$$\Delta x_{i+1} = \frac{f''(\alpha)}{2f'(\alpha)}(\Delta x_i)^2 + O(\Delta x_i)^3 \,,$$

where

$$\Delta x_i \triangleq x_i - \alpha \,.$$

If the derivative is 0 at α, then the convergence is usually only linear. Specifically, if f is twice continuously differentiable, $f'(\alpha) = 0$ and $f''(\alpha) \neq 0$, then there exists a neighborhood of α such that for all starting values x_0 in that neighborhood, the sequence of iterates converges linearly, with rate 1/2 [5] Alternatively if $f'(\alpha) = 0$ and $f'(x) \neq 0$ for $x \neq \alpha$, x in a neighborhood U of α, α being a zero of multiplicity r, and if $f \in C^r(U)$ then there exists a neighborhood of α such that for all starting values x_0 in that neighborhood, the sequence of iterates converges linearly.

However, even linear convergence is not guaranteed in pathological situations.

In practice these results are local, and the neighborhood of convergence is not known in advance. But there are also some results on global convergence: for instance, given a right neighborhood U_+ of α, if f is twice differentiable in U_+ and if $f' \neq 0, f \cdot f'' > 0$ in U_+, then, for each x_0 in U_+ the sequence x_k is monotonically decreasing to α.

Proof of quadratic convergence for Newton's iterative method

According to Taylor's theorem, any function $f(x)$ which has a continuous second derivative can be represented by an expansion about a point that is close to a root of $f(x)$. Suppose this root is α. Then the expansion of $f(\alpha)$ about x_n is:

$$f(\alpha) = f(x_n) + f'(x_n)(\alpha - x_n) + R_1 \qquad (1)$$

where the Lagrange form of the Taylor series expansion remainder is

$$R_1 = \frac{1}{2!} f''(\xi_n)(\alpha - x_n)^2 \,,$$

where ξ_n is in between x_n and α.

Since α is the root, (1) becomes:

$$0 = f(\alpha) = f(x_n) + f'(x_n)(\alpha - x_n) + \tfrac{1}{2} f''(\xi_n)(\alpha - x_n)^2 \qquad (2)$$

Dividing equation (2) by $f'(x_n)$ and rearranging gives

$$\frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) = \frac{-f''(\xi_n)}{2f'(x_n)}(\alpha - x_n)^2 \tag{3}$$

Remembering that x_{n+1} is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{4}$$

one finds that

$$\underbrace{\alpha - x_{n+1}}_{\varepsilon_{n+1}} = \frac{-f''(\xi_n)}{2f'(x_n)}(\underbrace{\alpha - x_n}_{\varepsilon_n})^2.$$

That is,

$$\varepsilon_{n+1} = \frac{-f''(\xi_n)}{2f'(x_n)} \cdot {\varepsilon_n}^2. \tag{5}$$

Taking absolute value of both sides gives

$$|\varepsilon_{n+1}| = \frac{|f''(\xi_n)|}{2|f'(x_n)|} \cdot {\varepsilon_n}^2. \tag{6}$$

Equation (6) shows that the rate of convergence is quadratic if the following conditions are satisfied:

1. $f'(x) \neq 0$; for all $x \in I$, where I is the interval $[\alpha - r, \alpha + r]$ for some $r \geq |\alpha - x_0|$;
2. $f''(x)$ is continuous, for all $x \in I$;
3. x_0 sufficiently close to the root α.

The term *sufficiently* close in this context means the following:

a. Taylor approximation is accurate enough such that we can ignore higher order terms;