STAT 433

Spring 2022
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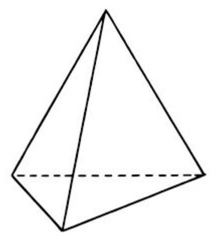
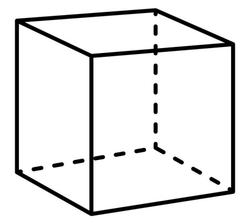
Examples for 02/17/2022 (3)

Example 0:

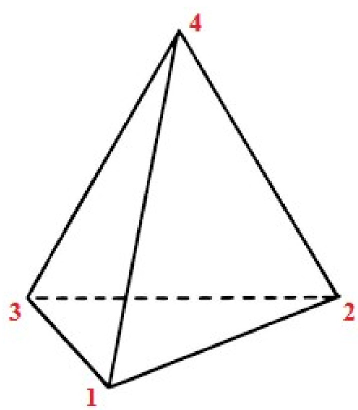
Consider a Markov chain with the state space being the vertices of a polyhedron that moves from one vertex to a different vertex along an edge with equal probability for the available edges, independently of the past. Then this Markov chain is irreducible. Find the period of this chain if …

a) the polyhedron is a tetrahedron; b) the polyhedron is a cube.





a)

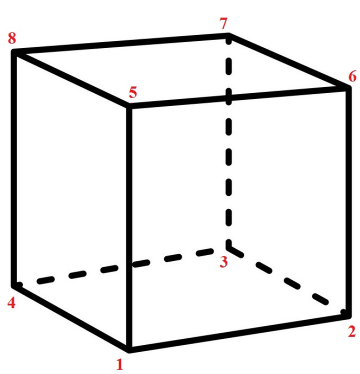


$1 \to 2 \to 1$ 2 steps

$1 \to 2 \to 3 \to 1$ 3 steps

$\Rightarrow$ period = **1**.

b)



$\big($ 1 or 3 or 6 or 8 $\big) \leftrightarrows \big($ 2 or 4 or 5 or 7 $\big)$.

On each step, the chain switches from Group 1 to Group 2 or from Group 2 to Group 1.

$\Rightarrow$ period = **2**.

Example 1:

Consider a Markov chain with the following transition probability matrix:

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
2 & 0 & 1 & 0 & 0 \\
3 & 0.6 & 0 & 0.4 & 0
\end{array}
$$

a) Determine the period for every state.

$1 \rightarrow 3 \rightarrow (0 \text{ or } 2) \rightarrow 1$

The chain is irreducible, all states have common period. $d = \mathbf{3}$.

b) Find a stationary distribution if it exists.

$\pi(0) = 0.6\,\pi(3).$

$\pi(1) = \pi(0) + \pi(2).$

$\pi(2) = 0.4\,\pi(3).$

$\pi(3) = \pi(1).$

$\pi(0) + \pi(1) + \pi(2) + \pi(3) = 1.$

$\Rightarrow \quad \pi(0) = \mathbf{\frac{3}{15}}, \qquad \pi(1) = \mathbf{\frac{1}{3}}, \qquad \pi(2) = \mathbf{\frac{2}{15}}, \qquad \pi(3) = \mathbf{\frac{1}{3}}.$

c) Find m_2, the expected number of steps needed to return to state 2.

$\pi(2) = \frac{1}{m_2} = \frac{2}{15}. \qquad \Rightarrow \qquad m_2 = \mathbf{7.5}.$

d) Find $\lim\limits_{n\to\infty} \mathrm{P}^{n}\,(0, y),\ \ y = 0, 1, 2, 3,$ if these limits exists.

If these limits do not exist, find the value of r (r depends on y) for which

$\lim\limits_{m\to\infty} \mathrm{P}^{m\,d+r}\,(0, y) > 0,\ \ y = 0, 1, 2, 3.$

Since the chain is periodic with period $d = 3$,

$\lim\limits_{n\to\infty} \mathrm{P}^{n}\,(0, y),\ \ y = 0, 1, 2, 3,$ do NOT exist.

For example, $\mathrm{P}^{n}(0, 1) = 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots$.

$y = 0$	$r = \mathbf{0}$	$\lim\limits_{m\to\infty} \mathrm{P}^{m\,d}\,(0, 0) = 0.6 = 3 \cdot \frac{3}{15} = d \cdot \pi(0).$
$y = 1$	$r = \mathbf{1}$	$\lim\limits_{m\to\infty} \mathrm{P}^{m\,d+1}\,(0, 1) = 1 = 3 \cdot \frac{1}{3} = d \cdot \pi(1).$
$y = 2$	$r = \mathbf{0}$	$\lim\limits_{m\to\infty} \mathrm{P}^{m\,d}\,(0, 2) = 0.4 = 3 \cdot \frac{2}{15} = d \cdot \pi(2).$
$y = 3$	$r = \mathbf{2}$	$\lim\limits_{m\to\infty} \mathrm{P}^{m\,d+2}\,(0, 3) = 1 = 3 \cdot \frac{1}{3} = d \cdot \pi(3).$

e) Find P^{d}.

$$\mathrm{P}^{3} = \begin{bmatrix} 0.6 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 2:

Consider a Markov chain with the following transition probability matrix:

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
2 & 0 & 0.8 & 0.2 & 0 \\
3 & 0.6 & 0 & 0.4 & 0
\end{array}
$$

a) Determine the period for every state.

$P(2,2) > 0 \qquad d_2 = 1.$

The chain is irreducible, all states have common period. $\Rightarrow \quad d = \mathbf{1}.$

b) Find a stationary distribution if it exists.

$\pi(0) = 0.6\,\pi(3).$

$\pi(1) = \pi(0) + 0.8\,\pi(2).$

$\pi(2) = 0.2\,\pi(2) + 0.4\,\pi(3).$

$\pi(3) = \pi(1).$

$\pi(0) + \pi(1) + \pi(2) + \pi(3) = 1.$

$\Rightarrow \quad \pi(0) = \mathbf{\frac{6}{31}}, \qquad \pi(1) = \mathbf{\frac{10}{31}}, \qquad \pi(2) = \mathbf{\frac{5}{31}}, \qquad \pi(3) = \mathbf{\frac{10}{31}}.$

c) Find m_2, the expected number of steps needed to return to state 2.

$\pi(2) = \frac{1}{m_2} = \frac{5}{31}. \qquad \Rightarrow \qquad m_2 = \mathbf{6.2}.$

d) Find $\lim\limits_{n\to\infty} \mathrm{P}^{n}\left(0, y \right)$, $y = 0, 1, 2, 3,$ if these limits exists.

If these limits do not exist, find the value of r (r depends on y) for which

$\lim\limits_{m\to\infty} \mathrm{P}^{m\,d+r}\left(0, y \right) > 0, \quad y = 0, 1, 2, 3.$

The chain is aperiodic.

$$\lim_{n\to\infty} \mathrm{P}^{n}\left(0, y \right) = \pi(y), \qquad y \in \mathrm{S}.$$

$y = 0$ $$\lim_{m\to\infty} \mathrm{P}^{n}\left(0, 0 \right) = \pi(0) = \frac{\mathbf{6}}{\mathbf{31}}.$$

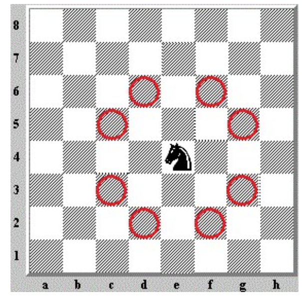
$y = 1$ $$\lim_{m\to\infty} \mathrm{P}^{n}\left(0, 1 \right) = \pi(1) = \frac{\mathbf{10}}{\mathbf{31}}.$$

$y = 2$ $$\lim_{m\to\infty} \mathrm{P}^{n}\left(0, 2 \right) = \pi(2) = \frac{\mathbf{5}}{\mathbf{31}}.$$

$y = 3$ $$\lim_{m\to\infty} \mathrm{P}^{n}\left(0, 3 \right) = \pi(3) = \frac{\mathbf{10}}{\mathbf{31}}.$$

Example 3:

A knight moves on a chess board, selecting the next spot at random, with equal probability from those available, independently of the past. This Markov chain has 64 states, and it is irreducible since you can find a path that takes the knight from any square to any square on the chess board. On each step, the knight moves to a square of the opposite color. You can return to the starting square only in even number of steps. Hence the period is 2. The chain is not aperiodic.



Stationary distribution:

I am not enthusiastic about trying to solve a system of 64 equations with 64 unknowns, even though there is a lot of symmetry here. It helps if we can make an educated guess.

Since the next spot is chosen at random with equal probability from those available, maybe $\pi(x)$ is proportional to the number $\mathrm{N}(x)$ of possible "neighbors" (not neighbors on the chess board, but possible next states from square x). That is, maybe $\pi(x) = C\,\mathrm{N}(x)$ for some C.

Need to show that $\displaystyle\sum_{x \in S} \pi(x) \times \mathrm{P}(x,y) = \pi(y), \quad y \in S.$

We have $\mathrm{P}(x,y) = \begin{cases} \dfrac{1}{\mathrm{N}(x)}, & \text{if } y \text{ is a "neighbor" of } x \\ 0, & \text{otherwise.} \end{cases}$

$$\text{Then} \quad \sum_{x \in S} \pi(x) \times \mathrm{P}(x,y) = \sum_{\substack{x \in S \\ y \text{ is a "neighbor" of } x}} C\,\mathrm{N}(x) \times \frac{1}{\mathrm{N}(x)}$$

$$= \sum_{\substack{x \in S \\ y \text{ is a "neighbor" of } x}} C = C\,\mathrm{N}(y) = \pi(y), \quad y \in S,$$

since y is a "neighbor" of x if and only if x is a "neighbor" of y.

To find C, we must have $\displaystyle\sum_{x \in S} \pi(x) = 1.$

The number of possible "neighbors" for each square of the chess board:

8	2	3	4	4	4	4	3	2
7	3	4	6	6	6	6	4	3
6	4	6	8	8	8	8	6	4
5	4	6	8	8	8	8	6	4
4	4	6	8	8	8	8	6	4
3	4	6	8	8	8	8	6	4
2	3	4	6	6	6	6	4	3
1	2	3	4	4	4	4	3	2
	a	b	c	d	e	f	g	h

2 "neighbors" – 4 out of 64,
3 "neighbors" – 8 out of 64,
4 "neighbors" – 20 out of 64,
6 "neighbors" – 16 out of 64,
8 "neighbors" – 16 out of 64.

$$2 \cdot 4 + 3 \cdot 8 + 4 \cdot 20 + 6 \cdot 16 + 8 \cdot 16 = 336. \qquad C = \frac{1}{336}.$$

$$\pi(\mathrm{a1}) = \frac{2}{336},\ \pi(\mathrm{b1}) = \frac{3}{336},\ \pi(\mathrm{c1}) = \frac{4}{336},\ \ldots,\ \pi(\mathrm{c2}) = \frac{6}{336},\ \ldots,\ \pi(\mathrm{c3}) = \frac{8}{336},\ \ldots .$$