Examples for 02/10/2022

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If $\pi(x)$, $x \in S$, are nonnegative numbers summing to one, and if

$$\sum_{x \in S} \pi(x) \times \mathrm{P}(x,y) = \pi(y), \qquad y \in \mathrm{S},$$

then π is called a **stationary distribution**. That is, a stationary distribution is a probability vector π such that

$$\pi\,\mathbf{P} = \pi.$$

1. Consider a Markov chain on $\{0, 1, 2, 3, 4, \dots\}$

with $\mathrm{P}(x,y) = \dfrac{1}{x+2}$ for $0 \le y \le x+1$

and $\mathrm{P}(x,y) = 0$ for $y > x+1$.

Find a stationary distribution if it exists.

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & \cdots \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$\pi\,\mathbf{P} = \pi.$

$$\pi(0) = \frac{1}{2}\pi(0) + \frac{1}{3}\pi(1) + \frac{1}{4}\pi(2) + \frac{1}{5}\pi(3) + \frac{1}{6}\pi(4) + \frac{1}{7}\pi(5) + \frac{1}{8}\pi(6) + \dots$$

$$\pi(1) = \frac{1}{2}\pi(0) + \frac{1}{3}\pi(1) + \frac{1}{4}\pi(2) + \frac{1}{5}\pi(3) + \frac{1}{6}\pi(4) + \frac{1}{7}\pi(5) + \frac{1}{8}\pi(6) + \dots$$

$$\pi(2) = \frac{1}{3}\pi(1) + \frac{1}{4}\pi(2) + \frac{1}{5}\pi(3) + \frac{1}{6}\pi(4) + \frac{1}{7}\pi(5) + \frac{1}{8}\pi(6) + \dots$$

$$\pi(3) = \frac{1}{4}\pi(2) + \frac{1}{5}\pi(3) + \frac{1}{6}\pi(4) + \frac{1}{7}\pi(5) + \frac{1}{8}\pi(6) + \dots$$

$$\pi(4) = \frac{1}{5}\pi(3) + \frac{1}{6}\pi(4) + \frac{1}{7}\pi(5) + \frac{1}{8}\pi(6) + \dots$$

…

First, $\pi(0) = \pi(1)$.

Second,
$$\pi(k) = \frac{1}{k+1}\pi(k-1) + \frac{1}{k+2}\pi(k) + \frac{1}{k+3}\pi(k+1) + \frac{1}{k+4}\pi(k+2) + \dots$$

$$\pi(k+1) = \frac{1}{k+2}\pi(k) + \frac{1}{k+3}\pi(k+1) + \frac{1}{k+4}\pi(k+2) + \dots$$

$$\Rightarrow \qquad \pi(k) = \frac{1}{k+1}\,\pi(k-1) + \pi(k+1).$$

$$\Rightarrow \qquad \pi(k+1) = \pi(k) - \frac{1}{k+1}\,\pi(k-1).$$

Claim: $\qquad \pi(n) = \dfrac{1}{n\,!}\,\pi(0), \qquad\qquad n \ge 0.$

Proof: $\qquad$ Mathematical Induction:

Since $\pi(k+1) = \pi(k) - \dfrac{1}{k+1}\,\pi(k-1)$, need not only $n = 0$ for the "Base" step, but both $n = 0$ and $n = 1$.

Base: $\qquad \pi(0) = \dfrac{1}{0!}\,\pi(0), \qquad\qquad \pi(1) = \pi(0) = \dfrac{1}{1!}\,\pi(0).$ ✓

Step: $\qquad$ Assume $\qquad \pi(k) = \dfrac{1}{k\,!}\,\pi(0)$ and $\pi(k-1) = \dfrac{1}{(k-1)\,!}\,\pi(0).$

Then

$$\begin{aligned}
\pi(k+1) &= \pi(k) - \frac{1}{k+1}\,\pi(k-1) \\
&= \frac{1}{k\,!}\,\pi(0) - \frac{1}{k+1}\cdot\frac{1}{(k-1)\,!}\,\pi(0) \\
&= \left[\,\frac{1}{k} - \frac{1}{k+1}\,\right]\cdot\frac{1}{(k-1)\,!}\,\pi(0) = \frac{1}{(k+1)\,!}\,\pi(0).
\end{aligned}$$ ✓

$$\pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4) + \pi(5) + \pi(6) + \ldots = 1.$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} \frac{1}{n\,!}\,\pi(0) = e\,\pi(0) = 1. \qquad\qquad \Rightarrow \qquad \pi(0) = \frac{1}{e}.$$

$$\Rightarrow \qquad \pi(n) = \frac{1}{n\,!}\,\pi(0) = \frac{1}{e\cdot n\,!} = \frac{1^n\cdot e^{-1}}{n\,!}, \qquad\qquad n \ge 0.$$

The stationary distribution is a Poisson distribution with mean $\lambda = 1$.

2. Consider a birth and death Markov chain on $\{1, 2, \dots, N\}$ having a transition function

$$P(x, x-1) = {}^{1}\!/_{x}, \qquad 1 < x \le N,$$

$$P(x, x+1) = {}^{1}\!/_{x}, \qquad 1 \le x < N,$$

$$P(x, x) = 1 - {}^{2}\!/_{x}, \qquad 1 < x < N,$$

$$P(N, N) = 1 - {}^{1}\!/_{N}.$$

Find the stationary distribution.

$$\pi(1) = \frac{1}{2}\,\pi(2).$$

$$\pi(2) = \pi(1) + \frac{1}{3}\,\pi(3). \qquad \Rightarrow \qquad \frac{1}{2}\,\pi(2) = \frac{1}{3}\,\pi(3).$$

$$\pi(3) = \frac{1}{2}\,\pi(2) + \frac{1}{3}\,\pi(3) + \frac{1}{4}\,\pi(4). \qquad \Rightarrow \qquad \frac{1}{3}\,\pi(3) = \frac{1}{4}\,\pi(4).$$

Assume $\dfrac{1}{x-1}\,\pi(x-1) = \dfrac{1}{x}\,\pi(x)$. Then

$$\pi(x) = \frac{1}{x-1}\,\pi(x-1) + \left(1 - \frac{2}{x}\right)\pi(x) + \frac{1}{x+1}\,\pi(x+1).$$

$$\Rightarrow \qquad \frac{1}{x}\,\pi(x) = \frac{1}{x+1}\,\pi(x+1).$$

By induction, $\dfrac{1}{x}\,\pi(x) = \dfrac{1}{x+1}\,\pi(x+1), \qquad 1 \le x < N.$

$$\Rightarrow \qquad \pi(x) = x\,\pi(1), \qquad 1 \le x \le N.$$

$$1 = \sum_{x=1}^{N} \pi(x) = \pi(1) \sum_{x=1}^{N} x = \pi(1)\,\frac{N(N+1)}{2}.$$

$$\Rightarrow \qquad \pi(x) = \frac{2x}{N(N+1)}, \qquad 1 \le x \le N.$$

Birth and death Markov chains:

$$\mathrm{P}(x,y) = \begin{cases} q_x & y = x-1 \qquad\qquad q_0 = 0 \\ r_x & y = x \\ p_x & y = x+1 \qquad\qquad p_d = 0 \end{cases} \qquad\qquad 0 \le x \le d$$

OR

$$\mathrm{P}(x,y) = \begin{cases} q_x & y = x-1 \qquad\qquad q_0 = 0 \\ r_x & y = x \\ p_x & y = x+1 \end{cases} \qquad\qquad x \ge 0$$

$$q_x + r_x + p_x = 1, \qquad\qquad x \in S.$$

$$\pi(0)\, r_0 + \pi(1)\, q_1 = \pi(0)$$

$$\pi(y-1)\, p_{y-1} + \pi(y)\, r_y + \pi(y+1)\, q_{y+1} = \pi(y) \qquad\qquad y \ge 1$$

$$\Rightarrow \qquad \pi(x) = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x}\, \pi(0), \qquad\qquad x \in S.$$

Set $\qquad \pi_0 = 1, \qquad \pi_x = \dfrac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x}, \qquad x \ge 1.$

$$\pi(x) = \pi_x\, \pi(0), \qquad\qquad x \in S.$$

Must have $\qquad \displaystyle\sum_{x \in S} \pi(x) = 1.$

If $S = \{0, 1, 2, \dots, d\}$, $\qquad$ then $\qquad \pi(x) = \dfrac{\pi_x}{\displaystyle\sum_{z=0}^{d} \pi_z}, \qquad x \in S.$

Suppose $S = \{0, 1, 2, 3, \dots\}$.

$\Rightarrow$ If $\sum_{x=0}^{\infty} \pi_x = \infty$, there is no stationary distribution.

Intuition: $\sum_{x=0}^{\infty} \pi_x = \infty \quad \Leftrightarrow$ there are a lot of "births" and not a lot of "deaths"

the Markov chain gravitates towards infinity

If $\sum_{x=0}^{\infty} \pi_x < \infty$, then $\pi(x) = \dfrac{\pi_x}{\sum_{z=0}^{\infty} \pi_z}$, $\quad x \in S$.

3. Example 2 from 02/03/2022.

Find the stationary distribution if it exists.

a) $P(0,0) = r_0 = 0.8, \qquad P(0,1) = p_0 = 0.2,$

$P(x, x-1) = q_x = 0.3,$

$P(x, x) = r_x = 0.5, \qquad x \ge 1.$

$P(x, x+1) = p_x = 0.2,$

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{0.2 \cdot 0.2 \cdot \ldots \cdot 0.2}{0.3 \cdot 0.3 \cdot \ldots \cdot 0.3} = \left(\frac{2}{3}\right)^x, \qquad x \ge 1.$$

$$\sum_{x=0}^{\infty} \pi_x = 1 + \sum_{x=1}^{\infty} \left(\frac{2}{3}\right)^x = 1 + \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 3 < \infty.$$

$$\Rightarrow \qquad \pi(0) = \frac{1}{3},$$

$$\pi(x) = \pi_x \, \pi(0) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^x, \qquad x \ge 1.$$

a¼) $P(0,1) = p_0 = 1,$

$$\left.\begin{aligned} P(x,x-1) &= q_x = 0.3, \\ P(x,x) &= r_x = 0.5, \\ P(x,x+1) &= p_x = 0.2, \end{aligned}\right. \qquad x \geq 1.$$

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{1 \cdot 0.2 \cdot \ldots \cdot 0.2}{0.3 \cdot 0.3 \cdot \ldots \cdot 0.3} = 5 \cdot \left(\frac{2}{3}\right)^x, \qquad x \geq 1.$$

$$\sum_{x=0}^{\infty} \pi_x = 1 + \sum_{x=1}^{\infty} 5 \cdot \left(\frac{2}{3}\right)^x = 1 + \frac{5 \cdot \frac{2}{3}}{1 - \frac{2}{3}} = 11 < \infty.$$

$$\Rightarrow \qquad \pi(0) = \frac{1}{11},$$

$$\pi(x) = \pi_x \, \pi(0) = \frac{5}{11} \cdot \left(\frac{2}{3}\right)^x, \qquad x \geq 1.$$

OR

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{1 \cdot 0.2 \cdot \ldots \cdot 0.2}{0.3 \cdot 0.3 \cdot \ldots \cdot 0.3} = \frac{10}{3} \cdot \left(\frac{2}{3}\right)^{x-1}, \qquad x \geq 1.$$

$$\sum_{x=0}^{\infty} \pi_x = 1 + \sum_{x=1}^{\infty} \frac{10}{3} \cdot \left(\frac{2}{3}\right)^{x-1} = 1 + \frac{\frac{10}{3}}{1 - \frac{2}{3}} = 11 < \infty.$$

$$\Rightarrow \qquad \pi(0) = \frac{1}{11},$$

$$\pi(x) = \pi_x \, \pi(0) = \frac{5}{11} \cdot \left(\frac{2}{3}\right)^x, \qquad x \geq 1.$$

a½) $P(0,1) = p_0 = 1,$

$P(x,x-1) = q_x = 0.3,$

$P(x,x) = r_x = 0.5, \qquad 1 \le x \le 4.$

$P(x,x+1) = p_x = 0.2,$

$P(5,4) = q_5 = 0.3, \qquad P(5,5) = r_5 = 0.7.$

$$\pi_x = \frac{p_0 \ldots p_{x-1}}{q_1 \ldots q_x} = \frac{1 \cdot 0.2 \cdot \ldots \cdot 0.2}{0.3 \cdot 0.3 \cdot \ldots \cdot 0.3} = 5 \cdot \left(\frac{2}{3}\right)^x, \qquad 1 \le x \le 5.$$

$$\sum_{x=0}^{5} \pi_x = 1 + 5 \cdot \frac{2}{3} + 5 \cdot \frac{4}{9} + 5 \cdot \frac{8}{27} + 5 \cdot \frac{16}{81} + 5 \cdot \frac{32}{243} = \frac{2353}{243}.$$

$\Rightarrow$ $$\pi(0) = \frac{243}{2353}, \qquad \pi(1) = \frac{810}{2353}, \qquad \pi(2) = \frac{540}{2353},$$

$$\pi(3) = \frac{360}{2353}, \qquad \pi(4) = \frac{240}{2353}, \qquad \pi(5) = \frac{160}{2353}.$$

b) $P(0,0) = r_0 = 0.7,$ $\quad P(0,1) = p_0 = 0.3,$

$$\left.\begin{aligned} P(x,x-1) &= q_x = 0.2, \\ P(x,x) &= r_x = 0.5, \\ P(x,x+1) &= p_x = 0.3, \end{aligned}\right\} \quad x \geq 1.$$

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{0.3 \cdot 0.3 \cdot \ldots \cdot 0.3}{0.2 \cdot 0.2 \cdot \ldots \cdot 0.2} = 1.5^x, \qquad x \geq 1.$$

$$\sum_{x=0}^{\infty} \pi_x = 1 + \sum_{x=1}^{\infty} 1.5^x = \infty.$$

$\Rightarrow$ there is no stationary distribution.

b½) $P(0,1) = p_0 = 1,$

$$\left.\begin{aligned} P(x,x-1) &= q_x = 0.2, \\ P(x,x) &= r_x = 0.5, \\ P(x,x+1) &= p_x = 0.3, \end{aligned}\right\} \quad 1 \leq x \leq 3.$$

$P(4,3) = q_4 = 0.2,$ $\quad P(4,4) = r_4 = 0.8.$

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{1 \cdot 0.3 \cdot \ldots \cdot 0.3}{0.2 \cdot 0.2 \cdot \ldots \cdot 0.2} = 5 \cdot 1.5^{x-1}, \qquad 1 \leq x \leq 4.$$

$$\sum_{x=0}^{4} \pi_x = 1 + 5 + \frac{15}{2} + \frac{45}{4} + \frac{135}{8} = \frac{8}{8} + \frac{40}{8} + \frac{60}{8} + \frac{90}{8} + \frac{135}{8} = \frac{333}{8}.$$

$$\Rightarrow \qquad \pi(0) = \frac{8}{333}, \qquad \pi(1) = \frac{40}{333}, \qquad \pi(2) = \frac{60}{333},$$

$$\pi(3) = \frac{90}{333}, \qquad \pi(4) = \frac{135}{333}.$$

c) $\quad P(0,0) = r_0 = 0.7, \qquad P(0,1) = p_0 = 0.3,$

$$P(x, x-1) = q_x = 0.2,$$
$$P(x, x) = r_x = 0.5, \qquad\qquad 1 \le x \le 99.$$
$$P(x, x+1) = p_x = 0.3,$$

$$P(100, 100) = r_{100} = 0.8, \qquad P(100, 99) = q_{100} = 0.2,$$

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{0.3 \cdot 0.3 \cdot \ldots \cdot 0.3}{0.2 \cdot 0.2 \cdot \ldots \cdot 0.2} = 1.5^x, \qquad 1 \le x \le 100.$$

$$\sum_{x=0}^{100} \pi_x = 1 + \sum_{x=1}^{100} 1.5^x = \frac{1.5^{101} - 1}{1.5 - 1} = 2(1.5^{101} - 1).$$

$$\pi(x) = \frac{1.5^x}{2\left(1.5^{101} - 1\right)}, \qquad 0 \le x \le 100.$$

$$\ldots, \quad \pi(97) \approx \frac{8}{81}, \quad \pi(98) \approx \frac{4}{27}, \quad \pi(99) \approx \frac{2}{9}, \quad \pi(100) \approx \frac{1}{3}.$$

d) Consider the birth and death chain on $\{0, 1, 2, \ldots\}$ defined by $p_x = (x+2)/(2x+2)$ and $q_x = x/(2x+2)$, $x \ge 0$ (i.e., there is a little bias to the right). Any state leads to any other state, so this chain is irreducible. (~ Example 13, p. 33 HPS)

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{1 \cdot \frac{3}{4} \cdot \frac{4}{6} \cdot \ldots \cdot \frac{x+1}{2x}}{\frac{1}{4} \cdot \frac{2}{6} \cdot \frac{3}{8} \cdot \ldots \cdot \frac{x}{2x+2}} = (x+1)^2, \qquad x \ge 1.$$

$$\sum_{x=0}^{\infty} \pi_x = 1 + \sum_{x=1}^{\infty} (x+1)^2 = \infty.$$

$\Rightarrow$ there is no stationary distribution.

e) Consider the birth and death chain on $\{0, 1, 2, \ldots\}$ defined by $p_x = (x+1)/(2x+1)$ and $q_x = x/(2x+1)$, $x \ge 0$ (i.e., there is a little bias to the right). Any state leads to any other state, so this chain is irreducible.

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{1 \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot \ldots \cdot \frac{x}{2x-1}}{\frac{1}{3} \cdot \frac{2}{5} \cdot \frac{3}{7} \cdot \ldots \cdot \frac{x}{2x+1}} = 2x+1, \qquad x \ge 1.$$

$$\sum_{x=0}^{\infty} \pi_x = 1 + \sum_{x=1}^{\infty} (2x+1) = \infty.$$

$\Rightarrow$ there is no stationary distribution.