the principal directions are $\boldsymbol{U}_1 = \begin{bmatrix}3 & 1\end{bmatrix}'$ and $\boldsymbol{U}_2 = \begin{bmatrix}-1 & 3\end{bmatrix}'$, with corresponding eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 2$, respectively. Images of these vectors are $\boldsymbol{V}_1 = \boldsymbol{A}\boldsymbol{U}_1 = \begin{bmatrix}12 & 4\end{bmatrix}' = 4\begin{bmatrix}3 & 1\end{bmatrix}'$ and $\boldsymbol{V}_2 = \boldsymbol{A}\boldsymbol{U}_2 = \begin{bmatrix}-2 & 6\end{bmatrix}' = 2\begin{bmatrix}-1 & 3\end{bmatrix}'$. This transformation stretches the quarter-circle shown in Figure 11.1(a) into the quarter ellipse shown in Figure 11.11(b).

11.1 Homogeneous Systems: Eigenvalue Problem

Background

We will now review some ideas from linear algebra. Proofs of the theorems are either left as exercises or can be found in any standard text on linear algebra.

In Chapter 3 we saw how to solve n linear equations in n unknowns. It was assumed that the determinant of the matrix was nonzero and hence that the solution was unique. In the case of a homogeneous system $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{0}$, if $\det(\boldsymbol{A}) \neq 0$, the unique solution is the trivial solution $\boldsymbol{X} = \boldsymbol{0}$. If $\det(\boldsymbol{A}) = 0$, there exist nontrivial solutions to $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{0}$. Suppose that $\det(\boldsymbol{A}) = 0$, and consider solutions to the homogeneous linear system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad &\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0.
\end{aligned} \tag{1}
$$

The system of equations (1) always has the trivial solution $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. Gaussian elimination can be used to obtain a solution by forming a set of relationships between the variables.

Example 11.1. Find the nontrivial solutions to the homogeneous system

$$
\begin{aligned}
x_1 + 2x_2 - x_3 &= 0 \\
2x_1 + \ x_2 + x_3 &= 0 \\
5x_1 + 4x_2 + x_3 &= 0.
\end{aligned}
$$

Use Gaussian elimination to eliminate x_1 and the result is

$$
\begin{aligned}
x_1 + 2x_2 - \ x_3 &= 0 \\
-3x_2 + 3x_3 &= 0 \\
-6x_2 + 6x_3 &= 0.
\end{aligned}
$$

Since the third equation is a multiple of the second equation, this system reduces to two equations in three unknowns:

$$\begin{aligned} x_1 + x_2 \qquad &= 0 \\ -x_2 + x_3 &= 0. \end{aligned}$$

We can select one unknown and use it as a parameter. For instance, let $x_3 = t$; then the second equation implies that $x_2 = t$ and the first equation is used to compute $x_1 = -t$. Therefore, the solution can be expressed as the set of relations:

$$\begin{aligned} x_1 &= -t \\ x_2 &= \ t \\ x_3 &= \ t \end{aligned} \qquad \text{or} \qquad \boldsymbol{X} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

where t is any real number. ■

Definition 11.1. The vectors $\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_n$ are said to be ***linearly independent*** if the equation

$$c_1\boldsymbol{U}_1 + c_2\boldsymbol{U}_2 + \cdots + c_n\boldsymbol{U}_n = \boldsymbol{0} \tag{2}$$

implies that $c_1 = 0, c_2 = 0, \ldots, c_n = 0$. If the vectors are not linearly independent they are said to be linearly dependent. In other words, the vectors are ***linearly dependent*** if there exists a set of numbers $\{c_1, c_2, \ldots, c_n\}$ not all zero, such that equation (2) holds. ▲

Two vectors in $\Re^2$ are linearly independent if and only if they are not parallel. Three vectors in $\Re^3$ are linearly independent if and only if they do not lie in the same plane.

Theorem 11.1. The vectors $\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_n$ are linearly dependent if and only if at least one of them is a linear combination of the others.

A desirable feature for a vector space is the ability to express each vector as a linear combination of vectors chosen from a small subset of vectors. This motivates the next definition.

Definition 11.2. Suppose that $S = \{\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_m\}$ is a set of m vectors in $\Re^n$. The set S is called a ***basis*** for $\Re^n$ if for every vector $\boldsymbol{X}$ in $\Re^n$ there exists a unique set of scalars $\{c_1, c_2, \ldots, c_m\}$ so that $\boldsymbol{X}$ can be expressed as the linear combination

$$\boldsymbol{X} = c_1\boldsymbol{U}_1 + c_2\boldsymbol{U}_2 + \cdots + c_m\boldsymbol{U}_m. \tag{3}$$ ▲

Theorem 11.2. In $\Re^n$, any set of n linearly independent vectors forms a basis of $\Re^n$. Each vector $\boldsymbol{X}$ in $\Re^n$ is uniquely expressed as a linear combination of the basis vectors, as shown in equation (3).

Theorem 11.4 is usually applied for hand computations in the following manner. The eigenvalue λ of multiplicity $r \geq 1$ is substituted into the equation

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{V} = \boldsymbol{0}. \tag{14}$$

Then Gaussian elimination can be performed to obtain the Gauss reduced form, which will involve $n - k$ equations in n unknowns, where $1 \leq k \leq r$. Hence there are k free variables to choose. The free variables can be selected in a judicious manner to produce k linearly independent solution vectors $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_k$ that correspond to λ.

Example 11.2. Find the eigenpairs $\lambda_j, \boldsymbol{V}_j$ for the matrix

$$\boldsymbol{A} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Also, show that the eigenvectors are linearly independent.

The characteristic equation $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$ is

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = -\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0, \tag{15}$$

which can be written as $-(\lambda - 1)(\lambda - 3)(\lambda - 4) = 0$. Therefore, the three eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 4$.

Case (i): Substitute $\lambda_1 = 1$ into equation (14) and obtain

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 0. \end{aligned}$$

Since the sum of the first equation plus two times the second equation plus the third equation is identically zero, the system can be reduced to two equations in three unknowns:

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_2 + 2x_3 &= 0. \end{aligned}$$

Choose $x_2 = 2a$, where a is an arbitrary constant; then the first and second equations are used to compute $x_1 = a$ and $x_3 = a$, respectively. Thus the first eigenpair is $\lambda_1 = 1$, $\boldsymbol{V}_1 = \begin{bmatrix} a & 2a & a \end{bmatrix}' = a\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}'$.

Case (ii): Substitute $\lambda_2 = 3$ into equation (14) and obtain

$$\begin{aligned} -x_2 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \\ -x_2 &= 0. \end{aligned}$$

This is equivalent to the system of two equations

$$\begin{aligned} x_1 \qquad + x_3 &= 0 \\ x_2 \qquad &= 0. \end{aligned}$$

Choose $x_1 = b$, where b is an arbitrary constant, and compute $x_3 = -b$. Hence the second eigenpair is $\lambda_2 = 3$, $\boldsymbol{V}_2 = \begin{bmatrix} b & 0 & -b \end{bmatrix}' = b\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}'$.

Case (iii): Substitute $\lambda_3 = 4$ into (14); the result is

$$\begin{aligned} -x_1 - \ x_2 \qquad &= 0 \\ -x_1 - 2x_2 - x_3 &= 0 \\ -x_2 - x_3 &= 0. \end{aligned}$$

This is equivalent to the two equations

$$\begin{aligned} x_1 + x_2 \qquad &= 0 \\ x_2 + x_3 &= 0. \end{aligned}$$

Choose $x_3 = c$, where c is a constant, then use the second equation to compute $x_2 = -c$. Then use the first equation to get $x_1 = c$. Thus the third eigenpair is $\lambda_3 = 4$, $\boldsymbol{V}_3 = \begin{bmatrix} c & -c & c \end{bmatrix}' = c\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}'$.

To prove that the vectors are linearly independent, it suffices to apply Theorem 11.5. However, it is beneficial to review techniques from linear algebra and use Theorem 11.3. Form the determinant

$$\det\left(\begin{bmatrix} \boldsymbol{V}_1 & \boldsymbol{V}_2 & \boldsymbol{V}_3 \end{bmatrix}\right) = \begin{vmatrix} a & b & c \\ 2a & 0 & -c \\ a & -b & c \end{vmatrix} = -6abc.$$

Since $\det\left(\begin{bmatrix} \boldsymbol{V}_1 & \boldsymbol{V}_2 & \boldsymbol{V}_3 \end{bmatrix}\right) \neq 0$, Theorem 11.3 implies that the vectors $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, and $\boldsymbol{V}_3$ are linearly independent. ■

Example 11.2 shows how hand computations are used to find eigenvalues when the dimension n is small: (1) find the coefficients of the characteristic polynomial; (2) find its roots; (3) find the nonzero solutions of the homogeneous linear system $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{V} = \boldsymbol{0}$. We will take the prevalent approach of studying the power and Jacobi methods and the QR algorithm. The QR algorithm and its improvements are used in professional software packages such as EISPACK and MATLAB.

Since $\boldsymbol{V}$ in (12) is multiplied on the right side of the matrix $\boldsymbol{A}$, it is called a ***right eigenvector*** corresponding to λ. There also exists a left eigenvector $\boldsymbol{Y}$ such that

$$\boldsymbol{Y}'\boldsymbol{A} = \lambda \boldsymbol{Y}'. \tag{16}$$

In general, the left eigenvector $\boldsymbol{Y}$ is not equal to the right eigenvector $\boldsymbol{V}$. However, if $\boldsymbol{A}$ is real and symmetric ($\boldsymbol{A}' = \boldsymbol{A}$), then

$$\begin{aligned} (\boldsymbol{A}\boldsymbol{V})' &= \boldsymbol{V}'\boldsymbol{A}' = \boldsymbol{V}'\boldsymbol{A}, \\ (\lambda \boldsymbol{V})' &= \lambda \boldsymbol{V}' \end{aligned} \tag{17}$$

Overview of Methods

For problems involving moderate-sized symmetric matrices, it is safe to use Jacobi's method. For problems involving large symmetric matrices (for n up to several hundred), it is best to use Householder's method to produce a tridiagonal form, followed by the QR algorithm. Unlike real symmetric matrices, real unsymmetric matrices can have complex eigenvalues and eigenvectors.

For matrices that possess a dominant eigenvalue, the power method can be used to find the dominant eigenvector. Deflation techniques can be used thereafter to find the first few subdominant eigenvectors. For real unsymmetric matrices, Householder's method is used to produce a Hessenberg matrix, followed by the LR or QR algorithm.

Exercises for Homogeneous Systems: Eigenvalue Problem

1. For each of the following matrices, find **(i)** the characteristic polynomial $p(\lambda)$, **(ii)** the eigenvalues, and **(iii)** an eigenvector for each eigenvalue.

(a) $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ **(b)** $\boldsymbol{A} = \begin{bmatrix} 1 & 6 \\ 9 & 2 \end{bmatrix}$ **(c)** $\boldsymbol{A} = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}$

(d) $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$ **(e)** $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

2. Determine the spectral radius of each of the matrices in Exercise 1.

3. Determine the $\|\boldsymbol{A}\|_2$ and $\|\boldsymbol{A}\|_\infty$ norms of each of the matrices in Exercise 1.

4. Determine which, if any, of the matrices in Exercise 1 are diagonalizable. For each diagonalizable matrix in Exercise 1, find the matrices $\boldsymbol{V}$ and $\boldsymbol{D}$ from Theorem 11.8 and carry out the matrix product in (24).

5. **(a)** For any fixed θ, show that

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is an orthogonal matrix.

Remark. The matrix $\boldsymbol{R}$ is called a *rotation matrix*.

(b) Determine all values of θ for which all the eigenvalues of $\boldsymbol{R}$ are real.

6. In Section 3.2 the plane rotations $\boldsymbol{R}_x(\alpha)$, $\boldsymbol{R}_y(\beta)$, and $\boldsymbol{R}_z(\gamma)$ were introduced.

(a) For any fixed α, β, and γ, show that $\boldsymbol{R}_x(\alpha)$, $\boldsymbol{R}_y(\beta)$, and $\boldsymbol{R}_z(\gamma)$, respectively, are orthogonal matrices.

(b) Determine all values of α, β, and γ for which all the eigenvalues of $\boldsymbol{R}_x(\alpha)$, $\boldsymbol{R}_y(\beta)$, and $\boldsymbol{R}_z(\gamma)$, respectively, are real.

Definition 11.11. An eigenvector $\boldsymbol{V}$ is said to be ***normalized*** if the coordinate of largest magnitude is equal to unity (i.e., the largest coordinate in the vector $\boldsymbol{V}$ is the number 1). ▲

It is easy to normalize an eigenvector $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}'$ by forming a new vector $\boldsymbol{V} = (1/c)\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}'$, where $c = v_j$ and $|v_j| = \max_{1 \le i \le n}\{|v_i|\}$.

Suppose that the matrix $\boldsymbol{A}$ has a dominant eigenvalue λ and that there is a unique normalized eigenvector $\boldsymbol{V}$ that corresponds to λ. This eigenpair λ, $\boldsymbol{V}$ can be found by the following iterative procedure called the ***power method***. Start with the vector

$$\boldsymbol{X}_0 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}'. \tag{1}$$

Generate the sequence $\{\boldsymbol{X}_k\}$ recursively, using

$$\begin{aligned} \boldsymbol{Y}_k &= \boldsymbol{A}\boldsymbol{X}_k, \\ \boldsymbol{X}_{k+1} &= \frac{1}{c_{k+1}}\boldsymbol{Y}_k, \end{aligned} \tag{2}$$

where c_{k+1} is the coordinate of $\boldsymbol{Y}_k$ of largest magnitude (in the case of a tie, choose the coordinate that comes first). The sequences $\{\boldsymbol{X}_k\}$ and $\{c_k\}$ will converge to $\boldsymbol{V}$ and λ, respectively:

$$\lim_{k\to\infty} \boldsymbol{X}_k = \boldsymbol{V} \qquad \text{and} \qquad \lim_{k\to\infty} c_k = \lambda. \tag{3}$$

Remark. If $\boldsymbol{X}_0$ is an eigenvector and $\boldsymbol{X}_0 \neq \boldsymbol{V}$, then some other starting vector must be chosen.

Example 11.5. Use the power method to find the dominant eigenvalue and eigenvector for the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}.$$

Start with $\boldsymbol{X}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$ and use the formulas in (2) to generate the sequence of vectors $\{\boldsymbol{X}_k\}$ and constants $\{c_k\}$. The first iteration produces

$$\begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix} = 12 \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \\ 1 \end{bmatrix} = c_1 \boldsymbol{X}_1.$$

The second iteration produces

$$\begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{10}{3} \\ \frac{16}{3} \end{bmatrix} = \frac{16}{3} \begin{bmatrix} \frac{7}{16} \\ \frac{5}{8} \\ 1 \end{bmatrix} = c_2 \boldsymbol{X}_2.$$

Table 11.1 Power Method Used in Example 11.5 to Find the Normalized Dominant Eigenvector $\boldsymbol{V} = \left[\frac{2}{5} \quad \frac{3}{5} \quad 1\right]'$ and Corresponding Eigenvalue $\lambda = 4$

$\boldsymbol{AX}_k =$	$\boldsymbol{Y}_k$	$=$	$c_{k+1}\boldsymbol{X}_{k+1}$
$\boldsymbol{AX}_0 =$	[6.000000 8.000000 12.00000]′	$=$	12.00000[0.500000 0.666667 1]′ $= c_1\boldsymbol{X}_1$
$\boldsymbol{AX}_1 =$	[2.333333 3.333333 5.333333]′	$=$	5.333333[0.437500 0.625000 1]′ $= c_2\boldsymbol{X}_2$
$\boldsymbol{AX}_2 =$	[1.875000 2.750000 4.500000]′	$=$	4.500000[0.416667 0.611111 1]′ $= c_3\boldsymbol{X}_3$
$\boldsymbol{AX}_3 =$	[1.722222 2.555556 4.222222]′	$=$	4.222222[0.407895 0.605263 1]′ $= c_4\boldsymbol{X}_4$
$\boldsymbol{AX}_4 =$	[1.657895 2.473684 4.105263]′	$=$	4.105263[0.403846 0.602564 1]′ $= c_5\boldsymbol{X}_5$
$\boldsymbol{AX}_5 =$	[1.628205 2.435897 4.051282]′	$=$	4.051282[0.401899 0.601266 1]′ $= c_6\boldsymbol{X}_6$
$\boldsymbol{AX}_6 =$	[1.613924 2.417722 4.025316]′	$=$	4.025316[0.400943 0.600629 1]′ $= c_7\boldsymbol{X}_7$
$\boldsymbol{AX}_7 =$	[1.606918 2.408805 4.012579]′	$=$	4.012579[0.400470 0.600313 1]′ $= c_8\boldsymbol{X}_8$
$\boldsymbol{AX}_8 =$	[1.603448 2.404389 4.006270]′	$=$	4.006270[0.400235 0.600156 1]′ $= c_9\boldsymbol{X}_9$
$\boldsymbol{AX}_9 =$	[1.601721 2.402191 4.003130]′	$=$	4.003130[0.400117 0.600078 1]′ $= c_{10}\boldsymbol{X}_{10}$
$\boldsymbol{AX}_{10} =$	[1.600860 2.401095 4.001564]′	$=$	4.001564[0.400059 0.600039 1]′ $= c_{11}\boldsymbol{X}_{11}$

Iteration generates the sequence $\{\boldsymbol{X}_k\}$ (where $\boldsymbol{X}_k$ is a normalized vector):

$$12\begin{bmatrix}\frac{1}{2}\\ \frac{2}{3}\\ 1\end{bmatrix}, \frac{16}{3}\begin{bmatrix}\frac{7}{16}\\ \frac{5}{8}\\ 1\end{bmatrix}, \frac{9}{2}\begin{bmatrix}\frac{5}{12}\\ \frac{11}{18}\\ 1\end{bmatrix}, \frac{38}{9}\begin{bmatrix}\frac{31}{76}\\ \frac{23}{38}\\ 1\end{bmatrix}, \frac{78}{19}\begin{bmatrix}\frac{21}{52}\\ \frac{47}{78}\\ 1\end{bmatrix}, \frac{158}{39}\begin{bmatrix}\frac{127}{316}\\ \frac{95}{158}\\ 1\end{bmatrix}, \cdots$$

The sequence of vectors converges to $\boldsymbol{V} = \left[\frac{2}{3} \quad \frac{3}{5} \quad 1\right]'$, and the sequence of constants converges to $\lambda = 4$ (see Table 11.1). It can be proved that the rate of convergence is linear. ■

Theorem 11.18 (Power Method). Assume that the $n \times n$ matrix $\boldsymbol{A}$ has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and that they are ordered in decreasing magnitude; that is,

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|. \tag{4}$$

If $\boldsymbol{X}_0$ is chosen appropriately, then the sequences $\left\{\boldsymbol{X}_k = \left[x_1^{(k)} \quad x_2^{(k)} \quad \ldots \quad x_n^{(k)}\right]'\right\}$ and $\{c_k\}$ generated recursively by

$$\boldsymbol{Y}_k = \boldsymbol{AX}_k \tag{5}$$

and

$$\boldsymbol{X}_{k+1} = \frac{1}{c_{k+1}}\boldsymbol{Y}_k, \tag{6}$$

where

$$c_{k+1} = x_j^{(k)} \quad \text{and} \quad x_j^{(k)} = \max_{1 \leq i \leq n}\{|x_i^{(k)}|\}, \tag{7}$$

Table 11.3 Shifted-Inverse Power Method for the Matrix $(\boldsymbol{A} - 4.2\boldsymbol{I})^{-1}$ in Example 11.6: Convergence to the Eigenvector $\boldsymbol{V} = \left[\frac{2}{5} \quad \frac{3}{5} \quad 1\right]'$ and $\mu_1 = -5$

$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_k =$	$c_{k+1}\boldsymbol{X}_{k+1}$			
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_0 = -23.18181818$	$[0.4117647059$	0.6078431373	$1]' = c_1\boldsymbol{X}_1$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_1 = -5.356506239$	$[0.4009983361$	0.6006655574	$1]' = c_2\boldsymbol{X}_2$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_2 = -5.030252609$	$[0.4000902120$	0.6000601413	$1]' = c_3\boldsymbol{X}_3$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_3 = -5.002733697$	$[0.4000081966$	0.6000054644	$1]' = c_4\boldsymbol{X}_4$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_4 = -5.000248382$	$[0.4000007451$	0.6000004967	$1]' = c_5\boldsymbol{X}_5$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_5 = -5.000022579$	$[0.4000000677$	0.6000000452	$1]' = c_6\boldsymbol{X}_6$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_6 = -5.000002053$	$[0.4000000062$	0.6000000041	$1]' = c_7\boldsymbol{X}_7$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_7 = -5.000000187$	$[0.4000000006$	0.6000000004	$1]' = c_8\boldsymbol{X}_8$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_8 = -5.000000017$	$[0.4000000001$	0.6000000000	$1]' = c_9\boldsymbol{X}_9$

Example 11.6. Employ the shifted-inverse power method to find the eigenpairs of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}.$$

Use the fact that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 4$, $\lambda_2 = 2$, and $\lambda_3 = 1$, and select an appropriate α and starting vector for each case.

Case (i): For the eigenvalue $\lambda_1 = 4$, we select $\alpha = 4.2$ and the starting vector $\boldsymbol{X}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$. First, form the matrix $\boldsymbol{A} - 4.2\boldsymbol{I}$, compute the solution to

$$\begin{bmatrix} -4.2 & 11 & -5 \\ -2 & 12.8 & -7 \\ -4 & 26 & -14.2 \end{bmatrix} \boldsymbol{Y}_0 = \boldsymbol{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and get the vector $\boldsymbol{Y}_0 = \begin{bmatrix} -9.545454545 & -14.09090909 & -23.18181818 \end{bmatrix}'$. Then compute $c_1 = -23.18181818$ and $\boldsymbol{X}_1 = \begin{bmatrix} 0.4117647059 & 0.6078431373 & 1 \end{bmatrix}'$. Iteration generates the values given in Table 11.3. The sequence $\{c_k\}$ converges to $\mu_1 = -5$, which is the dominant eigenvalue of $(\boldsymbol{A} - 4.2\boldsymbol{I})^{-1}$, and $\{\boldsymbol{X}_k\}$ converges to $\boldsymbol{V}_1 = \left[\frac{2}{5} \quad \frac{3}{5} \quad 1\right]'$. The eigenvalue λ_1 of $\boldsymbol{A}$ is given by the computation $\lambda_1 = 1/\mu_1 + \alpha = 1/(-5) + 4.2 = -0.2 + 4.2 = 4$.

Case (ii): For the eigenvalue $\lambda_2 = 2$, we select $\alpha = 2.1$ and the starting vector $\boldsymbol{X}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$. Form the matrix $\boldsymbol{A} - 2.1\boldsymbol{I}$, compute the solution to

$$\begin{bmatrix} -2.1 & 11 & -5 \\ -2 & 14.9 & -7 \\ -4 & 26 & -12.1 \end{bmatrix} \boldsymbol{Y}_0 = \boldsymbol{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and obtain the vector $\boldsymbol{Y}_0 = \begin{bmatrix} 11.05263158 & 21.57894737 & 42.63157895 \end{bmatrix}'$. Then $c_1 = 42.63157895$ and vector $\boldsymbol{X}_1 = \begin{bmatrix} 0.2592592593 & 0.5061728395 & 1 \end{bmatrix}'$. Iteration produces the

Table 11.4 Shifted-Inverse Power Method for the Matrix $(\boldsymbol{A} - 2.1\boldsymbol{I})^{-1}$ in Example 11.6: Convergence to the Dominant Eigenvector $\boldsymbol{V} = \left[\frac{1}{4} \quad \frac{1}{2} \quad 1\right]$ and $\mu_1 = -10$

$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_k =$	$c_{k+1}\boldsymbol{X}_{k+1}$			
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_0 =$	42.63157895	[0.2592592593	0.5061728395	$1]' = c_1\boldsymbol{X}_1$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_1 =$	−9.350227420	[0.2494788047	0.4996525365	$1]' = c_2\boldsymbol{X}_2$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_2 =$	−10.03657511	[0.2500273314	0.5000182209	$1]' = c_3\boldsymbol{X}_3$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_3 =$	−9.998082009	[0.2499985612	0.4999990408	$1]' = c_4\boldsymbol{X}_4$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_4 =$	−10.00010097	[0.2500000757	0.5000000505	$1]' = c_5\boldsymbol{X}_5$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_5 =$	−9.999994686	[0.2499999960	0.4999999973	$1]' = c_6\boldsymbol{X}_6$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_6 =$	−10.00000028	[0.2500000002	0.5000000001	$1]' = c_7\boldsymbol{X}_7$

Table 11.5 Shifted-Inverse Power Method for the Matrix $(\boldsymbol{A} - 0.875\boldsymbol{I})^{-1}$ in Example 11.6: Convergence to the Dominant Eigenvector $\boldsymbol{V} = \left[\frac{1}{2} \quad \frac{1}{2} \quad 1\right]'$ and $\mu_1 = 8$

$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_k =$	$c_{k+1}\boldsymbol{X}_{k+1}$			
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_0 =$	−30.40000000	[0.5052631579	0.4947368421	$1]' = c_1\boldsymbol{X}_1$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_1 =$	8.404210526	[0.5002004008	0.4997995992	$1]' = c_2\boldsymbol{X}_2$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_2 =$	8.015390782	[0.5000080006	0.4999919994	$1]' = c_3\boldsymbol{X}_3$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_3 =$	8.000614449	[0.5000003200	0.4999996800	$1]' = c_4\boldsymbol{X}_4$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_4 =$	8.000024576	[0.5000000128	0.4999999872	$1]' = c_5\boldsymbol{X}_5$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_5 =$	8.000000983	[0.5000000005	0.4999999995	$1]' = c_6\boldsymbol{X}_6$
$(\boldsymbol{A} - \alpha\boldsymbol{I})^{-1}\boldsymbol{X}_6 =$	8.000000039	[0.5000000000	0.5000000000	$1]' = c_7\boldsymbol{X}_7$

values given in Table 11.4. The dominant eigenvalue of $(\boldsymbol{A} - 2.1\boldsymbol{I})^{-1}$ is $\mu_1 = -10$, and the eigenpair of the matrix $\boldsymbol{A}$ is $\lambda_2 = 1/(-10) + 2.1 = -0.1 + 2.1 = 2$ and $\boldsymbol{V}_2 = \left[\frac{1}{4} \quad \frac{1}{2} \quad 1\right]'$.

Case (iii): For the eigenvalue $\lambda_3 = 1$, we select $\alpha = 0.875$ and the starting vector $\boldsymbol{X}_0 = \begin{bmatrix}0 & 1 & 1\end{bmatrix}'$. Iteration produces the values given in Table 11.5. The dominant eigenvalue of $(\boldsymbol{A} - 0.875\boldsymbol{I})^{-1}$ is $\mu_1 = 8$, and the eigenpair of matrix $\boldsymbol{A}$ is $\lambda_3 = 1/8 + 0.875 = 0.125 + 0.875 = 1$ and $\boldsymbol{V}_3 = \left[\frac{1}{2} \quad \frac{1}{2} \quad 1\right]'$. The sequence $\{\boldsymbol{X}_k\}$ of vectors with the starting vector $\begin{bmatrix}0 & 1 & 1\end{bmatrix}'$ converged in seven iterations. (Computational difficulties were encountered when $\boldsymbol{X}_0 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}'$ was used, and convergence took significantly longer.) ■

Program 11.1 (Power Method). To compute the dominant eigenvalue λ_1 and its associated eigenvector $\boldsymbol{V}_1$ for the $n \times n$ matrix $\boldsymbol{A}$. It is assumed that the n eigenvalues have the dominance property $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| > 0$.

```
function [lambda,V]=power1(A,X,epsilon,max1)
```

```
%Input  - A is an nxn matrix
%       - X is the nx1 starting vector
%       - epsilon is the tolerance
%       - max1 is the maximum number of iterations
%Output - lambda is the dominant eigenvalue
%       - V is the dominant eigenvector
%Initialize parameters
lambda=0;
cnt=0;
err=1;
state=1;
while ((cnt<=max1)&(state==1))
   Y=A*X;
   %Normalize Y
   [m j]=max(abs(Y));
   c1=m;
   dc=abs(lambda-c1);
   Y=(1/c1)*Y;
   %Update X and lambda and check for convergence
   dv=norm(X-Y);
   err=max(dc,dv);
   X=Y;
   lambda=c1;
   state=0;
   if(err>epsilon)
      state=1;
   end
   cnt=cnt+1;
end
V=X;
```

Program 11.2 (Shifted-Inverse Power Method). To compute the dominant eigenvalue λ_j and its associated eigenvector $\boldsymbol{V}_j$ for the $n \times n$ matrix $\boldsymbol{A}$. It is assumed that the n eigenvalues have the property $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and that α is a real number such that $|\lambda_j - \alpha| < |\lambda_i - \alpha|$, for each $i = 1, 2, \ldots, j-1, j+1, \ldots, n$.

```
function [lambda,V]=invpow(A,X,alpha,epsilon,max1)
%Input  - A is an nxn matrix
%       - X is the nx1 starting vector
%       - alpha is the given shift
%       - epsilon is the tolerance
%       - max1 is the maximum number of iterations
%Output - lambda is the dominant eigenvalue
```

```
%       - V is the dominant eigenvector
%Initialize the matrix A-alphaI and parameters
[n n]=size(A);
A=A-alpha*eye(n);
lambda=0;
cnt=0;
err=1;
state=1;
while ((cnt<=max1)&(state==1))
   %Solve system AY=X
   Y=A\X;
   %Normalize Y
   [m j]=max(abs(Y));
   c1=m;
   dc=abs(lambda-c1);
   Y=(1/c1)*Y;
   %Update X and lambda and check for convergence
   dv=norm(X-Y);
   err=max(dc,dv);
   X=Y;
   lambda=c1;
   state=0;
   if (err>epsilon)
      state=1;
   end
   cnt=cnt+1;
end
lambda=alpha+1/c1;
V=X;
```

Exercises for Power Method

1. Let λ, $\boldsymbol{V}$ be an eigenpair of $\boldsymbol{A}$. If α is any constant, show that $\lambda - \alpha$, $\boldsymbol{V}$ is an eigenpair of the matrix $\boldsymbol{A} - \alpha \boldsymbol{I}$.

2. Let λ, $\boldsymbol{V}$ be an eigenpair of $\boldsymbol{A}$. If $\lambda \neq 0$, show that $1/\lambda$, $\boldsymbol{V}$ is an eigenpair of the matrix $\boldsymbol{A}^{-1}$.

3. Let λ, $\boldsymbol{V}$ be an eigenpair of $\boldsymbol{A}$. If $\alpha \neq \lambda$, show that $1/(\lambda - \alpha)$, $\boldsymbol{V}$ is an eigenpair of the matrix $(\boldsymbol{A} - \alpha \boldsymbol{I})^{-1}$.

4. *Deflation techniques.* Suppose that $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{A}$ with associated eigenvectors $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3, \ldots, \boldsymbol{V}_n$ and that λ_1 has multiplicity 1. If $\boldsymbol{X}$ is

any vector with the property that $\boldsymbol{X}'\boldsymbol{V}_1 = 1$, prove that the matrix

$$\boldsymbol{B} = \boldsymbol{A} - \lambda_1 \boldsymbol{V}_1 \boldsymbol{X}'$$

has eigenvalues $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ with associated eigenvectors $\boldsymbol{V}_1, \boldsymbol{W}_2, \boldsymbol{W}_3, \ldots, \boldsymbol{W}_n$, where $\boldsymbol{V}_j$ and $\boldsymbol{W}_j$ are related by the equation

$$\boldsymbol{V}_j = (\lambda - \lambda_1)\boldsymbol{W}_j + \lambda_1(\boldsymbol{X}'\boldsymbol{W}_j)\boldsymbol{V}_1 \quad \text{for each } j = 2, 3, \ldots, n.$$

5. *Markov processes and eigenvalues.* A Markov process can be described by a square matrix $\boldsymbol{A}$ whose entries are all positive and the column sums all equal 1. For illustration, let $\boldsymbol{P}_0 = \begin{bmatrix} x^{(0)} & y^{(0)} \end{bmatrix}'$ record the number of people in a certain city who use brands X and Y, respectively. Each month people decide to keep using the same brand or switch brands. The probability that a user of brand X will switch to brand Y is 0.3. The probability that a user of brand Y will switch to brand X is 0.2. The transition matrix for this process is

$$\boldsymbol{P}_{k+1} = \boldsymbol{A}\boldsymbol{P}_k = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix}.$$

If $\boldsymbol{A}\boldsymbol{P}_j = \boldsymbol{P}_j$ for some j, then $\boldsymbol{P}_j = \boldsymbol{V}$ is said to be the steady-state distribution for the Markov process. Thus, if there is a steady-state distribution, then $\lambda = 1$ must be an eigenvalue of $\boldsymbol{A}$. Additionally, the steady-state distribution $\boldsymbol{V}$ is an eigenvector associated with $\lambda = 1$ (i.e., solve $(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{V} = \boldsymbol{0}$).

(a) For the example given above; verify that $\lambda = 1$ is an eigenvalue of the transition matrix $\boldsymbol{A}$.

(b) Verify that the set of eigenvectors associated with $\lambda = 1$ is $\{t\begin{bmatrix} 3/2 & 1 \end{bmatrix}' : t \in \Re, t \neq 0\}$.

(c) Assume that the population of the city was 50,000. Use your results from part (b) to verify that the steady-state distribution is $\begin{bmatrix} 30{,}000 & 20{,}000 \end{bmatrix}'$.

Algorithms and Programs

In Problems 1 through 4, use:
(a) Program 11.1 to find the dominant eigenpair of the given matrices.
(b) Program 11.2 to find the other eigenpairs.

1. $\boldsymbol{A} = \begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix}$.

2. $\boldsymbol{A} = \begin{bmatrix} -14 & -30 & 42 \\ 24 & 49 & -66 \\ 12 & 24 & -32 \end{bmatrix}$.

3. $\boldsymbol{A} = \begin{bmatrix} 2.5 & -2.5 & 3.0 & 0.5 \\ 0.0 & 5.0 & -2.0 & 2.0 \\ -0.5 & -0.5 & 4.0 & 2.5 \\ -2.5 & -2.5 & 5.0 & 3.5 \end{bmatrix}$.

4. $\boldsymbol{A} = \begin{bmatrix} 2.5 & -2.0 & 2.5 & 0.5 \\ 0.5 & 5.0 & -2.5 & -0.5 \\ -1.5 & 1.0 & 3.5 & -2.5 \\ 2.0 & 3.0 & -5.0 & 3.0 \end{bmatrix}$.

11.3 Jacobi's Method

Jacobi's method is an easily understood algorithm for finding all eigenpairs for a symmetric matrix. It is a reliable method that produces uniformly accurate answers for the results. For matrices of order up to 10, the algorithm is competitive with more sophisticated ones. If speed is not a major consideration, it is quite acceptable for matrices up to order 20.

A solution is guaranteed for all real symmetric matrices when Jacobi's method is used. This limitation is not severe since many practical problems of applied mathematics and engineering involve symmetric matrices. From a theoretical viewpoint, the method embodies techniques that are found in more sophisticated algorithms. For instructive purposes, it is worthwhile to investigate the details of Jacobi's method.

Plane Rotations

We start with some geometrical background about coordinate transformations. Let $\boldsymbol{X}$ denote a vector in n-dimensional space and consider the linear transformation $\boldsymbol{Y} = \boldsymbol{R}\boldsymbol{X}$, where $\boldsymbol{R}$ is an $n \times n$ matrix:

$$\boldsymbol{R} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & \cos\phi & \cdots & \sin\phi & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & -\sin\phi & \cdots & \cos\phi & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{row } p \\ \\ \leftarrow \text{row } q \\ \\ \\ \end{matrix}$$

$$\begin{matrix} & & \uparrow & & \uparrow & & \\ & & \text{col } p & & \text{col } q & & \end{matrix}$$

Here all off-diagonal elements of $\boldsymbol{R}$ are zero except for the values $\pm\sin\phi$, and all diagonal elements are 1 except for $\cos\phi$. The effect of the transformation $\boldsymbol{Y} = \boldsymbol{R}\boldsymbol{X}$ is easy to grasp:

$$\begin{aligned} y_j &= x_j \qquad \text{when } \ j \neq p \ \text{ and } \ j \neq q, \\ y_p &= x_p \cos\phi + x_q \sin\phi, \\ y_q &= -x_p \sin\phi + x_q \cos\phi. \end{aligned}$$

The transformation is seen to be a rotation of n-dimensional space in the $x_p x_q$-plane through the angle ϕ. By selecting an appropriate angle ϕ, we could make either $y_p = 0$ or $y_q = 0$ in the image. The inverse transformation $\boldsymbol{X} = \boldsymbol{R}^{-1}\boldsymbol{Y}$ rotates space in the same $x_p x_q$-plane through the angle $-\phi$. Observe that $\boldsymbol{R}$ is an orthogonal matrix; that is,

$$\boldsymbol{R}^{-1} = \boldsymbol{R}' \qquad \text{or} \qquad \boldsymbol{R}'\boldsymbol{R} = \boldsymbol{I}.$$

Similarity and Orthogonal Transformations

Consider the eigenproblem

$$\boldsymbol{AX} = \lambda \boldsymbol{X}. \tag{1}$$

Suppose that $\boldsymbol{K}$ is a nonsingular matrix and that $\boldsymbol{B}$ is defined by

$$\boldsymbol{B} = \boldsymbol{K}^{-1}\boldsymbol{AK}. \tag{2}$$

Multiply both members of (2) on the right side by the quantity $\boldsymbol{K}^{-1}\boldsymbol{X}$. This produces

$$\begin{aligned} \boldsymbol{BK}^{-1}\boldsymbol{X} &= \boldsymbol{K}^{-1}\boldsymbol{AKK}^{-1}\boldsymbol{X} = \boldsymbol{K}^{-1}\boldsymbol{AX} \\ &= \boldsymbol{K}^{-1}\lambda\boldsymbol{X} = \lambda\boldsymbol{K}^{-1}\boldsymbol{X}. \end{aligned} \tag{3}$$

We define the change of variable

$$\boldsymbol{Y} = \boldsymbol{K}^{-1}\boldsymbol{X} \quad \text{or} \quad \boldsymbol{X} = \boldsymbol{KY}. \tag{4}$$

When (4) is used in (3), the new eigenproblem is

$$\boldsymbol{BY} = \lambda \boldsymbol{Y}. \tag{5}$$

Comparing (1) and (5), we see that the similarity transformation (2) preserved the eigenvalue λ and that the eigenvectors are different, but are related by the change of variable in (4).

Suppose that the matrix $\boldsymbol{R}$ is an orthogonal matrix (i.e., $\boldsymbol{R}^{-1} = \boldsymbol{R}'$) and that $\boldsymbol{D}$ is defined by

$$\boldsymbol{D} = \boldsymbol{R}'\boldsymbol{AR}. \tag{6}$$

Multiply both terms in (6) on the right by $\boldsymbol{R}'\boldsymbol{X}$ to obtain

$$\boldsymbol{DR}'\boldsymbol{X} = \boldsymbol{R}'\boldsymbol{ARR}'\boldsymbol{X} = \boldsymbol{R}'\boldsymbol{AX} = \boldsymbol{R}'\lambda\boldsymbol{X} = \lambda\boldsymbol{R}'\boldsymbol{X}. \tag{7}$$

We define the change of variable

$$\boldsymbol{Y} = \boldsymbol{R}'\boldsymbol{X} \quad \text{or} \quad \boldsymbol{X} = \boldsymbol{RY}. \tag{8}$$

Now use (8) in (7) to obtain a new eigenproblem,

$$\boldsymbol{DY} = \lambda \boldsymbol{Y}. \tag{9}$$

As before, the eigenvalues of (1) and (9) are the same. However, for equation (9) the change of variable (8) makes it easier to convert $\boldsymbol{X}$ to $\boldsymbol{Y}$ and $\boldsymbol{Y}$ back into $\boldsymbol{X}$ because $\boldsymbol{R}^{-1} = \boldsymbol{R}'$.

In addition, suppose that $\boldsymbol{A}$ is a symmetric matrix (i.e., $\boldsymbol{A} = \boldsymbol{A}'$). Then we find that

$$\boldsymbol{D}' = (\boldsymbol{R}'\boldsymbol{AR})' = \boldsymbol{R}'\boldsymbol{A}(\boldsymbol{R}')' = \boldsymbol{R}'\boldsymbol{AR} = \boldsymbol{D}. \tag{10}$$

Hence $\boldsymbol{D}$ is a symmetric matrix. Therefore, we conclude that if $\boldsymbol{A}$ is a symmetric matrix and $\boldsymbol{R}$ is an orthogonal matrix, the transformation of $\boldsymbol{A}$ to $\boldsymbol{D}$ given by (6) preserves symmetry as well as eigenvalues. The relationship between their eigenvectors is given by the change of variable (8).

Jacobi Series of Transformations

Start with the real symmetric matrix $\boldsymbol{A}$. Then construct the sequence of orthogonal matrices $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_n$ as follows:

$$\begin{aligned} &\boldsymbol{D}_0 = \boldsymbol{A}, \\ &\boldsymbol{D}_j = \boldsymbol{R}_j' \boldsymbol{D}_{j-1} \boldsymbol{R}_j \qquad \text{for } j = 1,\ 2,\ \ldots. \end{aligned} \tag{11}$$

We will show how to construct the sequence $\{\boldsymbol{R}_j\}$ so that

$$\lim_{j \to \infty} \boldsymbol{D}_j = \boldsymbol{D} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n). \tag{12}$$

In practice we will stop when the off-diagonal elements are close to zero. Then we will have

$$\boldsymbol{D}_n \approx \boldsymbol{D}. \tag{13}$$

The construction produces

$$\boldsymbol{D}_n = \boldsymbol{R}_n' \boldsymbol{R}_{n-1}' \cdots \boldsymbol{R}_1' \boldsymbol{A} \boldsymbol{R}_1 \boldsymbol{R}_2 \cdots \boldsymbol{R}_{n-1} \boldsymbol{R}_n. \tag{14}$$

If we define

$$\boldsymbol{R} = \boldsymbol{R}_1 \boldsymbol{R}_2 \cdots \boldsymbol{R}_{n-1} \boldsymbol{R}_n, \tag{15}$$

then $\boldsymbol{R}^{-1} \boldsymbol{A} \boldsymbol{R} = \boldsymbol{D}_k$, which implies that

$$\boldsymbol{A}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{D}_k \approx \boldsymbol{R} \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n). \tag{16}$$

Let the columns of $\boldsymbol{R}$ be denoted by the vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$. Then $\boldsymbol{R}$ can be expressed as a row vector of column vectors:

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{X}_2 & \cdots & \boldsymbol{X}_n \end{bmatrix}. \tag{17}$$

The columns of the products in (16) now take on the form

$$\begin{bmatrix} \boldsymbol{A}\boldsymbol{X}_1 & \boldsymbol{A}\boldsymbol{X}_2 & \cdots & \boldsymbol{A}\boldsymbol{X}_n \end{bmatrix} \approx \begin{bmatrix} \lambda_1 \boldsymbol{X}_1 & \lambda_2 \boldsymbol{X}_2 & \cdots & \lambda_n \boldsymbol{X}_n \end{bmatrix}. \tag{18}$$

From (17) and (18) we see that the vector $\boldsymbol{X}_j$, which is the jth column of $\boldsymbol{R}$, is an eigenvector that corresponds to the eigenvalue λ_j.

General Step

Each step in the Jacobi iteration will accomplish the limited objective of reduction of the two off-diagonal elements d_{pq} and d_{qp} to zero. Let $\boldsymbol{R}_1$ denote the first orthogonal matrix used. Suppose that

$$\boldsymbol{D}_1 = \boldsymbol{R}_1' \boldsymbol{A} \boldsymbol{R}_1 \tag{19}$$

reduces the elements d_{pq} and d_{qp} to zero, where $\boldsymbol{R}_1$ has the form

$$\boldsymbol{R}_1 = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{row } p \\ \\ \leftarrow \text{row } q \\ \\ \\ \end{matrix} \tag{20}$$

$$\begin{matrix} & & \uparrow & & \uparrow & & \\ & & \text{col } p & & \text{col } q & & \end{matrix}$$

Here all off-diagonal elements of $\boldsymbol{R}_1$ are zero except for the element s located in row p, column q, and the element $-s$ located in row q, column p. Also note that all diagonal elements are 1 except for the element c, which appears at two locations, in row p, column p, and in row q, column q. The matrix is a plane rotation where we have used the notation $c = \cos\phi$ and $s = \sin\phi$.

We must verify that the transformation (19) will produce a change only to rows p and q and columns p and q. Consider postmultiplication of $\boldsymbol{A}$ by $\boldsymbol{R}_1$ and the product $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{R}_1$:

(21)

$$\boldsymbol{B} = \begin{bmatrix} a_{11} & \cdots & a_{1p} & \cdots & a_{1q} & \cdots & a_{1n} \\ a_{p1} & \cdots & a_{pp} & \cdots & a_{pq} & \cdots & a_{pn} \\ a_{q1} & \cdots & a_{qp} & \cdots & a_{qq} & \cdots & a_{qn} \\ a_{n1} & \cdots & a_{np} & \cdots & a_{nq} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

The row by column rule for multiplication applies, and we observe that there is no change to columns 1 to $p-1$ and $p+1$ to $q-1$ and $q+1$ to n. Hence only columns p and q are altered.

$$\begin{aligned} b_{jk} &= a_{jk} && \text{when } k \neq p \text{ and } k \neq q, \\ b_{jp} &= ca_{jp} - sa_{jq} && \text{for } j = 1, 2, \ldots, n, \\ b_{jq} &= sa_{jp} + ca_{jq} && \text{for } j = 1, 2, \ldots, n. \end{aligned} \tag{22}$$

A similar argument shows that premultiplication of $\boldsymbol{A}$ by $\boldsymbol{R}_1'$ will only alter rows p and q. Therefore, the transformation

$$\boldsymbol{D}_1 = \boldsymbol{R}_1' \boldsymbol{A} \boldsymbol{R}_1 \tag{23}$$

will alter only columns p and q and rows p and q of $\boldsymbol{A}$. The elements d_{jk} of $\boldsymbol{D}_1$ are computed with the formulas

$$\begin{aligned}
d_{jp} &= ca_{jp} - sa_{jq} \qquad \text{when } j \neq p \text{ and } j \neq q,\\
d_{jq} &= sa_{jp} + ca_{jq} \qquad \text{when } j \neq p \text{ and } j \neq q,\\
d_{pp} &= c^2 a_{pp} + s^2 a_{qq} - 2csa_{pq},\\
d_{qq} &= s^2 a_{pp} + c^2 a_{qq} + 2csa_{pq},\\
d_{pq} &= (c^2 - s^2)a_{pq} + cs(a_{pp} - a_{qq}),
\end{aligned} \tag{24}$$

and the other elements of $\boldsymbol{D}_1$ are found by symmetry.

Zeroing Out d_{pq} and d_{qp}

The goal for each step of Jacobi's iteration is to make the two off-diagonal elements d_{pq} and d_{qp} zero. The obvious strategy would be to observe the fact that

$$c = \cos\phi \qquad \text{and} \qquad s = \sin\phi, \tag{25}$$

where ϕ is the angle of rotation that produces the desired effect. However, some ingenious maneuvers with trigonometric identities are now required. The identity for $\cot\phi$ is used with (25) to define

$$\theta = \cot 2\phi = \frac{c^2 - s^2}{2cs}. \tag{26}$$

Suppose that $a_{pq} \neq 0$ and we want to produce $d_{pq} = 0$. Then using the last equation in (24), we obtain

$$0 = (c^2 - s^2)a_{pq} + cs(a_{pp} - a_{qq}). \tag{27}$$

This can be rearranged to yield $(c^2 - s^2)/(cs) = (a_{qq} - a_{pp})/a_{pq}$, which is used in (26) to solve for θ:

$$\theta = \frac{a_{qq} - a_{pp}}{2a_{pq}}. \tag{28}$$

Although we can use (28) with formulas (25) and (26) to compute c and s, less round-off error is propagated if we compute $\tan\phi$ and use it in later computations. So we define

$$t = \tan\phi = \frac{s}{c}. \tag{29}$$

Now divide the numerator and denominator in (26) by c^2 to obtain

$$\theta = \frac{1 - s^2/c^2}{2s/c} = \frac{1 - t^2}{2t},$$

which yields the equation

$$t^2 + 2t\theta - 1 = 0. \tag{30}$$

Since $t = \tan\phi$, the smaller root of (30) corresponds to the smaller angle of rotation with $|\phi| \leq \pi/4$. The special form of the quadratic formula for finding this root is

$$t = -\theta \pm (\theta^2 + 1)^{1/2} = \frac{\text{sign}(\theta)}{|\theta| + (\theta^2 + 1)^{1/2}}, \tag{31}$$

where $\text{sign}(\theta) = 1$ when $\theta \geq 0$ and $\text{sign}(\theta) = -1$ when $\theta < 0$. Then c and s are computed with the formulas

$$\begin{aligned} c &= \frac{1}{(t^2 + 1)^{1/2}} \\ s &= ct. \end{aligned} \tag{32}$$

Summary of the General Step

We can now outline the calculations required to zero out the element d_{pq}. First, select row p and column q for which $a_{pq} \neq 0$. Second, form the preliminary quantities

$$\begin{aligned} \theta &= \frac{a_{qq} - a_{pp}}{2a_{pq}}, \\ t &= \frac{\text{sign}(\theta)}{|\theta| + (\theta^2 + 1)^{1/2}}, \\ c &= \frac{1}{(t^2 + 1)^{1/2}}, \\ s &= ct. \end{aligned} \tag{33}$$

Third, to construct $\boldsymbol{D} = \boldsymbol{D}_1$, use

$$
\begin{aligned}
&d_{pq} = 0;\\
&d_{qp} = 0;\\
&d_{pp} = c^2 a_{pp} + s^2 a_{qq} - 2csa_{pq};\\
&d_{qq} = s^2 a_{pp} + c^2 a_{qq} + 2csa_{pq};
\end{aligned}
$$

(34)

```
for    j = 1 : n
   if    (j ~= p)    and    (j ~= q)
      d_jp = c*a_jp - s*a_jq;
      d_pj = d_jp;
      d_jq = c*a_jq + s*a_jp;
      d_qj = d_jq;
   end
end
```

Updating the Matrix of Eigenvectors

We need to keep track of the matrix product $\boldsymbol{R}_1\boldsymbol{R}_2\cdots\boldsymbol{R}_n$. When we stop at the nth iteration, we will have computed

$$\boldsymbol{V}_n = \boldsymbol{R}_1\boldsymbol{R}_2\cdots\boldsymbol{R}_n, \tag{35}$$

where $\boldsymbol{V}_n$ is an orthogonal matrix. We need only keep track of the current matrix $\boldsymbol{V}_j$, for $j = 1, 2, \ldots, n$. Start by initializing $\boldsymbol{V} = \boldsymbol{I}$. Use the vector variables **XP** and **XQ** to store columns p and q of $\boldsymbol{V}$, respectively. Then for each step perform the calculation

(36)

```
for    j = 1 : n
   XP_j = v_jp;
   XQ_j = v_jq;
end
for    j = 1 : n
   v_jp = c*XP_j - s*XQ_j;
   v_jq = s*XP_j + c*XQ_j;
end
```

Strategy for Eliminating a_{pq}

The speed of convergence of Jacobi's method is seen by considering the sum of the squares of the off-diagonal elements:

$$S_1 = \sum_{\substack{j,k=1 \\ k \neq j}}^{n} |a_{jk}|^2 \tag{37}$$

$$S_2 = \sum_{\substack{j,k=1 \\ k \neq j}}^{n} |d_{jk}|^2, \qquad \text{where} \qquad \boldsymbol{D}_1 = \boldsymbol{R}'\boldsymbol{A}\boldsymbol{R}. \tag{38}$$

The reader can verify that the equations given in (34) can be used to prove that

$$S_2 = S_1 - 2|a_{pq}|^2. \tag{39}$$

At each step we let S_j denote the sum of the squares of the off-diagonal elements of $\boldsymbol{D}_j$. Then the sequence $\{S_j\}$ decreases monotonically and is bounded below by zero. Jacobi's original algorithm of 1846 selected, at each step, the off-diagonal element a_{pq} of largest magnitude to zero out and involved a search to compute the value

$$\max\{\boldsymbol{A}\} = \max_{p<q}\{|a_{pq}|\}. \tag{40}$$

This choice will guarantee that $\{S_j\}$ converges to zero. As a consequence, this proves that $\{\boldsymbol{D}_j\}$ converges to $\boldsymbol{D}$ and $\{\boldsymbol{V}_j\}$ converges to the matrix $\boldsymbol{V}$ of eigenvectors.

Jacobi's search can become time consuming since it requires an order of $(n^2-n)/2$ comparisons in a loop. It is prohibitive for larger values of n. A better strategy is the cyclic Jacobi method, where one annihilates elements in a strict order across the rows. A tolerance value ϵ is selected; then a sweep is made throughout the matrix, and if an element a_{pq} is found to be larger than ϵ, it is zeroed out. For one sweep through the matrix the elements are checked in row 1, $a_{12}, a_{13}, \ldots, a_{1n}$; then row 2, $a_{23}, a_{24}, \ldots, a_{2n}$; and so on. It has been proved that the convergence rate is quadratic for both the original and cyclic Jacobi methods. An implementation of the cyclic Jacobi method starts by observing that the sum of the squares of the diagonal elements increases with each iteration; that is, if

$$T_0 = \sum_{j=1}^{n} |a_{jj}|^2 \tag{41}$$

and

$$T_1 = \sum_{j=1}^{n} |d_{jj}|^2,$$

then

$$T_1 = T_0 + 2|a_{pq}|^2.$$

Consequently, the sequence $\{\boldsymbol{D}_j\}$ converges to the diagonal matrix $\boldsymbol{D}$. Notice that the average size of a diagonal element can be computed with the formula $(T_0/n)^{1/2}$. The magnitudes of the off-diagonal elements are compared to $\epsilon(T_0/n)^{1/2}$, where ϵ is the preassigned tolerance. Therefore, the element a_{pq} is zeroed out if

$$|a_{pq}| > \epsilon \left(\frac{T_0}{n}\right)^{1/2}. \tag{42}$$

Another variation of the method, called the *threshold Jacobi method*, is left for the reader to investigate.

Example 11.7. Use Jacobi iteration to transform the following symmetric matrix into diagonal form.

$$\begin{bmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{bmatrix}$$

The computational details are left for the reader. The first rotation matrix that will zero out $a_{13} = 3$ is

$$\boldsymbol{R}_1 = \begin{bmatrix} 0.763020 & 0.000000 & 0.646375 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ -0.646375 & 0.000000 & 0.763020 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}.$$

Calculation reveals that $\boldsymbol{A}_2 = \boldsymbol{R}_1\boldsymbol{A}_1\boldsymbol{R}_1$ is

$$\boldsymbol{A}_2 = \begin{bmatrix} 5.458619 & -2.055770 & 0.000000 & -1.409395 \\ -2.055770 & 6.000000 & 0.879665 & 0.000000 \\ 0.000000 & 0.879665 & 11.541381 & 0.116645 \\ -1.409395 & 0.000000 & 0.116645 & 7.000000 \end{bmatrix}.$$

Next, the element $a_{12} = -2.055770$ is zeroed out and we get

$$\boldsymbol{A}_3 = \begin{bmatrix} 3.655795 & 0.000000 & 0.579997 & -1.059649 \\ 0.000000 & 7.802824 & 0.661373 & 0.929268 \\ 0.579997 & 0.661373 & 11.541381 & 0.116645 \\ -1.059649 & 0.929268 & 0.116645 & 7.000000 \end{bmatrix}.$$

After 10 iterations we arrive at

$$\boldsymbol{A}_{10} = \begin{bmatrix} 3.295870 & 0.002521 & 0.037859 & 0.000000 \\ 0.002521 & 8.405210 & -0.004957 & 0.066758 \\ 0.037859 & -0.004957 & 11.704123 & -0.001430 \\ 0.000000 & 0.066758 & -0.001430 & 6.594797 \end{bmatrix}.$$

It will take six more iterations for the diagonal elements to get close to the diagonal matrix

$$\boldsymbol{D} = \text{diag}(3.295699, 8.407662, 11.704301, 6.592338).$$

However, the off-diagonal elements are not small enough, and it will take three more iterations for them to be less than 10^{-6} in magnitude. Then the eigenvectors are the columns of the matrix $\boldsymbol{V} = \boldsymbol{R}_1\boldsymbol{R}_2\cdots\boldsymbol{R}_{18}$, which is

$$\boldsymbol{V} = \begin{bmatrix} 0.528779 & -0.573042 & 0.582298 & 0.230097 \\ 0.591967 & 0.472301 & 0.175776 & -0.628975 \\ -0.536039 & 0.282050 & 0.792487 & -0.071235 \\ 0.287454 & 0.607455 & 0.044680 & 0.739169 \end{bmatrix}. \qquad \blacksquare$$

Program 11.3 (Jacobi Iteration for Eigenvalues and Eigenvectors). To compute the full set of eigenpairs $\{\lambda_j, \boldsymbol{V}_j\}_{j=1}^n$ of the $n \times n$ real symmetric matrix $\boldsymbol{A}$. Jacobi iteration is used to find all eigenpairs.

```
function [V,D]=jacobi1(A,epsilon)
%Input  - A is an nxn matrix
%       - epsilon is the tolerance
%Output - V is the nxn matrix of eigenvectors
%       - D is the diagonal nxn matrix of eigenvalues
%Initialize V,D,and parameters
D=A;
[n,n]=size(A);
V=eye(n);
state=1;
%Calculate row p and column q of the off-diagonal element
%of greatest magnitude in A
[m1 p]=max(abs(D-diag(diag(D))));
[m2 q]=max(m1);
p=p(q);
while(state==1)
   %Zero out Dpq and Dqp
   t=D(p,q)/(D(q,q)-D(p,p));
   c=1/sqrt(t^2+1);
   s=c*t;
   R=[c s;-s c];
   D([p q],:)=R'*D([p q],:);
   D(:,[p q])=D(:,[p q])*R;
   V(:,[p q])=V(:,[p q])*R;
```

```
   [m1 p]=max(abs(D-diag(diag(D))));
   [m2 q]=max(m1);
   p=p(q);
   if (abs(D(p,q))<epsilon*sqrt(sum(diag(D).^2)/n))
      state=0;
   end
end
D=diag(diag(D));
```

Exercises for Jacobi's Method

1. *Mass-spring systems.* Consider the undamped mass-spring system shown in Figure 11.3. The mathematical model describing the displacements from static equilibrium is

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(a) Use the substitutions $x_j(t) = v_j \sin(\omega t + \theta)$ for $j = 1, 2, 3$, where θ is a constant, and show that the solution to the mathematical model can be reformulated as follows:

$$\begin{bmatrix} \dfrac{k_1+k_2}{m_1} & \dfrac{-k_2}{m_1} & 0 \\ \dfrac{-k_2}{m_2} & \dfrac{k_2+k_3}{m_2} & \dfrac{-k_3}{m_2} \\ 0 & \dfrac{-k_3}{m_3} & \dfrac{k_3}{m_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \omega^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

(b) Set $\lambda = \omega^2$; then the three solutions to part (a) are the eigenpairs λ_j, $\boldsymbol{V}_j = \begin{bmatrix} v_1^{(j)} & v_2^{(j)} & v_3^{(j)} \end{bmatrix}'$, for $j = 1, 2, 3$. Show that they are used to form the three

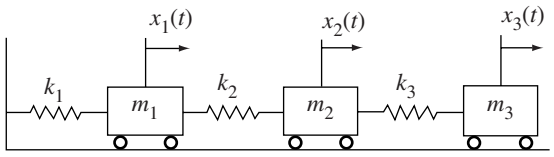

Figure 11.3 An undamped mass-spring system.

Algorithms and Programs

1. Use Program 11.3 to find the eigenpairs of the given matrix with a tolerance of $\epsilon = 10^{-7}$. Compare your results with those obtained from the MATLAB command `eig` by entering `[V,D]=eig(A)` in the MATLAB command window.

(a) $\boldsymbol{A} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

(b) $\boldsymbol{A} = \begin{bmatrix} 2.25 & -0.25 & -1.25 & 2.75 \\ -0.25 & 2.25 & 2.75 & 1.25 \\ -1.25 & 2.75 & 2.25 & -0.25 \\ 2.75 & 1.25 & -0.25 & 2.25 \end{bmatrix}$

(c) $\boldsymbol{A} = [a_{ij}]$, where $a_{ij} = \begin{cases} i + j & i = j \\ ij & i \neq j \end{cases}$ and $i, j = 1, 2, \ldots, 30$.

(d) $\boldsymbol{A} = [a_{ij}]$, where $a_{ij} = \begin{cases} \cos(\sin(i + j)) & i = j \\ i + ij + j & i \neq j \end{cases}$ and $i, j = 1, 2, \ldots, 40$.

2. Use the technique outlined in Exercise 1 and Program 11.3 to find the eigenpairs and the three principal modes of vibration for the undamped mass-spring systems with the following coefficients.

(a) $k_1 = 3, k_2 = 2, k_3 = 1, m_1 = 1, m_2 = 1, m_3 = 1$

(b) $k_1 = \frac{1}{2}, k_2 = \frac{1}{4}, k_3 = \frac{1}{4}, m_1 = 4, m_2 = 4, m_3 = 4$

(c) $k_1 = 0.2, k_2 = 0.4, k_3 = 0.3, m_1 = 2.5, m_2 = 2.5, m_3 = 2.5$

3. Use the technique outlined in Exercise 2 and Program 11.3 to find the general solution of the given homogeneous system of differential equations.

(a)
$$\begin{aligned} x_1' &= 4x_1 + 3x_2 + 2x_3 + x_4 \\ x_2' &= 3x_1 + 4x_2 + 3x_3 + 2x_4 \\ x_3' &= 2x_1 + 3x_2 + 4x_3 + 3x_4 \\ x_4' &= x_1 + 2x_2 + 3x_3 + 4x_4 \end{aligned}$$

(b)
$$\begin{aligned} x_1' &= 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \\ x_2' &= 4x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 \\ x_3' &= 3x_1 + 4x_2 + 5x_3 + 4x_4 + 3x_5 \\ x_4' &= 2x_1 + 3x_2 + 4x_3 + 5x_4 + 4x_5 \\ x_5' &= x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \end{aligned}$$

4. Modify Program 11.3 to implement the "cyclic" Jacobi method.

5. Use your program from Problem 4 on the symmetric matrices in Problem 1. In particular, compare the number of iterations required by your cyclic program and Program 11.3 to satisfy the given tolerance.