STAT 433 Spring 2022
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Examples for 03/24/2022

Stationary distribution: $$\sum_x \pi(x) q_{xy} = 0, \quad y \in S.$$

1. Suppose the time spent in Anytown Emergency Room is exponentially distributed with mean 4 hours (rate $q_{\text{E}} = 6$). Suppose also that $1/3$ of the ER patients are admitted to the Anytown Hospital, and $2/3$ are discharged. The time spent in the Hospital is exponentially distributed with mean 2 days (rate $q_{\text{H}} = 0.5$). People in Anytown are extremely accident-prone; the time until a person goes to the Emergency Room is exponentially distributed with mean 20 days (rate $q_{\text{D}} = 0.05$). Consider a Markov pure jump process $\mathrm{X}(t)$ with three states $\{$ E(mergency Room), H(ospital), D(ischarged) $\}$. Find the long-term distribution of $\mathrm{X}(t)$.

$$\begin{tikzcd} \mathrm{E} \arrow[rr, shift right] \arrow[dr] & & \mathrm{D} \arrow[ll, shift right] \\ & \mathrm{H} \arrow[ur] & \end{tikzcd}$$

2. Sue's sewing machine is very old, and it malfunctions often. When a machine fails, it needs either a small repair (which happens with probability 0.75) or a large repair (probability 0.25). If the machine needs a small repair, the time of the repair is exponentially distributed with mean 3 minutes (rate = 20). If the machine needs a large repair, the time of the repair is exponential with mean 6 minutes (rate = 10). After a repair, the machine works for an exponentially distributed time with mean 15 minutes (rate = 4). Assume that all times are independent. Consider a Markov pure jump process $X(t)$ with three states $\{$ W(orking), S(mall repair), L(arge repair) $\}$. Find the long-term distribution of $X(t)$.

$q_{W} = 4$	$q_{S} = 20$	$q_{L} = 10$
$Q_{WS} = 0.75, \quad Q_{WL} = 0.25$	$Q_{SW} = 1$	$Q_{LW} = 1$

$q_{WW} = -4$	$q_{WS} = 3$	$q_{WL} = 1$
$q_{SW} = 20$	$q_{SS} = -20$	$q_{SL} = 0$
$q_{LW} = 10$	$q_{LS} = 0$	$q_{LL} = -10$

3. The Department of Statistics has two photocopy machines. The time to breakdown for each machine has an exponential distribution with parameter λ. The time to repair for each machine has an exponential distribution with parameter μ. (The two machines could be undergoing repairs at the same time.) Assume that all times to breakdown and all times to repair are independent. For each machine, let 1 denote the *working* condition, and 0 denote the *broken* condition. Then the status of both machines can be described using 4 states, i.e.,

$$0 = (0, 0) \qquad 1 = (1, 0) \qquad 2 = (0, 1) \qquad 3 = (1, 1).$$

Let $X(t)$ denote the conditions of both machines at time t.

a) Identify all infinitesimal parameters of $X(t)$.

b) Find the stationary distribution for $X(t)$.

4. The Department of Statistics has three printers. Each printer breaks down independently at rate μ, then it is sent to the repair shop. The repair shop can only repair one printer at a time and each printer takes an exponential amount of time with parameter λ to repair. Let $X(t)$ denote the number of working printers.

a) Identify all infinitesimal parameters of $X(t)$.

b) Find the stationary distribution for $X(t)$.

Birth and death process:

$$\pi_0 = 1, \qquad \pi_x = \frac{\lambda_0 \dots \lambda_{x-1}}{\mu_1 \dots \mu_x}, \quad x \ge 1. \qquad\qquad \pi(x) = \frac{\pi_x}{\sum_y \pi_y}, \quad x \ge 0.$$

Birth and death process:

$$\gamma_0 = 1, \qquad \gamma_x = \frac{\mu_1 \dots \mu_x}{\lambda_1 \dots \lambda_x}, \quad x \geq 1.$$

$$\sum_{x=0}^{\infty} \gamma_x = \infty. \qquad \sum_{x=0}^{\infty} \pi_x < \infty. \qquad \Rightarrow \qquad \text{Positive recurrent.}$$

$$\sum_{x=0}^{\infty} \gamma_x = \infty. \qquad \sum_{x=0}^{\infty} \pi_x = \infty. \qquad \Rightarrow \qquad \text{Null recurrent.}$$

$$\sum_{x=0}^{\infty} \gamma_x < \infty. \qquad \sum_{x=0}^{\infty} \pi_x = \infty. \qquad \Rightarrow \qquad \text{Transient.}$$

5. HPS 3.16

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(a) $\lambda_x = x + 1$, $x \geq 0$;
(b) $\lambda_x = x + 2$, $x \geq 0$.