Examples for 02/17/2022 (1)

Birth and death Markov chains:

$$\gamma_0 = 1. \qquad \gamma_y = \frac{q_1 \cdots q_y}{p_1 \cdots p_y}, \qquad y \ge 1.$$

$$\pi_0 = 1, \qquad \pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x}, \qquad x \ge 1.$$

$\sum_{y=0}^{\infty} \gamma_y < \infty$	$\sum_{x=0}^{\infty} \pi_x = \infty$	Transient
$\sum_{y=0}^{\infty} \gamma_y = \infty$	$\sum_{x=0}^{\infty} \pi_x = \infty$	Null Recurrent
$\sum_{y=0}^{\infty} \gamma_y = \infty$	$\sum_{x=0}^{\infty} \pi_x < \infty$	Positive Recurrent

Recall Example 2(e) from Examples for 02/03/2022:

Consider the birth and death chain on $\{0, 1, 2, \ldots\}$ defined by $p_x = (x+1)/(2x+1)$ and $q_x = x/(2x+1)$, $x \ge 0$ (i.e., there is a little bias to the right). Any state leads to any other state, so this chain is irreducible. We showed that this chain is recurrent. Determine whether the chain positive recurrent or null recurrent.

$$\pi_0 = 1. \qquad \pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{1 \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot \ldots \cdot \frac{x}{2x-1}}{\frac{1}{3} \cdot \frac{2}{5} \cdot \frac{3}{7} \cdot \ldots \cdot \frac{x}{2x+1}} = 2x+1, \qquad x \ge 1.$$

$\sum_{x=0}^{\infty} \pi_x = \infty.$ $\quad\Rightarrow\quad$ Stationary distribution does NOT exist.

$\Rightarrow$ The chain is **null** recurrent.

Let $C \subseteq S$.

π is concentrated on C, if $\pi(x) = 0$, $x \notin C$.

Theorem Let C be an irreducible closed set of positive recurrent states. Then the Markov chain has a unique stationary distribution π, concentrated on C, given by

$$\pi(x) = \frac{1}{m_x}, \qquad x \in C.$$

$$\pi(x) = 0, \quad x \notin C.$$

If C_1 and C_2 are two distinct irreducible closed sets of positive recurrent states, the Markov chain has a stationary distribution π_1 concentrated on C_1 and a different stationary distribution π_2 concentrated on C_2.

Then the distributions π_α defined for $0 \le \alpha \le 1$ by

$$\pi_\alpha(x) = (1-\alpha)\,\pi_1(x) + \alpha\,\pi_2(x), \qquad x \in S,$$

are distinct stationary distributions.

Let S_P denote the set of positive recurrent states of a Markov chain.

(i) If S_P is empty, the chain has no stationary distributions.

(ii) If S_P is a nonempty irreducible set, the chain has a unique stationary distribution.

(iii) If S_P is a nonempty but not irreducible, the chain has an infinite number of distinct stationary distributions.

Example: Example 3 from Examples for 01/27/2022:

$$
\begin{array}{c|ccccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
1 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 \\
2 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 \\
3 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\
4 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\
5 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$S_R = \{0, 2, 4, 5, 6\} = \{0, 2\} \cup \{4, 5\} \cup \{6\} = C_1 \cup C_2 \cup C_3.$$

$$S_T = \{1, 3\}.$$

Consider the stationary distribution π_1 concentrated on $C_1 = \{0, 2\}$.

$$\pi_1(0) = \frac{1}{2}\,\pi_1(0) + \frac{1}{4}\,\pi_1(2)$$

$$\pi_1(2) = \frac{1}{2}\,\pi_1(0) + \frac{3}{4}\,\pi_1(2)$$

$$\pi_1(0) + \pi_1(2) = 1$$

$$\Rightarrow \qquad \pi_1(0) = \frac{1}{3}, \qquad \pi_1(2) = \frac{2}{3}.$$

Consider the stationary distribution π_2 concentrated on $C_2 = \{4, 5\}$.

$$\pi_2(4) = \frac{2}{3}\,\pi_2(4) + \frac{1}{2}\,\pi_2(5)$$

$$\pi_2(5) = \frac{1}{3}\,\pi_2(4) + \frac{1}{2}\,\pi_2(5)$$

$$\pi_2(4) + \pi_2(5) = 1$$

$$\Rightarrow \qquad \pi_2(4) = \frac{3}{5}, \qquad \pi_2(5) = \frac{2}{5}.$$

Consider the stationary distribution π_3 concentrated on $C_3 = \{6\}$.

$$\pi_3(6) = 1.$$

$$\pi_1 = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \pi_1 = \pi_1 \mathbf{P}$$

$$\pi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{2}{5} & 0 \end{bmatrix} \qquad \pi_2 = \pi_2 \mathbf{P}$$

$$\pi_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \pi_3 = \pi_3 \mathbf{P}$$

Let $\alpha \ge 0$, $\beta \ge 0$, $\alpha + \beta \le 1$.

$$\pi_{\alpha\beta} = \alpha\,\pi_1 + \beta\,\pi_2 + (1 - \alpha - \beta)\,\pi_3$$

$$= \begin{bmatrix} \alpha\frac{1}{3} & 0 & \alpha\frac{2}{3} & 0 & \beta\frac{3}{5} & \beta\frac{2}{5} & (1-\alpha-\beta) \end{bmatrix}$$

$\pi_{\alpha\beta} = \pi_{\alpha\beta}\,\mathbf{P}$ $\qquad$ $\pi_{\alpha\beta}$ is a stationary distribution.

```
> P = rbind( c(1/2, 0 ,1/2, 0 , 0 , 0 , 0 ),
+            c(1/5,2/5,1/5,1/5, 0 , 0 , 0 ),
+            c(1/4, 0 ,3/4, 0 , 0 , 0 , 0 ),
+            c( 0 ,1/5,1/5,1/5,1/5, 0 ,1/5),
+            c( 0 , 0 , 0 , 0 ,2/3,1/3, 0 ),
+            c( 0 , 0 , 0 , 0 ,1/2,1/2, 0 ),
+            c( 0 , 0 , 0 , 0 , 0 , 0 , 1 ) )
> eigen(t(P))
$values
[1] 1.0000000 1.0000000 1.0000000 0.5236068 0.2500000 0.1666667 0.0763932

$vectors
              [,1] [,2]          [,3]        [,4]          [,5]          [,6]
[1,] -4.362761e-01    0  2.352993e-01  0.18159762  7.071068e-01  6.984940e-16
[2,]  1.394071e-16    0 -5.110122e-16 -0.66446390 -6.782113e-16  5.341425e-16
[3,] -8.725523e-01    0  4.705987e-01  0.54871887 -7.071068e-01 -1.068285e-15
[4,]  2.466433e-16    0 -3.341234e-16 -0.41066127 -6.563335e-17  5.341425e-16
[5,] -8.725523e-02    0  6.589466e-01  0.01140223 -1.465811e-15  7.071068e-01
[6,] -5.817015e-02    0  4.392977e-01  0.16100211  1.662712e-15 -7.071068e-01
[7,] -1.931796e-01    1  3.098031e-01  0.17240434  0.000000e+00 -1.448348e-16
           [,7]
[1,]  0.3252181
[2,] -0.3148529
[3,] -0.2991760
[4,]  0.5094427
[5,] -0.5176557
[6,]  0.4073398
[7,] -0.1103159
```

`eigen(t(P))$vectors[,2]` is π_3.

```
> eigen(t(P))$vectors[,1]/sum(eigen(t(P))$vectors[,1])
[1]  2.648217e-01 -8.462078e-17  5.296434e-01 -1.497137e-16  5.296434e-02
[6]  3.530956e-02  1.172610e-01
> eigen(t(P))$vectors[,3]/sum(eigen(t(P))$vectors[,3])
[1]  1.113081e-01 -2.417339e-16  2.226163e-01 -1.580568e-16  3.117141e-01
[6]  2.078094e-01  1.465521e-01
```

Both are probability vectors.

Both are $\left[\ \alpha\frac{1}{3}\quad 0\quad \alpha\frac{2}{3}\quad 0\quad \beta\frac{3}{5}\quad \beta\frac{2}{5}\quad (1-\alpha-\beta)\ \right]$ for some $\alpha > 0,\ \ \beta > 0,\ \ \alpha+\beta < 1.$

Strangely, I also got a different result: **? ? ?**

```
> P = rbind( c(1/2, 0 ,1/2, 0 , 0 , 0 , 0 ),
+            c(1/5,2/5,1/5,1/5, 0 , 0 , 0 ),
+            c(1/4, 0 ,3/4, 0 , 0 , 0 , 0 ),
+            c( 0 ,1/5,1/5,1/5,1/5, 0 ,1/5),
+            c( 0 , 0 , 0 , 0 ,2/3,1/3, 0 ),
+            c( 0 , 0 , 0 , 0 ,1/2,1/2, 0 ),
+            c( 0 , 0 , 0 , 0 , 0 , 0 , 1 ) )
> eigen(t(P))
$values
[1] 1.0000000 1.0000000 1.0000000 0.5236068 0.2500000 0.1666667 0.0763932

$vectors
     [,1]          [,2]          [,3]        [,4]          [,5]          [,6]
[1,]    0 -3.907799e-02 -4.446498e-01  0.18159762  7.071068e-01 -1.232637e-16
[2,]    0  1.868384e-17  1.311533e-16 -0.66446390 -4.813112e-16 -4.108788e-17
[3,]    0 -7.815598e-02 -8.892995e-01  0.54871887 -7.071068e-01  1.438076e-16
[4,]    0 -2.685190e-17 -8.743556e-17 -0.41066127  0.000000e+00 -3.697910e-16
[5,]    0  8.281497e-01 -8.892995e-02  0.01140223 -3.500445e-16  7.071068e-01
[6,]    0  5.520998e-01 -5.928664e-02  0.16100211  3.828612e-16 -7.071068e-01
[7,]    1  4.145536e-02  3.082374e-03  0.17240434 -2.922379e-17  6.239019e-17
           [,7]
[1,]  0.3252181
[2,] -0.3148529
[3,] -0.2991760
[4,]  0.5094427
[5,] -0.5176557
[6,]  0.4073398
[7,] -0.1103159
```

`eigen(t(P))$vectors[,1]` is π_3.

```
> eigen(t(P))$vectors[,2]
[1] -3.907799e-02  1.868384e-17 -7.815598e-02 -2.685190e-17  8.281497e-01
[6]  5.520998e-01  4.145536e-02
> eigen(t(P))$vectors[,3]
[1] -4.446498e-01  1.311533e-16 -8.892995e-01 -8.743556e-17 -8.892995e-02
[6] -5.928664e-02  3.082374e-03
```

Both are $\left[\ \alpha\frac{1}{3}\quad 0\quad \alpha\frac{2}{3}\quad 0\quad \beta\frac{3}{5}\quad \beta\frac{2}{5}\quad \gamma\ \right]$ for some $\alpha,\ \beta,\ \gamma$.

Neither one can be turned into a probability vector easily. ☹