Examples for 03/01/2022

0. Let $T_1, T_2, \dots, T_k$ be independent Exponential random variables.

Suppose $E(T_i) = \dfrac{1}{\lambda_i}$, $i = 1, 2, \dots, k$.

That is, $f_{T_i}(t) = \lambda_i e^{-\lambda_i t}$, $t > 0$, $i = 1, 2, \dots, k$.

Denote $T_{\min} = \min(T_1, T_2, \dots, T_k)$.

a) Show that $T_{\min}$ also has an Exponential distribution. What is the mean of $T_{\min}$?

Hint: Consider $P(T_{\min} > t) = P(T_1 > t \text{ AND } T_2 > t \text{ AND } \dots \text{ AND } T_k > t)$.

Since $T_1, T_2, \dots, T_k$ are independent,

$$\begin{aligned} P(T_{\min} > t) &= P(T_1 > t \text{ AND } T_2 > t \text{ AND } \dots \text{ AND } T_k > t) \\ &= P(T_1 > t) \times P(T_2 > t) \times \dots \times P(T_k > t) \\ &= e^{-\lambda_1 t} \times e^{-\lambda_2 t} \times \dots \times e^{-\lambda_k t} \\ &= e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_k) t}, \qquad t > 0. \end{aligned}$$

$$F_{T_{\min}}(t) = P(T_{\min} \le t) = 1 - P(T_{\min} > t) = 1 - e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_k) t}, \qquad t > 0.$$

$$f_{T_{\min}}(t) = (\lambda_1 + \lambda_2 + \dots + \lambda_k)\, e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_k) t}, \qquad t > 0.$$

$\Rightarrow$ $T_{\min}$ has an Exponential distribution with mean $\dfrac{1}{\lambda_1 + \lambda_2 + \dots + \lambda_k}$.

b) Find $P(T_1 = T_{\min}) = P(T_1 \text{ is the smallest of } T_1, T_2, \dots, T_k)$

$= P(T_1 < T_2 \text{ AND } \dots \text{ AND } T_1 < T_k)$.

Since $T_1, T_2, \dots, T_k$ are independent, their joint probability density function is

$$f(t_1, t_2, \dots, t_k) = \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \dots \lambda_k e^{-\lambda_k t_k},$$

$$t_1 > 0, \quad t_2 > 0, \dots, \quad t_k > 0.$$

$$P(T_1 = T_{\min}) = P(T_1 < T_2 \text{ AND } \dots \text{ AND } T_1 < T_k)$$

$$= \int_0^\infty \left(\int_{t_1}^\infty \dots \int_{t_1}^\infty \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \dots \lambda_k e^{-\lambda_k t_k} \, dt_2 \dots dt_k \right) dt_1$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} \left(\int_{t_1}^\infty \lambda_2 e^{-\lambda_2 t_2} \, dt_2 \right) \dots \left(\int_{t_1}^\infty \lambda_k e^{-\lambda_k t_k} \, dt_k \right) dt_1$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 t_1} e^{-\lambda_2 t_1} \dots e^{-\lambda_k t_1} \, dt_1$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \dots + \lambda_k}.$$

$\{T_1 > t\}, \{T_2 > t\}, \dots, \{T_k > t\}$ are independent

(t is a (non-random) real-valued variable).

$\{T_1 < T_2\}, \dots, \{T_1 < T_k\}$ are **NOT** independent, since they all involve T_1.

For Exponential distribution, $\text{mean} = \dfrac{1}{\text{rate}}$, $\text{rate} = \dfrac{1}{\text{mean}}$, $\theta = \dfrac{1}{\lambda}$, $\lambda = \dfrac{1}{\theta}$.

1. An office has two clerks. Three people, Alex, Bob, and Carl, enter simultaneously. Alex and Bob begin service at the two clerks, while Carl waits for the first available clerk. Assume that the service times for Alex, Bob, and Carl are independent and have Exponential distributions with rates λ_A, λ_B, and λ_C, respectively.

a) What is the probability that __________ is done first?

i) Alex

$$P(\text{Alex is done first}) = P(\text{Alex is done before Bob}) = P(T_A < T_B) = \frac{\lambda_A}{\lambda_A + \lambda_B}.$$

ii) Bob

$$P(\text{Bob is done first}) = P(\text{Bob is done before Alex}) = P(T_A > T_B) = \frac{\lambda_B}{\lambda_A + \lambda_B}.$$

iii) Carl

$P(\text{Carl is done first}) = 0,$ since Carl waits for the first available clerk, and either Alex or Bob has to be done with his service before Carl can start his.

b) What is the expected time before __________ is done?

i) Alex

Alex begins service right away. $E(T_A) = \dfrac{1}{\lambda_A}.$

ii) Bob

Bob begins service right away. $E(T_B) = \dfrac{1}{\lambda_B}$.

iii) Carl (i.e., Carl's combined waiting and service time)?

First, Carl waits for either Alex or Bob to be done. Let $T_{AB} = \min(T_A, T_B)$. After that, his service time is T_C. Carl's combined waiting and service time is $T_{AB} + T_C$.

$$E(T_{AB}) + E(T_C) = \frac{1}{\lambda_A + \lambda_B} + \frac{1}{\lambda_C}.$$

c) What is the probability that …

i) Alex is done before Bob?

$$P(\text{Alex is done before Bob}) = P(T_A < T_B) = \frac{\lambda_A}{\lambda_A + \lambda_B}.$$

ii) Carl is done before Alex?

First, Bob is done before Alex. Then Carl is done before Alex.

$$\begin{aligned} P(\text{Carl is done before Alex}) &= P(T_B < T_A) \times P(T_C < T_A) \\ &= \frac{\lambda_B}{\lambda_A + \lambda_B} \cdot \frac{\lambda_C}{\lambda_A + \lambda_C}. \end{aligned}$$

Note that even though Alex has already been served for some time when Bob is done, it does not affect the second probability because of the memoryless property of the Exponential distribution:

memoryless property: $\qquad P(T > t + s \mid T > t) = P(T > s), \qquad t, s > 0.$

$$P(T > t + s \mid T > t) = \frac{P(\,T > t + s \;\cap\; T > t\,)}{P(\,T > t\,)} = \frac{P(\,T > t + s\,)}{P(\,T > t\,)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(T > s).$$

That is, an Exponential waiting time could "restart" itself at any time t, independently from its past.

iii) Bob is done before Carl?

Bob is done before Alex OR Alex is done before Bob, then Bob is done before Carl

$$P(\text{Bob is done before Carl}) = P(T_B < T_A) + P(T_B > T_A) \times P(T_B < T_C)$$

$$= \frac{\lambda_B}{\lambda_A + \lambda_B} + \frac{\lambda_A}{\lambda_A + \lambda_B} \cdot \frac{\lambda_B}{\lambda_B + \lambda_C}.$$

Again, even though Bob has already been served for some time when Alex is done, it does not affect the second probability in the second term. When Alex is done, Bob's service time "restarts" itself.

2. Three mice, Alex, Bob, and Carl, live in a house. The owner of the house sets a mousetrap with a piece of sharp cheddar. All three mice will independently try to go for the cheese, but they need to develop some courage first. Suppose that mouse Alex will go for the cheese after an exponentially distributed time with mean 120 minutes, mouse Bob will do that after an exponential time with mean 60 minutes, and mouse Carl will do that after an exponentially distributed time with mean 40 minutes.

rates: $\lambda_A = \frac{1}{120}$, $\lambda_B = \frac{1}{60}$, $\lambda_C = \frac{1}{40}$.

a) What is the expected time until someone goes for the cheese?

$T_{ABC} = \min(T_A, T_B, T_C) \sim$ Exponential with rate $\lambda_{ABC} = \lambda_A + \lambda_B + \lambda_C$.

$$E(T_{ABC}) = \frac{1}{\lambda_A + \lambda_B + \lambda_C} = \frac{1}{\frac{1}{120} + \frac{1}{60} + \frac{1}{40}} = \mathbf{20} \text{ minutes.}$$

b) What is the probability that mouse Alex will be the first one to go for the cheese?

$$P(T_A = T_{ABC}) = \frac{\lambda_A}{\lambda_A + \lambda_B + \lambda_C} = \frac{\frac{1}{120}}{\frac{1}{120} + \frac{1}{60} + \frac{1}{40}} = \mathbf{\frac{1}{6}}.$$

OR

$T_{BC} = \min(T_B, T_C) \sim$ Exponential with rate $\lambda_{BC} = \lambda_B + \lambda_C = \frac{1}{60} + \frac{1}{40} = \frac{5}{120}$.

$$P(T_A < T_{BC}) = \frac{\lambda_A}{\lambda_A + \lambda_{BC}} = \frac{\frac{1}{120}}{\frac{1}{120} + \frac{5}{120}} = \mathbf{\frac{1}{6}}.$$

c) The first mouse that goes for the cheese will get trapped. The second mouse will be able to get the cheese without any danger. What is the probability that mouse Alex will get the cheese?

Bob is first and is trapped, then Alex beats Carl

OR Carl is first and is trapped, then Alex beats Bob

$$\frac{\lambda_B}{\lambda_A+\lambda_B+\lambda_C} \times \frac{\lambda_A}{\lambda_A+\lambda_C} + \frac{\lambda_C}{\lambda_A+\lambda_B+\lambda_C} \times \frac{\lambda_A}{\lambda_A+\lambda_B}$$

$$= \frac{\frac{1}{60}}{\frac{1}{120}+\frac{1}{60}+\frac{1}{40}} \times \frac{\frac{1}{120}}{\frac{1}{120}+\frac{1}{40}} + \frac{\frac{1}{40}}{\frac{1}{120}+\frac{1}{60}+\frac{1}{40}} \times \frac{\frac{1}{120}}{\frac{1}{120}+\frac{1}{60}}$$

$$= \frac{1}{3} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{3} = \mathbf{\frac{1}{4}} = \mathbf{0.25}.$$

OR

$T_1 \sim$ Exponential with rate λ_1, $T_2 \sim$ Exponential with rate λ_2, $T_3 \sim$ Exponential with rate λ_3,

T_1, T_2, T_3 are independent.

$$P(T_1 < T_2 < T_3) = \int_0^\infty \left(\int_x^\infty \left(\int_y^\infty \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \lambda_3 e^{-\lambda_3 z} \, dz \right) dy \right) dx$$

$$= \int_0^\infty \left(\int_x^\infty \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} e^{-\lambda_3 y} \, dy \right) dx$$

$$= \int_0^\infty \lambda_1 e^{-\lambda_1 x} \frac{\lambda_2}{\lambda_2+\lambda_3} e^{-\lambda_2 x} e^{-\lambda_3 x} \, dx$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3}.$$

Note that $P(T_1 < T_2 < T_3) \neq P(T_1 < T_2) \cdot P(T_2 < T_3)$,

since $\{T_1 < T_2\}$ and $\{T_2 < T_3\}$ are NOT independent, they all involve T_2.

$$P(T_B < T_A < T_C) + P(T_C < T_A < T_B)$$

$$= \frac{\lambda_B}{\lambda_A + \lambda_B + \lambda_C} \cdot \frac{\lambda_A}{\lambda_A + \lambda_C} + \frac{\lambda_C}{\lambda_A + \lambda_B + \lambda_C} \cdot \frac{\lambda_A}{\lambda_A + \lambda_B}$$

$$= \frac{\frac{1}{60}}{\frac{1}{120} + \frac{1}{60} + \frac{1}{40}} \cdot \frac{\frac{1}{120}}{\frac{1}{120} + \frac{1}{40}} + \frac{\frac{1}{40}}{\frac{1}{120} + \frac{1}{60} + \frac{1}{40}} \cdot \frac{\frac{1}{120}}{\frac{1}{120} + \frac{1}{60}}$$

$$= \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} = \mathbf{\frac{1}{4}} = \mathbf{0.25}.$$

(No animals were harmed in making of this problem.)

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Time until the next jump τ:

Exponential with rate q_x

$$f_x(t) = q_x e^{-q_x t}, \quad t > 0.$$

Transition probabilities:

$$Q_{xy} = P_x(X(\tau) = y), \quad x, y \in S, \quad x \neq y.$$

$$Q_{xx} = 0. \qquad \sum_{y \in S} Q_{xy} = 1, \quad x \in S.$$

Transition functions: $\quad P_{xy}(t) = P_x(X(t) = y), \quad t \geq 0, \quad x, y \in S.$

$$\sum_{y \in S} P_{xy}(t) = 1, \quad t \geq 0, \quad x \in S. \qquad P_{xy}(0) = \delta_{xy}, \quad x, y \in S.$$

Chapman-Kolmogorov equation:

$$P_{xy}(t+s) = \sum_{z \in S} P_{xz}(t)\, P_{zy}(s), \quad s, t \geq 0, \quad x, y \in S.$$