Examples for 02/17/2022 (2)

Period d_x = greatest common divisor $\{ n \ge 1 : \mathrm{P}^n(x, x) > 0 \}$.

The states in an irreducible Markov chain have a common period d.

The chain is **periodic with period d** if $d > 1$.

The chain is **aperiodic** if $d = 1$.

Theorem *Let X_n, $n \ge 0$, be an irreducible positive recurrent Markov chain having stationary distribution π. If the chain is aperiodic,*

$$\lim_{n \to \infty} P^n(x, y) = \pi(y), \qquad x, y \in \mathcal{S}.$$

If the chain is periodic with period d, then for each pair x, y of states in $\mathcal{S}$ there is an integer r, $0 \le r < d$, such that $P^n(x, y) = 0$ unless $n = md + r$ for some nonnegative integer m, and

$$\lim_{m \to \infty} P^{md+r}(x, y) = d\pi(y).$$

Example 1: Winter weather in Central Illinois.

$$\begin{array}{c} \\ \mathrm{N} \\ \mathrm{R} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{N} & \mathrm{R} & \mathrm{S} \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

Possible path:

N $\to$ R $\to$ S $\to$ N

$\Rightarrow$ The chain is irreducible.

$\mathrm{P}(\mathrm{N}, \mathrm{N}) = \mathrm{P}^1(\mathrm{N}, \mathrm{N}) = 0$

N $\to$ R $\to$ N

$\mathrm{P}^2(\mathrm{N}, \mathrm{N}) > 0$

N $\to$ R $\to$ S $\to$ N

$\mathrm{P}^3(\mathrm{N}, \mathrm{N}) > 0$

The greatest common divisor of 2 and 3 is 1. $\Rightarrow$ $d_{\mathrm{N}} = 1$.

The chain is irreducible. $\Rightarrow$ $d = \mathbf{1}$.

OR $\mathrm{P}(\mathrm{R}, \mathrm{R}) = \mathrm{P}^1(\mathrm{R}, \mathrm{R}) > 0.$ $\Rightarrow$ $d_{\mathrm{R}} = 1.$ $\Rightarrow$ $d = \mathbf{1}$.

Example 2: Consider a Markov chain on $\{0, 1\}$ having transition probability matrix

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \end{array} \qquad \left[\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right] \qquad ⓪ \;\overset{\rightarrow}{\leftarrow}\; ①$$

$0 \rightarrow 1 \rightarrow 0$ 2 steps

$P^{2m}(0, 0) = 1, \qquad P^{2m+1}(0, 0) = 0.$

$P^{2m}(1, 1) = 1, \qquad P^{2m+1}(1, 1) = 0. \qquad d = \mathbf{2}.$

$\pi(0) = \pi(1) \qquad \pi(1) = \pi(0)$

$\pi(0) + \pi(1) = 1$

$\Rightarrow \quad \pi(0) = \mathbf{0.50}, \qquad \pi(1) = \mathbf{0.50}.$

$P^{2m}(0, 0) = 1 = 2 \times 0.50 = d \times \pi(0),$ $\qquad r = 0$

$P^{2m+1}(0, 0) = 0.$

$P^{2m}(0, 1) = 0,$

$P^{2m+1}(0, 1) = 1 = 2 \times 0.50 = d \times \pi(1).$ $\qquad r = 1$

$P^{2m}(1, 1) = 1 = 2 \times 0.50 = d \times \pi(1),$ $\qquad r = 0$

$P^{2m+1}(1, 1) = 0.$

$P^{2m}(1, 0) = 0,$

$P^{2m+1}(1, 0) = 1 = 2 \times 0.50 = d \times \pi(0).$ $\qquad r = 1$

Example 3: HPS 2.22

22 Consider a Markov chain on $\{0, 1, 2\}$ having transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}.$$

(a) Show that the chain is irreducible.
(b) Find the period.
(c) Find the stationary distribution.

a) A possible path: $0 \to 2 \to 1 \to 0$

$\Rightarrow$ One can reach any state from any state. The chain is irreducible.

b) $0 \to 2 \to 1 \to 0$ $\quad$ $0 \to 2 \to 0$

$P^3(0,0) > 0$ $\quad$ $P^2(0,0) > 0$ $\quad \Rightarrow \quad d_0 = 1$

The chain is irreducible. $\quad \Rightarrow \quad d_0 = d_1 = d_2 = \mathbf{1}$

The chain is aperiodic.

c) $\pi(0) = \pi(1) + 0.5\,\pi(2)$

$\pi(1) = 0.5\,\pi(2)$

$\pi(2) = \pi(0)$

$\pi(0) + \pi(1) + \pi(2) = 1$

$\Rightarrow \quad \pi(0) = \mathbf{0.40}, \qquad \pi(1) = \mathbf{0.20}, \qquad \pi(2) = \mathbf{0.40}.$

$\pi(0) = 0.4 = \dfrac{1}{m_0} \quad \Rightarrow \quad m_0 = E_0(T_0) = 2.5.$

$\pi(1) = 0.2 = \dfrac{1}{m_1} \quad \Rightarrow \quad m_1 = E_1(T_1) = 5.$

Example 4: HPS 2.23

23 Consider a Markov chain on $\{0, 1, 2, 3, 4\}$ having transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

(a) Show that the chain is irreducible.
(b) Find the period.
(c) Find the stationary distribution.

a) A possible path: $0 \to 1 \to 3 \to 0 \to 2 \to 4 \to 0$

$\Rightarrow$ One can reach any state from any state. The chain is irreducible.

b) $0 \to (1 \text{ or } 2) \to (3 \text{ or } 4) \to 0$ $\qquad d = \mathbf{3}$

c) $\pi(0) = \pi(3) + \pi(4)$

$\pi(1) = \frac{1}{3}\pi(0)$

$\pi(2) = \frac{2}{3}\pi(0)$

$\pi(3) = \frac{1}{4}\pi(1) + \frac{1}{4}\pi(2)$

$\pi(4) = \frac{3}{4}\pi(1) + \frac{3}{4}\pi(2)$

$\pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4) = 1$

$\Rightarrow$ $\pi(0) = \pi(3) + \pi(4) = \pi(1) + \pi(2)$

$\Rightarrow$ $\pi(0) = \mathbf{\frac{1}{3}}, \qquad \pi(1) = \mathbf{\frac{1}{9}}, \qquad \pi(2) = \mathbf{\frac{2}{9}},$

$\pi(3) = \mathbf{\frac{1}{12}}, \qquad \pi(4) = \mathbf{\frac{1}{4}}.$

$\lim\limits_{n\to\infty} \mathrm{P}^{n}\left(0,\ 0\right)$ does NOT exist.

$\mathrm{P}^{3m}\left(0,\ 0\right) = 1 = 3 \times \frac{1}{3} = d \times \pi(0).$ $\qquad r = 0$

$\mathrm{P}^{3m+1}\left(0,\ 0\right) = 0.$ $\qquad \mathrm{P}^{3m+2}\left(0,\ 0\right) = 0.$

$\lim\limits_{n\to\infty} \mathrm{P}^{n}\left(0,\ 1\right)$ does NOT exist.

$\mathrm{P}^{3m+1}\left(0,\ 1\right) = \frac{1}{3} = 3 \times \frac{1}{9} = d \times \pi(1).$ $\qquad r = 1$

$\mathrm{P}^{3m}\left(0,\ 1\right) = 0.$ $\qquad \mathrm{P}^{3m+2}\left(0,\ 1\right) = 0.$

$\lim\limits_{n\to\infty} \mathrm{P}^{n}\left(1,\ 0\right)$ does NOT exist.

$\mathrm{P}^{3m}\left(1,\ 0\right) = 0.$ $\qquad \mathrm{P}^{3m+1}\left(1,\ 0\right) = 0.$

$\mathrm{P}^{3m+2}\left(1,\ 0\right) = 1 = 3 \times \frac{1}{3} = d \times \pi(0).$ $\qquad r = 2$

$\lim\limits_{n\to\infty} \mathrm{P}^{n}\left(1,\ 1\right)$ does NOT exist.

$\mathrm{P}^{3m}\left(1,\ 1\right) = \frac{1}{3} = 3 \times \frac{1}{9} = d \times \pi(1),$ since $\mathrm{P}\left(0,\ 1\right) = \frac{1}{3}.$ $\qquad r = 0$

$\mathrm{P}^{3m+1}\left(1,\ 1\right) = 0.$ $\qquad \mathrm{P}^{3m+2}\left(1,\ 1\right) = 0.$

$\pi(0) = \frac{1}{3} = \frac{1}{m_0} \quad \Rightarrow \quad m_0 = \mathrm{E}_0(\mathrm{T}_0) = 3.$ $\qquad \left(\mathrm{P}_0(\mathrm{T}_0 = 3) = 1\right)$

$\pi(1) = \frac{1}{9} = \frac{1}{m_1} \quad \Rightarrow \quad m_1 = \mathrm{E}_1(\mathrm{T}_1) = 9.$

$\pi(2) = \frac{2}{9} = \frac{1}{m_2} \quad \Rightarrow \quad m_2 = \mathrm{E}_2(\mathrm{T}_2) = 4.5.$

$\pi(3) = \frac{1}{12} = \frac{1}{m_3} \quad \Rightarrow \quad m_3 = \mathrm{E}_3(\mathrm{T}_3) = 12.$

$\pi(4) = \frac{1}{4} = \frac{1}{m_4} \quad \Rightarrow \quad m_4 = \mathrm{E}_4(\mathrm{T}_4) = 4.$