Brownian Motion and Stationary Processes

10.1. Brownian Motion

Let us start by considering the symmetric random walk, which in each time unit is equally likely to take a unit step either to the left or to the right. That is, it is a Markov chain with $P_{i,i+1} = \frac{1}{2} = P_{i,i-1}$, $i = 0, \pm 1, \ldots$. Now suppose that we speed up this process by taking smaller and smaller steps in smaller and smaller time intervals. If we now go to the limit in the right manner what we obtain is Brownian motion.

More precisely, suppose that each Δt time unit we take a step of size Δx either to the left or the right with equal probabilities. If we let $X(t)$ denote the position at time t then

$$X(t) = \Delta x (X_1 + \cdots + X_{[t/\Delta t]}) \tag{10.1}$$

where

$$X_i = \begin{cases} +1, & \text{if the } i\text{th step of length } \Delta x \text{ is to the right} \\ -1, & \text{if it is to the left} \end{cases}$$

and $[t/\Delta t]$ is the largest integer less than or equal to $t/\Delta t$, and where the X_i are assumed independent with

$$P\{X_i = 1\} = P\{X_i = -1\} = \tfrac{1}{2}$$

As $E[X_i] = 0$, $\mathrm{Var}(X_i) = E[X_i^2] = 1$, we see from Equation (10.1) that

$$\begin{aligned} E[X(t)] &= 0, \\ \mathrm{Var}(X(t)) &= (\Delta x)^2 \left[\frac{t}{\Delta t}\right] \end{aligned} \tag{10.2}$$

We shall now let Δx and Δt go to 0. However, we must do it in a way such that the resulting limiting process is nontrivial (for instance, if we let $\Delta x = \Delta t$ and let $\Delta t \to 0$, then from the preceding we see that $E[X(t)]$ and $\mathrm{Var}(X(t))$ would both converge to 0 and thus $X(t)$ would equal 0 with probability 1). If we let $\Delta x = \sigma\sqrt{\Delta t}$ for some positive constant σ then from Equation (10.2) we see that as $\Delta t \to 0$

$$E[X(t)] = 0,$$
$$\mathrm{Var}(X(t)) \to \sigma^2 t$$

We now list some intuitive properties of this limiting process obtained by taking $\Delta x = \sigma\sqrt{\Delta t}$ and then letting $\Delta t \to 0$. From Equation (10.1) and the central limit theorem the following seems reasonable:

(i) $X(t)$ is normal with mean 0 and variance $\sigma^2 t$. In addition, because the changes of value of the random walk in nonoverlapping time intervals are independent, we have

(ii) $\{X(t), t \geqslant 0\}$ has independent increments, in that for all $t_1 < t_2 < \cdots < t_n$

$$X(t_n) - X(t_{n-1}), X(t_{n-1}) - X(t_{n-2}), \ldots, X(t_2) - X(t_1), X(t_1)$$

are independent. Finally, because the distribution of the change in position of the random walk over any time interval depends only on the length of that interval, it would appear that

(iii) $\{X(t),\ t \geqslant 0\}$ has stationary increments, in that the distribution of $X(t+s) - X(t)$ does not depend on t. We are now ready for the following formal definition.

Definition 10.1 A stochastic process $\{X(t),\ t \geqslant 0\}$ is said to be a *Brownian motion* process if

(i) $X(0) = 0$;
(ii) $\{X(t), t \geqslant 0\}$ has stationary and independent increments;
(iii) for every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

The Brownian motion process, sometimes called the Wiener process, is one of the most useful stochastic processes in applied probability theory. It originated in physics as a description of Brownian motion. This phenomenon, named after the English botanist Robert Brown who discovered it, is the motion exhibited by a small particle which is totally immersed in a liquid or gas. Since then, the process has been used beneficially in such areas as statistical testing of goodness of fit, analyzing the price levels on the stock market, and quantum mechanics.

The first explanation of the phenomenon of Brownian motion was given by Einstein in 1905. He showed that Brownian motion could be explained by assuming that the immersed particle was continually being subjected to bombardment by the molecules of the surrounding medium. However, the preceding concise definition of this stochastic process underlying Brownian motion was given by Wiener in a series of papers originating in 1918.

When $\sigma = 1$, the process is called *standard Brownian motion*. Because any Brownian motion can be converted to the standard process by letting $B(t) = X(t)/\sigma$ we shall, unless otherwise stated, suppose throughout this chapter that $\sigma = 1$.

The interpretation of Brownian motion as the limit of the random walks [Equation (10.1)] suggests that $X(t)$ should be a continuous function of t. This turns out to be the case, and it may be proven that, with probability 1, $X(t)$ is indeed a continuous function of t. This fact is quite deep, and no proof shall be attempted.

As $X(t)$ is normal with mean 0 and variance t, its density function is given by

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

To obtain the joint density function of $X(t_1), X(t_2), \dots, X(t_n)$ for $t_1 < \cdots < t_n$, note first that the set of equalities

$$\begin{aligned} X(t_1) &= x_1, \\ X(t_2) &= x_2, \\ &\vdots \\ X(t_n) &= x_n \end{aligned}$$

is equivalent to

$$\begin{aligned} X(t_1) &= x_1, \\ X(t_2) - X(t_1) &= x_2 - x_1, \\ &\vdots \\ X(t_n) - X(t_{n-1}) &= x_n - x_{n-1} \end{aligned}$$

However, by the independent increment assumption it follows that $X(t_1)$, $X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$, are independent and, by the stationary increment assumption, that $X(t_k) - X(t_{k-1})$ is normal with mean 0 and variance $t_k - t_{k-1}$. Hence, the joint density of $X(t_1), \ldots, X(t_n)$ is given by

$$f(x_1, x_2, \ldots, x_n) = f_{t_1}(x_1) f_{t_2-t_1}(x_2 - x_1) \cdots f_{t_n-t_{n-1}}(x_n - x_{n-1})$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right]\right\}}{(2\pi)^{n/2}[t_1(t_2 - t_1)\cdots(t_n - t_{n-1})]^{1/2}} \tag{10.3}$$

From this equation, we can compute in principle any desired probabilities. For instance, suppose we require the conditional distribution of $X(s)$ given that $X(t) = B$ where $s < t$. The conditional density is

$$\begin{aligned}
f_{s|t}(x|B) &= \frac{f_s(x) f_{t-s}(B - x)}{f_t(B)} \\
&= K_1 \exp\{-x^2/2s - (B - x)^2/2(t - s)\} \\
&= K_2 \exp\left\{-x^2\left(\frac{1}{2s} + \frac{1}{2(t - s)}\right) + \frac{Bx}{t - s}\right\} \\
&= K_2 \exp\left\{-\frac{t}{2s(t - s)}\left(x^2 - 2\frac{sB}{t}x\right)\right\} \\
&= K_3 \exp\left\{-\frac{(x - Bs/t)^2}{2s(t - s)/t}\right\}
\end{aligned}$$

where K_1, K_2, and K_3 do not depend on x. Hence, we see from the preceding that the conditional distribution of $X(s)$ given that $X(t) = B$ is, for $s < t$, normal with mean and variance given by

$$\begin{aligned}
E[X(s)|X(t) = B] &= \frac{s}{t}B, \\
\mathrm{Var}[X(s)|X(t) = B] &= \frac{s}{t}(t - s)
\end{aligned} \tag{10.4}$$

Example 10.1 In a bicycle race between two competitors, let $Y(t)$ denote the amount of time (in seconds) by which the racer that started in the inside position is ahead when $100t$ percent of the race has been completed, and suppose that $\{Y(t),\ 0 \leqslant t \leqslant 1\}$ can be effectively modeled as a Brownian motion process with variance parameter σ^2.

(a) If the inside racer is leading by σ seconds at the midpoint of the race, what is the probability that she is the winner?
(b) If the inside racer wins the race by a margin of σ seconds, what is the probability that she was ahead at the midpoint?

Solution:

(a)
$$
\begin{aligned}
&P\{Y(1) > 0 | Y(1/2) = \sigma\}\\
&\quad = P\{Y(1) - Y(1/2) > -\sigma | Y(1/2) = \sigma\}\\
&\quad = P\{Y(1) - Y(1/2) > -\sigma\} \qquad \text{by independent increments}\\
&\quad = P\{Y(1/2) > -\sigma\} \qquad \text{by stationary increments}\\
&\quad = P\left\{\frac{Y(1/2)}{\sigma/\sqrt{2}} > -\sqrt{2}\right\}\\
&\quad = \Phi(\sqrt{2})\\
&\quad \approx 0.9213
\end{aligned}
$$

where $\Phi(x) = P\{N(0,1) \leqslant x\}$ is the standard normal distribution function.
(b) Because we must compute $P\{Y(1/2) > 0 | Y(1) = \sigma\}$, let us first determine the conditional distribution of $Y(s)$ given that $Y(t) = C$, when $s < t$. Now, since $\{X(t),\ t \geqslant 0\}$ is standard Brownian motion when $X(t) = Y(t)/\sigma$, we obtain from Equation (10.4) that the conditional distribution of $X(s)$, given that $X(t) = C/\sigma$, is normal with mean $sC/t\sigma$ and variance $s(t-s)/t$. Hence, the conditional distribution of $Y(s) = \sigma X(s)$ given that $Y(t) = C$ is normal with mean sC/t and variance $\sigma^2 s(t-s)/t$. Hence,

$$
\begin{aligned}
P\{Y(1/2) > 0 | Y(1) = \sigma\} &= P\{N(\sigma/2, \sigma^2/4) > 0\}\\
&= \Phi(1)\\
&\approx 0.8413 \quad \blacksquare
\end{aligned}
$$

10.2. Hitting Times, Maximum Variable, and the Gambler's Ruin Problem

Let T_a denote the first time the Brownian motion process hits a. When $a > 0$ we will compute $P\{T_a \leqslant t\}$ by considering $P\{X(t) \geqslant a\}$ and conditioning on whether or not $T_a \leqslant t$. This gives

$$
\begin{aligned}
P\{X(t) \geqslant a\} = {}& P\{X(t) \geqslant a | T_a \leqslant t\} P\{T_a \leqslant t\}\\
&+ P\{X(t) \geqslant a | T_a > t\} P\{T_a > t\} \qquad (10.5)
\end{aligned}
$$

Now if $T_a \leqslant t$, then the process hits a at some point in $[0, t]$ and, by symmetry, it is just as likely to be above a or below a at time t. That is

$$
P\{X(t) \geqslant a | T_a \leqslant t\} = \tfrac{1}{2}
$$

As the second right-hand term of Equation (10.5) is clearly equal to 0 (since, by continuity, the process value cannot be greater than a without having yet hit a), we see that

$$\begin{aligned} P\{T_a \leqslant t\} &= 2P\{X(t) \geqslant a\} \\ &= \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t}\, dx \\ &= \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-y^2/2}\, dy, \qquad a > 0 \end{aligned} \tag{10.6}$$

For $a < 0$, the distribution of T_a is, by symmetry, the same as that of T_{-a}. Hence, from Equation (10.6) we obtain

$$P\{T_a \leqslant t\} = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^\infty e^{-y^2/2}\, dy \tag{10.7}$$

Another random variable of interest is the maximum value the process attains in $[0, t]$. Its distribution is obtained as follows: For $a > 0$

$$\begin{aligned} P\left\{\max_{0 \leqslant s \leqslant t} X(s) \geqslant a\right\} &= P\{T_a \leqslant t\} \qquad \text{by continuity} \\ &= 2P\{X(t) \geqslant a\} \qquad \text{from (10.6)} \\ &= \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-y^2/2}\, dy \end{aligned}$$

Let us now consider the probability that Brownian motion hits A before $-B$ where $A > 0$, $B > 0$. To compute this we shall make use of the interpretation of Brownian motion as being a limit of the symmetric random walk. To start let us recall from the results of the gambler's ruin problem (see Section 4.5.1) that the probability that the symmetric random walk goes up A before going down B when each step is equally likely to be either up or down a distance Δx is [by Equation (4.14) with $N = (A + B)/\Delta x$, $i = B/\Delta x$] equal to $B\Delta x/(A + B)\Delta x = B/(A + B)$.

Hence, upon letting $\Delta x \to 0$, we see that

$$P\{\text{up } A \text{ before down } B\} = \frac{B}{A + B}$$

10.3. Variations on Brownian Motion

10.3.1. Brownian Motion with Drift

We say that $\{X(t),\ t \geqslant 0\}$ is a Brownian motion process with drift coefficient μ and variance parameter σ^2 if

(i) $X(0) = 0$;
(ii) $\{X(t),\ t \geqslant 0\}$ has stationary and independent increments;
(iii) $X(t)$ is normally distributed with mean μt and variance $t\sigma^2$.

An equivalent definition is to let $\{B(t),\ t \geqslant 0\}$ be standard Brownian motion and then define

$$X(t) = \sigma B(t) + \mu t$$

10.3.2. Geometric Brownian Motion

If $\{Y(t),\ t \geqslant 0\}$ is a Brownian motion process with drift coefficient μ and variance parameter σ^2, then the process $\{X(t),\ t \geqslant 0\}$ defined by

$$X(t) = e^{Y(t)}$$

is called *geometric Brownian motion*.

For a geometric Brownian motion process $\{X(t)\}$, let us compute the expected value of the process at time t given the history of the process up to time s. That is, for $s < t$, consider $E[X(t)|X(u),\ 0 \leqslant u \leqslant s]$. Now,

$$\begin{aligned}
E[X(t)|X(u),\ 0 \leqslant u \leqslant s] &= E\left[e^{Y(t)}|Y(u),\ 0 \leqslant u \leqslant s\right] \\
&= E\left[e^{Y(s)+Y(t)-Y(s)}|Y(u),\ 0 \leqslant u \leqslant s\right] \\
&= e^{Y(s)}E\left[e^{Y(t)-Y(s)}|Y(u),\ 0 \leqslant u \leqslant s\right] \\
&= X(s)E\left[e^{Y(t)-Y(s)}\right]
\end{aligned}$$

where the next to last equality follows from the fact that $Y(s)$ is given, and the last equality from the independent increment property of Brownian motion. Now, the moment generating function of a normal random variable W is given by

$$E[e^{aW}] = e^{aE[W]+a^2\,\mathrm{Var}(W)/2}$$

Hence, since $Y(t) - Y(s)$ is normal with mean $\mu(t-s)$ and variance $(t-s)\sigma^2$, it follows by setting $a = 1$ that

$$E\left[e^{Y(t)-Y(s)}\right] = e^{\mu(t-s)+(t-s)\sigma^2/2}$$

Thus, we obtain

$$E[X(t)|X(u),\ 0 \leqslant u \leqslant s] = X(s)e^{(t-s)(\mu+\sigma^2/2)} \tag{10.8}$$

Geometric Brownian motion is useful in the modeling of stock prices over time when you feel that the percentage changes are independent and identically distributed. For instance, suppose that X_n is the price of some stock at time n. Then, it might be reasonable to suppose that X_n/X_{n-1}, $n \geqslant 1$, are independent and identically distributed. Let

$$Y_n = X_n/X_{n-1}$$

and so

$$X_n = Y_n X_{n-1}$$

Iterating this equality gives

$$\begin{aligned} X_n &= Y_n Y_{n-1} X_{n-2} \\ &= Y_n Y_{n-1} Y_{n-2} X_{n-3} \\ &\ \vdots \\ &= Y_n Y_{n-1} \cdots Y_1 X_0 \end{aligned}$$

Thus,

$$\log(X_n) = \sum_{i=1}^{n} \log(Y_i) + \log(X_0)$$

Since $\log(Y_i)$, $i \geqslant 1$ are independent and identically distributed, $\{\log(X_n)\}$ will, when suitably normalized, approximately be Brownian motion with a drift, and so $\{X_n\}$ will be approximately geometric Brownian motion.

10.4. Pricing Stock Options

10.4.1. An Example in Options Pricing

In situations in which money is to be received or paid out in differing time periods, we must take into account the time value of money. That is, to be given the amount v a time t in the future is not worth as much as being given v immediately. The reason for this is that if we were immediately given v, then it could be loaned out

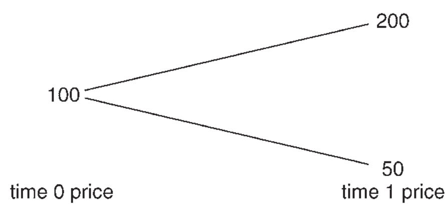

Figure 10.1.

with interest and so be worth more than v at time t. To take this into account, we will suppose that the time 0 value, also called the *present value*, of the amount v to be earned at time t is $ve^{-\alpha t}$. The quantity α is often called the discount factor. In economic terms, the assumption of the discount function $e^{-\alpha t}$ is equivalent to the assumption that we can earn interest at a continuously compounded rate of 100α percent per unit time.

We will now consider a simple model for pricing an option to purchase a stock at a future time at a fixed price.

Suppose the present price of a stock is \$100 per unit share, and suppose we know that after one time period it will be, in present value dollars, either \$200 or \$50 (see Figure 10.1). It should be noted that the prices at time 1 are the present value (or time 0) prices. That is, if the discount factor is α, then the actual possible prices at time 1 are either $200e^{\alpha}$ or $50e^{\alpha}$. To keep the notation simple, we will suppose that all prices given are time 0 prices.

Suppose that for any y, at a cost of cy, you can purchase at time 0 the option to buy y shares of the stock at time 1 at a (time 0) cost of \$150 per share. Thus, for instance, if you do purchase this option and the stock rises to \$200, then you would exercise the option at time 1 and realize a gain of $\$200 - 150 = \$50$ for each of the y option units purchased. On the other hand, if the price at time 1 was \$50, then the option would be worthless at time 1. In addition, at a cost of $100x$ you can purchase x units of the stock at time 0, and this will be worth either $200x$ or $50x$ at time 1.

We will suppose that both x or y can be either positive or negative (or zero). That is, you can either buy or sell both the stock and the option. For instance, if x were negative then you would be selling $-x$ shares of the stock, yielding you a return of $-100x$, and you would then be responsible for buying $-x$ shares of the stock at time 1 at a cost of either \$200 or \$50 per share.

We are interested in determining the appropriate value of c, the unit cost of an option. Specifically, we will show that unless $c = 50/3$ there will be a combination of purchases that will always result in a positive gain.

To show this, suppose that at time 0 we

buy x units of stock, and
buy y units of options

where x and y (which can be either positive or negative) are to be determined. The value of our holding at time 1 depends on the price of the stock at that time; and it is given by the following

$$\text{value} = \begin{cases} 200x + 50y, & \text{if price is } 200 \\ 50x, & \text{if price is } 50 \end{cases}$$

The preceding formula follows by noting that if the price is 200 then the x units of the stock are worth $200x$, and the y units of the option to buy the stock at a unit price of 150 are worth $(200 - 150)y$. On the other hand, if the stock price is 50, then the x units are worth $50x$ and the y units of the option are worthless. Now, suppose we choose y to be such that the preceding value is the same no matter what the price at time 1. That is, we choose y so that

$$200x + 50y = 50x$$

or

$$y = -3x$$

(Note that y has the opposite sign of x, and so if x is positive and as a result x units of the stock are purchased at time 0, then $3x$ units of stock options are also *sold* at that time. Similarly, if x is negative, then $-x$ units of stock are sold and $-3x$ units of stock options are purchased at time 0.)

Thus, with $y = -3x$, the value of our holding at time 1 is

$$\text{value} = 50x$$

Since the original cost of purchasing x units of the stock and $-3x$ units of options is

$$\text{original cost} = 100x - 3xc,$$

we see that our gain on the transaction is

$$\text{gain} = 50x - (100x - 3xc) = x(3c - 50)$$

Thus, if $3c = 50$, then the gain is 0; on the other hand if $3c \neq 50$, we can guarantee a positive gain (no matter what the price of the stock at time 1) by letting x be positive when $3c > 50$ and letting it be negative when $3c < 50$.

For instance, if the unit cost per option is $c = 20$, then purchasing 1 unit of the stock $(x = 1)$ and simultaneously selling 3 units of the option $(y = -3)$ initially costs us $100 - 60 = 40$. However, the value of this holding at time 1 is 50 whether the stock goes up to 200 or down to 50. Thus, a guaranteed profit of 10 is attained. Similarly, if the unit cost per option is $c = 15$, then selling 1 unit of the stock ($x = -1$) and buying 3 units of the option ($y = 3$) leads to an initial gain of $100 - 45 = 55$. On the other hand, the value of this holding at time 1 is -50. Thus, a guaranteed profit of 5 is attained.

A sure win betting scheme is called an *arbitrage*. Thus, as we have just seen, the only option cost c that does not result in an arbitrage is $c = 50/3$.

10.4.2. The Arbitrage Theorem

Consider an experiment whose set of possible outcomes is $S = \{1, 2, \ldots, m\}$. Suppose that n wagers are available. If the amount x is bet on wager i, then the return $xr_i(j)$ is earned if the outcome of the experiment is j. In other words, $r_i(\cdot)$ is the return function for a unit bet on wager i. The amount bet on a wager is allowed to be either positive or negative or zero.

A betting scheme is a vector $\mathbf{x} = (x_1, \ldots, x_n)$ with the interpretation that x_1 is bet on wager 1, x_2 on wager $2, \ldots,$ and x_n on wager n. If the outcome of the experiment is j, then the return from the betting scheme $\mathbf{x}$ is

$$\text{return from } \mathbf{x} = \sum_{i=1}^{n} x_i r_i(j)$$

The following theorem states that either there exists a probability vector $\mathbf{p} = (p_1, \ldots, p_m)$ on the set of possible outcomes of the experiment under which each of the wagers has expected return 0, or else there is a betting scheme that guarantees a positive win.

Theorem 10.1 (The Arbitrage Theorem) Exactly one of the following is true: Either

(i) there exists a probability vector $\mathbf{p} = (p_1, \ldots, p_m)$ for which

$$\sum_{j=1}^{m} p_j r_i(j) = 0, \qquad \text{for all } i = 1, \ldots, n$$

or

(ii) there exists a betting scheme $\mathbf{x} = (x_1, \ldots, x_n)$ for which

$$\sum_{i=1}^{n} x_i r_i(j) > 0, \qquad \text{for all } j = 1, \ldots, m$$

In other words, if X is the outcome of the experiment, then the arbitrage theorem states that either there is a probability vector $\mathbf{p}$ for X such that

$$E_{\mathbf{p}}[r_i(X)] = 0, \qquad \text{for all } i = 1, \ldots, n$$

or else there is a betting scheme that leads to a sure win.

Remark This theorem is a consequence of the (linear algebra) theorem of the separating hyperplane, which is often used as a mechanism to prove the duality theorem of linear programming.

The theory of linear programming can be used to determine a betting strategy that guarantees the greatest return. Suppose that the absolute value of the amount bet on each wager must be less than or equal to 1. To determine the vector $\mathbf{x}$ that yields the greatest guaranteed win—call this win v—we need to choose $\mathbf{x}$ and v so as to maximize v, subject to the constraints

$$\sum_{i=1}^{n} x_i r_i(j) \geqslant v, \qquad \text{for } j = 1, \ldots, m$$

$$-1 \leqslant x_i \leqslant 1, \qquad i = 1, \ldots, n$$

This optimization problem is a linear program and can be solved by standard techniques (such as by using the simplex algorithm). The arbitrage theorem yields that the optimal v will be positive unless there is a probability vector $\mathbf{p}$ for which $\sum_{j=1}^{m} p_j r_i(j) = 0$ for all $i = 1, \ldots, n$.

Example 10.2 In some situations, the only types of wagers allowed are to choose one of the outcomes $i, i = 1, \ldots, m$, and bet that i is the outcome of the experiment. The return from such a bet is often quoted in terms of “odds.” If the odds for outcome i are o_i (often written as “o_i to 1”) then a 1 unit bet will return o_i if the outcome of the experiment is i and will return -1 otherwise. That is,

$$r_i(j) = \begin{cases} o_i, & \text{if } j = i \\ -1 & \text{otherwise} \end{cases}$$

Suppose the odds $o_1, \ldots, o_m$ are posted. In order for there not to be a sure win there must be a probability vector $\mathbf{p} = (p_1, \ldots, p_m)$ such that

$$0 \equiv E_{\mathbf{p}}[r_i(X)] = o_i p_i - (1 - p_i)$$

That is, we must have

$$p_i = \frac{1}{1 + o_i}$$

Since the p_i must sum to 1, this means that the condition for there not to be an arbitrage is that

$$\sum_{i=1}^{m}(1+o_i)^{-1}=1$$

Thus, if the posted odds are such that $\sum_i (1+o_i)^{-1} \neq 1$, then a sure win is possible. For instance, suppose there are three possible outcomes and the odds are as follows:

Outcome	Odds
1	1
2	2
3	3

That is, the odds for outcome 1 are $1-1$, the odds for outcome 2 are $2-1$, and that for outcome 3 are $3-1$. Since

$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}>1$$

a sure win is possible. One possibility is to bet -1 on outcome 1 (and so you either win 1 if the outcome is not 1 and lose 1 if the outcome is 1) and bet -0.7 on outcome 2, and -0.5 on outcome 3. If the experiment results in outcome 1, then we win $-1+0.7+0.5=0.2$; if it results in outcome 2, then we win $1-1.4+0.5=0.1$; if it results in outcome 3, then we win $1+0.7-1.5=0.2$. Hence, in all cases we win a positive amount. ■

Remark If $\sum_i (1+o_i)^{-1} \neq 1$, then the betting scheme

$$x_i=\frac{(1+o_i)^{-1}}{1-\sum_i (1+o_i)^{-1}}, \qquad i=1,\ldots,n$$

will always yield a gain of exactly 1.

Example 10.3 Let us reconsider the option pricing example of the previous section, where the initial price of a stock is 100 and the present value of the price at time 1 is either 200 or 50. At a cost of c per share we can purchase at time 0 the option to buy the stock at time 1 at a present value price of 150 per share. The problem is to set the value of c so that no sure win is possible.

In the context of this section, the outcome of the experiment is the value of the stock at time 1. Thus, there are two possible outcomes. There are also two different wagers: to buy (or sell) the stock, and to buy (or sell) the option. By the arbitrage theorem, there will be no sure win if there is a probability vector $(p, 1-p)$ that makes the expected return under both wagers equal to 0.

Now, the return from purchasing 1 unit of the stock is

$$\text{return} = \begin{cases} 200 - 100 = 100, & \text{if the price is 200 at time 1} \\ 50 - 100 = -50, & \text{if the price is 50 at time 1} \end{cases}$$

Hence, if p is the probability that the price is 200 at time 1, then

$$E[\text{return}] = 100p - 50(1 - p)$$

Setting this equal to 0 yields that

$$p = \tfrac{1}{3}$$

That is, the only probability vector $(p, 1 - p)$ for which wager 1 yields an expected return 0 is the vector $(\frac{1}{3}, \frac{2}{3})$.

Now, the return from purchasing one share of the option is

$$\text{return} = \begin{cases} 50 - c, & \text{if price is 200} \\ -c, & \text{if price is 50} \end{cases}$$

Hence, the expected return when $p = \frac{1}{3}$ is

$$\begin{aligned} E[\text{return}] &= (50 - c)\tfrac{1}{3} - c\tfrac{2}{3} \\ &= \tfrac{50}{3} - c \end{aligned}$$

Thus, it follows from the arbitrage theorem that the only value of c for which there will not be a sure win is $c = \frac{50}{3}$, which verifies the result of section 10.4.1. ■

10.4.3. The Black-Scholes Option Pricing Formula

Suppose the present price of a stock is $X(0) = x_0$, and let $X(t)$ denote its price at time t. Suppose we are interested in the stock over the time interval 0 to T. Assume that the discount factor is α (equivalently, the interest rate is 100α percent compounded continuously), and so the present value of the stock price at time t is $e^{-\alpha t}X(t)$.

We can regard the evolution of the price of the stock over time as our experiment, and thus the outcome of the experiment is the value of the function $X(t)$, $0 \leqslant t \leqslant T$. The types of wagers available are that for any $s < t$ we can observe the process for a time s and then buy (or sell) shares of the stock at price $X(s)$ and then sell (or buy) these shares at time t for the price $X(t)$. In addition, we will suppose that we may purchase any of N different options at time 0. Option i, costing c_i per share, gives us the option of purchasing shares of the stock at time t_i for the fixed price of K_i per share, $i = 1, \ldots, N$.

Suppose we want to determine values of the c_i for which there is no betting strategy that leads to a sure win. Assuming that the arbitrage theorem can be generalized (to handle the preceding situation, where the outcome of the experiment is a function), it follows that there will be no sure win if and only if there exists a probability measure over the set of outcomes under which all of the wagers have expected return 0. Let **P** be a probability measure on the set of outcomes. Consider first the wager of observing the stock for a time s and then purchasing (or selling) one share with the intention of selling (or purchasing) it at time t, $0 \leqslant s < t \leqslant T$. The present value of the amount paid for the stock is $e^{-\alpha s}X(s)$, whereas the present value of the amount received is $e^{-\alpha t}X(t)$. Hence, in order for the expected return of this wager to be 0 when **P** is the probability measure on $X(t)$, $0 \leqslant t \leqslant T$, we must have that

$$E_{\mathbf{P}}[e^{-\alpha t}X(t)|X(u), 0 \leqslant u \leqslant s] = e^{-\alpha s}X(s) \tag{10.9}$$

Consider now the wager of purchasing an option. Suppose the option gives us the right to buy one share of the stock at time t for a price K. At time t, the worth of this option will be as follows:

$$\text{worth of option at time } t = \begin{cases} X(t) - K, & \text{if } X(t) \geqslant K \\ 0, & \text{if } X(t) < K \end{cases}$$

That is, the time t worth of the option is $(X(t) - K)^+$. Hence, the present value of the worth of the option is $e^{-\alpha t}(X(t) - K)^+$. If c is the (time 0) cost of the option, we see that, in order for purchasing the option to have expected (present value) return 0, we must have that

$$E_{\mathbf{P}}[e^{-\alpha t}(X(t) - K)^+] = c \tag{10.10}$$

By the arbitrage theorem, if we can find a probability measure **P** on the set of outcomes that satisfies Equation (10.9), then if c, the cost of an option to purchase one share at time t at the fixed price K, is as given in Equation (10.10), then no arbitrage is possible. On the other hand, if for given prices c_i, $i = 1, \ldots, N$, there is no probability measure **P** that satisfies both (10.9) and the equality

$$c_i = E_{\mathbf{P}}[e^{-\alpha t_i}(X(t_i) - K_i)^+], \qquad i = 1, \ldots, N$$

then a sure win is possible.

We will now present a probability measure **P** on the outcome $X(t)$, $0 \leqslant t \leqslant T$, that satisfies Equation (10.9).

Suppose that

$$X(t) = x_0 e^{Y(t)}$$

where $\{Y(t), t \geqslant 0\}$ is a Brownian motion process with drift coefficient μ and variance parameter σ^2. That is, $\{X(t), t \geqslant 0\}$ is a geometric Brownian motion process (see Section 10.3.2). From Equation (10.8) we have that, for $s < t$,

$$E[X(t)|X(u), 0 \leqslant u \leqslant s] = X(s)e^{(t-s)(\mu+\sigma^2/2)}$$

Hence, if we choose μ and σ^2 so that

$$\mu + \sigma^2/2 = \alpha$$

then Equation (10.9) will be satisfied. That is, by letting $\mathbf{P}$ be the probability measure governing the stochastic process $\{x_0 e^{Y(t)}, 0 \leqslant t \leqslant T\}$, where $\{Y(t)\}$ is Brownian motion with drift parameter μ and variance parameter σ^2, and where $\mu + \sigma^2/2 = \alpha$, Equation (10.9) is satisfied.

It follows from the preceding that if we price an option to purchase a share of the stock at time t for a fixed price K by

$$c = E_{\mathbf{P}}[e^{-\alpha t}(X(t) - K)^+]$$

then no arbitrage is possible. Since $X(t) = x_0 e^{Y(t)}$, where $Y(t)$ is normal with mean μt and variance $t\sigma^2$, we see that

$$\begin{aligned} ce^{\alpha t} &= \int_{-\infty}^{\infty} (x_0 e^y - K)^+ \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-(y-\mu t)^2/2t\sigma^2}\, dy \\ &= \int_{\log(K/x_0)}^{\infty} (x_0 e^y - K) \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-(y-\mu t)^2/2t\sigma^2}\, dy \end{aligned}$$

Making the change of variable $w = (y - \mu t)/(\sigma t^{1/2})$ yields

$$ce^{\alpha t} = x_0 e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{\sigma w\sqrt{t}} e^{-w^2/2}\, dw - K \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-w^2/2}\, dw \qquad (10.11)$$

where

$$a = \frac{\log(K/x_0) - \mu t}{\sigma\sqrt{t}}$$

Now,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{\sigma w\sqrt{t}} e^{-w^2/2}\, dw &= e^{t\sigma^2/2} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-(w-\sigma\sqrt{t})^2/2}\, dw \\ &= e^{t\sigma^2/2} P\big\{N(\sigma\sqrt{t}, 1) \geqslant a\big\} \\ &= e^{t\sigma^2/2} P\big\{N(0,1) \geqslant a - \sigma\sqrt{t}\big\} \\ &= e^{t\sigma^2/2} P\big\{N(0,1) \leqslant -\big(a - \sigma\sqrt{t}\big)\big\} \\ &= e^{t\sigma^2/2} \phi\big(\sigma\sqrt{t} - a\big) \end{aligned}$$

where $N(m, v)$ is a normal random variable with mean m and variance v, and ϕ is the standard normal distribution function.

Thus, we see from Equation (10.11) that

$$ce^{\alpha t} = x_0 e^{\mu t + \sigma^2 t/2} \phi\big(\sigma\sqrt{t} - a\big) - K\phi(-a)$$

Using that

$$\mu + \sigma^2/2 = \alpha$$

and letting $b = -a$, we can write this as follows:

$$c = x_0 \phi\big(\sigma\sqrt{t} + b\big) - Ke^{-\alpha t}\phi(b) \tag{10.12}$$

where

$$b = \frac{\alpha t - \sigma^2 t/2 - \log(K/x_0)}{\sigma\sqrt{t}}$$

The option price formula given by Equation (10.12) depends on the initial price of the stock x_0, the option exercise time t, the option exercise price K, the discount (or interest rate) factor α, and the value σ^2. Note that for any value of σ^2, if the options are priced according to the formula of Equation (10.12) then no arbitrage is possible. However, as many people believe that the price of a stock actually follows a geometric Brownian motion—that is, $X(t) = x_0 e^{Y(t)}$ where $Y(t)$ is Brownian motion with parameters μ and σ^2—it has been suggested that it is natural to price the option according to the formula (10.12) with the parameter σ^2 taken equal to the estimated value (see the remark that follows) of the variance parameter under the assumption of a geometric Brownian motion model. When this is done, the formula (10.12) is known as the Black–Scholes option cost valuation. It is interesting that this valuation does not depend on the value of the drift parameter μ but only on the variance parameter σ^2.

If the option itself can be traded, then the formula of Equation (10.12) can be used to set its price in such a way so that no arbitrage is possible. If at time s the price of the stock is $X(s) = x_s$, then the price of a (t, K) option—that is, an option to purchase one unit of the stock at time t for a price K—should be set by replacing t by $t - s$ and x_0 by x_s in Equation (10.12).

Remark If we observe a Brownian motion process with variance parameter σ^2 over any time interval, then we could theoretically obtain an arbitrarily precise estimate of σ^2. For suppose we observe such a process $\{Y(s)\}$ for a time t. Then, for fixed h, let $N = [t/h]$ and set

$$\begin{aligned} W_1 &= Y(h) - Y(0), \\ W_2 &= Y(2h) - Y(h), \\ &\vdots \\ W_N &= Y(Nh) - Y(Nh - h) \end{aligned}$$

Then random variables $W_1, \ldots, W_N$ are independent and identically distributed normal random variables having variance $h\sigma^2$. We now use the fact (see Section 3.6.4) that $(N-1)S^2/(\sigma^2 h)$ has a chi-squared distribution with $N-1$ degrees of freedom, where S^2 is the sample variance defined by

$$S^2 = \sum_{i=1}^{N} (W_i - \bar{W})^2/(N-1)$$

Since the expected value and variance of a chi-squared with k degrees of freedom are equal to k and $2k$, respectively, we see that

$$E[(N-1)S^2/(\sigma^2 h)] = N - 1$$

and

$$\mathrm{Var}[(N-1)S^2/(\sigma^2 h)] = 2(N-1)$$

From this, we see that

$$E[S^2/h] = \sigma^2$$

and

$$\mathrm{Var}[S^2/h] = 2\sigma^4/(N-1)$$

Hence, as we let h become smaller (and so $N = [t/h]$ becomes larger) the variance of the unbiased estimator of σ^2 becomes arbitrarily small. ■

Equation (10.12) is not the only way in which options can be priced so that no arbitrage is possible. Let $\{X(t), 0 \leqslant t \leqslant T\}$ be any stochastic process satisfying, for $s < t$,

$$E[e^{-\alpha t}X(t)|X(u),\ 0 \leqslant u \leqslant s] = e^{-\alpha s}X(s) \tag{10.13}$$

[that is, Equation (10.9) is satisfied]. By setting c, the cost of an option to purchase one share of the stock at time t for price K, equal to

$$c = E[e^{-\alpha t}(X(t) - K)^{+}] \tag{10.14}$$

it follows that no arbitrage is possible.

Another type of stochastic process, aside from geometric Brownian motion, that satisfies Equation (10.13) is obtained as follows. Let $Y_1, Y_2, \dots$ be a sequence of independent random variables having a common mean μ, and suppose that this process is independent of $\{N(t), t \geqslant 0\}$, which is a Poisson process with rate λ. Let

$$X(t) = x_0 \prod_{i=1}^{N(t)} Y_i$$

Using the identity

$$X(t) = x_0 \prod_{i=1}^{N(s)} Y_i \prod_{j=N(s)+1}^{N(t)} Y_j$$

and the independent increment assumption of the Poisson process, we see that, for $s < t$,

$$E[X(t)|X(u),\ 0 \leqslant u \leqslant s] = X(s)\ E\left[\prod_{j=N(s)+1}^{N(t)} Y_j\right]$$

Conditioning on the number of events between s and t yields

$$\begin{aligned} E\left[\prod_{j=N(s)+1}^{N(t)} Y_j\right] &= \sum_{n=0}^{\infty} \mu^n e^{-\lambda(t-s)}[\lambda(t-s)]^n/n! \\ &= e^{-\lambda(t-s)(1-\mu)} \end{aligned}$$

Hence,

$$E[X(t)|X(u),\ 0 \leqslant u \leqslant s] = X(s)e^{-\lambda(t-s)(1-\mu)}$$

Thus, if we choose λ and μ to satisfy

$$\lambda(1-\mu) = -\alpha$$

then Equation (10.13) is satisfied. Therefore, if for any value of λ we let the Y_i have any distributions with a common mean equal to $\mu = 1 + \alpha/\lambda$ and then price the options according to Equation (10.14), then no arbitrage is possible.

Remark If $\{X(t),\ t \geqslant 0\}$ satisfies Equation (10.13), then the process $\{e^{-\alpha t}X(t),\, t \geqslant 0\}$ is called a Martingale. Thus, any pricing of options for which the expected gain on the option is equal to 0 when $\{e^{-\alpha t}X(t)\}$ follows the probability law of some Martingale will result in no arbitrage possibilities.

That is, if we choose any Martingale process $\{Z(t)\}$ and let the cost of a (t, K) option be

$$\begin{aligned} c &= E[e^{-\alpha t}(e^{\alpha t}Z(t) - K)^+] \\ &= E[(Z(t) - Ke^{-\alpha t})^+] \end{aligned}$$

then there is no sure win.

In addition, while we did not consider the type of wager where a stock that is purchased at time s is sold not at a fixed time t but rather at some random time that depends on the movement of the stock, it can be shown using results about Martingales that the expected return of such wagers is also equal to 0.

Remark A variation of the arbitrage theorem was first noted by de Finetti in 1937. A more general version of de Finetti's result, of which the arbitrage theorem is a special case, is given in Reference 3.

10.5. White Noise

Let $\{X(t), t \geqslant 0\}$ denote a standard Brownian motion process and let f be a function having a continuous derivative in the region $[a, b]$. The stochastic integral $\int_a^b f(t)\,dX(t)$ is defined as follows:

$$\int_a^b f(t)\,dX(t) \equiv \lim_{\substack{n\to\infty \\ \max(t_i - t_{i-1})\to 0}} \sum_{i=1}^{n} f(t_{i-1})[X(t_i) - X(t_{i-1})] \tag{10.15}$$

where $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of the region $[a, b]$. Using the identity (the integration by parts formula applied to sums)

$$\begin{aligned} &\sum_{i=1}^{n} f(t_{i-1})[X(t_i) - X(t_{i-1})] \\ &\quad = f(b)X(b) - f(a)X(a) - \sum_{i=1}^{n} X(t_i)[f(t_i) - f(t_{i-1})] \end{aligned}$$

we see that

$$\int_a^b f(t)\,dX(t) = f(b)X(b) - f(a)X(a) - \int_a^b X(t)\,df(t) \tag{10.16}$$

Equation (10.16) is usually taken as the definition of $\int_a^b f(t)\,dX(t)$.

By using the right side of Equation (10.16) we obtain, upon assuming the interchangeability of expectation and limit, that

$$E\left[\int_a^b f(t)\,dX(t)\right] = 0$$

Also,

$$\begin{aligned}\operatorname{Var}\left(\sum_{i=1}^{n} f(t_{i-1})[X(t_i) - X(t_{i-1})]\right) &= \sum_{i=1}^{n} f^2(t_{i-1}) \operatorname{Var}[X(t_i) - X(t_{i-1})] \\ &= \sum_{i=1}^{n} f^2(t_{i-1})(t_i - t_{i-1})\end{aligned}$$

where the top equality follows from the independent increments of Brownian motion. Hence, we obtain from Equation (10.15) upon taking limits of the preceding that

$$\operatorname{Var}\left[\int_a^b f(t)\,dX(t)\right] = \int_a^b f^2(t)\,dt$$

Remark The preceding gives operational meaning to the family of quantities $\{dX(t),\ 0 \leqslant t < \infty\}$ by viewing it as an operator that carries functions f into the values $\int_a^b f(t)\,dX(t)$. This is called a white noise transformation, or more loosely $\{dX(t),\ 0 \leqslant t < \infty\}$ is called white noise since it can be imagined that a time varying function f travels through a white noise medium to yield the output (at time b) $\int_a^b f(t)\,dX(t)$.

Example 10.4 Consider a particle of unit mass that is suspended in a liquid and suppose that, due to the liquid, there is a viscous force that retards the velocity of the particle at a rate proportional to its present velocity. In addition, let us suppose that the velocity instantaneously changes according to a constant multiple of white noise. That is, if $V(t)$ denotes the particle's velocity at t, suppose that

$$V'(t) = -\beta V(t) + \alpha X'(t)$$

where $\{X(t),\ t \geqslant 0\}$ is standard Brownian motion. This can be written as follows:

$$e^{\beta t}[V'(t) + \beta V(t)] = \alpha e^{\beta t} X'(t)$$

or

$$\frac{d}{dt}[e^{\beta t} V(t)] = \alpha e^{\beta t} X'(t)$$

Hence, upon integration, we obtain

$$e^{\beta t} V(t) = V(0) + \alpha \int_0^t e^{\beta s} X'(s)\, ds$$

or

$$V(t) = V(0)e^{-\beta t} + \alpha \int_0^t e^{-\beta(t-s)}\, dX(s)$$

Hence, from Equation (10.16),

$$V(t) = V(0)e^{-\beta t} + \alpha\left[X(t) - \int_0^t X(s)\beta e^{-\beta(t-s)}\, ds\right] \quad \blacksquare$$

10.6. Gaussian Processes

We start with the following definition.

Definition 10.2 A stochastic process $X(t)$, $t \geqslant 0$ is called a *Gaussian*, or a *normal*, process if $X(t_1), \ldots, X(t_n)$ has a multivariate normal distribution for all $t_1, \ldots, t_n$.

If $\{X(t), t \geqslant 0\}$ is a Brownian motion process, then because each of $X(t_1)$, $X(t_2), \ldots, X(t_n)$ can be expressed as a linear combination of the independent normal random variables $X(t_1),\ X(t_2) - X(t_1),\ X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$ it follows that Brownian motion is a Gaussian process.

Because a multivariate normal distribution is completely determined by the marginal mean values and the covariance values (see Section 2.6) it follows that standard Brownian motion could also be defined as a Gaussian process having $E[X(t)] = 0$ and, for $s \leqslant t$,

$$\begin{aligned}
\text{Cov}(X(s), X(t)) &= \text{Cov}(X(s), X(s) + X(t) - X(s)) \\
&= \text{Cov}(X(s), X(s)) + \text{Cov}(X(s), X(t) - X(s)) \\
&= \text{Cov}(X(s), X(s)) \qquad \text{by independent increments} \\
&= s \qquad \text{since } \text{Var}(X(s)) = s
\end{aligned} \tag{10.17}$$

Let $\{X(t),\ t \geqslant 0\}$ be a standard Brownian motion process and consider the process values between 0 and 1 conditional on $X(1) = 0$. That is, consider the conditional stochastic process $\{X(t),\ 0 \leqslant t \leqslant 1 | X(1) = 0\}$. Since the conditional distribution of $X(t_1), \ldots, X(t_n)$ is multivariate normal it follows that this conditional process, known as the *Brownian bridge* (as it is tied down both at 0 and at 1), is a Gaussian process. Let us compute its covariance function. As, from Equation (10.4),

$$E[X(s)|X(1) = 0] = 0, \qquad \text{for } s < 1$$

we have that, for $s < t < 1$,

$$\begin{aligned}
&\text{Cov}[(X(s), X(t))|X(1) = 0] \\
&\quad = E[X(s)X(t)|X(1) = 0] \\
&\quad = E[E[X(s)X(t)|X(t), X(1) = 0]|X(1) = 0] \\
&\quad = E[X(t)E[X(s)|X(t)]|X(1) = 0] \\
&\quad = E\left[X(t)\frac{s}{t}X(t)|X(1) = 0\right] \qquad \text{by (10.4)} \\
&\quad = \frac{s}{t}E[X^2(t)|X(1) = 0] \\
&\quad = \frac{s}{t}t(1-t) \qquad \text{by (10.4)} \\
&\quad = s(1-t)
\end{aligned}$$

Thus, the Brownian bridge can be defined as a Gaussian process with mean value 0 and covariance function $s(1-t)$, $s \leqslant t$. This leads to an alternative approach to obtaining such a process.

Proposition 10.1 If $\{X(t),\ t \geqslant 0\}$ is standard Brownian motion, then $\{Z(t),\ 0 \leqslant t \leqslant 1\}$ is a Brownian bridge process when $Z(t) = X(t) - tX(1)$.

Proof As it is immediate that $\{Z(t),\ t \geqslant 0\}$ is a Gaussian process, all we need verify is that $E[Z(t)] = 0$ and $\mathrm{Cov}(Z(s), Z(t)) = s(1-t)$, when $s \leqslant t$. The former is immediate and the latter follows from

$$\begin{aligned}
\mathrm{Cov}(Z(s), Z(t)) &= \mathrm{Cov}(X(s) - sX(1), X(t) - tX(1)) \\
&= \mathrm{Cov}(X(s), X(t)) - t\,\mathrm{Cov}(X(s), X(1)) \\
&\quad - s\,\mathrm{Cov}(X(1), X(t)) + st\,\mathrm{Cov}(X(1), X(1)) \\
&= s - st - st + st \\
&= s(1-t)
\end{aligned}$$

and the proof is complete. ■

If $\{X(t),\ t \geqslant 0\}$ is Brownian motion, then the process $\{Z(t),\ t \geqslant 0\}$ defined by

$$Z(t) = \int_0^t X(s)\,ds \tag{10.18}$$

is called *integrated Brownian motion*. As an illustration of how such a process may arise in practice, suppose we are interested in modeling the price of a commodity throughout time. Letting $Z(t)$ denote the price at t then, rather than assuming that $\{Z(t)\}$ is Brownian motion (or that $\log Z(t)$ is Brownian motion), we might want to assume that the rate of change of $Z(t)$ follows a Brownian motion. For instance, we might suppose that the rate of change of the commodity's price is the current inflation rate which is imagined to vary as Brownian motion. Hence,

$$\frac{d}{dt} Z(t) = X(t),$$

$$Z(t) = Z(0) + \int_0^t X(s)\,ds$$

It follows from the fact that Brownian motion is a Gaussian process that $\{Z(t),\ t \geqslant 0\}$ is also Gaussian. To prove this, first recall that $W_1, \ldots, W_n$ is said to have a multivariate normal distribution if they can be represented as

$$W_i = \sum_{j=1}^{m} a_{ij} U_j, \qquad i = 1, \ldots, n$$

where $U_j,\ j = 1, \ldots, m$ are independent normal random variables. From this it follows that any set of partial sums of $W_1, \ldots, W_n$ are also jointly normal. The fact that $Z(t_1), \ldots, Z(t_n)$ is multivariate normal can now be shown by writing the integral in Equation (10.18) as a limit of approximating sums.

As $\{Z(t),\ t \geqslant 0\}$ is Gaussian it follows that its distribution is characterized by its mean value and covariance function. We now compute these when $\{X(t),\ t \geqslant 0\}$ is standard Brownian motion.

$$\begin{aligned} E[Z(t)] &= E\left[\int_0^t X(s)\,ds\right] \\ &= \int_0^t E[X(s)]\,ds \\ &= 0 \end{aligned}$$

For $s \leqslant t$,

$$\begin{aligned} \text{Cov}[Z(s), Z(t)] &= E[Z(s)Z(t)] \\ &= E\left[\int_0^t X(y)\,dy \int_0^s X(u)\,du\right] \\ &= E\left[\int_0^s \int_0^t X(y)X(u)\,dy\,du\right] \\ &= \int_0^s \int_0^t E[X(y)X(u)]\,dy\,du \\ &= \int_0^s \int_0^t \min(y, u)\,dy\,du \qquad \text{by (10.17)} \\ &= \int_0^s \left(\int_0^u y\,dy + \int_u^t u\,dy\right)du = s^2\left(\frac{t}{2} - \frac{s}{6}\right) \quad \blacksquare \end{aligned}$$

10.7. Stationary and Weakly Stationary Processes

A stochastic process $\{X(t),\ t \geqslant 0\}$ is said to be a *stationary process* if for all $n, s, t, \ldots, t_n$ the random vectors $X(t_1), \ldots, X(t_n)$ and $X(t_1 + s), \ldots, X(t_n + s)$ have the same joint distribution. In other words, a process is stationary if, in choosing any fixed point s as the origin, the ensuing process has the same probability law. Two examples of stationary processes are:

(i) An ergodic continuous-time Markov chain $\{X(t),\ t \geqslant 0\}$ when

$$P\{X(0) = j\} = P_j, \qquad j \geqslant 0$$

where $\{P_j,\ j \geqslant 0\}$ are the limiting probabilities.

(ii) $\{X(t),\ t \geqslant 0\}$ when $X(t) = N(t + L) - N(t),\ t \geqslant 0$, where $L > 0$ is a fixed constant and $\{N(t),\ t \geqslant 0\}$ is a Poisson process having rate λ.

The first one of these processes is stationary for it is a Markov chain whose initial state is chosen according to the limiting probabilities, and it can thus be regarded as an ergodic Markov chain that we start observing at time ∞. Hence the continuation of this process at time s after observation begins is just the continuation of the chain starting at time $\infty + s$, which clearly has the same probability for all s. That the second example—where $X(t)$ represents the number of events of a Poisson process that occur between t and $t + L$—is stationary follows the stationary and independent increment assumption of the Poisson process which implies that the continuation of a Poisson process at any time s remains a Poisson process.

Example 10.5 (The Random Telegraph Signal Process) Let $\{N(t),\ t \geqslant 0\}$ denote a Poisson process, and let X_0 be independent of this process and be such that $P\{X_0 = 1\} = P\{X_0 = -1\} = \frac{1}{2}$. Defining $X(t) = X_0(-1)^{N(t)}$ then $\{X(t),\ t \geqslant 0\}$ is called *random telegraph signal* process. To see that it is stationary, note first that starting at any time t, no matter what the value of $N(t)$, as X_0 is equally likely to be either plus or minus 1, it follows that $X(t)$ is equally likely to be either plus or minus 1. Hence, because the continuation of a Poisson process beyond any time remains a Poisson process, it follows that $\{X(t),\ t \geqslant 0\}$ is a stationary process.

Let us compute the mean and covariance function of the random telegraph signal

$$
\begin{aligned}
E[X(t)] &= E\left[X_0(-1)^{N(t)}\right] \\
&= E\left[X_0\right]E[(-1)^{N(t)}] \qquad \text{by independence} \\
&= 0 \qquad \text{since } E[X_0] = 0, \\
\operatorname{Cov}[X(t), X(t+s)] &= E[X(t)X(t+s)] \\
&= E\left[X_0^2(-1)^{N(t)+N(t+s)}\right] \\
&= E[(-1)^{2N(t)}(-1)^{N(t+s)-N(t)}] \\
&= E[(-1)^{N(t+s)-N(t)}] \\
&= E[(-1)^{N(s)}] \\
&= \sum_{i=0}^{\infty}(-1)^i e^{-\lambda s}\frac{(\lambda s)^i}{i!} \\
&= e^{-2\lambda s} \qquad (10.19)
\end{aligned}
$$

For an application of the random telegraph signal consider a particle moving at a constant unit velocity along a straight line and suppose that collisions involving this particle occur at a Poisson rate λ. Also suppose that each time the particle suffers a collision it reverses direction. Therefore, if X_0 represents the initial velocity of the particle, then its velocity at time t—call it $X(t)$—is given by $X(t) = X_0(-1)^{N(t)}$, where $N(t)$ denotes the number of collisions involving the particle by time t. Hence, if X_0 is equally likely to be plus or minus 1, and is independent of $\{N(t),\ t \geqslant 0\}$, then $\{X(t),\ t \geqslant 0\}$ is a random telegraph signal process. If we now let

$$D(t) = \int_0^t X(s)\,ds$$

then $D(t)$ represents the displacement of the particle at time t from its position at time 0. The mean and variance of $D(t)$ are obtained as follows:

$$\begin{aligned}
E[D(t)] &= \int_0^t E[X(s)]\,ds = 0,\\
\operatorname{Var}[D(t)] &= E[D^2(t)]\\
&= E\left[\int_0^t X(y)\,dy \int_0^t X(u)\,du\right]\\
&= \int_0^t \int_0^t E[X(y)X(u)]\,dy\,du\\
&= 2 \iint\limits_{0<y<u<t} E[X(y)X(u)]\,dy\,du\\
&= 2\int_0^t \int_0^u e^{-2\lambda(u-y)}\,dy\,du \qquad \text{by (10.19)}\\
&= \frac{1}{\lambda}\left(t - \frac{1}{2\lambda} + \frac{1}{2\lambda}e^{-2\lambda t}\right) \quad \blacksquare
\end{aligned}$$

The condition for a process to be stationary is rather stringent and so we define the process $\{X(t),\ t \geqslant 0\}$ to be a *second-order stationary* or a *weakly stationary* process if $E[X(t)] = c$ and $\operatorname{Cov}[X(t), X(t+s)]$ does not depend on t. That is, a process is second-order stationary if the first two moments of $X(t)$ are the same for all t and the covariance between $X(s)$ and $X(t)$ depends only on $|t - s|$. For a second-order stationary process, let

$$R(s) = \operatorname{Cov}[X(t), X(t+s)]$$

As the finite dimensional distributions of a Gaussian process (being multivariate normal) are determined by their means and covariance, it follows that a second-order stationary Gaussian process is stationary.

Example 10.6 (The Ornstein–Uhlenbeck Process) Let $\{X(t),\ t \geqslant 0\}$ be a standard Brownian motion process, and define, for $\alpha > 0$,

$$V(t) = e^{-\alpha t/2} X(e^{\alpha t})$$

The process $\{V(t),\ t \geqslant 0\}$ is called the Ornstein–Uhlenbeck process. It has been proposed as a model for describing the velocity of a particle immersed in a liquid or gas, and as such is useful in statistical mechanics. Let us compute its mean and covariance function

$$\begin{aligned} E[V(t)] &= 0, \\ \text{Cov}[V(t), V(t+s)] &= e^{-\alpha t/2} e^{-\alpha(t+s)/2} \text{Cov}[X(e^{\alpha t}), X(e^{\alpha(t+s)})] \\ &= e^{-\alpha t} e^{-\alpha s/2} e^{\alpha t} \qquad \text{by Equation (10.17)} \\ &= e^{-\alpha s/2} \end{aligned}$$

Hence, $\{V(t),\ t \geqslant 0\}$ is weakly stationary and as it is clearly a Gaussian process (since Brownian motion is Gaussian) we can conclude that it is stationary. It is interesting to note that (with $\alpha = 4\lambda$) it has the same mean and covariance function as the random telegraph signal process, thus illustrating that two quite different processes can have the same second-order properties. (Of course, if two Gaussian processes have the same mean and covariance functions then they are identically distributed.) ■

As the following examples show, there are many types of second-order stationary processes that are not stationary.

Example 10.7 (An Autoregressive Process) Let $Z_0, Z_1, Z_2, \ldots$ be uncorrelated random variables with $E[Z_n] = 0$, $n \geqslant 0$ and

$$\text{Var}(Z_n) = \begin{cases} \sigma^2/(1-\lambda^2), & n = 0 \\ \sigma^2, & n \geqslant 1 \end{cases}$$

where $\lambda^2 < 1$. Define

$$\begin{aligned} X_0 &= Z_0, \\ X_n &= \lambda X_{n-1} + Z_n, \qquad n \geqslant 1 \end{aligned} \tag{10.20}$$

The process $\{X_n,\ n \geqslant 0\}$ is called a *first-order autoregressive process*. It says that the state at time n (that is, X_n) is a constant multiple of the state at time $n-1$ plus a random error term Z_n.

Iterating Equation (10.20) yields

$$\begin{aligned}
X_n &= \lambda(\lambda X_{n-2} + Z_{n-1}) + Z_n \\
&= \lambda^2 X_{n-2} + \lambda Z_{n-1} + Z_n \\
&\;\vdots \\
&= \sum_{i=0}^{n} \lambda^{n-i} Z_i
\end{aligned}$$

and so

$$\begin{aligned}
\mathrm{Cov}(X_n, X_{n+m}) &= \mathrm{Cov}\left(\sum_{i=0}^{n} \lambda^{n-i} Z_i, \sum_{i=0}^{n+m} \lambda^{n+m-i} Z_i \right) \\
&= \sum_{i=0}^{n} \lambda^{n-i} \lambda^{n+m-i}\, \mathrm{Cov}(Z_i, Z_i) \\
&= \sigma^2 \lambda^{2n+m} \left(\frac{1}{1-\lambda^2} + \sum_{i=1}^{n} \lambda^{-2i} \right) \\
&= \frac{\sigma^2 \lambda^m}{1-\lambda^2}
\end{aligned}$$

where the preceding uses the fact that Z_i and Z_j are uncorrelated when $i \neq j$. As $E[X_n] = 0$, we see that $\{X_n,\ n \geqslant 0\}$ is weakly stationary (the definition for a discrete time process is the obvious analog of that given for continuous time processes). ■

Example 10.8 If, in the random telegraph signal process, we drop the requirement that $P\{X_0 = 1\} = P\{X_0 = -1\} = \frac{1}{2}$ and only require that $E[X_0] = 0$, then the process $\{X(t),\ t \geqslant 0\}$ need no longer be stationary. (It will remain stationary if X_0 has a symmetric distribution in the sense that $-X_0$ has the same distribution as X_0.) However, the process will be weakly stationary since

$$\begin{aligned}
E[X(t)] &= E[X_0] E\big[(-1)^{N(t)}\big] = 0, \\
\mathrm{Cov}[X(t), X(t+s)] &= E[X(t)X(t+s)] \\
&= E\big[X_0^2\big] E\big[(-1)^{N(t)+N(t+s)}\big] \\
&= E\big[X_0^2\big] e^{-2\lambda s} \qquad \text{from (10.19)} \quad ■
\end{aligned}$$

Example 10.9 Let $W_0, W_1, W_2, \ldots$ be uncorrelated with $E[W_n] = \mu$ and $\text{Var}(W_n) = \sigma^2$, $n \geqslant 0$, and for some positive integer k define

$$X_n = \frac{W_n + W_{n-1} + \cdots + W_{n-k}}{k+1}, \qquad n \geqslant k$$

The process $\{X_n, n \geqslant k\}$, which at each time keeps track of the arithmetic average of the most recent $k+1$ values of the Ws, is called a moving average process. Using the fact that the $W_n, n \geqslant 0$ are uncorrelated, we see that

$$\text{Cov}(X_n, X_{n+m}) = \begin{cases} \dfrac{(k+1-m)\sigma^2}{(k+1)^2}, & \text{if } 0 \leqslant m \leqslant k \\ 0, & \text{if } m > k \end{cases}$$

Hence, $\{X_n,\ n \geqslant k\}$ is a second-order stationary process. ■

Let $\{X_n,\ n \geqslant 1\}$ be a second-order stationary process with $E[X_n] = \mu$. An important question is when, if ever, does $\bar{X}_n \equiv \sum_{i=1}^{n} X_i/n$ converge to μ? The following proposition, which we state without proof, shows that $E[(\bar{X}_n - \mu)^2] \to 0$ if and only if $\sum_{i=1}^{n} R(i)/n \to 0$. That is, the expected square of the difference between $\bar{X}_n$ and μ will converge to 0 if and only if the limiting average value of $R(i)$ converges to 0.

Proposition 10.2 Let $\{X_n, n \geqslant 1\}$ be a second-order stationary process having mean μ and covariance function $R(i) = \text{Cov}(X_n, X_{n+i})$, and let $\bar{X}_n \equiv \sum_{i=1}^{n} X_i/n$. Then $\lim_{n\to\infty} E[(\bar{X}_n - \mu)^2] = 0$ and only if $\lim_{n\to\infty} \sum_{i=1}^{n} R(i)/n = 0$.

10.8. Harmonic Analysis of Weakly Stationary Processes

Suppose that the stochastic processes $\{X(t),\ -\infty < t < \infty\}$ and $\{Y(t),\ -\infty < t < \infty\}$ are related as follows:

$$Y(t) = \int_{-\infty}^{\infty} X(t-s)h(s)\,ds \tag{10.21}$$

We can imagine that a signal, whose value at time t is $X(t)$, is passed through a physical system that distorts its value so that $Y(t)$, the received value at t, is given by Equation (10.21). The processes $\{X(t)\}$ and $\{Y(t)\}$ are called, respectively, the input and output processes. The function h is called the *impulse response* function. If $h(s) = 0$ whenever $s < 0$, then h is also called a weighting function since Equation (10.21) expresses the output at t as a weighted integral of all the inputs prior to t with $h(s)$ representing the weight given the input s time units ago.

The relationship expressed by Equation (10.21) is a special case of a time invariant linear filter. It is called a filter because we can imagine that the input process $\{X(t)\}$ is passed through a medium and then filtered to yield the output process $\{Y(t)\}$. It is a linear filter because if the input processes $\{X_i(t)\}$, $i = 1, 2$, result in the output processes $\{Y_i(t)\}$—that is, if $Y_i(t) = \int_0^\infty X_i(t-s)h(s)\,ds$—then the output process corresponding to the input process $\{aX_1(t) + bX_2(t)\}$ is just $\{aY_1(t) + bY_2(t)\}$. It is called time invariant since lagging the input process by a time τ—that is, considering the new input process $\bar{X}(t) = X(t+\tau)$—results in a lag of τ in the output process since

$$\int_0^\infty \bar{X}(t-s)h(s)\,ds = \int_0^\infty X(t+\tau-s)h(s)\,ds = Y(t+\tau)$$

Let us now suppose that the input process $\{X(t),\ -\infty < t < \infty\}$ is weakly stationary with $E[X(t)] = 0$ and covariance function

$$R_X(s) = \text{Cov}[X(t), X(t+s)].$$

Let us compute the mean value and covariance function of the output process $\{Y(t)\}$.

Assuming that we can interchange the expectation and integration operations (a sufficient condition being that $\int |h(s)| < \infty$* and, for some $M < \infty$, $E|X(t)| < M$ for all t) we obtain

$$E[Y(t)] = \int E[X(t-s)]h(s)\,ds = 0$$

Similarly,

$$\begin{aligned}
\text{Cov}[Y(t_1), Y(t_2)] &= \text{Cov}\left[\int X(t_1 - s_1)h(s_1)\,ds_1,\ \int X(t_2 - s_2)h(s_2)\,ds_2\right] \\
&= \iint \text{Cov}[X(t_1 - s_1), X(t_2 - s_2)]h(s_1)h(s_2)\,ds_1\,ds_2 \\
&= \iint R_X(t_2 - s_2 - t_1 + s_1)h(s_1)h(s_2)\,ds_1\,ds_2 \qquad (10.22)
\end{aligned}$$

Hence, $\text{Cov}[Y(t_1),\ Y(t_2)]$ depends on t_1, t_2 only through $t_2 - t_1$; thus, showing that $\{Y(t)\}$ is also weakly stationary.

*The range of all integrals in this section is from $-\infty$ to $+\infty$.

The preceding expression for $R_Y(t_2 - t_1) = \text{Cov}[Y(t_1),\ Y(t_2)]$ is, however, more compactly and usefully expressed in terms of Fourier transforms of R_X and R_Y. Let, for $i = \sqrt{-1}$,

$$\widetilde{R}_X(w) = \int e^{-iws} R_X(s)\, ds$$

and

$$\widetilde{R}_Y(w) = \int e^{-iws} R_Y(s)\, ds$$

denote the Fourier transforms, respectively, of R_X and R_Y. The function $\widetilde{R}_X(w)$ is also called the *power spectral density* of the process $\{X(t)\}$. Also, let

$$\widetilde{h}(w) = \int e^{-iws} h(s)\, ds$$

denote the Fourier transform of the function h. Then, from Equation (10.22),

$$\begin{aligned}\widetilde{R}_Y(w) &= \iiint e^{iws} R_X(s - s_2 + s_1) h(s_1) h(s_2)\, ds_1\, ds_2\, ds \\ &= \iiint e^{iw(s-s_2+s_1)} R_X(s - s_2 + s_1)\, ds e^{-iws_2} h(s_2)\, ds_2 e^{iws_1} h(s_1)\, ds_1 \\ &= \widetilde{R}_X(w) \widetilde{h}(w) \widetilde{h}(-w) \end{aligned} \tag{10.23}$$

Now, using the representation

$$\begin{aligned} e^{ix} &= \cos x + i \sin x, \\ e^{-ix} &= \cos(-x) + i \sin(-x) = \cos x - i \sin x \end{aligned}$$

we obtain

$$\begin{aligned} \widetilde{h}(w)\widetilde{h}(-w) &= \left[\int h(s)\cos(ws)\, ds - i \int h(s) \sin(ws)\, ds\right] \\ &\quad \times \left[\int h(s)\cos(ws)\, ds + i \int h(s)\sin(ws)\, ds\right] \\ &= \left[\int h(s)\cos(ws)\, ds\right]^2 + \left[\int h(s)\sin(ws)\, ds\right]^2 \\ &= \left|\int h(s) e^{-iws}\, ds\right|^2 = |\widetilde{h}(w)|^2 \end{aligned}$$

Hence, from Equation (10.23) we obtain

$$\widetilde{R}_Y(w) = \widetilde{R}_X(w)|\widetilde{h}(w)|^2$$

In words, the Fourier transform of the covariance function of the output process is equal to the square of the amplitude of the Fourier transform of the impulse function multiplied by the Fourier transform of the covariance function of the input process.

Exercises

In the following exercises $\{B(t),\ t \geqslant 0\}$ is a standard Brownian motion process and T_a denotes the time it takes this process to hit a.

***1.** What is the distribution of $B(s) + B(t)$, $s \leqslant t$?

2. Compute the conditional distribution of $B(s)$ given that $B(t_1) = A$, $B(t_2) = B$, where $0 < t_1 < s < t_2$.

***3.** Compute $E[B(t_1)B(t_2)B(t_3)]$ for $t_1 < t_2 < t_3$.

4. Show that

$$P\{T_a < \infty\} = 1,$$
$$E[T_a] = \infty, \qquad a \neq 0$$

***5.** What is $P\{T_1 < T_{-1} < T_2\}$?

6. Suppose you own one share of a stock whose price changes according to a standard Brownian motion process. Suppose that you purchased the stock at a price $b + c$, $c > 0$, and the present price is b. You have decided to sell the stock either when it reaches the price $b + c$ or when an additional time t goes by (whichever occurs first). What is the probability that you do not recover your purchase price?

7. Compute an expression for

$$P\left\{\max_{t_1 \leqslant s \leqslant t_2} B(s) > x\right\}$$

8. Consider the random walk which in each Δt time unit either goes up or down the amount $\sqrt{\Delta t}$ with respective probabilities p and $1 - p$ where $p = \frac{1}{2}(1 + \mu\sqrt{\Delta t})$.

(a) Argue that as $\Delta t \to 0$ the resulting limiting process is a Brownian motion process with drift rate μ.

(b) Using part (a) and the results of the gambler's ruin problem (Section 4.5.1), compute the probability that a Brownian motion process with drift rate μ goes up A before going down B, $A > 0$, $B > 0$.

9. Let $\{X(t),\ t \geqslant 0\}$ be a Brownian motion with drift coefficient μ and variance parameter σ^2. What is the joint density function of $X(s)$ and $X(t)$, $s < t$?

***10.** Let $\{X(t),\ t \geqslant 0\}$ be a Brownian motion with drift coefficient μ and variance parameter σ^2. What is the conditional distribution of $X(t)$ given that $X(s) = c$ when

(a) $s < t$?
(b) $t < s$?

11. Consider a process whose value changes every h time units; its new value being its old value multiplied either by the factor $e^{\sigma\sqrt{h}}$ with probability $p = \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{h})$, or by the factor $e^{-\sigma\sqrt{h}}$ with probability $1 - p$. As h goes to zero, show that this process converges to geometric Brownian motion with drift coefficient μ and variance parameter σ^2.

12. A stock is presently selling at a price of \$50 per share. After one time period, its selling price will (in present value dollars) be either \$150 or \$25. An option to purchase y units of the stock at time 1 can be purchased at cost cy.

(a) What should c be in order for there to be no sure win?
(b) If $c = 4$, explain how you could guarantee a sure win.
(c) If $c = 10$, explain how you could guarantee a sure win.
(d) Use the arbitrage theorem to verify your answer to part (a).

13. Verify the statement made in the remark following Example 10.2.

14. The present price of a stock is 100. The price at time 1 will be either 50, 100, or 200. An option to purchase y shares of the stock at time 1 for the (present value) price ky costs cy.

(a) If $k = 120$, show that an arbitrage opportunity occurs if and only if $c > 80/3$.
(b) If $k = 80$, show that there is not an arbitrage opportunity if and only if $20 \leqslant c \leqslant 40$.

15. The current price of a stock is 100. Suppose that the logarithm of the price of the stock changes according to a Brownian motion with drift coefficient $\mu = 2$ and variance parameter $\sigma^2 = 1$. Give the Black-Scholes cost of an option to buy the stock at time 10 for a cost of

(a) 100 per unit.
(b) 120 per unit.
(c) 80 per unit.

Assume that the continuously compounded interest rate is 5 percent.

A stochastic process $\{Y(t),\ t \geqslant 0\}$ is said to be a *Martingale* process if, for $s < t$,

$$E[Y(t)|Y(u),\ 0 \leqslant u \leqslant s] = Y(s)$$

16. If $\{Y(t),\ t \geqslant 0\}$ is a Martingale, show that

$$E[Y(t)] = E[Y(0)]$$

17. Show that standard Brownian motion is a Martingale.

18. Show that $\{Y(t),\ t \geqslant 0\}$ is a Martingale when

$$Y(t) = B^2(t) - t$$

What is $E[Y(t)]$?

Hint: First compute $E[Y(t)|B(u),\ 0 \leqslant u \leqslant s]$.

***19.** Show that $\{Y(t), t \geqslant 0\}$ is a Martingale when

$$Y(t) = \exp\{cB(t) - c^2t/2\}$$

where c is an arbitrary constant. What is $E[Y(t)]$?

An important property of a Martingale is that if you continually observe the process and then stop at some time T, then, subject to some technical conditions (which will hold in the problems to be considered),

$$E[Y(T)] = E[Y(0)]$$

The time T usually depends on the values of the process and is known as a *stopping time* for the Martingale. This result, that the expected value of the stopped Martingale is equal to its fixed time expectation, is known as the *Martingale stopping theorem*.

***20.** Let

$$T = \mathrm{Min}\{t:\ \ B(t) = 2 - 4t\}$$

That is, T is the first time that standard Brownian motion hits the line $2 - 4t$. Use the Martingale stopping theorem to find $E[T]$.

21. Let $\{X(t),\ t \geqslant 0\}$ be Brownian motion with drift coefficient μ and variance parameter σ^2. That is,

$$X(t) = \sigma B(t) + \mu t$$

Let $\mu > 0$, and for a positive constant x let

$$\begin{aligned} T &= \text{Min}\{t:\ X(t) = x\} \\ &= \text{Min}\left\{t:\ B(t) = \frac{x - \mu t}{\sigma}\right\} \end{aligned}$$

That is, T is the first time the process $\{X(t),\ t \geqslant 0\}$ hits x. Use the Martingale stopping theorem to show that

$$E[T] = x/\mu$$

22. Let $X(t) = \sigma B(t) + \mu t$, and for given positive constants A and B, let p denote the probability that $\{X(t),\ t \geqslant 0\}$ hits A before it hits $-B$.

(a) Define the stopping time T to be the first time the process hits either A or $-B$. Use this stopping time and the Martingale defined in Exercise 19 to show that

$$E[\exp\{c(X(T) - \mu T)/\sigma - c^2 T/2\}] = 1$$

(b) Let $c = -2\mu/\sigma$, and show that

$$E[\exp\{-2\mu X(T)/\sigma\}] = 1$$

(c) Use part (b) and the definition of T to find p.

Hint: What are the possible values of $\exp\{-2\mu X(T)/\sigma^2\}$?

23. Let $X(t) = \sigma B(t) + \mu t$, and define T to be the first time the process $\{X(t),\ t \geqslant 0\}$ hits either A or $-B$, where A and B are given positive numbers. Use the Martingale stopping theorem and part (c) of Exercise 22 to find $E[T]$.

***24.** Let $\{X(t),\ t \geqslant 0\}$ be Brownian motion with drift coefficient μ and variance parameter σ^2. Suppose that $\mu > 0$. Let $x > 0$ and define the stopping time T (as in Exercise 21) by

$$T = \text{Min}\{t:\ X(t) = x\}$$

Use the Martingale defined in Exercise 18, along with the result of Exercise 21, to show that

$$\text{Var}(T) = x\sigma^2/\mu^3$$

25. Compute the mean and variance of

(a) $\int_0^1 t\,dB(t)$.
(b) $\int_0^1 t^2\,dB(t)$.

26. Let $Y(t) = tB(1/t)$, $t > 0$ and $Y(0) = 0$.

(a) What is the distribution of $Y(t)$?
(b) Compare $\text{Cov}(Y(s), Y(t))$.
(c) Argue that $\{Y(t),\ t \geqslant 0\}$ is a standard Brownian motion process.

***27.** Let $Y(t) = B(a^2 t)/a$ for $a > 0$. Argue that $\{Y(t)\}$ is a standard Brownian motion process.

28. For $s < t$, argue that $B(s) - \frac{s}{t}B(t)$ and $B(t)$ are independent.

29. Let $\{Z(t),\ t \geqslant 0\}$ denote a Brownian bridge process. Show that if

$$Y(t) = (t+1)Z(t/(t+1))$$

then $\{Y(t),\ t \geqslant 0\}$ is a standard Brownian motion process.

30. Let $X(t) = N(t+1) - N(t)$ where $\{N(t),\ t \geqslant 0\}$ is a Poisson process with rate λ. Compute

$$\text{Cov}[X(t), X(t+s)]$$

***31.** Let $\{N(t), t \geqslant 0\}$ denote a Poisson process with rate λ and define $Y(t)$ to be the time from t until the next Poisson event.

(a) Argue that $\{Y(t),\ t \geqslant 0\}$ is a stationary process.
(b) Compute $\text{Cov}[Y(t), Y(t+s)]$.

32. Let $\{X(t), -\infty < t < \infty\}$ be a weakly stationary process having covariance function $R_X(s) = \text{Cov}[X(t), X(t+s)]$.

(a) Show that

$$\text{Var}(X(t+s) - X(t)) = 2R_X(0) - 2R_X(t)$$

(b) If $Y(t) = X(t+1) - X(t)$ show that $\{Y(t),\ -\infty < t < \infty\}$ is also weakly stationary having a covariance function $R_Y(s) = \text{Cov}[Y(t), Y(t+s)]$ that satisfies

$$R_Y(s) = 2R_X(s) - R_X(s-1) - R_X(s+1)$$

33. Let Y_1 and Y_2 be independent unit normal random variables and for some constant w set

$$X(t) = Y_1 \cos wt + Y_2 \sin wt, \qquad -\infty < t < \infty$$

(a) Show that $\{X(t)\}$ is a weakly stationary process.
(b) Argue that $\{X(t)\}$ is a stationary process.

34. Let $\{X(t),\ -\infty < t < \infty\}$ be weakly stationary with covariance function $R(s) = \text{Cov}(X(t), X(t+s))$ and let $\widetilde{R}(w)$ denote the power spectral density of the process.

(i) Show that $\widetilde{R}(w) = \widetilde{R}(-w)$. It can be shown that

$$R(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{R}(w) e^{iws}\, dw$$

(ii) Use the preceding to show that

$$\int_{-\infty}^{\infty} \widetilde{R}(w)\, dw = 2\pi E[X^2(t)]$$

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Chapter 10

1. $B(s) + B(t) = 2B(s) + B(t) - B(s)$. Now $2B(s)$ is normal with mean 0 and variance $4s$ and $B(t) - B(s)$ is normal with mean 0 and variance $t - s$. Because $B(s)$ and $B(t) - B(s)$ are independent, it follows that $B(s) + B(t)$ is normal with mean 0 and variance $4s + t - s = 3s + t$.

3.
$$\begin{aligned}
E[B(t_1)B(t_2)B(t_3)] &= E[E[B(t_1)B(t_2)B(t_3)|B(t_1),\ B(t_2)]] \\
&= E[B(t_1)B(t_2)E[B(t_3) \mid B(t_1), B(t_2)]] \\
&= E[B(t_1)B(t_2)B(t_2)] \\
&= E[E[B(t_1)B^2(t_2) \mid B(t_1)]] \\
&= E[B(t_1)E[B^2(t_2) \mid B(t_1)]] \\
&= E[B(t_1)\{(t_2 - t_1) + B^2(t_1)\}] \qquad (*) \\
&= E[B^3(t_1)] + (t_2 - t_1)E[B(t_1)] \\
&= 0
\end{aligned}$$

where the equality $(*)$ follows since given $B(t_1)$, $B(t_2)$ is normal with mean $B(t_1)$ and variance $t_2 - t_1$. Also, $E[B^3(t)] = 0$ since $B(t)$ is normal with mean 0.

5.
$$\begin{aligned}
P\{T_1 < T_{-1} < T_2\} &= P\{\text{hit 1 before } -1 \text{ before } 2\} \\
&= P\{\text{hit 1 before } -1\} \\
&\quad \times P\{\text{hit } -1 \text{ before } 2 \mid \text{hit 1 before } -1\} \\
&= \tfrac{1}{2} P\{\text{down 2 before up 1}\} \\
&= \frac{1}{2}\frac{1}{3} = \frac{1}{6}
\end{aligned}$$

The next to last equality follows by looking at the Brownian motion when it first hits 1.

10. (a) Writing $X(t) = X(s) + X(t) - X(s)$ and using independent increments, we see that given $X(s) = c$, $X(t)$ is distributed as $c + X(t) - X(s)$. By stationary increments this has the same distribution as $c + X(t - s)$, and is thus normal with mean $c + \mu(t - s)$ and variance $(t - s)\sigma^2$.
(b) Use the representation $X(t) = \sigma B(t) + \mu t$, where $\{B(t)\}$ is standard Brownian motion. Using Equation (1.4), but reversing s and t, we see that the conditional distribution of $B(t)$ given that $B(s) = (c - \mu s)/\sigma$ is normal with mean $t(c - \mu s)/(\sigma s)$ and variance $t(s - t)/s$. Thus, the conditional

distribution of $X(t)$ given that $X(s) = c, s > t$, is normal with mean

$$\sigma\left[\frac{t(c-\mu s)}{\sigma s}\right] + \mu t = \frac{(c-\mu s)t}{s} + \mu t$$

and variance

$$\frac{\sigma^2 t(s-t)}{s}$$

19. Since knowing the value of $Y(t)$ is equivalent to knowing $B(t)$, we have

$$\begin{aligned} E[Y(t) \mid Y(u),\ 0 \leqslant u \leqslant s] &= e^{-c^2t/2} E[e^{cB(t)} \mid B(u), 0 \leqslant u \leqslant s] \\ &= e^{-c^2t/2} E[e^{cB(t)} \mid B(s)] \end{aligned}$$

Now, given $B(s)$, the conditional distribution of $B(t)$ is normal with mean $B(s)$ and variance $t - s$. Using the formula for the moment generating function of a normal random variable we see that

$$\begin{aligned} e^{-c^2t/2} E[e^{cB(t)} | B(s)] &= e^{-c^2t/2} e^{cB(s)+(t-s)c^2/2} \\ &= e^{-c^2s/2} e^{cB(s)} \\ &= Y(s) \end{aligned}$$

Thus $\{Y(t)\}$ is a Martingale.

$$E[Y(t)] = E[Y(0)] = 1$$

20. By the Martingale stopping theorem

$$E[B(T)] = E[B(0)] = 0$$

However, $B(T) = 2 - 4T$ and so $2 - 4E[T] = 0$, or $E[T] = \frac{1}{2}$.

24. It follows from the Martingale stopping theorem and the result of Exercise 18 that

$$E[B^2(T) - T] = 0$$

where T is the stopping time given in this problem and

$$B(t) = \frac{X(t) - \mu t}{\sigma}$$

Therefore,

$$E\left[\frac{(X(T)-\mu T)^2}{\sigma^2} - T\right] = 0$$

However, $X(T) = x$ and so the preceding gives that

$$E[(x-\mu T)^2] = \sigma^2 E[T]$$

But, from Exercise 21, $E[T] = x/\mu$ and so the preceding is equivalent to

$$\mathrm{Var}(\mu T) = \sigma^2 \frac{x}{\mu} \quad \text{or} \quad \mathrm{Var}(T) = \sigma^2 \frac{x}{\mu^3}$$

27. $E[X(a^2 t)/a] = (1/a)E[X(a^2 t)] = 0$. For $s < t$,

$$\begin{aligned}\mathrm{Cov}\big(Y(s), Y(t)\big) &= \frac{1}{a^2}\mathrm{Cov}\big(X(a^2 s), X(a^2 t)\big) \\ &= \frac{1}{a^2} a^2 s = s\end{aligned}$$

Because $\{Y(t)\}$ is clearly Gaussian, the result follows.

30. (a) Starting at any time t the continuation of the Poisson process remains a Poisson process with rate λ.

(b) $$\begin{aligned}E[Y(t)Y(t+s)] &= \int_0^\infty E[Y(t)Y(t+s)|Y(t)=y]\lambda e^{-\lambda y}\,dy \\ &= \int_0^s yE[Y(t+s)|Y(t)=y]\lambda e^{-\lambda y}\,dy \\ &\quad + \int_s^\infty y(y-s)\lambda e^{-\lambda y}\,dy \\ &= \int_0^s y\frac{1}{\lambda}\lambda e^{-\lambda y}\,dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y}\,dy\end{aligned}$$

where the preceding used that

$$E[Y(t)Y(t+s)|Y(t)=y] = \begin{cases} yE(Y(t+s)) = \dfrac{y}{\lambda}, & \text{if } y < s \\ y(y-s), & \text{if } y > s \end{cases}$$

Hence,

$$\mathrm{Cov}(Y(t), Y(t+s)) = \int_0^s ye^{-\lambda y}\,dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y}\,dy - \frac{1}{\lambda^2}$$