a given day?

This example is from
D.P. Bertsekas and J.N. Tsitsiklis.
Introduction to Probability, 2nd Edition, Athena Scientific, Belmont, Massachusetts, 2008.

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3. Anna needs to go to the top floor of the Ice Tower to see her sister Elsa. The Ice Tower has 99 floors. The Ice Elevator goes up or down one floor at any given time. When Anna is on the first floor, she always goes to the 2nd floor. But when there is a choice, Anna randomly decides whether she will go up or down next with equal probability. Anna starts on the first floor. How many floors, on average, must she travel to reach the 99th floor for the first time?

This example is from http://www.aquatutoring.org/ExpectedValueMarkovChains.pdf

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The state is $\{ 0, 1, 2, 3, \dots \}$ (all nonnegative integers).
[OR $\{ 0, 1, 2, \dots, d \}$ if Y has finitely many possible values and d is the largest possible value of Y, but we will assume that Y countably infinitely many possible values of Y, for generality.]

The transition probability function is

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Then the chain is irreducible. Once the chain leaves 0, it moves back one step at a time, and will get back to 0 eventually.

The assumption $\mathrm{E}(\mathrm{Y}) < \infty$ makes the chain positive recurrent (we will discuss this later), and then this chain has a unique stationary distribution.

Sneak preview:

Since this Markov chain moves towards 0 one step at a time from $y \ge 1$, it would take y steps to get to 0 from $y \ge 1$. (That is, $\mathrm{P}_y(\mathrm{T}_0 = y) = 1, \ y \ge 1.$) Then $m_{y0} = \mathrm{E}_y(\mathrm{T}_0) = y, \ y \ge 1.$

$$m_0 = \mathrm{E}_0(\mathrm{T}_0) = m_{00} = 1 + \sum_{y=1}^{\infty} \mathrm{P}(0, y) \cdot m_{y0}$$

$$= 1 + \sum_{y=1}^{\infty} p(y) \cdot y = 1 + \mathrm{E}(\mathrm{Y}).$$

Need $m_0 < \infty$ for positive recurrent. $\quad\Rightarrow\quad$ Need $\mathrm{E}(\mathrm{Y}) < \infty$.

a) Find the stationary distribution of the number of apples Alex has.

Hint 1: We have infinitely many states, we should try to find some patterns.
Consider considering $\quad \pi(y) = ☺_y \cdot \pi(0), \quad y \ge 0.$
Now you need a good "guess" for $☺_y$, and then the "classy" thing to do would be to use mathematical induction to show that your "guess" holds for all $y \ge 0$.

Hint 2: $1 = P(Y \geq 0)$, $P(Y \geq y) = p(y) + P(Y \geq y+1), \quad y \geq 0.$

Hint 3: $E(Y) = \sum_{y=1}^{\infty} P(Y \geq y).$

Hint 4: You have a constitutionally protected right not to follow hints !

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \cdots \end{array} &
\begin{bmatrix}
p(0) & p(1) & p(2) & p(3) & p(4) & \cdots \\
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\end{array}
$$

$$\pi(y) = p(y)\,\pi(0) + \pi(y+1), \quad y \geq 0.$$

Claim: $\pi(y) = P(Y \geq y)\,\pi(0), \quad y \geq 0.$

Proof: Mathematical induction:

Base: $\pi(0) = P(Y \geq 0)\,\pi(0).$

Step: Assume $\pi(y) = P(Y \geq y)\,\pi(0).$

$$\pi(y) = p(y)\,\pi(0) + \pi(y+1).$$

$$\Rightarrow \quad P(Y \geq y)\,\pi(0) = p(y)\,\pi(0) + \pi(y+1).$$

$$\Rightarrow \quad \pi(y+1) = [P(Y \geq y) - p(y)]\,\pi(0) = P(Y \geq y+1)\,\pi(0).$$

Last thing to do: find $\pi(0)$.

$$1 = \sum_{y=0}^{\infty} \pi(y) = \pi(0) + \sum_{y=1}^{\infty} \pi(y)$$

$$= \pi(0) + \sum_{y=1}^{\infty} \mathrm{P}(\mathrm{Y} \ge y)\, \pi(0)$$

$$= \pi(0)\left[1 + \mathrm{E}(\mathrm{Y})\right].$$

$$\pi(0) = \frac{1}{1 + \mathrm{E}(\mathrm{Y})}.$$

OR

$$m_0 = 1 + \mathrm{E}(\mathrm{Y}). \qquad \Rightarrow \qquad \pi(0) = \frac{1}{m_0} = \frac{1}{1 + \mathrm{E}(\mathrm{Y})}.$$

$$\Rightarrow \qquad \pi(y) = \frac{\mathrm{P}(\mathrm{Y} \ge y)}{1 + \mathrm{E}(\mathrm{Y})}, \quad y \ge 0.$$

b) Suppose $p(0) = 0.10$, $p(1) = 0.30$, $p(2) = 0.60$.

Find the stationary distribution of the number of apples Alex has.

$\mathrm{E}(\mathrm{Y}) = 0 \times 0.10 + 1 \times 0.30 + 2 \times 0.60 = 1.50.$ $\qquad 1 + \mathrm{E}(\mathrm{Y}) = 2.5.$

$\mathrm{P}(\mathrm{Y} \ge 0) = 1,$ $\qquad \mathrm{P}(\mathrm{Y} \ge 1) = 0.90,$

$\mathrm{P}(\mathrm{Y} \ge 2) = 0.60,$ $\qquad \mathrm{P}(\mathrm{Y} \ge y) = 0, \quad y \ge 3.$

$\pi(0) = \mathbf{0.40},$ $\qquad \pi(1) = \mathbf{0.36},$

$\pi(2) = \mathbf{0.24},$ $\qquad \pi(y) = 0, \quad y \ge 3.$

2. An absent-minded professor has two umbrellas that she uses when commuting from home to office and back. If it rains and an umbrella is available in her location, she takes it. If it is not raining, she always forgets to take an umbrella. Suppose that it rains with probability p each time she commutes, independently of other times.

a) Consider a Markov chain with three states – 0, 1, 2 – the number of umbrellas available at the current location. Write a transition probability matrix.

$x = 0$: Regardless of whether it is raining or not, the professor travels from the location with no umbrellas to the location with two umbrellas. $P(0, 2) = 1$.

$x = 1$: One umbrella at the current location, one umbrella at the other location.
If it is raining, the professor takes the umbrella available at the current location and travels to the other location, where there are now two umbrellas. $P(1, 2) = p$.
If it is not raining, she does not take an umbrella and travels to the other location, where there is still one umbrella. $P(1, 1) = 1 - p$.

$x = 2$: Two umbrellas at the current location, no umbrellas at the other location.
If it is raining, the professor takes an umbrella available at the current location and travels to the other location, where there is now one umbrellas. $P(2, 1) = p$.
If it is not raining, she does not take an umbrella and travels to the other location, where there are still no umbrellas. $P(2, 0) = 1 - p$.

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} \end{array}
$$

b) Find a stationary distribution π.

$$\boldsymbol{\pi} = \boldsymbol{\pi} \times \mathbf{P} \qquad \pi(0) = (1-p)\,\pi(2)$$

$$\pi(1) = (1-p)\,\pi(1) + p\,\pi(2)$$

$$\pi(2) = \pi(0) + p\,\pi(1)$$

$$\pi(1) = (1-p)\,\pi(1) + p\,\pi(2) \qquad \Rightarrow \qquad \pi(1) = \pi(2)$$

$$1 = \pi(0) + \pi(1) + \pi(2) = (1-p)\,\pi(2) + \pi(2) + \pi(2) = (3-p)\,\pi(2)$$

$$\Rightarrow \qquad \pi(1) = \pi(2) = \frac{1}{3-p}. \qquad \pi(0) = (1-p)\,\pi(2) = \frac{1-p}{3-p}.$$

c) What is the steady-state probability she gets wet on a given day?

Spoiler: This Markov chain is aperiodic.
The stationary distribution is the steady-state distribution.

$$\mathrm{P}(\text{wet}) = \pi(0)\,\mathrm{P}(\text{rain}) = \pi(0)\,p = \frac{(1-p)\,p}{3-p}.$$

IF $p = 1$, it is always raining. The professor would always carry an umbrella with her and never get wet (after possibly getting wet once on her very first trip if both umbrellas were at the other location).

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Another way to look at this:

$$⓪ \underset{1-p}{\overset{1}{\rightleftarrows}} ② \underset{p}{\overset{p}{\rightleftarrows}} ①$$

$$\pi(0) = (1-p)\,\pi(2) \qquad\qquad p\,\pi(1) = p\,\pi(2)$$

$$1 = \pi(0) + \pi(1) + \pi(2) = (1-p)\,\pi(2) + \pi(2) + \pi(2) = (3-p)\,\pi(2)$$

$$\Rightarrow \qquad \pi(1) = \pi(2) = \frac{1}{3-p} = \frac{1}{2+(1-p)}. \qquad \pi(0) = \frac{1-p}{3-p} = \frac{1-p}{2+(1-p)}.$$

4 umbrellas:

$$⓪ \underset{1-p}{\overset{1}{\rightleftarrows}} ④ \underset{p}{\overset{p}{\rightleftarrows}} ① \underset{1-p}{\overset{1-p}{\rightleftarrows}} ③ \underset{p}{\overset{p}{\rightleftarrows}} ②$$

$$\Rightarrow \qquad \pi(0) = (1-p)\,\pi(4), \qquad\qquad \pi(4) = \pi(1) = \pi(3) = \pi(2).$$

$$1 = \pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4).$$

$$\Rightarrow \qquad \pi(1) = \pi(2) = \pi(3) = \pi(4) = \frac{1}{4+(1-p)}. \qquad \pi(0) = \frac{1-p}{4+(1-p)}.$$

N umbrellas:

$$\pi(1) = \pi(2) = \ldots = \pi(\mathrm{N}-1) = \pi(\mathrm{N}) = \frac{1}{\mathrm{N}+(1-p)}. \qquad \pi(0) = \frac{1-p}{\mathrm{N}+(1-p)}.$$

$$\mathrm{P}(\text{wet}) = \pi(0)\,\mathrm{P}(\text{rain}) = \pi(0)\,p = \frac{(1-p)\,p}{\mathrm{N}+(1-p)}.$$

Need 25 umbrellas to ensure at most 1% chance of getting wet.

3. Anna needs to go to the top floor of the Ice Tower to see her sister Elsa. The Ice Tower has 99 floors. The Ice Elevator goes up or down one floor at any given time. When Anna is on the first floor, she always goes to the 2nd floor. But when there is a choice, Anna randomly decides whether she will go up or down next with equal probability. Anna starts on the first floor. How many floors, on average, must she travel to reach the 99th floor for the first time?

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Obviously, 99 is an arbitrarily chosen number, large enough so no one would try dealing with 98×98 matrix Q, so you should look for patterns. Once you have a good guess for a pattern, mathematical induction is a cute way to prove your guess is correct (make your good guess "politically correct").

So you would solve this problem for N (general) floors, and then just plug 99 in for N.

N – 1 transient states: 1, 2, 3, … , N – 2, N – 1.

1 absorbing state: N.

	1	2	3	4	…	N – 3	N – 2	N – 1	N
1	0	1	0	0	…	0	0	0	0
2	0.5	0	0.5	0	…	0	0	0	0
3	0	0.5	0	0.5	…	0	0	0	0
…	…	…	…	…	…	…	…	…	…
N – 2	0	0	0	0	…	0.5	0	0.5	0
N – 1	0	0	0	0	…	0	0.5	0	0.5
N	0	0	0	0	…	0	0	0	1

```
> numberofsteps = function(n) {
+ ImQ = diag(n-1)
+ ImQ[1,2] = -1
+ for (i in 2:(n-2)) {
+     ImQ[i,i-1] = -0.5
+     ImQ[i,i+1] = -0.5
+ }
+ ImQ[n-1,n-2] = -0.5
+ F = solve(ImQ)
+ sum(F[1,])
+ }
>
> ### This function does not work for fewer than 4 floors.
> ### I need n to be at least 4 for (i in 2:(n-2)) to make
> ### sense.
>
> numberofsteps(4)
[1] 9
> numberofsteps(5)
[1] 16
> numberofsteps(6)
[1] 25
> numberofsteps(7)
[1] 36
> numberofsteps(8)
[1] 49
> numberofsteps(9)
[1] 64
> numberofsteps(10)
[1] 81
> numberofsteps(11)
[1] 100
> numberofsteps(12)
[1] 121
> numberofsteps(13)
[1] 144
> numberofsteps(99)
[1] 9604
```

I now have the answer... And I think I see a pattern... On to the proof!

$a_1 = 1 + a_2$ (1)

$a_2 = 1 + 0.5\, a_1 + 0.5\, a_3$ (2)

$a_3 = 1 + 0.5\, a_2 + 0.5\, a_4$ (3)

• • •

$a_k = 1 + 0.5\, a_{k-1} + 0.5\, a_{k+1}$ (k)

• • •

$a_{N-2} = 1 + 0.5\, a_{N-3} + 0.5\, a_{N-1}$ (N – 2)

$a_{N-1} = 1 + 0.5\, a_{N-2}$ (N – 1)

Claim: $a_{k+1} = a_1 - k^2,$ $0 \le k \le N-2.$

Mathematical Induction:

Base: 0, 1 $a_1 = a_1 - 0,$ (1) $\Rightarrow$ $a_2 = a_1 - 1.$

Step: $k-1, k \Rightarrow k+1$ (k) $\Rightarrow$ $a_k = 1 + 0.5\, a_{k-1} + 0.5\, a_{k+1}$

$$\Rightarrow \quad a_1 - (k-1)^2 = 1 + 0.5\, a_1 - 0.5\,(k-2)^2 + 0.5\, a_{k+1}$$

$$\Rightarrow \quad a_{k+1} = a_1 - 2\,(k-1)^2 + (k-2)^2 - 2$$

$$= a_1 - 2\,k^2 + 4\,k - 2 + k^2 - 4\,k + 4 - 2$$

$$= a_1 - k^2.$$

(N – 1) $a_{N-1} = 1 + 0.5\, a_{N-2}$

$$\Rightarrow \quad a_1 - (N-2)^2 = 1 + 0.5\, a_1 - 0.5\,(N-3)^2$$

$$\Rightarrow \quad a_1 = 2 + 2\,(N-2)^2 - (N-3)^2 = 2 + 2\,N^2 - 8\,N + 8 - N^2 + 6\,N - 9$$

$$= N^2 - 2\,N + 1 = (N-1)^2.$$

$a_1 = \mathbf{(N-1)^2}.$ $(99-1)^2 = 9{,}604.$