Examples for 02/01/2022

Let C be an irreducible closed set of recurrent states.

$\rho_C(x) = P_x(T_C < \infty), \qquad x \in S.$

(probability that the Markov chain that starts in x would be absorbed into C)

$\rho_C(x) = 1, \qquad x \in C.$

$\rho_C(x) = 0, \qquad x$ is recurrent, $\quad x \notin C,$

$$\rho_C(x) = \sum_{y \in C} P(x,y) + \sum_{y \in S_T} P(x,y) \cdot \rho_C(y), \qquad x \in S_T.$$

Suppose we have $m+k$ states – m absorbing states and k transient states.

Rearrange the states so that the first m states are absorbing and the last k states are transient.

Then

$$\mathbf{P} = \begin{array}{c|c} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ k \end{array} & \left[\begin{array}{c|c} \mathbf{I}_{m \times m} & \mathbf{O}_{m \times k} \\ \hline \mathbf{R}_{k \times m} & \mathbf{Q}_{k \times k} \end{array}\right] \end{array}$$

$\mathbf{F}_{k \times k} = (\mathbf{I}_{k \times k} - \mathbf{Q}_{k \times k})^{-1}$ – fundamental matrix.

$$\mathbf{P}^n \to \begin{array}{c|c} & \begin{array}{cc} m & k \end{array} \\ \begin{array}{c} m \\ k \end{array} & \left[\begin{array}{c|c} \mathbf{I} & \mathbf{O} \\ \hline \mathbf{F}\,\mathbf{R} & \mathbf{O} \end{array}\right] \end{array} \quad \text{as } n \to \infty.$$

The element in row i, column j of the product $\mathbf{F}\,\mathbf{R}$ gives the probability for the Markov chain to be absorbed into absorbing state j, starting from transient state i.

For a transient state i, let a_i denote the expected number of steps before a Markov chain that starts in state i is absorbed into S_R. Then

$$a_i = \sum_{j \in S_R} P(i,j) \cdot 1 + \sum_{k \in S_T} P(i,k) \cdot (1 + a_k).$$

(To be absorbed into S_R from a transient state i, we go to a recurrent state j in one step OR we go to a transient state k in one step and then will be absorbed into S_R from k in a_k steps (on average).)

$$a_i = \sum_{j \in S_R} P(i,j) \cdot 1 + \sum_{k \in S_T} P(i,k) \cdot (1 + a_k) = 1 + \sum_{k \in S_T} P(i,k) \cdot a_k,$$

since $\displaystyle\sum_{j \in S_R} P(i,j) + \sum_{k \in S_T} P(i,k) = 1.$

$$\Rightarrow \quad \vec{\mathbf{a}} = \vec{\mathbf{1}} + \mathbf{Q}\,\vec{\mathbf{a}}. \qquad \Rightarrow \quad \vec{\mathbf{a}} = (\mathbf{I} - \mathbf{Q})^{-1}\,\vec{\mathbf{1}} = \mathbf{F}\,\vec{\mathbf{1}}, \qquad \text{where } \vec{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

The element in row i, column j of the fundamental matrix $\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1}$ gives the number of visits to transient state j that are expected to occur before absorption, starting from transient state i. $\mathbf{F}_{ii}$ includes the initial "visit" to state i, starting from i.

<u>Example 3</u>:

$$a_1 = 1 + \frac{2}{5} a_1 + \frac{1}{5} a_3 \qquad \Rightarrow \qquad 3 a_1 = 5 + a_3$$

$$a_3 = 1 + \frac{1}{5} a_1 + \frac{1}{5} a_3 \qquad \Rightarrow \qquad 4 a_3 = 5 + a_1$$

$$\Rightarrow \qquad 12 a_1 - 20 = 5 + a_1$$

$$\Rightarrow \qquad a_1 = \frac{25}{11}, \qquad a_3 = \frac{20}{11}.$$

```
> Q = rbind( c(2/5,1/5), c(1/5,1/5) )
> F = solve( diag(2) - Q )
> F
          [,1]      [,2]
[1,] 1.8181818 0.4545455
[2,] 0.4545455 1.3636364
```

$$1 + G(1,1) = 1 + \frac{\rho_{11}}{1-\rho_{11}} = 1 + \frac{0.45}{1-0.45} = 1 + \frac{9}{11} = \frac{20}{11}$$ – the expected number of visits to 1 from 1.

("1 +" is for the initial "visit" from 1 to 1)

$$G(1,3) = \frac{\rho_{13}}{1-\rho_{33}} = \frac{\frac{1}{3}}{1-\frac{4}{15}} = \frac{5}{11}$$ – the expected number of visits to 1 from 3.

$$G(3,1) = \frac{\rho_{31}}{1-\rho_{11}} = \frac{0.25}{1-0.45} = \frac{5}{11}$$ – the expected number of visits to 3 from 1.

$$1 + G(3,3) = 1 + \frac{\rho_{33}}{1-\rho_{33}} = 1 + \frac{\frac{4}{15}}{1-\frac{4}{15}} = 1 + \frac{4}{11} = \frac{15}{11}$$ – the expected number of visits to 3 from 3.

("1 +" is for the initial "visit" from 3 to 3)

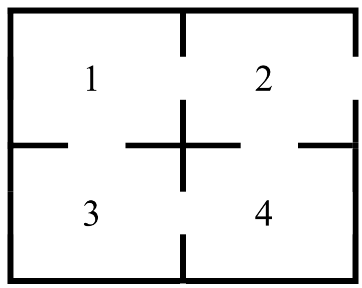
```
> F %*% c(1,1)
         [,1]
[1,] 2.272727
[2,] 1.818182
```

Starting from 1, the expected number of steps before being absorbed by either C_1, or C_2, or C_3 is $\frac{25}{11}$.

Starting from 3, the expected number of steps before being absorbed by either C_1, or C_2, or C_3 is $\frac{20}{11}$.

Example 1 from Examples for 01/20/2022 (1):

A rat runs through the following maze:



The rat starts in a given cell, and at each step, it moves to a neighboring cell (chosen with equal probability from those available, independently of the past). It continues moving between cells in this way until it escapes to the outside. Assume that the outside (freedom) is denoted by state 0 (and it can only be reached from cell 2). Assume also that once the rat has escaped, it remains escaped forever. Recall that the transition matrix for the appropriate Markov chain is as follows:

$$\begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

Find the expected number of steps taken to escape given that the rat starts in cell i, for $i = 1, 2, 3, 4$.

$$a_i = 1 + \sum_{k \in S_T} \mathrm{P}(i,k) \cdot a_k , \qquad i \in S_T .$$

$$a_1 = 1 + \frac{1}{2} a_2 + \frac{1}{2} a_3 \qquad (1)$$

$$a_2 = 1 + \frac{1}{3} a_1 + \frac{1}{3} a_4 \qquad (2)$$

$$a_3 = 1 + \frac{1}{2} a_1 + \frac{1}{2} a_4 \qquad (3)$$

$$a_4 = 1 + \frac{1}{2} a_2 + \frac{1}{2} a_3 \qquad (4)$$

$(1), (4) \quad \Rightarrow \quad a_1 = a_4 \qquad (5)$

$(2), (5) \quad \Rightarrow \quad a_2 = 1 + \frac{2}{3} a_4 \qquad (6)$

$(3), (5) \quad \Rightarrow \quad a_3 = 1 + a_4 \qquad (7)$

$(4), (6), (7) \quad \Rightarrow \quad a_4 = 1 + \frac{1}{2} + \frac{1}{3} a_4 + \frac{1}{2} + \frac{1}{2} a_4$

$\Rightarrow \quad \frac{1}{6} a_4 = 2 \quad \Rightarrow \quad a_4 = \mathbf{12} \qquad (8)$

$(5), (8) \quad \Rightarrow \quad a_1 = \mathbf{12}$

$(6), (8) \quad \Rightarrow \quad a_2 = \mathbf{9}$

$(7), (8) \quad \Rightarrow \quad a_3 = \mathbf{13}$

$a_1 = \mathbf{12} \qquad a_2 = \mathbf{9} \qquad a_3 = \mathbf{13} \qquad a_4 = \mathbf{12}$

```
> Q = rbind( c(0,1/2,1/2,0), c(1/3,0,0,1/3),
+            c(1/2,0,0,1/2), c(0,1/2,1/2,0) )
> Q
          [,1] [,2] [,3]      [,4]
[1,] 0.0000000  0.5  0.5 0.0000000
[2,] 0.3333333  0.0  0.0 0.3333333
[3,] 0.5000000  0.0  0.0 0.5000000
[4,] 0.0000000  0.5  0.5 0.0000000
> F = solve(diag(4) - Q)
> F
     [,1] [,2] [,3] [,4]
[1,]  3.5    3    3  2.5
[2,]  2.0    3    2  2.0
[3,]  3.0    3    4  3.0
[4,]  2.5    3    3  3.5
```

```
> F %*% c(1,1,1,1)
      [,1]
[1,]    12
[2,]     9
[3,]    13
[4,]    12
```

If cheese is placed and replaced in cell 3, the expected number of visits to cell 3 by the rat is

3 if the rat starts in cell 1 or 4,

2 if the rat starts in cell 2, and

4 (including the "visit" at the start) if the rat starts in cell 3.

Regardless of which cell is the starting point, the expected number of visits to cell 2 is 3.

Every time the rat is in cell 2 ("attempt"), the rat escapes the maze ("success") with probability $p = \frac{1}{3}$. Let Y denote the number of independent "attempts" needed for the first "success". Then Y has a Geometric distribution.

$E(Y) = \frac{1}{p} = 3.$ On average, 3 "attempts" are needed for the first "success".

Example 0: Y has a Geometric distribution with probability of "success" p.

$X_0 = F.$

$$\mathbf{P} = \begin{array}{c|cc} & S & F \\ \hline S & 1 & 0 \\ F & p & 1-p \end{array}$$

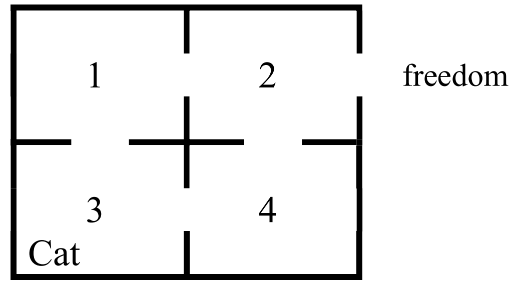
$\mathbf{R} = [\,p\,]$ $\qquad$ $\mathbf{Q} = [\,1-p\,]$

$\mathbf{F} = (\mathbf{I}-\mathbf{Q})^{-1} = [\,p\,]^{-1} = \left[\frac{1}{p}\right]$ $\qquad$ Recall: $E(Y) = \frac{1}{p}$

$\mathbf{F}\,\mathbf{R} = [\,1\,]$ $\qquad$ Recall: $P(Y < \infty) = 1$

Example 2 from Examples for 01/20/2022 (1):

A rat runs through the following maze:



The rat starts in a given cell, and at each step, it moves to a neighboring cell (chosen with equal probability from those available, independently of the past). It continues moving between cells in this way until it escapes to the outside or is eaten by the cat in cell 3. Assume that the outside (freedom) is denoted by state 0 (and it can only be reached from cell 2). Recall that the transition matrix for the appropriate Markov chain is as follows:

$$
\begin{array}{c}
 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0
\end{array}\right]
\end{array}
$$

Find the probability that the rat escapes (instead of being eaten) if it starts in cell

a) 1;

b) 2;

c) 4.

3 transient states: 1, 2, 4; 2 absorbing states: 0, 3.

$$\rho_C(x) = \sum_{y \in C} P(x,y) + \sum_{y \in S_T} P(x,y) \cdot \rho_C(y), \qquad x \in S_T.$$

$$\rho_{\{0\}}(1) = \frac{1}{2}\,\rho_{\{0\}}(2) \tag{1}$$

$$\rho_{\{0\}}(2) = \frac{1}{3} + \frac{1}{3}\,\rho_{\{0\}}(1) + \frac{1}{3}\,\rho_{\{0\}}(4) \tag{2}$$

$$\rho_{\{0\}}(4) = \frac{1}{2}\,\rho_{\{0\}}(2) \tag{3}$$

$$(1),(3) \quad \Rightarrow \quad \rho_{\{0\}}(1) = \rho_{\{0\}}(4) = \frac{1}{2}\,\rho_{\{0\}}(2) \qquad (4)$$

$$(2),(4) \quad \Rightarrow \quad \rho_{\{0\}}(2) = \frac{1}{3} + \frac{2}{3}\,\rho_{\{0\}}(1) = \frac{1}{3} + \frac{1}{3}\,\rho_{\{0\}}(2) \qquad (5)$$

$$(5) \quad \Rightarrow \quad \rho_{\{0\}}(2) = \mathbf{\frac{1}{2}} = \mathbf{0.50} \qquad (6)$$

$$(4),(6) \quad \Rightarrow \quad \rho_{\{0\}}(1) = \rho_{\{0\}}(4) = \mathbf{\frac{1}{4}} = \mathbf{0.25}$$

OR

$$\begin{array}{c} \\ 0 \\ 3 \\ 1 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 3 & 1 & 2 & 4 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right] \end{array}$$

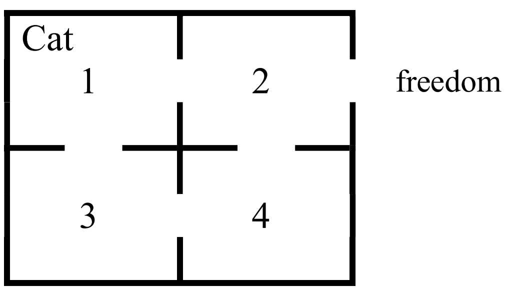
```
> Q = rbind( c(0,1/2,0), c(1/3,0,1/3), c(0,1/2,0) )
>
> R = cbind( c(0,1/3,0), c(1/2,0,1/2) )
>
> F = solve(diag(3) - Q)
>
> F %*% R
     [,1] [,2]
[1,] 0.25 0.75
[2,] 0.50 0.50
[3,] 0.25 0.75
```

```
> F %*% c(1,1,1)
      [,1]
[1,] 2.25
[2,] 2.50
[3,] 2.25
```

Example 2.5:

Repeat Example 2 if the cat is in cell 1, and the rat starts in cell

a) 2; b) 3; c) 4.



$$
\begin{array}{c|c}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left[\begin{array}{cc|ccc}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\hline
\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0
\end{array}\right]
\end{array}
$$

3 transient states: 2, 3, 4; 2 absorbing states: 0, 1.

$$\rho_C(x) = \sum_{y \in C} P(x,y) + \sum_{y \in S_T} P(x,y) \cdot \rho_C(y), \qquad x \in S_T.$$

$$\rho_{\{0\}}(2) = \frac{1}{3} + \frac{1}{3}\rho_{\{0\}}(4) \qquad (1)$$

$$\rho_{\{0\}}(3) = \frac{1}{2}\rho_{\{0\}}(4) \qquad (2)$$

$$\rho_{\{0\}}(4) = \frac{1}{2}\rho_{\{0\}}(2) + \frac{1}{2}\rho_{\{0\}}(3) \qquad (3)$$

(1), (2), (3) $\Rightarrow \quad \rho_{\{0\}}(4) = \frac{1}{6} + \frac{1}{6}\rho_{\{0\}}(4) + \frac{1}{4}\rho_{\{0\}}(4)$

$\Rightarrow \quad \frac{7}{12}\rho_{\{0\}}(4) = \frac{1}{6} \quad \Rightarrow \quad \rho_{\{0\}}(4) = \mathbf{\frac{2}{7}}$ (4)

(1), (4) $\Rightarrow \quad \rho_{\{0\}}(2) = \mathbf{\frac{3}{7}}$

(2), (4) $\Rightarrow \quad \rho_{\{0\}}(3) = \mathbf{\frac{1}{7}}$

OR

```
> Q = rbind( c(0,0,1/3), c(0,0,1/2), c(1/2,1/2,0) )
>
> R = cbind( c(1/3,0,0), c(1/3,1/2,0) )
>
> F = solve(diag(3) - Q)
>
> F %*% R
          [,1]      [,2]
[1,] 0.4285714 0.5714286
[2,] 0.1428571 0.8571429
[3,] 0.2857143 0.7142857
>
> F %*% c(1,1,1)
         [,1]
[1,] 2.142857
[2,] 2.714286
[3,] 3.428571
```

<u>Example 4</u> from Examples for 01/20/2022 (1):

A life insurance company wants to find out how much money to charge its clients.

The following 4-state Markov chain model is proposed for the changes in health for the 50+ age group: H – healthy, M – mildly ill, S – seriously ill, D – dead. The matrix of one-step (one-year) transition probabilities is given below:

$$
\begin{array}{c c}
 & \begin{array}{cccc} \mathrm{H} & \mathrm{M} & \mathrm{S} & \mathrm{D} \end{array} \\
\begin{array}{c} \mathrm{H} \\ \mathrm{M} \\ \mathrm{S} \\ \mathrm{D} \end{array} &
\begin{bmatrix}
0.90 & 0.05 & 0.03 & 0.02 \\
0.80 & 0.10 & 0.05 & 0.05 \\
0.08 & 0.16 & 0.50 & 0.26 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

a) According to this model, what is the expected number of years until death for an individual from the 50+ age group who is currently …

i) … healthy? ii) … seriously ill?

H, M, S – transient.

D – recurrent (absorbing).

$$a_i = 1 + \sum_{k \in S_T} \mathrm{P}(i,k) \cdot a_k \, .$$

$$a_{\mathrm{H}} = 1 + 0.90\, a_{\mathrm{H}} + 0.05\, a_{\mathrm{M}} + 0.03\, a_{\mathrm{S}} \qquad (1)$$

$$a_{\mathrm{M}} = 1 + 0.80\, a_{\mathrm{H}} + 0.10\, a_{\mathrm{M}} + 0.05\, a_{\mathrm{S}} \qquad (2)$$

$$a_{\mathrm{S}} = 1 + 0.08\, a_{\mathrm{H}} + 0.16\, a_{\mathrm{M}} + 0.50\, a_{\mathrm{S}} \qquad (3)$$

(3) $\Rightarrow$ $0.08\, a_{\mathrm{H}} = 0.50\, a_{\mathrm{S}} - 0.16\, a_{\mathrm{M}} - 1$

$$0.80\, a_{\mathrm{H}} = 5\, a_{\mathrm{S}} - 1.6\, a_{\mathrm{M}} - 10 \qquad (4)$$

(2), (4) $\Rightarrow$ $a_M = 1 + 5\,a_S - 1.6\,a_M - 10 + 0.10\,a_M + 0.05\,a_S$

$$2.5\,a_M = 5.05\,a_S - 9$$

$$a_M = 2.02\,a_S - 3.6 \qquad (5)$$

(4), (5) $\Rightarrow$ $a_H = 6.25\,a_S - 2\,a_M - 12.5$

$$= 6.25\,a_S - 4.04\,a_S + 7.2 - 12.5$$

$$= 2.21\,a_S - 5.3 \qquad (6)$$

(1), (5), (6) $\Rightarrow$ $0 = 100 - 10\,a_H + 5\,a_M + 3\,a_S$

$$= 100 - 22.1\,a_S + 53 + 10.1\,a_S - 18 + 3\,a_S$$

$$= 135 - 9\,a_S$$

$\Rightarrow$ $a_S = \mathbf{15}$ (7)

(6), (7) $\Rightarrow$ $a_H = \mathbf{27.85}$.

(5), (7) $\Rightarrow$ $a_M = 26.7$.

OR

```
> Q = rbind( c(0.90,0.05,0.03),
+            c(0.80,0.10,0.05),
+            c(0.08,0.16,0.50) )
> F = solve( diag(3) - Q )
> F %*% c(1,1,1)
      [,1]
[1,] 27.85
[2,] 26.70
[3,] 15.00
```

b) What is the probability that an individual from the 50+ age group who is currently healthy will not become seriously ill before death?

Hint: $\rho_{HS} = P_H(T_S < \infty)$ = the probability that a healthy individual WILL eventually become seriously ill.

OR Since we are only concerned about the first visit, pretend that S is also absorbing for this part only.

$$\rho_{xy} = P(x,y) + \sum_{\substack{z \in S_T \\ z \neq y}} P(x,z) \cdot \rho_{zy}, \qquad x, y \in S_T.$$

$$\rho_{HS} = 0.03 + 0.90\,\rho_{HS} + 0.05\,\rho_{MS} \qquad (1)$$

$$\rho_{MS} = 0.05 + 0.80\,\rho_{HS} + 0.10\,\rho_{MS} \qquad (2)$$

(1) $\Rightarrow$ $0.05\,\rho_{MS} = 0.10\,\rho_{HS} - 0.03$

$\Rightarrow$ $\rho_{MS} = 2\,\rho_{HS} - 0.6$ (3)

(2), (3) $\Rightarrow$ $2\,\rho_{HS} - 0.6 = 0.05 + 0.80\,\rho_{HS} + 0.20\,\rho_{HS} - 0.06$

$\Rightarrow$ $\rho_{HS} = 0.59.$

$P_H(\text{never go to S}) = 1 - \rho_{HS} = 1 - 0.59 = \mathbf{0.41}.$

OR

```
> Qb = rbind( c(0.90,0.05), c(0.80,0.10) )
> Fb = solve( diag(2) - Qb )
> Rb = rbind( c(0.03,0.02), c(0.05,0.05) )
> Fb %*% Rb
     [,1] [,2]
[1,] 0.59 0.41
[2,] 0.58 0.42
```

c) For an individual who is currently healthy, what is the expected number of years he/she will be seriously ill before death? That is, starting from state H, what is the expected number of visits to state S before entering the absorbing state D?

$$G(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}}, \qquad x, y \in S_T.$$

$$\rho_{xy} = P(x,y) + \sum_{\substack{z \in S_T \\ z \neq y}} P(x,z) \cdot \rho_{zy}, \qquad x, y \in S_T.$$

$$\rho_{HS} = 0.03 + 0.90\,\rho_{HS} + 0.05\,\rho_{MS} \qquad (1)$$

$$\rho_{MS} = 0.05 + 0.80\,\rho_{HS} + 0.10\,\rho_{MS} \qquad (2)$$

$$\rho_{SS} = 0.50 + 0.08\,\rho_{HS} + 0.16\,\rho_{MS} \qquad (3)$$

(1) $\Rightarrow$ $\rho_{HS} = 0.30 + 0.50\,\rho_{MS}$ (4)

(2), (4) $\Rightarrow$ $\rho_{MS} = 0.05 + 0.24 + 0.40\,\rho_{MS} + 0.10\,\rho_{MS}$

$\Rightarrow$ $\rho_{MS} = 0.58.$ (5)

(1), (5) $\Rightarrow$ $\rho_{HS} = 0.59.$ (6)

(3), (5), (6) $\Rightarrow$ $\rho_{SS} = 0.64.$

$$G(H,S) = \frac{\rho_{HS}}{1-\rho_{SS}} = \frac{0.59}{1-0.64} = \mathbf{\frac{59}{36}} \approx 1.638889.$$

OR

```
> F
         [,1]     [,2]     [,3]
[1,] 24.55556 1.655556 1.638889
[2,] 22.44444 2.644444 1.611111
[3,] 11.11111 1.111111 2.777778
```

<u>Example 5</u> from Examples for 01/20/2022 (1):

Consider the game of tennis when *deuce* is reached. If a player wins the next point, he has *advantage*. On the following point, he either wins the game or the game returns to deuce. Assume that for any point, player A has probability p of winning the point and player B has probability $(1-p)$ of winning the point.

$$\begin{array}{c|ccccc} & \text{A wins} & \text{adv A} & \text{Deuce} & \text{adv B} & \text{B wins} \\ \hline \text{A wins} & 1 & 0 & 0 & 0 & 0 \\ \text{adv A} & p & 0 & 1-p & 0 & 0 \\ \text{Deuce} & 0 & p & 0 & 1-p & 0 \\ \text{adv B} & 0 & 0 & p & 0 & 1-p \\ \text{B wins} & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{c|cc|ccc} & \text{A wins} & \text{B wins} & \text{adv A} & \text{adv B} & \text{Deuce} \\ \hline \text{A wins} & 1 & 0 & 0 & 0 & 0 \\ \text{B wins} & 0 & 1 & 0 & 0 & 0 \\ \hline \text{adv A} & p & 0 & 0 & 0 & 1-p \\ \text{adv B} & 0 & 1-p & 0 & 0 & p \\ \text{Deuce} & 0 & 0 & p & 1-p & 0 \end{array}$$

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 1-p \\ 0 & 0 & p \\ p & 1-p & 0 \end{bmatrix} \qquad \mathbf{I}-\mathbf{Q} = \begin{bmatrix} 1 & 0 & p-1 \\ 0 & 1 & -p \\ -p & p-1 & 1 \end{bmatrix}$$

$$\det(\mathbf{I}-\mathbf{Q}) = 1-2p(1-p) = p^2+(1-p)^2$$

$$\mathbf{F} = (\mathbf{I}-\mathbf{Q})^{-1} = \frac{1}{p^2+(1-p)^2} \cdot \begin{bmatrix} 1-p(1-p) & (1-p)^2 & 1-p \\ p^2 & 1-p(1-p) & p \\ p & 1-p & 1 \end{bmatrix}$$

$$\mathbf{F\,R} = \frac{1}{p^2+(1-p)^2}\cdot\begin{bmatrix} 1-p(1-p) & (1-p)^2 & 1-p \\ p^2 & 1-p(1-p) & p \\ p & 1-p & 1 \end{bmatrix}\cdot\begin{bmatrix} p & 0 \\ 0 & 1-p \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{p^2+(1-p)^2}\cdot\begin{bmatrix} p-p^2(1-p) & (1-p)^3 \\ p^3 & (1-p)-p(1-p)^2 \\ p^2 & (1-p)^2 \end{bmatrix}$$

$$\mathrm{P}_{\text{Deuce}}(\text{A wins}) = \frac{p^2}{p^2+(1-p)^2} \qquad \mathrm{P}_{\text{Deuce}}(\text{B wins}) = \frac{(1-p)^2}{p^2+(1-p)^2}$$

$$\mathrm{P}_{\text{adv A}}(\text{A wins}) = \frac{p-p^2(1-p)}{p^2+(1-p)^2} \qquad \mathrm{P}_{\text{adv A}}(\text{B wins}) = \frac{(1-p)^3}{p^2+(1-p)^2}$$

$$\mathrm{P}_{\text{adv B}}(\text{A wins}) = \frac{p^3}{p^2+(1-p)^2} \qquad \mathrm{P}_{\text{adv B}}(\text{B wins}) = \frac{(1-p)-p(1-p)^2}{p^2+(1-p)^2}$$

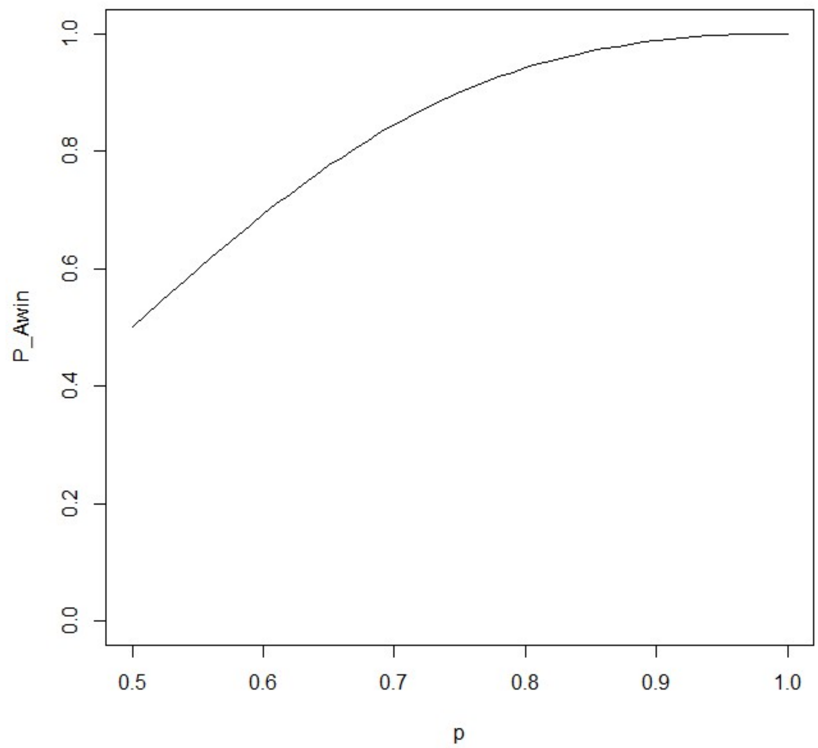
$$\mathbf{F}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{p^2+(1-p)^2}\cdot\begin{bmatrix} 1-p(1-p) & (1-p)^2 & 1-p \\ p^2 & 1-p(1-p) & p \\ p & 1-p & 1 \end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}$$

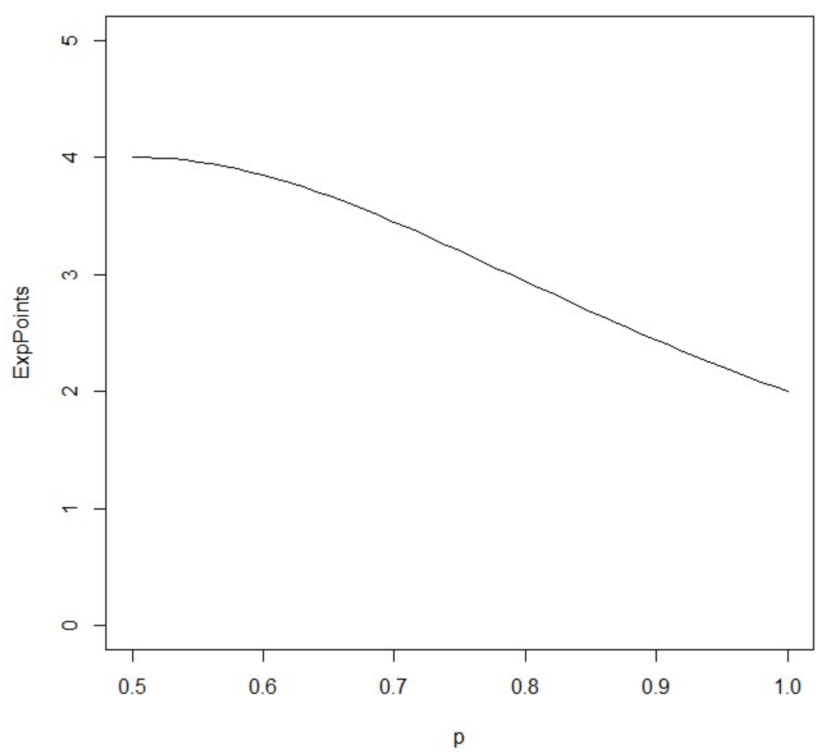
$$\frac{1}{p^2+(1-p)^2}\cdot\begin{bmatrix} 1+2(1-p)^2 \\ 1+2p^2 \\ 2 \end{bmatrix}$$

When *deuce* is reached, the average (expected) number of points needed to determine the winner is $\dfrac{2}{p^2+(1-p)^2}$.

When A has *advantage*, the average (expected) number of points needed to determine the winner is $\dfrac{1+2(1-p)^2}{p^2+(1-p)^2}$.

When B has *advantage*, the average (expected) number of points needed to determine the winner is $\dfrac{1+2p^2}{p^2+(1-p)^2}$.


P_Awin
1.0
0.8
0.6
0.4
0.2
0.0
0.5
0.6
0.7
0.8
0.9
1.0
p


ExpPoints
5
4
3
2
1
0
0.5
0.6
0.7
0.8
0.9
1.0
p