STAT 433

Examples for 01/20/2022 (2)

Spring 2022
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Let $A \subset S$. The *hitting time* T_A of A is defined by $\quad T_A = \min\{\, n > 0 : X_n \in A \,\}$.

T_A is the first (positive) time the Markov chain hits A.

$T_A = \infty$ if $X_n \notin A$ for all $n > 0$ (if the Markov chain never hits A).

Let $a \in S$. Let $T_a = T_{\{a\}}$.

Notation: $\quad P_x(\ldots) = P(\ldots \mid X_0 = x)$.

$$P_x(X_n = y) = P^n(x, y) = \sum_{m=1}^{n} P_x\left(T_y = m\right) \times P^{n-m}(y, y), \qquad n \geq 1.$$

(To go from x to y in n steps, it takes m steps to hit y for the first time, and then $n - m$ steps to go from y to y.)

$\Rightarrow$ IF y is absorbing, then $\quad P_x(X_n = y) = P_x(T_y \leq n)$.

(Since we cannot leave y, if we hit y for the first time in n steps or before that, then we will be in y after n steps.)

$$P_x(T_y = 1) = P(X_1 = y \mid X_0 = x) = P(x, y).$$

$$P_x(T_y = 2) = \sum_{z \neq y} P(x, z) \times P(z, y).$$

$$P_x(T_y = n+1) = \sum_{z \neq y} P(x, z) \times P_z\left(T_y = n\right), \qquad n \geq 1.$$

(To hit y from x for the first time in n +1 steps, move from x to $z \neq y$, and then hit y from z for the first time in n steps.)

For states i and j, let $m_{ij} = \mathrm{E}_i(\mathrm{T}_j)$ denote the expected number of steps it takes a Markov chain starting in state i to visit state j for the first time. Then

$$m_{ij} = \mathrm{P}(i,j)\cdot 1 + \sum_{k \neq j} \mathrm{P}(i,k)\cdot\left(1+m_{kj}\right) = 1 + \sum_{k \neq j} \mathrm{P}(i,k)\cdot m_{kj},$$

$$\uparrow \quad \text{since} \quad \mathrm{P}(i,j) + \sum_{k \neq j} \mathrm{P}(i,k) = 1.$$

Intuition:

$$m_{ij} = \mathrm{P}(i,j)\cdot 1 + \sum_{k \neq j} \mathrm{P}(i,k)\cdot\left(1+m_{kj}\right).$$

Starting from state i, on the first step,
we go to state j with probability $\mathrm{P}(i,j)$ (and then our journey is over in 1 step)
OR
we go to some other state k (different from j) with probability $\mathrm{P}(i,k)$, and then (after 1 step) it would take additional m_{kj} steps (on average) to get to state j for the first time, now starting from state k [and our total journey will be $(1 + m_{kj})$ steps (on average)].

$$m_{ij} = 1 + \sum_{k \neq j} \mathrm{P}(i,k)\cdot m_{kj}.$$

Starting from state i, we take the first step into the unknown, the number of steps is $1 + ???$.
If we are in state j, our journey is over, we are done, nothing else needs to be added.
If we are in some other state k (different from j), which would happen with probability $\mathrm{P}(i,k)$, then [the objective is still to get to j] additional m_{kj} steps (on average) will be required to get to state j for the first time, now starting from state k.

<u>Example</u>: Winter weather in Central Illinois.

$$\begin{array}{c c} & \begin{array}{ccc} \mathrm{N} & \mathrm{R} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{N} \\ \mathrm{R} \\ \mathrm{S} \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$