STAT 433 | Spring 2022

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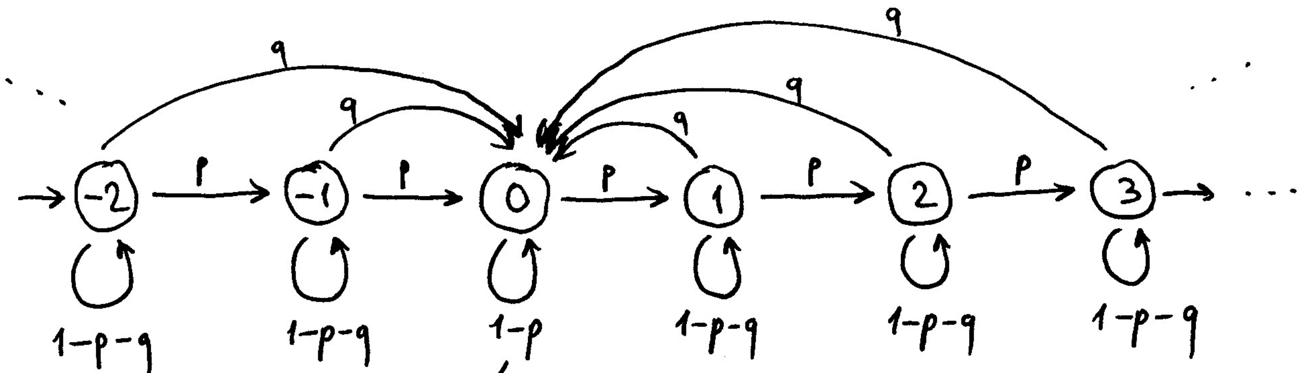
Examples for 01/27/2022

Example 1: Consider p and q such that $p > 0$, $q > 0$, $p + q < 1$.

Consider a Markov Chain with the states being all integer numbers (i.e., …, – 2, – 1, 0, 1, 2, …), having transition function $\mathrm{P}(k, k+1) = p > 0$, $\mathrm{P}(k, 0) = q > 0$, and $\mathrm{P}(k, k) = 1 - p - q > 0$. For each state, determine whether it is recurrent or transient.



should have written P(0,0) = 1-p

State 0 is recurrent, at any step we have non-zero chance q of going to 0, eventually will do this.

State 0 leads to 1, 2, 3, …, so they are recurrent too.

Once you leave a state k < 0 (and this will eventually happen, as at any step there is non-zero chance p+q of doing that), there is no way to return. Thus, all states -1, -2, -3, … are transient.

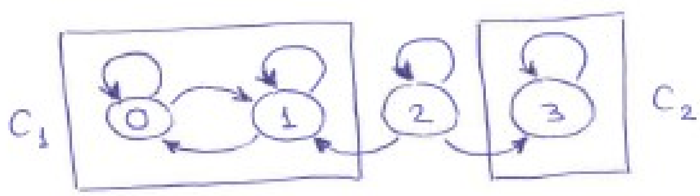
This example is from my brother, Mikhail Stepanov.

Example 2: Consider a Markov chain with the following transition probability matrix:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

a) Determine which states are recurrent and which are transient. Identify all irreducible closed sets of recurrent states.

b) Find ρ_{xy}, $x = 0, 1, 2, 3$, $y = 0, 1, 2, 3$.



a) $S_R = \{0, 1, 3\} = \{0, 1\} \cup \{3\} = C_1 \cup C_2$. $\quad S_T = \{2\}$.

b) $\rho_{00} = \rho_{01} = 1$, $\qquad \rho_{02} = \rho_{03} = 0$.

$\rho_{10} = \rho_{11} = 1$, $\qquad \rho_{12} = \rho_{13} = 0$.

There is no way out of "box" $\{0, 1\}$.

$\rho_{33} = 1$, $\qquad \rho_{30} = \rho_{31} = \rho_{32} = 0$.

There is no way out of "box" $\{3\}$.

$\rho_{22} = P(2, 2) = \frac{2}{5}$ $\qquad$ (if the chain leaves 2, it never comes back)

(if we return from 2 to 2, it has to be on the very next step)

$$\rho_{23} = \frac{2}{5} + \frac{2}{5}\,\frac{2}{5} + \frac{2}{5}\,\frac{2}{5}\,\frac{2}{5} + \ldots = \frac{\frac{2}{5}}{1-\frac{2}{5}} = \frac{2}{3}$$

$$(\; 2 \to 3 \quad \text{or} \quad 2 \to 2 \to 3 \quad \text{or} \quad 2 \to 2 \to 2 \to 3 \quad \text{or} \quad \ldots \;)$$

$$\rho_{21} = \frac{1}{5} + \frac{2}{5}\,\frac{1}{5} + \frac{2}{5}\,\frac{2}{5}\,\frac{1}{5} + \ldots = \frac{\frac{1}{5}}{1-\frac{2}{5}} = \frac{1}{3}$$

$$(\; 2 \to 1 \quad \text{or} \quad 2 \to 2 \to 1 \quad \text{or} \quad 2 \to 2 \to 2 \to 1 \quad \text{or} \quad \ldots \;)$$

$$\rho_{20} = \rho_{21} = \frac{1}{3} \qquad\qquad (\text{ if we get from 2 to 1, then we will get to 0 too })$$

OR

Let C be an irreducible closed set of recurrent states.

$$\rho_C(x) = P_x(T_C < \infty), \qquad x \in S.$$

(probability that the Markov chain that starts in x would be absorbed into C)

$$\rho_C(x) = 1, \qquad x \in C.$$

$$\rho_C(x) = 0, \qquad x \text{ is recurrent}, \qquad x \notin C,$$

$$\rho_C(x) = \sum_{y \in C} P(x,y) + \sum_{y \in S_T} P(x,y) \cdot \rho_C(y), \qquad x \in S_T.$$

$$\rho_{C_1}(2) = \frac{1}{5} + \frac{2}{5}\,\rho_{C_1}(2) \qquad \Rightarrow \qquad \rho_{C_1}(2) = \frac{1}{3}$$

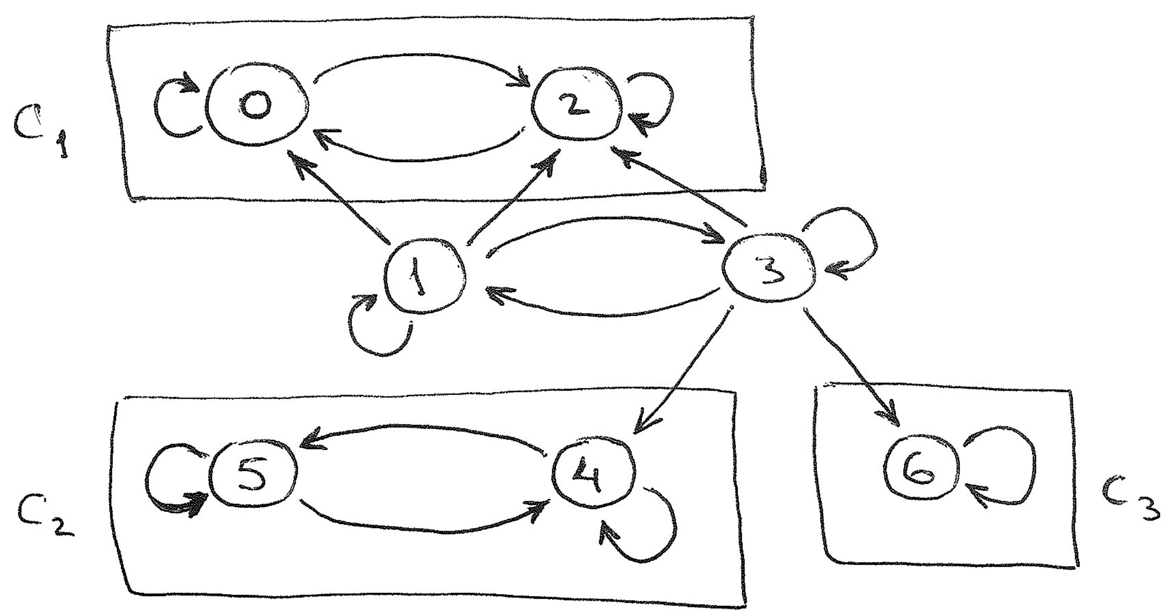
$$\rho_{C_2}(2) = \frac{2}{5} + \frac{2}{5}\,\rho_{C_2}(2) \qquad \Rightarrow \qquad \rho_{C_2}(2) = \frac{2}{3}$$

$$\rho_{20} = \rho_{21} = \rho_{C_1}(2) = \frac{1}{3} \qquad\qquad \rho_{23} = \rho_{C_2}(2) = \frac{2}{3}$$

Example 3: Consider a Markov chain with the following transition probability matrix:

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{5} & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 \\
\frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\
0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

a) Determine which states are recurrent and which are transient. Identify all irreducible closed sets of recurrent states.



$S_R = \{0, 2, 4, 5, 6\} = \{0, 2\} \cup \{4, 5\} \cup \{6\} = C_1 \cup C_2 \cup C_3.$

$S_T = \{1, 3\}.$

b) Find $\rho_C(x)$ for each transient state x and each irreducible closed set C of recurrent states.

$$\rho_C(x) = \sum_{y \in C} P(x,y) + \sum_{y \in S_T} P(x,y) \cdot \rho_C(y), \qquad x \in S_T.$$

$$\rho_{C_1}(1) = \left(\frac{1}{5} + \frac{1}{5}\right) + \frac{2}{5}\rho_{C_1}(1) + \frac{1}{5}\rho_{C_1}(3)$$

$$\rho_{C_1}(3) = \left(0 + \frac{1}{5}\right) + \frac{1}{5}\rho_{C_1}(1) + \frac{1}{5}\rho_{C_1}(3)$$

$$\Rightarrow \quad \rho_{C_1}(1) = \frac{9}{11}, \qquad \rho_{C_1}(3) = \frac{5}{11}.$$

$$\rho_{C_2}(1) = (0 + 0) + \frac{2}{5}\rho_{C_2}(1) + \frac{1}{5}\rho_{C_2}(3)$$

$$\rho_{C_2}(3) = \left(\frac{1}{5} + 0\right) + \frac{1}{5}\rho_{C_2}(1) + \frac{1}{5}\rho_{C_2}(3)$$

$$\Rightarrow \quad \rho_{C_2}(1) = \frac{1}{11}, \qquad \rho_{C_2}(3) = \frac{3}{11}.$$

$$\rho_{C_3}(1) = 0 + \frac{2}{5}\rho_{C_3}(1) + \frac{1}{5}\rho_{C_3}(3)$$

$$\rho_{C_3}(3) = \frac{1}{5} + \frac{1}{5}\rho_{C_3}(1) + \frac{1}{5}\rho_{C_3}(3)$$

$$\Rightarrow \quad \rho_{C_3}(1) = \frac{1}{11}, \qquad \rho_{C_3}(3) = \frac{3}{11}.$$

$$\rho_{C_1}(1) = \frac{9}{11}, \qquad \rho_{C_2}(1) = \frac{1}{11}, \qquad \rho_{C_3}(1) = \frac{1}{11}.$$

$$\rho_{C_1}(3) = \frac{5}{11}, \qquad \rho_{C_2}(3) = \frac{3}{11}, \qquad \rho_{C_3}(3) = \frac{3}{11}.$$

c) Find ρ_{xy} for each transient state x and $y = 0, 1, 2, 3, 4, 5, 6$.

$$\rho_{10} = \rho_{12} = \rho_{C_1}(1) = \frac{9}{11},$$

$$\rho_{14} = \rho_{15} = \rho_{C_2}(1) = \frac{1}{11},$$

$$\rho_{16} = \rho_{C_3}(1) = \frac{1}{11}.$$

$$\rho_{30} = \rho_{32} = \rho_{C_1}(3) = \frac{5}{11},$$

$$\rho_{34} = \rho_{35} = \rho_{C_2}(3) = \frac{3}{11},$$

$$\rho_{36} = \rho_{C_3}(3) = \frac{3}{11}.$$

$$\rho_{xy} = \mathrm{P}(x,y) + \sum_{\substack{z \in S_T \\ z \neq y}} \mathrm{P}(x,z) \cdot \rho_{zy}, \qquad x, y \in S_T.$$

$$\rho_{11} = \frac{2}{5} + \frac{1}{5}\rho_{31}$$

$$\rho_{31} = \frac{1}{5} + \frac{1}{5}\rho_{31} \qquad \Rightarrow \qquad \rho_{31} = 0.25.$$

$$\Rightarrow \qquad \rho_{11} = 0.45.$$

OR $\quad 1 \to 1$ or $1 \to 3 \to 1$ or $1 \to 3 \to 3 \to 1$ or $1 \to 3 \to 3 \to 3 \to 1$ or …

$$\rho_{11} = \frac{2}{5} + \frac{1}{5}\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{5}\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{5}\cdot\frac{1}{5}\cdot\frac{1}{5} + \ldots = \frac{2}{5} + \frac{\frac{1}{5}\cdot\frac{1}{5}}{1-\frac{1}{5}} = 0.45.$$

$$\rho_{13} = \frac{1}{5} + \frac{2}{5}\rho_{13} \qquad \Rightarrow \qquad \rho_{13} = \frac{1}{3}.$$

$$\rho_{33} = \frac{1}{5} + \frac{1}{5}\rho_{13} \qquad \Rightarrow \qquad \rho_{33} = \frac{4}{15}.$$

Suppose we have $m + k$ states – m absorbing states and k transient states.

Rearrange the states so that the first m states are absorbing and the last k states are transient.

Then

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ k \end{array} & \left[\begin{array}{c|c} \mathbf{I}_{m \times m} & \mathbf{O}_{m \times k} \\ \hline \mathbf{R}_{k \times m} & \mathbf{Q}_{k \times k} \end{array}\right] \end{array}$$

$$\mathbf{P}^2 = \begin{array}{c c} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ k \end{array} & \left[\begin{array}{c|c} \mathbf{I} & \mathbf{O} \\ \hline \mathbf{R} + \mathbf{Q}\,\mathbf{R} & \mathbf{Q}^2 \end{array}\right] \end{array}$$

$$\mathbf{P}^3 = \begin{array}{c c} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ k \end{array} & \left[\begin{array}{c|c} \mathbf{I} & \mathbf{O} \\ \hline \mathbf{R} + \mathbf{Q}\,\mathbf{R} + \mathbf{Q}^2\,\mathbf{R} & \mathbf{Q}^3 \end{array}\right] \end{array}$$

$$\mathbf{P}^n = \begin{array}{c c} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ k \end{array} & \left[\begin{array}{c|c} \mathbf{I} & \mathbf{O} \\ \hline \mathbf{R} + \mathbf{Q}\,\mathbf{R} + \mathbf{Q}^2\,\mathbf{R} + \ldots + \mathbf{Q}^{n-1}\,\mathbf{R} & \mathbf{Q}^n \end{array}\right] \end{array}$$

$\mathbf{F}_{k \times k} = (\mathbf{I}_{k \times k} - \mathbf{Q}_{k \times k})^{-1}$ – fundamental matrix.

$$\mathbf{P}^n \quad \rightarrow \quad \begin{array}{c c} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ k \end{array} & \left[\begin{array}{c|c} \mathbf{I} & \mathbf{O} \\ \hline \mathbf{F}\,\mathbf{R} & \mathbf{O} \end{array}\right] \end{array} \qquad \text{as } n \rightarrow \infty.$$

The element in row i, column j of the product $\mathbf{F}\,\mathbf{R}$ gives the probability for the Markov chain to be absorbed into absorbing state j, starting from transient state i.

Back to Example 3:

Collapse each irreducible closed sets of recurrent states into an imaginary absorbing state:

$$
\begin{array}{c}
\\ C_1 \\ C_2 \\ C_3 \\ 1 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{ccccc} C_1 & C_2 & C_3 & 1 & 3 \end{array} \\
\left[\begin{array}{ccc|cc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\hline
\frac{2}{5} & 0 & 0 & \frac{2}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5}
\end{array}\right]
\end{array}
$$

$$\mathbf{Q} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}, \qquad \mathbf{I}-\mathbf{Q} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix},$$

$$\det(\mathbf{I}-\mathbf{Q}) = \frac{11}{25}.$$

$$\mathbf{F} = (\mathbf{I}-\mathbf{Q})^{-1} = \frac{25}{11}\cdot\begin{bmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{20}{11} & \frac{5}{11} \\ \frac{5}{11} & \frac{15}{11} \end{bmatrix}.$$

$$\mathbf{F}\,\mathbf{R} = \begin{bmatrix} \frac{20}{11} & \frac{5}{11} \\ \frac{5}{11} & \frac{15}{11} \end{bmatrix} \times \begin{bmatrix} \frac{2}{5} & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{9}{11} & \frac{1}{11} & \frac{1}{11} \\ \frac{5}{11} & \frac{3}{11} & \frac{3}{11} \end{bmatrix}$$

$$= \begin{bmatrix} \rho_{C_1}(1) & \rho_{C_2}(1) & \rho_{C_3}(1) \\ \rho_{C_1}(3) & \rho_{C_2}(3) & \rho_{C_3}(3) \end{bmatrix}.$$

```
> Q = rbind( c(2/5,1/5), c(1/5,1/5) )
> Q
     [,1] [,2]
[1,]  0.4  0.2
[2,]  0.2  0.2
>
> R = rbind( c(2/5, 0 , 0 ), c(1/5,1/5,1/5) )
> R
     [,1] [,2] [,3]
[1,]  0.4  0.0  0.0
[2,]  0.2  0.2  0.2
>
> F = solve( diag(2) - Q )
> F
          [,1]      [,2]
[1,] 1.8181818 0.4545455
[2,] 0.4545455 1.3636364
>
> F %*% R
          [,1]       [,2]       [,3]
[1,] 0.8181818 0.09090909 0.09090909
[2,] 0.4545455 0.27272727 0.27272727
```

$$\rho_{C_1}(1) = \frac{9}{11}, \qquad \rho_{C_2}(1) = \frac{1}{11}, \qquad \rho_{C_3}(1) = \frac{1}{11}.$$

$$\rho_{C_1}(3) = \frac{5}{11}, \qquad \rho_{C_2}(3) = \frac{3}{11}, \qquad \rho_{C_3}(3) = \frac{3}{11}.$$