STAT 433 Spring 2022
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Examples for 04/05/2022

Brownian motion:

1) $B(0) = 0.$

2) If $0 \le s_1 \le t_1 \le s_2 \le t_2$, $B(t_1) - B(s_1)$ and $B(t_2) - B(s_2)$ are independent.

3) $B(t) - B(s)$ has a Normal distribution with mean 0 and variance $\sigma^2 (t - s)$,
$0 \le s \le t.$

$\left(B(t) \,|\, B(s) = b \right)$ has a Normal distribution with mean with mean b
and variance $\sigma^2 (t - s)$, $0 \le s \le t.$

$\left(B(s) \,|\, B(t) = b \right)$ has a Normal distribution with mean with mean $\frac{s}{t}\, b$
and variance $\frac{s\,(t-s)}{t}\, \sigma^2$, $0 \le s \le t.$

$r_B(s,t) = \operatorname{Cov}(B(s), B(t)) = \sigma^2 \min(s, t),$ $s, t \ge 0.$

1. Let $B(t)$ be a Brownian motion with parameter $\sigma = 5.$

a) Find the probability $P(B(10) < 18)$.

$$P(B(10) < 18) = P\left(Z < \frac{(18-0)-0}{\sqrt{5^2\,(10-0)}} \right) \approx P(Z < 1.14) = \mathbf{0.8729}.$$

b) Find the probability $P(B(10) < 18 \,|\, B(6) = 8)$.

$$P(B(10) < 18 \,|\, B(6) = 8) = P\left(Z < \frac{18-8}{\sqrt{5^2\,(10-6)}} \right) = P(Z < 1.00) = \mathbf{0.8413}.$$

c) Find the probability $P(B(6) < 8 \mid B(10) = 18)$.

$(B(s) \mid B(t) = b)$ has a Normal distribution with mean with mean $\frac{s}{t}\,b$

and variance $\frac{s\,(t-s)}{t}\,\sigma^2$, $0 \le s \le t$.

$$P(B(6) < 8 \mid B(10) = 18) = P\left(Z < \frac{8 - \frac{6}{10} \cdot 18}{\sqrt{\frac{6 \cdot (10-6)}{10} \cdot 5^2}} \right) = P\left(Z < \frac{8 - 10.8}{\sqrt{60}} \right)$$

$$\approx P(Z < -0.36) = \mathbf{0.3594}.$$

d) Find the probability $P(B(10) - B(6) < 18)$.

$$P(B(10) - B(6) < 18) = P\left(Z < \frac{18 - 0}{\sqrt{5^2\,(10-6)}} \right) = P(Z < 1.80) = \mathbf{0.9641}.$$

e) Find the probability $P(B(10) + B(6) < 18)$.

$$E(B(10) + B(6)) = 0.$$

$$\operatorname{Var}(B(10) + B(6)) = \operatorname{Var}(B(10)) + 2\operatorname{Cov}(B(10), B(6)) + \operatorname{Var}(B(6))$$

$$= 10\,\sigma^2 + 2 \times 6\,\sigma^2 + 6\,\sigma^2 = 28\,\sigma^2 = 700.$$

$$P(B(10) + B(6) < 18) = P\left(Z < \frac{18 - 0}{\sqrt{700}} \right) \approx P(Z < 0.68) = \mathbf{0.7517}.$$

Let $\mathrm{B}(t)$ be a Brownian motion with parameter σ.

Let T_a denote the first time the Brownian motion process hits $a > 0$.

$$\mathrm{P}(\mathrm{B}(t) \geq a) = \mathrm{P}(\mathrm{B}(t) \geq a \mid \mathrm{T}_a \leq t) \cdot \mathrm{P}(\mathrm{T}_a \leq t) + \mathrm{P}(\mathrm{B}(t) \geq a \mid \mathrm{T}_a > t) \cdot \mathrm{P}(\mathrm{T}_a > t).$$

By symmetry, $\quad \mathrm{P}(\mathrm{B}(t) \geq a \mid \mathrm{T}_a \leq t) = \dfrac{1}{2}.$

By continuity, $\quad \mathrm{P}(\mathrm{B}(t) \geq a \mid \mathrm{T}_a > t) = 0.$

Thus $\quad \mathrm{F}_{\mathrm{T}_a}(t) = \mathrm{P}(\mathrm{T}_a \leq t) = 2\,\mathrm{P}(\mathrm{B}(t) \geq a) = 2\,\mathrm{P}\left(\mathrm{Z} \geq \dfrac{a}{\sigma\sqrt{t}}\right)$

$$= \int_{\frac{a}{\sigma\sqrt{t}}}^{\infty} \frac{2}{\sqrt{2\pi}} e^{-z^2/2}\,dz, \qquad t > 0.$$

$$f_{\mathrm{T}_a}(t) = \mathrm{F}'_{\mathrm{T}_a}(t) = \frac{2}{\sqrt{2\pi}} e^{-a^2/\left(2\sigma^2 t\right)} \cdot \left(\frac{a}{2\sigma\sqrt{t^3}}\right)$$

$$= \frac{a}{\sqrt{2\pi}\,\sigma\, t^{3/2}} e^{-a^2/\left(2\sigma^2 t\right)}, \qquad t > 0.$$

If W has a Normal distribution with mean 0 and variance $\dfrac{\sigma^2}{a^2}$,

then T_a has the same distribution as $\dfrac{1}{\mathrm{W}^2}$.

$$\mathrm{P}(\mathrm{T}_a > t) = 1 - \mathrm{F}_{\mathrm{T}_a}(t) = \mathrm{P}\left(-\frac{a}{\sigma\sqrt{t}} < \mathrm{Z} < \frac{a}{\sigma\sqrt{t}}\right) \to 0 \quad \text{as } t \to \infty.$$

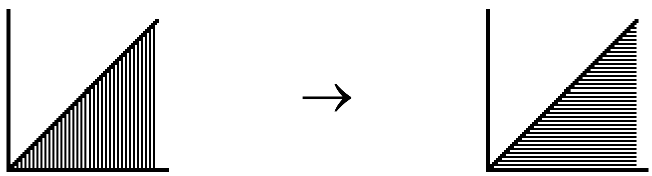
$$\mathrm{P}(\mathrm{T}_a = \infty) = \lim_{t\to\infty} \mathrm{P}(\mathrm{T}_a > t) = 0. \qquad \Rightarrow \qquad \mathrm{P}(\mathrm{T}_a < \infty) = 1.$$

Fact: Let X be a nonnegative continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. Then

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty \left(1 - F(x)\right) dx .$$

Proof:

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty x\, f(x)\, dx = \int_0^\infty \left(\int_0^x dy \right) f(x)\, dx = \int_0^\infty \left(\int_0^x f(x)\, dy \right) dx$$



$$\int_0^\infty \left(\int_0^x f(x)\, dy \right) dx = \int_0^\infty \left(\int_y^\infty f(x)\, dx \right) dy$$

$$\Rightarrow \quad \mathrm{E}(\mathrm{X}) = \int_0^\infty \left(\int_y^\infty f(x)\, dx \right) dy = \int_0^\infty \mathrm{P}(\mathrm{X} > y)\, dy = \int_0^\infty \left(1 - F(y)\right) dy .$$

$$\mathrm{E}(\mathrm{T}_a) = \int_0^\infty \left(1 - \mathrm{F}_{\mathrm{T}_a}(t) \right) dt = \int_0^\infty \int_0^{\frac{a}{\sigma\sqrt{t}}} \frac{2}{\sqrt{2\pi}}\, e^{-z^2/2} dz\, dt$$

$$= \int_0^\infty \int_0^{\frac{a^2}{\sigma^2 z^2}} \frac{2}{\sqrt{2\pi}}\, e^{-z^2/2} dt\, dz = \int_0^\infty \frac{2}{\sqrt{2\pi}} \frac{a^2}{\sigma^2 z^2}\, e^{-z^2/2} dz = \infty.$$

This is not surprising, even though $\mathrm{P}(\mathrm{T}_a < \infty) = 1$, since a Brownian motion is the limit of a (null recurrent) symmetric random walk.

Let $M(t) = \max_{0 \le s \le t} B(s),\ t > 0.$

By continuity, $P(M(t) \ge a) = P(\max_{0 \le s \le t} B(s) \ge a) = P(T_a \le t),\ a > 0,\ t \ge 0.$

$$F_{M(t)}(a) = 1 - P(\max_{0 \le s \le t} B(s) \ge a) = 1 - P(T_a \le t) = 1 - \int_{\frac{a}{\sigma\sqrt{t}}}^{\infty} \frac{2}{\sqrt{2\pi}} e^{-z^2/2}\, dz,$$

$a > 0.$

$$f_{M(t)}(a) = \frac{2}{\sqrt{2\pi}} e^{-a^2/(2\sigma^2 t)} \cdot \left(\frac{1}{\sigma\sqrt{t}}\right), \qquad a > 0,\ t \ge 0.$$

$\max_{0 \le s \le t} B(t)$ has the same probability distribution as $|B(t)|$.

Example: Let $B(t)$ be a Brownian motion with parameter $\sigma^2 = 1$. Then the process $X(t) = |B(t)|,\ t \ge 0,$ is called Brownian motion reflected at the origin.

a) Find the density function of $X(t),\ t \ge 0.$

$$F_{X(t)}(x) = P(X(t) \le x) = P(-x \le B(t) \le x) = F_{B(t)}(x) - F_{B(t)}(-x).$$

$x > 0.$

$$f_{X(t)}(x) = F'_{X(t)}(x) = f_{B(t)}(x) + f_{B(t)}(-x)$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} + \frac{1}{\sqrt{2\pi t}} e^{-(-x)^2/2t}$$

$$= \frac{2}{\sqrt{2\pi t}} e^{-x^2/2t}, \qquad x > 0.$$

b) Find $E[X(t)]$ and $\mathrm{Var}[X(t)]$.

Hint: Compute $E[X(t)]$ directly after $u = x^2$ and $E[(X(t))^2]$ by comparing the integral with the integral representing the variance of a random variable that is $N(0, t)$.

$$E[X(t)] = \int_0^{\infty} x \frac{2}{\sqrt{2\pi t}} e^{-x^2/2t} dx = \ldots$$

$$u = \frac{x^2}{2t} \qquad du = \frac{1}{t} x \, dx$$

$$\ldots = \int_0^{\infty} \frac{2\sqrt{t}}{\sqrt{2\pi}} e^{-u} du = \frac{2\sqrt{t}}{\sqrt{2\pi}} = \frac{\sqrt{2t}}{\sqrt{\pi}}.$$

$$E[(X(t))^2] = \int_0^{\infty} x^2 \frac{2}{\sqrt{2\pi t}} e^{-x^2/2t} dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

$$= E(W^2) = t, \qquad \text{where } W \sim N(0, t).$$

OR

$$E[(X(t))^2] = \int_0^{\infty} x^2 \frac{2}{\sqrt{2\pi t}} e^{-x^2/2t} dx = \ldots$$

$$u = \frac{x^2}{2t} \qquad du = \frac{1}{t} x \, dx$$

$$\ldots = \int_0^{\infty} \frac{2t}{\sqrt{\pi}} \sqrt{u}\, e^{-u} du = \frac{2t}{\sqrt{\pi}} \int_0^{\infty} u^{1/2} e^{-u} du = \frac{2t}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{2t}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = t.$$

$$\mathrm{Var}[X(t)] = E[(X(t))^2] - [E(X(t))]^2 = t - \frac{2t}{\pi} = t\left(1 - \frac{2}{\pi}\right).$$

2. Let $B(t)$ be a Brownian motion.

a) $P(T_2 < T_{-1})$.

$$P(T_2 < T_{-1}) = P(\text{up } A = 2 \text{ before down } B = 1) = \frac{B}{A+B} = \frac{1}{2+1} = \mathbf{\frac{1}{3}}.$$

b) $P(T_2 < T_{-1} < T_3)$.

$$P(T_2 < T_{-1} < T_3) = P(\text{first, up 2 before down 1,}$$
$$\text{then (from 2) down 3 (to } -1\text{) before up 1 (to 3)})$$
$$= \frac{1}{2+1} \times \frac{1}{1+3} = \mathbf{\frac{1}{12}}.$$