STAT 433 | **Examples for 02/15/2022 (1)** | Spring 2022 A. Stepanov

Recall:

Let $A \subset S$. The *hitting time* T_A of A is defined by $T_A = \min\{ n > 0 : X_n \in A \}$.

T_A is the first (positive) time the Markov chain hits A.

$T_A = \infty$ if $X_n \notin A$ for all $n > 0$ (if the Markov chain never hits A).

Let $a \in S$. Let $T_a = T_{\{a\}}$.

$\rho_{xy} = P_x(T_y < \infty)$.

(The probability that the Markov chain eventually visits y starting from x.)

State y is **recurrent** if $\rho_{yy} = 1$.

($\rho_{yy} = P_y(T_y < \infty) = 1 \Rightarrow$ the Markov chain is guaranteed to return to y starting from y.)

State y is **transient** if $\rho_{yy} < 1$.

($\rho_{yy} < 1 \Rightarrow$ the Markov chain is NOT guaranteed to return to y starting from y.)

Let y be a recurrent state.

$m_y = E_y(T_y)$ – mean return time to y for a Markov chain starting at y.

$E_x(1_y(X_n)) = P_x(X_n = y) = P^n(x, y)$.

$N_n(y) = \sum_{m=1}^{n} 1_y(X_m)$ – the number of times the Markov chain visits state y up to time n.

$G_n(x, y) = E_x(N_n(y)) = \sum_{m=1}^{n} P^m(x, y)$ – expected number of visits to y starting at x up to time n.

Let y be a transient state.

$$\Rightarrow \quad \mathrm{N}(y) = \lim_{n\to\infty} \mathrm{N}_n(y) < \infty \quad \text{with probability 1,}$$

$$\mathrm{G}(x,y) = \lim_{n\to\infty} \mathrm{G}_n(x,y) < \infty, \qquad x \in S.$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\mathrm{N}_n(y)}{n} = 0 \quad \text{with probability 1.}$$

$$\lim_{n\to\infty} \frac{\mathrm{G}_n(x,y)}{n} = 0.$$

Theorem Let y be a recurrent state.

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\mathrm{N}_n(y)}{n} = \frac{1\{\mathrm{T}_y < \infty\}}{m_y} \quad \text{with probability 1,}$$

$$\lim_{n\to\infty} \frac{\mathrm{G}_n(x,y)}{n} = \frac{\rho_{xy}}{m_y}, \qquad x \in S.$$

$\dfrac{\mathrm{N}_n(y)}{n}$ = proportion of the first n units of time that the chain is in state y.

$\dfrac{\mathrm{G}_n(x,y)}{n}$ = expected value of this proportion for a chain starting at x.

Corollary Let C be an irreducible closed set of recurrent states.

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\mathrm{G}_n(x,y)}{n} = \frac{1}{m_y}, \qquad x, y \in C.$$

If $\mathrm{P}(\mathrm{X}_0 \in C) = 1$, then with probability 1

$$\lim_{n\to\infty} \frac{\mathrm{N}_n(y)}{n} = \frac{1}{m_y}, \qquad y \in C.$$

If the chain returns to y every m_y steps (on average), then the proportion of time the chain is in y in the long run is about $\dfrac{1}{m_y}$.

y is **positive recurrent** if $m_y < \infty$.

y is **null recurrent** if $m_y = \infty$.

Theorem x is positive recurrent, x leads to y

$\Rightarrow$ y is positive recurrent.

An irreducible Markov chain (it is possible to get to any state from any state) is either a transient chain (all of its states are transient), or a null recurrent chain (all of its states are null recurrent), or a positive recurrent chain (all of its states are positive recurrent).

An irreducible Markov chain having a finite number of states is positive recurrent (all of its states are positive recurrent).

Silly example **1.** Let $0 < p < 1$. Consider a Markov Chain with the states being all nonnegative integer numbers, having transition function

$$\mathrm{P}(k, k+1) = 1 - p, \qquad \mathrm{P}(k, 0) = p, \qquad k \geq 0.$$

Determine whether state 0 (and the entire chain) is transient, null recurrent, or positive recurrent.

$$\mathrm{P}_0(\mathrm{T}_0 \geq m) = \mathrm{P}(0, 1) \times \mathrm{P}(1, 2) \times \mathrm{P}(2, 3) \times \ldots \times \mathrm{P}(m-3, m-2) \times \mathrm{P}(m-2, m-1)$$
$$= (1-p)^{m-1}.$$

$\Rightarrow$ $\mathrm{P}_0(\mathrm{T}_0 = \infty) = \lim\limits_{m \to \infty} \mathrm{P}_0(\mathrm{T}_0 \geq m) = 0.$

$\Rightarrow$ $\rho_{00} = \mathrm{P}_0(\mathrm{T}_0 < \infty) = 1.$ $\Rightarrow$ 0 is recurrent.

$$\mathrm{E}_0(\mathrm{T}_0) = \sum_{m=1}^{\infty} \mathrm{P}(\mathrm{T}_0 \geq m) = \sum_{m=1}^{\infty} (1-p)^{m-1} = \frac{1}{1-(1-p)} = \frac{1}{p} < \infty.$$

$\Rightarrow$ 0 is positive recurrent.

The Markov chain is irreducible (it is possible to get to any state from any state).

$\Rightarrow$ The Markov chain is positive recurrent.

Spoiler: Markov chain is irreducible, positive recurrent.

$\Rightarrow$ There is stationary distribution:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \cdots \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{array} \\
\left[\begin{array}{ccccccccc}
p & 1-p & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
p & 0 & 1-p & 0 & 0 & 0 & 0 & 0 & \cdots \\
p & 0 & 0 & 1-p & 0 & 0 & 0 & 0 & \cdots \\
p & 0 & 0 & 0 & 1-p & 0 & 0 & 0 & \cdots \\
p & 0 & 0 & 0 & 0 & 1-p & 0 & 0 & \cdots \\
p & 0 & 0 & 0 & 0 & 0 & 1-p & 0 & \cdots \\
p & 0 & 0 & 0 & 0 & 0 & 0 & 1-p & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right]
\end{array}$$

$\pi(0) = p\,\pi(0) + p\,\pi(1) + p\,\pi(2) + p\,\pi(3) + p\,\pi(4) + p\,\pi(5) + p\,\pi(6) + \ldots,$

$\pi(1) = (1-p)\,\pi(0),$

$\pi(2) = (1-p)\,\pi(1) = (1-p)^2\,\pi(0),$

$\pi(3) = (1-p)\,\pi(2) = (1-p)^3\,\pi(0),$

$\pi(4) = (1-p)\,\pi(3) = (1-p)^4\,\pi(0),$

$\pi(5) = (1-p)\,\pi(4) = (1-p)^5\,\pi(0),$

$\pi(6) = (1-p)\,\pi(5) = (1-p)^6\,\pi(0),$

$\ldots$

$\pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4) + \pi(5) + \pi(6) + \ldots = 1.$

$\Rightarrow$ $\pi(0) = p,\quad \pi(1) = (1-p)\,p,\quad \pi(2) = (1-p)^2\,p,\quad \pi(3) = (1-p)^3\,p,\quad \ldots$

That is, $\pi(x) = (1-p)^x\,p, \quad x \geq 0.$

Note that $m_0 = \dfrac{1}{p}$ and $\pi(0) = p = \dfrac{1}{m_0}.$

Silly example **2.** Consider a Markov Chain with the states being all nonnegative integer numbers, having transition function

$$\mathrm{P}(k, k+1) = \frac{k+1}{k+2}, \qquad \mathrm{P}(k, 0) = \frac{1}{k+2}, \qquad k \geq 0.$$

Determine whether state 0 (and the entire chain) is transient, null recurrent, or positive recurrent.

$$\mathrm{P}_0(\mathrm{T}_0 \geq m) = \mathrm{P}(0, 1) \times \mathrm{P}(1, 2) \times \mathrm{P}(2, 3) \times \ldots \times \mathrm{P}(m-3, m-2) \times \mathrm{P}(m-2, m-1)$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{m-2}{m-1} \cdot \frac{m-1}{m} = \frac{1}{m}.$$

$\Rightarrow$ $\quad \mathrm{P}_0(\mathrm{T}_0 = \infty) = \lim\limits_{m \to \infty} \mathrm{P}_0(\mathrm{T}_0 \geq m) = 0.$

$\Rightarrow$ $\quad \rho_{00} = \mathrm{P}_0(\mathrm{T}_0 < \infty) = 1.$ $\qquad \Rightarrow$ 0 is recurrent.

$$\mathrm{E}_0(\mathrm{T}_0) = \sum_{m=1}^{\infty} \mathrm{P}(\mathrm{T}_0 \geq m) = \sum_{m=1}^{\infty} \frac{1}{m} = \infty.$$

$\Rightarrow$ 0 is null recurrent.

The Markov chain is irreducible (it is possible to get to any state from any state).

$\Rightarrow$ The Markov chain is null recurrent.

Spoiler: Markov chain is irreducible, null recurrent.

$\Rightarrow$ There is no stationary distribution:

Silly example **2.5.** Consider a Markov Chain with the states being all nonnegative integer numbers, having transition function

$$\mathrm{P}(k, k+1) = \frac{1}{k+2}, \qquad \mathrm{P}(k, 0) = \frac{k+1}{k+2}, \qquad k \geq 0.$$

Determine whether state 0 (and the entire chain) is transient, null recurrent, or positive recurrent.

$$\mathrm{P}_0(\mathrm{T}_0 \geq m) = \mathrm{P}(0,1) \times \mathrm{P}(1,2) \times \mathrm{P}(2,3) \times \ldots \times \mathrm{P}(m-3, m-2) \times \mathrm{P}(m-2, m-1)$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \ldots \cdot \frac{1}{m-1} \cdot \frac{1}{m} = \frac{1}{m!}.$$

$\Rightarrow \quad \mathrm{P}_0(\mathrm{T}_0 = \infty) = \lim\limits_{m \to \infty} \mathrm{P}_0(\mathrm{T}_0 \geq m) = 0.$

$\Rightarrow \quad \rho_{00} = \mathrm{P}_0(\mathrm{T}_0 < \infty) = 1. \qquad \Rightarrow \qquad$ 0 is recurrent.

$$\mathrm{E}_0(\mathrm{T}_0) = \sum_{m=1}^{\infty} \mathrm{P}(\mathrm{T}_0 \geq m) = \sum_{m=1}^{\infty} \frac{1}{m!} = e - 1 < \infty.$$

$\Rightarrow$ 0 is positive recurrent.

The Markov chain is irreducible (it is possible to get to any state from any state).

$\Rightarrow$ The Markov chain is positive recurrent.

Spoiler: Markov chain is irreducible, positive recurrent.

$\Rightarrow$ There is stationary distribution:

$$\begin{array}{c|ccccccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\hline
0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 3/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & \cdots \\
3 & 4/5 & 0 & 0 & 0 & 1/5 & 0 & 0 & 0 & \cdots \\
4 & 5/6 & 0 & 0 & 0 & 0 & 1/6 & 0 & 0 & \cdots \\
5 & 6/7 & 0 & 0 & 0 & 0 & 0 & 1/7 & 0 & \cdots \\
6 & 7/8 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

$$\pi(0) = \frac{1}{2}\,\pi(0) + \frac{2}{3}\,\pi(1) + \frac{3}{4}\,\pi(2) + \frac{4}{5}\,\pi(3) + \frac{5}{6}\,\pi(4) + \frac{6}{7}\,\pi(5) + \dots,$$

$$\pi(1) = \frac{1}{2}\,\pi(0) = \frac{1}{2!}\,\pi(0),$$

$$\pi(2) = \frac{1}{3}\,\pi(1) = \frac{1}{3!}\,\pi(0),$$

$$\pi(3) = \frac{1}{4}\,\pi(2) = \frac{1}{4!}\,\pi(0),$$

$$\pi(4) = \frac{1}{5}\,\pi(3) = \frac{1}{5!}\,\pi(0),$$

$$\pi(5) = \frac{1}{6}\,\pi(4) = \frac{1}{6!}\,\pi(0),$$

$\dots$

$$\pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4) + \pi(5) + \pi(6) + \dots = 1.$$

$$\Rightarrow \qquad 1 = \pi(0) + \frac{1}{2!}\,\pi(0) + \frac{1}{3!}\,\pi(0) + \frac{1}{4!}\,\pi(0) + \frac{1}{5!}\,\pi(0) + \frac{1}{6!}\,\pi(0) + \dots$$

$$= \pi(0) \cdot \sum_{k=1}^{\infty} \frac{1}{k!} = \pi(0)\,(e-1).$$

$$\Rightarrow \qquad \pi(0) = \frac{1}{e-1}. \qquad\qquad \pi(x) = \frac{1}{(x+1)!} \cdot \pi(0) = \frac{1}{(x+1)!} \cdot \frac{1}{e-1}, \quad x \ge 1.$$

That is, $\pi(x) = \dfrac{1}{(x+1)!} \cdot \dfrac{1}{e-1}, \quad x \ge 0.$

Note that $m_0 = e-1$ and $\pi(0) = \dfrac{1}{e-1} = \dfrac{1}{m_0}.$

Silly example **3.** Consider a Markov Chain with the states being all nonnegative integer numbers, having transition function

$$\mathrm{P}(k, k+1) = \frac{\left(1+\frac{1}{k+1}\right)^{k+1}}{\left(1+\frac{1}{k+2}\right)^{k+2}}, \qquad \mathrm{P}(k, 0) = 1 - \frac{\left(1+\frac{1}{k+1}\right)^{k+1}}{\left(1+\frac{1}{k+2}\right)^{k+2}}, \qquad k \ge 0.$$

Determine whether state 0 (and the entire chain) is transient, null recurrent, or positive recurrent.

$$\mathrm{P}_0(\mathrm{T}_0 \ge m) = \mathrm{P}(0,1) \times \mathrm{P}(1,2) \times \mathrm{P}(2,3) \times \ldots \times \mathrm{P}(m-3, m-2) \times \mathrm{P}(m-2, m-1)$$

$$= \frac{2}{\frac{9}{4}} \cdot \frac{\frac{9}{4}}{\frac{64}{27}} \cdot \frac{\frac{64}{27}}{\frac{625}{256}} \cdot \ldots \cdot \frac{\left(1+\frac{1}{m-2}\right)^{m-2}}{\left(1+\frac{1}{m-1}\right)^{m-1}} \cdot \frac{\left(1+\frac{1}{m-1}\right)^{m-1}}{\left(1+\frac{1}{m}\right)^{m}} = \frac{2}{\left(1+\frac{1}{m}\right)^{m}}.$$

$\Rightarrow$ $\mathrm{P}_0(\mathrm{T}_0 = \infty) = \lim_{m\to\infty} \mathrm{P}_0(\mathrm{T}_0 \ge m) = \frac{2}{e} > 0.$

$\Rightarrow$ $\rho_{00} = \mathrm{P}_0(\mathrm{T}_0 < \infty) = 1 - \frac{2}{e} < 1.$

$\Rightarrow$ 0 is transient.

The Markov chain is irreducible (it is possible to get to any state from any state).

$\Rightarrow$ The Markov chain is transient.

Spoiler: Markov chain is irreducible, transient.

$\Rightarrow$ There is no stationary distribution:

Theorem Let π be a stationary distribution.

If x is transient or null recurrent, then $\pi(x) = 0$.

Theorem An irreducible positive recurrent Markov chain has a unique stationary distribution π, given by

$$\pi(x) = \frac{1}{m_x}, \qquad x \in S.$$

Example: Winter weather in Central Illinois.

$$\begin{array}{c|c} & \begin{array}{ccc} \mathrm{N} & \mathrm{R} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{N} \\ \mathrm{R} \\ \mathrm{S} \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

Recall (Examples for 02/08/2022 (2)):

$$\pi(\mathrm{N}) = \mathbf{\frac{2}{9}}, \qquad \pi(\mathrm{R}) = \mathbf{\frac{4}{9}}, \qquad \pi(\mathrm{S}) = \mathbf{\frac{3}{9}}.$$

Recall (Examples for 01/20/2022 (2)): $m_{ij} = \mathrm{E}_i(\mathrm{T}_j)$

$$m_{\mathrm{N\,N}} = \mathbf{\frac{9}{2}}, \qquad m_{\mathrm{R\,R}} = \mathbf{\frac{9}{4}}, \qquad m_{\mathrm{S\,S}} = \mathbf{3}.$$

$m_y = \mathrm{E}_y(\mathrm{T}_y) = m_{yy}.$

$$m_{\mathrm{N}} = \frac{9}{2}, \qquad m_{\mathrm{R}} = \frac{9}{4}, \qquad m_{\mathrm{S}} = 3 = \frac{9}{3}.$$

$$\pi(x) = \frac{1}{m_x}, \qquad x \in S.$$