STAT 433 Spring 2022
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Examples for 03/10/2022

Example 1:

HPS 3.3

3 Find the conditional probability that there are m events in the first s units of time, given that there are n events in the first t units of time, where $0 \le m \le n$ and $0 \le s \le t$.

That is, Find $\mathrm{P}(\mathrm{X}(s) = m \mid \mathrm{X}(t) = n)$ for $0 \le m \le n$ and $0 \le s \le t$, where $\mathrm{X}(t)$ is a Poisson process with rate λ.

Def $\mathrm{P}(\mathrm{A} \mid \mathrm{B}) = \dfrac{\mathrm{P}(\mathrm{A} \cap \mathrm{B})}{\mathrm{P}(\mathrm{B})}$, provided $\mathrm{P}(\mathrm{B}) > 0$.

$$\mathrm{P}(\mathrm{X}(s) = m \mid \mathrm{X}(t) = n) = \frac{\mathrm{P}(\mathrm{X}(s) = m \cap \mathrm{X}(t) = n)}{\mathrm{P}(\mathrm{X}(t) = n)}$$

$$= \frac{\mathrm{P}(\mathrm{X}(s) = m \cap \mathrm{X}(t) - \mathrm{X}(s) = n - m)}{\mathrm{P}(\mathrm{X}(t) = n)}$$

since $\mathrm{X}(s)$ and $\mathrm{X}(t) - \mathrm{X}(s)$ are independent

$$= \frac{\mathrm{P}(\mathrm{X}(s) = m) \times \mathrm{P}(\mathrm{X}(t) - \mathrm{X}(s) = n - m)}{\mathrm{P}(\mathrm{X}(t) = n)}$$

$$= \frac{\dfrac{(\lambda s)^m e^{-\lambda s}}{m!} \times \dfrac{(\lambda(t - s))^{n-m} e^{-\lambda(t-s)}}{(n-m)!}}{\dfrac{(\lambda t)^n e^{-\lambda t}}{n!}}$$

$$= \frac{n!}{m!(n-m)!} \times \left(\frac{s}{t}\right)^m \times \left(1 - \frac{s}{t}\right)^{n-m}, \qquad 0 \le m \le n, \quad 0 \le s \le t.$$

Given $\mathrm{X}(t) = n$, $\mathrm{X}(s)$ has a $\text{Binomial}(n, p = \frac{s}{t})$ distribution.

Example 2:

Let $X(t)$ be a Poisson process with rate λ. Let T_1 be the time of the first occurrence.

Find $P(T_1 < s \mid X(t) = 1)$ for $0 \le s \le t$.

$$P(T_1 \le s \mid X(t) = 1) = P(X(s) \ge 1 \mid X(t) = 1)$$

$$= P(X(s) = 1 \mid X(t) = 1)$$

$$= \frac{s}{t}, \qquad 0 \le s \le t.$$

That is, given that there is one occurrence between 0 and t, the time of that occurrence has a Uniform distribution between 0 and t.

Example 3:

HPS 3.6

6 Find $P(T_1 \le s \mid X(t) = n)$ for $0 \le s \le t$ and n a positive integer.

$$P(T_1 < s \mid X(t) = n) = P(X(s) \ge 1 \mid X(t) = n)$$

$$= 1 - P(X(s) = 0 \mid X(t) = n)$$

$$= 1 - \left(1 - \frac{s}{t}\right)^n, \qquad 0 \le s \le t.$$

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Let $X_1(t)$ and $X_2(t)$ be independent Poisson processes with parameters λ_1 and λ_2, respectively. Consider the processes $X(t) = X_1(t) + X_2(t)$.

1) $X(0) = X_1(0) + X_2(0) = 0.$

2) Let $0 \le s < t$. Then

$$X(t) - X(s) = \{X_1(t) - X_1(s)\} + \{X_2(t) - X_2(s)\}$$

has a Poisson distribution with mean

$$\lambda_1(t-s) + \lambda_2(t-s) = (\lambda_1 + \lambda_2)(t-s).$$

Let X and Y be independent Poisson random variables with means μ_1 and μ_2, respectively.

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = e^{\mu_1(e^t - 1)} \cdot e^{\mu_2(e^t - 1)} = e^{(\mu_1 + \mu_2)(e^t - 1)}.$$

OR

$$P(X + Y = n) = \sum_{k=0}^{n} P(X = k) \cdot P(Y = n - k) = \sum_{k=0}^{n} \frac{\mu_1^k \cdot e^{-\mu_1}}{k!} \cdot \frac{\mu_2^{n-k} \cdot e^{-\mu_2}}{(n-k)!}$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k! \cdot (n-k)!} \cdot \mu_1^k \cdot \mu_2^{n-k} = \frac{(\mu_1 + \mu_2)^n \cdot e^{-(\mu_1+\mu_2)}}{n!}.$$

Therefore, X + Y has a Poisson distribution with mean $\mu_1 + \mu_2$.

3) Let $0 \le s_1 < t_1 \le s_2 < t_2$. Then

$$X(t_1) - X(s_1) = \{X_1(t_1) - X_1(s_1)\} + \{X_2(t_1) - X_2(s_1)\}$$

and

$$X(t_2) - X(s_2) = \{X_1(t_2) - X_1(s_2)\} + \{X_2(t_2) - X_2(s_2)\}$$

are independent.

Let T_1 be the time until the next occurrence in $X_1(t)$.

Then T_1 has an Exponential distribution with rate λ_1.

Let T_2 be the time until the next occurrence in $X_2(t)$.

Then T_2 has an Exponential distribution with rate λ_2.

The next occurrence in $X(t)$ will happen at time $T = \min(T_1, T_2)$.

Then $T = \min(T_1, T_2)$ has an Exponential distribution with rate $\lambda_1 + \lambda_2$.

$X(t)$ is a Poisson process with parameter $\lambda_1 + \lambda_2$.

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Let $X(t)$ be a Poisson process with parameter λ.

Suppose that each occurrence in $X(t)$ is independently either "observed" with probability p or "missed" with probability $1-p$.
Let $Y(t)$ denote the number of "observed" occurrences on $[0, t]$.
Let $W(t)$ denote the number of "missed" occurrences on $[0, t]$.

Suppose that for each occurrence in $X(t)$, a coin with $P(H) = p$ and $P(T) = 1-p$ is tossed independently.
Let $Y(t)$ denote the number of H's on $[0, t]$.
Let $W(t)$ denote the number of T's on $[0, t]$.

Let m be a nonnegative integer.

$$P(Y(t) = m) = \sum_{n=m}^{\infty} P\left(X(t) = n \cap Y(t) = m\right)$$

$$= \sum_{n=m}^{\infty} P\left(X(t) = n\right) \times P\left(Y(t) = m \mid X(t) = n\right)$$

$$= \sum_{n=m}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \times \frac{n!}{m!(n-m)!} \times p^m \times (1-p)^{n-m}$$

$$= \frac{(\lambda p t)^m e^{-\lambda p t}}{m!} \times \sum_{n=m}^{\infty} \frac{(\lambda(1-p)t)^{n-m} e^{-\lambda(1-p)t}}{(n-m)!}$$

$$= \frac{(\lambda p t)^m e^{-\lambda p t}}{m!} \times \sum_{k=0}^{\infty} \frac{(\lambda(1-p)t)^{k} e^{-\lambda(1-p)t}}{k!}$$

$$= \frac{(\lambda p t)^m e^{-\lambda p t}}{m!}.$$

$\mathrm{Y}(t)$ is a Poisson process with parameter λp.

Similarly, $\mathrm{W}(t)$ is a Poisson process with parameter $\lambda(1-p)$.

Let m, k be nonnegative integers.

$$\mathrm{P}(\mathrm{Y}(t)=m \cap \mathrm{W}(t)=k) = \mathrm{P}(\mathrm{Y}(t)=m \cap \mathrm{X}(t)=m+k)$$

$$= \mathrm{P}(\mathrm{X}(t)=m+k) \times \mathrm{P}(\mathrm{Y}(t)=m \mid \mathrm{X}(t)=m+k)$$

$$= \frac{(\lambda t)^{m+k} e^{-\lambda t}}{(m+k)!} \times \frac{(m+k)!}{m!\,k!} \times p^m \times (1-p)^k$$

$$= \frac{(\lambda p t)^m e^{-\lambda p t}}{m!} \times \frac{(\lambda(1-p)t)^{k} e^{-\lambda(1-p)t}}{k!}$$

$$= \mathrm{P}(\mathrm{Y}(t)=m) \times \mathrm{P}(\mathrm{W}(t)=k).$$

$\mathrm{Y}(t)$ and $\mathrm{W}(t)$ are two independent Poisson processes with parameters λp and $\lambda(1-p)$, respectively.

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1. On a highway, cars pass according to a Poisson process with rate 5 per minute. Trucks pass according to a Poisson process with rate 3 per minute. The two processes are independent. Let $N_C(t)$ and $N_T(t)$ denote the number of cars and trucks that pass in t minutes, respectively. Then $N(t) = N_C(t) + N_T(t)$ is the number of vehicles that pass in t minutes.

a) Find $P(N_C(3) = 20)$.

$N_C(3)$ has a Poisson distribution with mean $5 \cdot 3 = 15$.

$$P(N_C(3) = 20) = \frac{15^{20} e^{-15}}{20!} = \mathbf{0.0418}.$$

b) Find $P(N(3) = 20)$.

$N(3)$ has a Poisson distribution with mean $(5+3) \cdot 3 = 24$.

$$P(N(3) = 20) = \frac{24^{20} e^{-24}}{20!} = \mathbf{0.0624}.$$

c) Find $P(N(3) = 20 \mid N(1) = 8)$.

For a Poisson process $X(t)$ with rate λ, $0 \le s \le t$,
$X(t) - X(s)$ has a Poisson distribution with mean $\lambda(t-s)$,
$X(t) - X(s)$ and $X(s)$ are independent.

$$P(N(3) = 20 \mid N(1) = 8) = P(N(3) - N(1) = 12) = \frac{16^{12} e^{-16}}{12!} = \mathbf{0.0661}.$$

d) Find $P(N(1) = 8 \mid N(3) = 20)$.

For a Poisson process $X(t)$ with rate λ, $0 \le s \le t$,

$X(s) \mid X(t) = n$ has a Binomial$\left(n,\ p = \frac{s}{t}\right)$ distribution.

$$P(N(1) = 8 \mid N(3) = 20) = \binom{20}{8}\left(\frac{1}{3}\right)^{8}\left(\frac{2}{3}\right)^{12} = \mathbf{0.1480}.$$

e) Find $P(N_T(3) = 7 \mid N(3) = 20)$.

We know that if X and Y are independent Poisson random variables with means μ_1 and μ_2, respectively, then $X + Y$ has a Poisson distribution with mean $\mu_1 + \mu_2$.

$$P(X = k \mid X + Y = n) = \frac{P(X = k \cap X + Y = n)}{P(X + Y = n)} = \frac{P(X = k \cap Y = n - k)}{P(X + Y = n)}$$

$$= \frac{\dfrac{\mu_1^k \cdot e^{-\mu_1}}{k!} \cdot \dfrac{\mu_2^{n-k} \cdot e^{-\mu_2}}{(n-k)!}}{\dfrac{(\mu_1 + \mu_2)^n \cdot e^{-(\mu_1+\mu_2)}}{n!}} = \frac{n!}{k! \cdot (n-k)!} \cdot \left(\frac{\mu_1}{\mu_1 + \mu_2}\right)^{k} \cdot \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^{n-k}.$$

$\Rightarrow$ $X \mid X + Y = n$ has a Binomial distribution, $p = \dfrac{\mu_1}{\mu_1 + \mu_2}$.

$$P(N_T(3) = 7 \mid N(3) = 20) = \binom{20}{7}\left(\frac{3}{5+3}\right)^{7}\left(\frac{5}{5+3}\right)^{13}$$

$$= \binom{20}{7}\left(\frac{3}{8}\right)^{7}\left(\frac{5}{8}\right)^{13} = \mathbf{0.1795}.$$

f) Find $E(N(4) \mid N_T(3) = 7)$.

Hint: $N_T(4) = \{N_T(4) - N_T(3)\} + N_T(3)$.

$$E(N(4) \mid N_T(3) = 7) = E(N_C(4) + N_T(4) \mid N_T(3) = 7)$$

$$= E(N_C(4) \mid N_T(3) = 7) + E(\{N_T(4) - N_T(3)\} + N_T(3) \mid N_T(3) = 7)$$

$$= E(N_C(4) \mid N_T(3) = 7) + E(\{N_T(4) - N_T(3)\} \mid N_T(3) = 7)$$
$$+ E(N_T(3) \mid N_T(3) = 7)$$

$N_C(4)$ and $N_T(3)$ are independent.

$\{N_T(4) - N_T(3)\}$ and $N_T(3)$ are independent.

$$= E(N_C(4)) + E(\{N_T(4) - N_T(3)\}) + E(N_T(3) \mid N_T(3) = 7)$$

$$= 5 \cdot 4 + 3 \cdot (4 - 3) + 7 = 20 + 3 + 7 = \mathbf{30}.$$

Compound Poisson Process:

2. Let $X(t)$ be a Poisson process with rate λ.

Let $S(t) = \sum_{i=1}^{X(t)} Y_i$, where $Y_1, Y_2, \ldots$ are independent, identically distributed random variables (independent of $X(t)$) with mean μ and variance σ^2.

a) Find the mean and the variance of $S(t)$.

Hint 1: $$E[(S(t))^k] = \sum_{x=1}^{\infty} P(X(t)=x) \cdot E\left[\left(\sum_{i=1}^{x} Y_i\right)^k\right], \quad k = 1, 2.$$

Hint 2: $$\sigma^2 = \operatorname{Var}(Y) = E(Y^2) - [E(Y)]^2 = E(Y^2) - \mu^2.$$

$$E[S(t)] = \sum_{x=1}^{\infty} P(X(t)=x) \cdot E\left[\sum_{i=1}^{x} Y_i\right] = \sum_{x=1}^{\infty} P(X(t)=x) \cdot x\mu = \mu E[X(t)] = \mu\lambda t.$$

$$E\left[\left(\sum_{i=1}^{x} Y_i\right)^2\right] = \sum_{i=1}^{x} E\left[Y_i^2\right] + \sum_{i=1}^{x} \sum_{j \neq i} E\left[Y_i Y_j\right] = x(\sigma^2 + \mu^2) + x(x-1)\mu^2$$

$$= x\sigma^2 + x^2\mu^2.$$

$$E[(S(t))^2] = \sum_{x=1}^{\infty} P(X(t)=x) \cdot \left(x\sigma^2 + x^2\mu^2\right) = \sigma^2 E[X(t)] + \mu^2 E[(X(t))^2]$$

$$= \sigma^2 \lambda t + \mu^2 [(\lambda t)^2 + \lambda t].$$

$$\operatorname{Var}[S(t)] = \sigma^2 \lambda t + \mu^2 [(\lambda t)^2 + \lambda t] - (\mu\lambda t)^2 = (\sigma^2 + \mu^2)\lambda t. = \lambda t E(Y^2).$$

b) A person makes shopping trips according to a Poisson process with rate λ. The number of purchases he makes during each shopping trip is distributed according to a Geometric distribution with probability of "success" p. What are the mean and variance of the number of purchases made by time t?

$$Y = \text{Geometric}, \qquad \mu = \frac{1}{p}, \qquad \sigma^2 = \frac{1-p}{p^2}.$$

$$E[S(t)] = \mu \lambda t = \frac{\lambda t}{p},$$

$$\operatorname{Var}[S(t)] = (\sigma^2 + \mu^2) \lambda t = \frac{2-p}{p^2} \lambda t.$$

c) Suppose that cars arrive to a fair according to a Poisson process with rate λ. The number of passengers (in addition to the driver) in a car has a Binomial$(n = 3, p)$ distribution. What are the mean and the variance of the number of people who have arrived by time t?

Hint: $Y = 1 + \text{Binomial}(n = 3, p) = \text{driver} + \text{passengers}$.

$$Y = 1 + \text{Binomial}, \qquad \mu = 1 + 3p, \qquad \sigma^2 = 3p(1-p).$$

$$E[S(t)] = \mu \lambda t = (1 + 3p) \lambda t,$$

$$\operatorname{Var}[S(t)] = (\sigma^2 + \mu^2) \lambda t = [3p(1-p) + (1+3p)^2] \lambda t = [1 + 9p + 6p^2] \lambda t.$$