Examples for 02/08/2022 (2)

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If $\pi(x)$, $x \in S$, are nonnegative numbers summing to one, and if

$$\sum_{x \in S} \pi(x) \times \mathrm{P}(x, y) = \pi(y), \qquad y \in \mathrm{S},$$

then π is called a **stationary distribution**. That is, a stationary distribution is a probability vector π such that

$$\pi\,\mathbf{P} = \pi.$$

If a stationary distribution π exists and $\lim\limits_{n \to \infty} \mathrm{P}^{n}(x, y) = \pi(y), \quad y \in \mathrm{S},$
then regardless of the initial distribution of the Markov chain, the distribution of X_n approaches π as $n \to \infty$. In such cases, π is sometimes called **steady state** distribution.

1. Winter weather in Central Illinois.

$$\begin{array}{c} \\ \mathrm{N} \\ \mathrm{R} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{N} & \mathrm{R} & \mathrm{S} \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

Find a stationary distribution π.

$\pi\,\mathbf{P} = \pi.$

$$\pi(\mathrm{N}) = \frac{1}{4}\pi(\mathrm{R}) + \frac{1}{3}\pi(\mathrm{S})$$

$$\pi(\mathrm{R}) = \frac{1}{2}\pi(\mathrm{N}) + \frac{1}{2}\pi(\mathrm{R}) + \frac{1}{3}\pi(\mathrm{S})$$

$$\pi(\mathrm{S}) = \frac{1}{2}\pi(\mathrm{N}) + \frac{1}{4}\pi(\mathrm{R}) + \frac{1}{3}\pi(\mathrm{S})$$

$$1 = \pi(\mathrm{N}) + \pi(\mathrm{R}) + \pi(\mathrm{S})$$

$$\Rightarrow \quad \pi(\mathrm{N}) = \mathbf{\frac{2}{9}}, \quad \pi(\mathrm{R}) = \mathbf{\frac{4}{9}}, \quad \pi(\mathrm{S}) = \mathbf{\frac{3}{9}}.$$

```
> P = rbind( c(0,1/2,1/2), c(1/4,1/2,1/4), c(1/3,1/3,1/3) )
>
> P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P%*%P
          [,1]      [,2]      [,3]
[1,] 0.2222222 0.4444444 0.3333333
[2,] 0.2222222 0.4444444 0.3333333
[3,] 0.2222222 0.4444444 0.3333333
>
> pi00 = c( 2/9, 4/9, 3/9 )
> t(pi00) %*% P
          [,1]      [,2]      [,3]
[1,] 0.2222222 0.4444444 0.3333333
```

eigen(A)

finds λ and $\boldsymbol{v}$ such that $\mathbf{A}\,\boldsymbol{v} = \lambda\,\boldsymbol{v}$.

We need $\pi = \pi\,\mathbf{P}$. $\Rightarrow$ $\mathbf{P}^{\mathrm{T}}\,\pi^{\mathrm{T}} = 1 \cdot \pi^{\mathrm{T}}$.

$\Rightarrow$ we need **eigen(t(P))** and $\lambda = 1$.

```
> eigen(t(P))
$values
[1]  1.0000000 -0.3038126  0.1371459

$vectors
           [,1]       [,2]       [,3]
[1,] -0.3713907 -0.8051731  0.1355099
[2,] -0.7427814  0.2852315 -0.7650553
[3,] -0.5570860  0.5199416  0.6295454
```

We need a probability vector.

```
> eigen(t(P))$vectors[,1]
[1] -0.3713907 -0.7427814 -0.5570860
```

is NOT a probability vector.

```
> eigen(t(P))$vectors[,1]/sum(eigen(t(P))$vectors[,1])
[1] 0.2222222 0.4444444 0.3333333
```