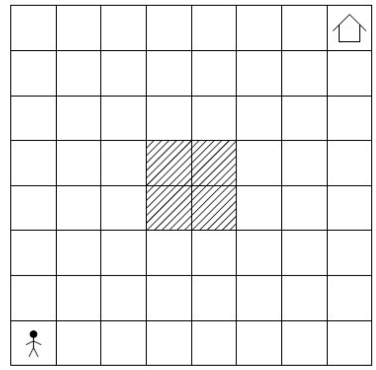
STAT 433 — Spring 2022 — A. Stepanov

Examples for 02/22/2022

0f. The state space $\{1, 2, \ldots, 8\}^2$ is a square with $64 = 8 \times 8$ cells. A drunk person starting from $(1, 1)$ tries to reach a house at $(8, 8)$. The person moves one cell up or one cell to the right with probabilities $1/2$ (or just up or just to the right if that is the only move possible). A crocodile lives in a lake which is a 2×2 square $\{4, 5\}^2$. If the person steps into the lake, then he or she is eaten by the crocodile. What is the probability that the person reaches the house safely?



As the person always gets closer to the house (in L 1 distance), the outcome is determined in finite number of steps (here the outcome is predetermined after 7 steps, and after 14 steps the person's position distribution doesn't change). Thus we can compute the probability of reaching the house $P_{\text{house}} = \rho_{(1,1),(8,8)} = P^{14}((1,1),(8,8))$ in finite number of steps. Below is the evolution $\boldsymbol{\pi}_{\boldsymbol{n}} = \boldsymbol{\pi}_{\boldsymbol{n-1}} \times \mathbf{P} = \boldsymbol{\pi}_{\mathbf{0}} \times \mathbf{P}^n$ (here we propagate forward in time). (Of course, on paper the whole calculation can be executed on just one panel.)

1							

$\frac{1}{2}$							
	$\frac{1}{2}$						

$\frac{1}{4}$							
	$\frac{2}{4}$						
		$\frac{1}{4}$					

$\frac{1}{8}$							
	$\frac{3}{8}$						
		$\frac{3}{8}$					
			$\frac{1}{8}$				

$\frac{1}{16}$							
	$\frac{4}{16}$						
		$\frac{6}{16}$					
			$\frac{4}{16}$				
				$\frac{1}{16}$			

$\frac{1}{32}$							
	$\frac{5}{32}$						
		$\frac{10}{32}$					
			$\frac{10}{32}$				
				$\frac{5}{32}$			
					$\frac{1}{32}$		

$\frac{1}{64}$							
	$\frac{6}{64}$						
		$\frac{15}{64}$					
			$\frac{20}{64}$				
				$\frac{15}{64}$			
					$\frac{6}{64}$		
						$\frac{1}{64}$	

$\frac{1}{128}$							
	$\frac{7}{128}$						
		$\frac{21}{128}$					
			$\frac{15}{128}$				
			$\frac{20}{64}$	$\frac{15}{128}$			
					$\frac{21}{128}$		
						$\frac{7}{128}$	
							$\frac{1}{128}$

	$\frac{9}{256}$						
		$\frac{28}{256}$					
			$\frac{21}{256}$				
			$\frac{15}{128}$				
			$\frac{20}{64}$	$\frac{15}{128}$	$\frac{21}{256}$		
						$\frac{28}{256}$	
							$\frac{9}{256}$

		$\frac{46}{512}$					
			$\frac{49}{512}$				
				$\frac{21}{512}$			
			$\frac{15}{128}$		$\frac{21}{512}$		
			$\frac{20}{64}$	$\frac{15}{128}$		$\frac{49}{512}$	
							$\frac{46}{512}$

			$\frac{141}{1024}$				
				$\frac{70}{1024}$			
					$\frac{42}{1024}$		
			$\frac{15}{128}$			$\frac{70}{1024}$	
			$\frac{20}{64}$	$\frac{15}{128}$			$\frac{141}{1024}$

				$\frac{22}{128}$			
					$\frac{7}{128}$		
						$\frac{7}{128}$	
			$\frac{15}{128}$				$\frac{22}{128}$
			$\frac{20}{64}$	$\frac{15}{128}$			

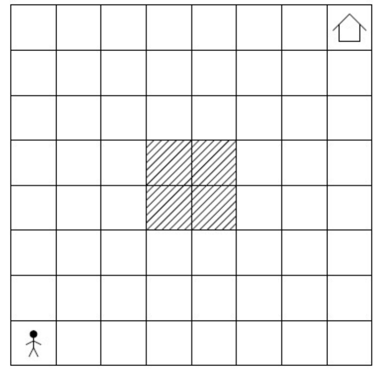
					$\frac{51}{256}$		
						$\frac{14}{256}$	
							$\frac{51}{256}$
			$\frac{15}{128}$				
			$\frac{20}{64}$	$\frac{15}{128}$			

						$\frac{29}{128}$	
							$\frac{29}{128}$
			$\frac{15}{128}$				
			$\frac{20}{64}$	$\frac{15}{128}$			

							$\frac{29}{64}$
			$\frac{15}{128}$				
			$\frac{20}{64}$	$\frac{15}{128}$			

$$P_{\text{house}} = \mathbf{\frac{29}{64}}.$$

0b. The state space $\{1, 2, \ldots, 8\}^2$ is a square with $64 = 8 \times 8$ cells. A drunk person starting from $(1, 1)$ tries to reach a house at $(8, 8)$. The person moves one cell up or one cell to the right with probabilities $1/2$ (or just up or just to the right if that is the only move possible). A crocodile lives in a lake which is a 2×2 square $\{4, 5\}^2$. If the person steps into the lake, then he or she is eaten by the crocodile. What is the probability that the person reaches the house safely?



We can calculate P_{house} by solving the equation for the function on the state space $\rho_{x,(8,8)}$. At each panel the ρ's which can be determined from the right hand side of the equation $\rho_{xy} = P(x,y) + \sum_{z \neq y} P(x,z) \cdot \rho_{zy}$ are indicated by question marks (here we propagate backward in time). In comparison to $P^{14}((1,1),(8,8))$, the calculation is somewhat less laborious.

						?	1
							?
			0	0			
			0	0			

					?	1	1
						?	1
							?
			0	0			
			0	0			

				?	1	1	1
					?	1	1
						?	1
			0	0			?
			0	0			

			?	1	1	1	1
				?	1	1	1
					?	1	1
			0	0		?	1
			0	0			?

		?	1	1	1	1	1
			?	1	1	1	1
				?	1	1	1
			0	0	?	1	1
			0	0		?	1
							?

	?	1	1	1	1	1	1
		?	1	1	1	1	1
			?	1	1	1	1
			0	0	1	1	1
			0	0	?	1	1
						?	1
							?

?	1	1	1	1	1	1	1
	?	1	1	1	1	1	1
		?	1	1	1	1	1
			0	0	1	1	1
			0	0	1	1	1
					?	1	1
						?	1
							?

1	1	1	1	1	1	1	1
?	1	1	1	1	1	1	1
	?	1	1	1	1	1	1
		?	0	0	1	1	1
			0	0	1	1	1
				?	1	1	1
					?	1	1
						?	1

1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
?	1	1	1	1	1	1	1
	?	$\frac{1}{2}$	0	0	1	1	1
		?	0	0	1	1	1
			?	$\frac{1}{2}$	1	1	1
				?	1	1	1
					?	1	1

1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
?	$\frac{3}{4}$	$\frac{1}{2}$	0	0	1	1	1
	?	$\frac{1}{4}$	0	0	1	1	1
		?	$\frac{1}{4}$	$\frac{1}{2}$	1	1	1
			?	$\frac{3}{4}$	1	1	1
				?	1	1	1

1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
$\frac{7}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	0	0	1	1	1
?	$\frac{1}{2}$	$\frac{1}{4}$	0	0	1	1	1
	?	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	1	1
		?	$\frac{1}{2}$	$\frac{3}{4}$	1	1	1
			?	$\frac{7}{8}$	1	1	1

1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
$\frac{7}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	0	0	1	1	1
$\frac{11}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0	1	1	1
?	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	1	1
	?	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	1	1	1
		?	$\frac{11}{16}$	$\frac{7}{8}$	1	1	1

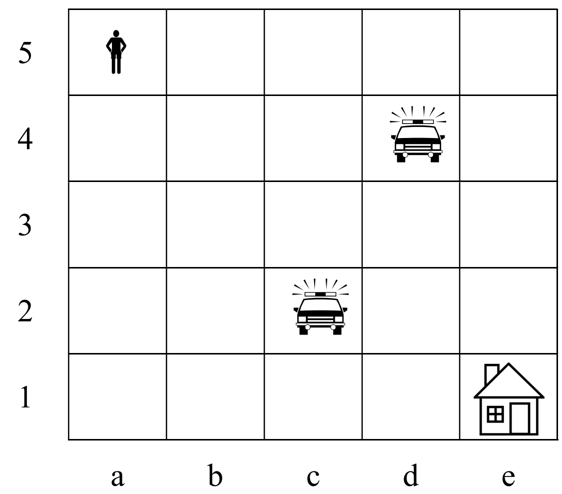
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
$\frac{7}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	0	0	1	1	1
$\frac{11}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0	1	1	1
$\frac{17}{32}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	1	1
?	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	1	1	1
	?	$\frac{17}{32}$	$\frac{11}{16}$	$\frac{7}{8}$	1	1	1

1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
$\frac{7}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	0	0	1	1	1
$\frac{11}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0	1	1	1
$\frac{17}{32}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	1	1
$\frac{29}{64}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	1	1	1
?	$\frac{29}{64}$	$\frac{17}{32}$	$\frac{11}{16}$	$\frac{7}{8}$	1	1	1

1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
$\frac{7}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	0	0	1	1	1
$\frac{11}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0	1	1	1
$\frac{17}{32}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	1	1
$\frac{29}{64}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	1	1	1
$\frac{29}{64}$	$\frac{29}{64}$	$\frac{17}{32}$	$\frac{11}{16}$	$\frac{7}{8}$	1	1	1

We should not be surprised that we get the same answer: $P_{\text{house}} = \mathbf{\frac{29}{64}}$.

1. The state space is a square with $25 = 5 \times 5$ cells. A student who celebrated Unofficial St. Patrick's Day a bit too excessively is trying to reach his house at e1 starting from a5. The student moves one cell down or one cell to the right at random with probabilities $1/2$ (or just down or just to the right if that is the only move possible) independently of the past. However, there are friendly and welcoming police officers located in cells c2 and d4. If the student comes in contact with the police officers, he would be transported to a cozy special facility where he would stay until no longer intoxicated. What is the probability that the student would avoid the police officers and reach his house?



Forward in time: $\pi_{n+1}(x) = P(X_{n+1} = x) = \sum_{y \in S} \pi_n(y) \cdot P(y, x)$

5	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
4	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{8}$	$\frac{4}{16}$	$\frac{2}{32}$
3	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{6}{16}$	$\frac{6}{32}$	$\frac{10}{64}$
2	$\frac{1}{8}$	$\frac{4}{16}$	$\frac{10}{32}$	$\frac{6}{64}$	$\frac{26}{128}$
1	$\frac{1}{16}$	$\frac{6}{32}$	$\frac{12}{64}$	$\frac{30}{128}$	$\frac{112}{256}$
	a	b	c	d	e

$\frac{112}{256} = \mathbf{\frac{7}{16}}$.

OR

$1 - P(\text{Police}) = 1 - \frac{4}{16} - \frac{10}{32} = \mathbf{\frac{7}{16}}$.

OR

Backward in time: $\rho_{xy} = P_x(T_y < \infty) = P(x,y) + \sum_{z \neq y} P(x,z) \cdot \rho_{zy}$

$y = e1 =$

5	$\mathbf{\frac{7}{16}}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$	1
4	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	0	1
3	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
2	$\frac{3}{4}$	$\frac{1}{2}$	0	1	1
1	1	1	1	1	1
	a	b	c	d	e

2. Consider a random walk (see HPS, 1.3, Example 1, page 7) with $P(x,y) = f(y-x)$, where $f(-1) = 2/3$, $f(2) = 1/3$, and $f(x) = 0$ if $(x+1)(x-2) \neq 0$. That is, $P(x, x-1) = 2/3$, $P(x, x+2) = 1/3$, $x \in \mathbb{Z}$.

a) Show that the chain is irreducible.

It is enough to show that any state x leads to $x-1$ and to $x+1$.
(Then we can reach any state y starting from x in $|x-y|$ such moves.)

We can reach $x-1$ from x with one step down.

We can reach $x+1$ from x with a step up (landing on $x+2$) and then a step down.

b) Find the period.

Consider three groups of states: $S_0 = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$, $S_1 = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$, and $S_2 = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$.

The transitions between these groups are deterministic: $S_0 \to S_2 \to S_1 \to S_0$.

Thus the chain can return to the initial state only after the number of steps which is divisible by 3, and it can return in 3 steps (for example, two steps down and one step up).

The period is **3**.

c) Find out whether the chain is transient, null recurrent, or positive recurrent.

On each step, we either go up by 2 or go down by 1. To go from x to x, we need to go up m times and go down $2m$ times.

$$G(x,x) = \sum_{n=1}^{\infty} P^n(x,y) = \sum_{m=1}^{\infty} P^{3m}(x,y).$$

$P^{3m}(x,x) = P(m \text{ steps up}, 2m \text{ steps down})$

$$= \binom{3m}{m}\left(\frac{1}{3}\right)^{m}\left(\frac{2}{3}\right)^{2m} = \frac{(3m)!}{m!\,(2m)!}\left(\frac{1}{3}\right)^{m}\left(\frac{2}{3}\right)^{2m}.$$

For large n, $\qquad n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} \qquad$ (Stirling's approximation)

For large m,

$$P^{3m}(x,x) \approx \frac{\sqrt{6\pi m}\left(\frac{3m}{e}\right)^{3m}}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^{m}\sqrt{4\pi m}\left(\frac{2m}{e}\right)^{2m}}\left(\frac{1}{3}\right)^{m}\left(\frac{2}{3}\right)^{2m} = \frac{\sqrt{3}}{2\sqrt{\pi m}}.$$

$$\Rightarrow \qquad \sum_{n=1}^{\infty} P^{n}(x,y) = \sum_{m=1}^{\infty} P^{3m}(x,y) \text{ diverges.}$$

$G(x,x) = \infty$. The chain is recurrent.

Since the chain is irreducible, IF the chain is positive recurrent, then there is a unique stationary distribution π. Then $\pi(x)$ must be the same for all $x \in \mathbb{Z}$, which is impossible since there are infinitely many states. There is no stationary distribution.

$\Big($ OR $\quad \lim_{m\to\infty} P^{3m}(x,x) = d\,\pi(x) = 3\,\pi(x)$, $\quad$ yet $\quad \lim_{m\to\infty} P^{3m}(x,x) = 0.$ $\Big)$

The chain is **null recurrent**.