STAT 433

Examples for 02/08/2022 (1)

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A. Stepanov

Branching chain:

Consider particles (e.g., bacteria) that can generate new particles of the same type. The initial set of particles is referred to as the 0th generation. Particles generated from the nth generation are said to belong to the $(n+1)$th generation.

Let X_n, $n \geq 0$, denote the number of particles in the nth generation.

Suppose that each particle (parent) gives rise to Y particles (children) in the next generation, where Y is a nonnegative integer-valued random variable with probability mass function

$$f(y), \quad y = 0, 1, 2, \ldots .$$

X_n, $n \geq 0$, is a Markov chain whose state space is the nonnegative integers.

0 is an absorbing state. (no parents $\Rightarrow$ no children)

If we have x particles-parents in the nth generation, and particle-parent $\# i$ has Y_i children (independently of the others), then the total number of children is $Y_1 + \ldots + Y_x$.

For $x \geq 1$, $$P(x, y) = P(Y_1 + \ldots + Y_x = y),$$

where $Y_1, \ldots, Y_x$ are independent random variables having common probability mass function f.

The extinction probability ρ of the chain is the probability that the descendants of a given particle eventually become extinct.

$$\rho = \rho_{10} = P_1(T_0 < \infty).$$

If $x \geq 1$ particles are present initially, then

$$\rho_{x0} = \rho^x, \qquad x = 1, 2, 3, \ldots ,$$

since particles produce children independently of the others.

If $f(1) = 1$, then every state is absorbing.

Suppose $f(1) < 1$.

0 is an absorbing state, all other states are transient. With probability 1, the branching chain is either absorbed at 0 or approaches $+\infty$.

$$\mathrm{P}_x\left(\lim_{n\to\infty} \mathrm{X}_n = \infty \right) = 1 - \rho^x, \qquad x = 1, 2, 3, \dots .$$

ρ satisfies $\qquad \Phi(\rho) = \rho, \qquad$ where

$$\Phi(t) = f(0) + \sum_{y=1}^{\infty} f(y) \cdot t^y, \qquad 0 \le t \le 1.$$

Let $\mu = \mu_{\mathrm{Y}}$ denote the expected number of offspring (children) of any given particle.

If $\mu \le 1$, then $\Phi(t) = t$ has no roots in $[0, 1)$.

$\Rightarrow \qquad \rho = 1.$

$\Rightarrow$ Ultimate extinction is certain.

If $\mu > 1$, then $\Phi(t) = t$ has a unique root ρ_0 in $[0, 1)$.

$\Rightarrow \qquad \rho = \rho_0.$

$\Rightarrow$ The probability of ultimate extinction is less than one.

For a nonnegative integer-valued random variable Y with probability mass function

$$f(y), \quad y = 0, 1, 2, 3, \dots,$$

$$\Phi(t) = \mathrm{E}(t^{\mathrm{Y}}) = \sum_{y=0}^{\infty} f(y) \cdot t^y$$

is the **probability generating function**.

$\Phi(1) = 1.$ $\Phi'(1) = E(Y).$

$\Phi(0) = f(0).$ $\Phi^{(y)}(0) = y! \cdot f(y),$ $y = 0, 1, 2, 3, \ldots .$

$\Phi(e^t) = M_Y(t)$ – moment-generating function of Y.

<u>Example 1</u>:

Suppose each individual in a branching process can have only 0, 1, or 2 offspring, with respective probabilities q, r, and p. Determine whether extinction is certain if

a) $q < p$ b) $q = p$ c) $q > p$

In any case in which extinction is not certain, find the probability that it will occur.

$\mu = 0 \times q + 1 \times r + 2 \times p = 1 + (p - q).$

a) $q < p$ $\Rightarrow$ $\mu > 1$ $\Rightarrow$ extinction is NOT certain.

$\rho = \rho^0 \times q + \rho^1 \times r + \rho^2 \times p.$

$\rho = q + r\rho + p\rho^2.$

$q - q\rho - p\rho + p\rho^2 = 0.$

$(\rho - 1)(p\rho - q) = 0.$

$\rho = \dfrac{q}{p} < 1.$

b) $q = p$ $\Rightarrow$ $\mu = 1$

$\Rightarrow$ extinction IS certain.

c) $q > p$ $\Rightarrow$ $\mu < 1$

$\Rightarrow$ extinction IS certain.

Example 2:

Every man has exactly two children, which independently have probability p of being a boy and $1-p$ of being a girl. The number of males in the nth generation forms a branching chain.

$$\Phi(t) = f(0) + \sum_{y=1}^{\infty} f(y)\cdot t^{y} = (1-p)^2 + 2p(1-p)t + p^2 t^2.$$

$t = (1-p)^2 + 2p(1-p)t + p^2 t^2$ has two solutions: 1 and $\dfrac{(1-p)^2}{p^2}$.

If $p \le \dfrac{1}{2}$, then $\mu = 2p \le 1$ and $\dfrac{(1-p)^2}{p^2} \ge 1$.

$\Rightarrow$ $\rho = 1$.

$\Rightarrow$ Ultimate extinction of the male line is certain.

If $p > \dfrac{1}{2}$, then $\mu = 2p > 1$ and $\dfrac{(1-p)^2}{p^2} < 1$.

$\Rightarrow$ $\rho = \dfrac{(1-p)^2}{p^2}$.

Three children and $p = \dfrac{1}{2}$ – Example 14 (page 35) in HPS $\rho = \sqrt{5} - 2 \approx 0.236$.

Example 3:

Four children and $p = \dfrac{1}{2}$. $\mu = 2$.

$$\rho = \frac{1}{16} + \frac{4}{16}\rho + \frac{6}{16}\rho^2 + \frac{4}{16}\rho^3 + \frac{1}{16}\rho^4.$$

$$\rho^4 + 4\rho^3 + 6\rho^2 - 12\rho + 1 = 0.$$

$$(\rho - 1)(\rho^3 + 5\rho^2 + 11\rho - 1) = 0.$$

$\Rightarrow$ $\rho \approx 0.087378$.

Example 4:

Before "Reply All" button became the horror of electronic mail, there were chain e-mail messages that promised good luck if you forward them to others and bad luck if you do not. Suppose that all recipients of such chain e-mail message independently forward it to a random number of individuals that follows the following probability distribution:

$$f(0) = 0.15, \qquad f(1) = 0.30, \qquad f(2) = 0.35, \qquad f(3) = 0.20.$$

Find the probability that this chain e-mail message would die out on its own.

$$\mu = 0 \times 0.15 + 1 \times 0.30 + 2 \times 0.35 + 3 \times 0.20 = 1.6 > 1.$$

Extinction is NOT certain.

$$\rho = 0.15 + 0.30\,\rho + 0.35\,\rho^2 + 0.20\,\rho^3$$

$$0 = 0.15 - 0.70\,\rho + 0.35\,\rho^2 + 0.20\,\rho^3$$

$$4\,\rho^3 + 7\,\rho^2 - 14\,\rho + 3 = 0$$

$$(\rho - 1)(4\,\rho^2 + 11\,\rho - 3) = 0$$

$$\rho = 1 \qquad \text{or} \qquad \rho = \frac{-11 \pm \sqrt{121 + 48}}{8} = \frac{-11 \pm 13}{8} = -3 \quad \text{or} \quad \frac{1}{4}.$$

Need $0 \le \rho < 1$. $\qquad \rho = \mathbf{\frac{1}{4}}.$

Example 5:

Suppose Phroggs reproduce by parthenogenesis. An adult Phrogg lays $N \gg 1$ eggs, and then dies. Due to predators preying on Phroggs' tadpoles, each egg, independently of others, eventually develops to an adult Phrogg with probability $\varepsilon \ll 1$. The number of offspring is distributed binomially with parameters N and ε. Let us denote the expected value of mature offspring as $\mu = \varepsilon N$. As N is large and ε is small, the distribution (in the limit $N \to \infty$, with $\mu = \varepsilon N$ being kept fixed) becomes the Poisson distribution with parameter μ, i.e.,

$$f(y) = \frac{\mu^y \cdot e^{-\mu}}{y\,!}, \qquad y = 0, 1, 2, 3, 4, \ldots .$$

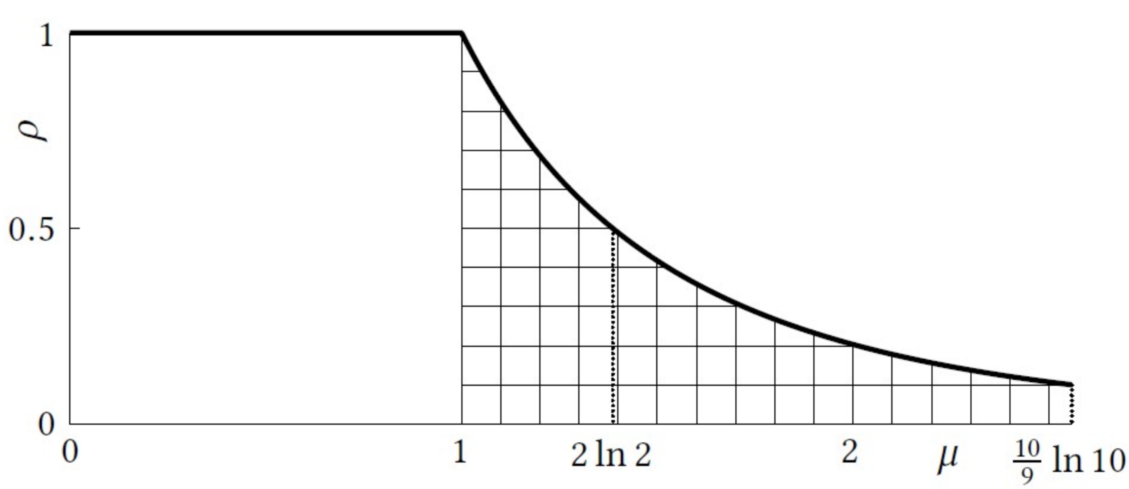
Graph the probability of extinction ρ as a function of μ.

Hint: It might be easier to find (and graph) μ as a function of ρ.

$$\Phi(t) = f(0) + \sum_{y=0}^{\infty} f(y) \cdot t^y = \sum_{y=0}^{\infty} \frac{\mu^y \cdot e^{-\mu}}{y\,!} \cdot t^y = e^{\mu t - \mu}.$$

$$\Phi(\rho) = \rho \qquad \Rightarrow \qquad e^{\mu\rho - \mu} = \rho.$$

$$\Rightarrow \qquad \mu = \frac{-\ln\rho}{(1-\rho)} \qquad \text{for } 0 < \rho < 1 \;\Leftrightarrow\; \mu > 1.$$



Example 6 from Examples for 01/20/2022 (1):

Suppose the number of Padawan Apprentices that a Jedi Knight would train over lifetime has the following probability distribution:

$$f(0) = 0.30, \quad f(1) = 0.25, \quad f(2) = 0.20, \quad f(3) = 0.15, \quad f(4) = 0.10.$$

(A Padawan Apprentice becomes a Jedi Knight when the training is complete.)

Suppose also that Jedi Knights train their Padawan Apprentices independently.

How many Jedi Knights are needed initially for the probability of the Jedi extinction to be less than 0.1% ?

$$\mu = 0 \times 0.30 + 1 \times 0.25 + 2 \times 0.20 + 3 \times 0.15 + 4 \times 0.10 = 1.5 > 1.$$

Extinction is NOT certain.

$$\rho = 0.30 + 0.25\,\rho + 0.20\,\rho^2 + 0.15\,\rho^3 + 0.10\,\rho^4.$$

$$0 = 6 - 15\,\rho + 4\,\rho^2 + 3\,\rho^3 + 2\,\rho^4.$$

$$(\rho - 1)(2\,\rho^3 + 5\,\rho^2 + 9\,\rho - 6) = 0.$$

$$(\rho - 1)(2\,\rho - 1)(\rho^2 + 3\,\rho + 6) = 0.$$

$$\rho = 1,\ \frac{1}{2},\ \frac{-3 \pm \sqrt{9-24}}{2} = \frac{-3 \pm i\sqrt{15}}{2}. \qquad \Rightarrow \qquad \rho = \frac{1}{2}.$$

$$\rho^n = \frac{1}{2^n} \le 0.001 = \frac{1}{1{,}000}. \qquad \Rightarrow \qquad n \ge \mathbf{10}.$$