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Discrete-time Zhang neural network of $O(\tau^3)$ pattern for time-varying matrix pseudoinversion with application to manipulator motion generation ☆

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ABSTRACT

In addition to the high-speed parallel-distributed processing property, neural networks can be readily implemented by hardware and thus have been applied widely in various fields. In this paper, via a new numerical-differentiation formula, two Taylor-type discrete-time Zhang neural network (ZNN) models (termed T-ZNN-K and T-ZNN-U models) are first proposed, developed and investigated for online time-varying matrix pseudoinversion. For comparison as well as for illustration, Euler-type discrete-time ZNN models (termed E-ZNN-K and E-ZNN-U models) and Newton iteration are presented. In addition, according to the criterion of whether the time-derivative information of time-varying matrix is explicitly known or not, these discrete-time ZNN models are classified into two categories: (1) models with time-derivative information known (i.e., T-ZNN-K and E-ZNN-K models), and (2) models with time-derivative information unknown (i.e., T-ZNN-U and E-ZNN-U models). Moreover, theoretical analyses show that, the maximal steady-state residual errors (MSSREs) of T-ZNN-K and T-ZNN-U models have an $O(\tau^3)$ pattern, the MSSREs of E-ZNN-K and E-ZNN-U models have an $O(\tau^2)$ pattern, whereas the MSSRE of Newton iteration has an $O(\tau)$ pattern, with τ denoting the sampling gap. Finally, two illustrative numerical experiments and an application example to manipulator motion generation are provided and analyzed to substantiate the efficacy of the proposed Taylor-type discrete-time ZNN models for online time-varying matrix pseudoinversion.

1. Introduction

The matrix pseudoinversion (also known as Moore–Penrose inversion) is considered to be one of the basic problems widely encountered in a variety of scientific and engineering fields, e.g., robotics [1], machine learning [2], associative memories [3], optimization [4]. Owing to its fundamental roles, much effort has been devoted to the fast solution and high accuracy of matrix pseudoinversion, and many algorithms have been put forward by researchers [5–8]. Huang and Zhang [5] showed that Newton iteration can be used to compute the weighted Moore–Penrose inverse of an arbitrary matrix. Courrieu [7] proposed an algorithm based on a full-rank Cholesky factorization for fast computation of Moore–Penrose generalized inverse matrices. When solving time-varying matrix pseudoinverse, many conventional algorithms generally assume the short-time invariance of the matrix and compute at each single time instant, where the change trend of the time-varying matrix is not exploited [1,9]. The computed results are then directly used for the next time instant, and lagging errors may thus be generated between the obtained solution and the theoretical solution. What is more, most numerical algorithms may not be efficient enough in large-scale online applications due to their serial-processing nature. Especially, when applied to online solution of time-varying matrix pseudoinverse, these related numerical algorithms should be fulfilled within every sampling period and the algorithms fail when the sampling rate is too high to allow the algorithms to complete the calculation in a single sampling period, not to mention more challenging situations.

In recent decades, with the characteristics of distributed-storage and high-speed parallel-processing natures, superior performance in large-scale online applications, and convenience of hardware implementations, neural networks have widely arisen in scientific computation and optimization, drawing extensive

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interest and investigation of researchers [10–14]. Due to the in-depth research in neural networks, the neural-dynamic approach based on recurrent neural networks is now regarded as a powerful alternative [11–15], which can solve many mathematical and engineering problems with time-varying coefficients.

Zhang neural network (ZNN) is a special class of recurrent neural networks, which originates from the research of Hopfield neural network; it is proposed as a systematic approach to solve online time-varying problems; and, it differs from conventional gradient-based recurrent neural networks in terms of problem to be solved, error function, design formula, dynamic equation, and the utilization of time derivatives [16–19]. Liao and Zhang proposed several continuous-time ZNN models for time-varying matrix pseudoinversion, which can be accelerated to finite-time convergence via Li activation functions [19]. However, in view of the variation of the time step, the continuous-time models are difficult for digital computer to be implemented directly, which often demand constant time step (i.e., the sampling gap τ is a constant in a certain calculating process). Thus, it is necessary to develop the corresponding discrete-time models for online time-varying matrix pseudoinversion.

For obtaining the first-order derivative approximation and then discretizing the continuous-time ZNN model, there exist many numerical differentiation methods we may consider. However, notice that the following three facts [20]. (1) The backward differentiation rules may not adapt to the fast variational rate of the first-order derivative of target point. (2) The central differentiation rules cannot approximate the first-order derivative of the target point without enough number of data points in either side. (3) A numerical differentiation formula does not necessarily generate a stable and convergent discrete-time ZNN model (not to mention a high-accuracy one). Thus, a new kind of effective numerical differentiation formula is needed. In light of this analysis, a Taylor-type numerical differentiation formula is thus constructed for the first-order derivative approximation. Furthermore, adopting the new Taylor-type numerical differentiation formula, we propose two Taylor-type discrete-time ZNN models for the online time-varying pseudoinversion. In more detail, these discrete-time models are derived from the continuous-time ZNN model with time-derivative information known (i.e., T-ZNN-K model), and with time-derivative information unknown but approximated by using numerical difference rule (i.e., T-ZNN-U model). It is theoretically proved that the two Taylor-type discrete-time ZNN models converge towards the time-varying theoretical solution of the time-varying pseudoinversion with $O(\tau^3)$ residual error pattern. Besides, for purposes of comparison, two Euler-type discrete-time ZNN models (i.e., E-ZNN-K and E-ZNN-U models) and the conventional Newton iteration, together with their convergence performance analyses, are also presented. Through the numerical experiments, the theoretical results and the efficacy of the proposed Taylor-type ZNN models for time-varying matrix pseudoinversion are well substantiated.

The rest of this paper is organized into six sections. The preliminaries, problem formulation and the continuous-time ZNN model of the time-varying matrix pseudoinversion are presented in Section 2. In Section 3, the Taylor-type numerical differentiation formula and the two corresponding discrete-time ZNN models are proposed. The convergence properties of the Taylor-type discrete-time ZNN models are analyzed and proved. For comparison, Section 4 develops and investigates the Euler-type discrete-time ZNN models and Newton iteration for the same solution task of time-varying matrix pseudoinversion. Section 5 provides two illustrative numerical examples to substantiate the efficacy and superiority of the proposed Taylor-type discrete-time ZNN models for online time-varying matrix pseudoinversion. As an application, the T-ZNN-U model is employed for the online motion generation of a five-link robot manipulator in Section 6. Section 7 concludes this paper with final remarks. Before ending this introductory section, it is worthwhile pointing out the main contributions of this paper as follows.

(1) This paper focuses on solving online discrete-time varying matrix pseudoinverse rather than conventionally investigated static ones, which is also quite different from Zhang et al.'s previous research on solving continuous-time varying matrix pseudoinverse.
(2) This paper presents and investigates a Taylor-type numerical differentiation formula to approximate the first-order derivative of the target point, thereby remedying the intrinsic weaknesses of the backward and central differentiation rules.
(3) Based on the presented Taylor-type numerical differentiation formula, two Taylor-type discrete-time ZNN models with $O(\tau^3)$ residual error pattern are derived for the first time for time-varying matrix pseudoinversion, of which the stability and convergence are proved theoretically.
(4) The stability and convergence of Euler-type discrete-time ZNN models and Newton iteration for time-varying matrix pseudoinversion are proved theoretically for the first time as well in this paper.
(5) Numerical experiment results, together with an application example to the motion generation of a five-link robot manipulator, are illustrated, which substantiate the efficacy of the proposed Taylor-type discrete-time ZNN models for time-varying matrix pseudoinversion.

2. Problem formulation and continuous-time ZNN model

In order to lay a basis for further investigation, the preliminaries and problem formulation of matrix pseudoinversion are presented in this section.

Definition 1 (*Wang [15], Liao and Zhang [19], Ben-Israel and Greville [21]*). For a given time-varying matrix $A(t)\in\mathbb{R}^{m\times n}$, if $X(t)\in\mathbb{R}^{n\times m}$ satisfies at least one of the following four Penrose equations:

$$A(t)X(t)A(t)=A(t),\quad X(t)A(t)X(t)=X(t),$$
$$(A(t)X(t))^{\mathrm{T}}=A(t)X(t),\quad (X(t)A(t))^{\mathrm{T}}=X(t)A(t),$$

where superscript $^{\mathrm{T}}$ denotes the transpose of a matrix, $X(t)$ is called the generalized inverse of $A(t)$. If matrix $X(t)$ satisfies all of the Penrose equations, then matrix $X(t)$ is called the pseudoinverse of matrix $A(t)$, which is often denoted by $A^{+}(t)$.

Note that the time-varying pseudoinverse $A^{+}(t)$ always exists and is unique [19]. Specially, if matrix $A(t)$ is full rank at any time instant t, i.e., $\mathrm{rank}(A(t))=\min\{m,n\}$, $\forall t\in[0,\infty)$, we have the following lemma to obtain the time-varying pseudoinverse of $A(t)$.

Lemma 1 (*Liao and Zhang [19], Ben-Israel and Greville [21]*). *For any time-varying matrix $A(t)\in\mathbb{R}^{m\times n}$, if $\mathrm{rank}(A(t))=\min\{m,n\}$, $\forall t\in[0,\infty)$, then the unique time-varying pseudoinverse $A^{+}(t)$ can be given as*

$$A^{+}(t)=\begin{cases}A^{\mathrm{T}}(t)(A(t)A^{\mathrm{T}}(t))^{-1} & \text{if } m<n,\\ (A^{\mathrm{T}}(t)A(t))^{-1}A^{\mathrm{T}}(t) & \text{if } m\geq n.\end{cases}\tag{1}$$

In the case of $m=n$, we have $A^{+}(t)=A^{-1}(t)=A^{\mathrm{T}}(t)((A(t)A^{\mathrm{T}}(t))^{-1}=(A^{\mathrm{T}}(t)A(t))^{-1}A^{\mathrm{T}}(t)$. Note that, for $m\geq n$, the procedure of obtaining the time-varying pseudoinverse of $A(t)$ is similar to that of $m<n$ and thus omitted.

2.1. Problem formulation

Let us consider the following online discrete-time varying matrix pseudoinversion with X_{k+1} to be computed at each computational time interval $[k\tau,(k+1)\tau) \subseteq [0,+\infty)$:

$$A_{k+1} - X^{+}_{k+1} = \mathbf{0} \in \mathbb{R}^{m\times n}, \tag{2}$$

where $A_{k+1} \in \mathbb{R}^{m\times n}$ is generated or measured from the smoothly time-varying coefficient matrix $A(t)$ by sampling at time instant $t=(k+1)\tau$ (which is denoted as t_{k+1}) and $X_{k+1} \in \mathbb{R}^{n\times m}$ is the unknown matrix need to be obtained during $[t_k, t_{k+1})$. Besides, $k=0$, 1, 2,… denotes the updating index. In the online solution process of discrete-time varying matrix pseudoinversion (2), computation has to be performed based on the present and/or previous data. For example, at time instant t_k, we can only use known information such as A_k and its derivative, instead of unknown information such as A_{k+1} and its derivative, for computing the unknown matrix X_{k+1} during the computational time interval $[k\tau,(k+1)\tau)$. Thus, the objective of this work is, through the present and/or previous data, to find the unknown matrix X_{k+1} during $[t_k, t_{k+1})$, such that (2) holds true at each time instant.

2.2. Continuous-time ZNN model

To develop discrete-time ZNN models effectively achieving (2) (which is an actually unknown matrix pseudoinversion), the continuous-time ZNN model can firstly be generated by exploiting the continuous-time ZNN method. Now, let us consider the following continuous-time matrix pseudoinversion as the continuation of (2): $A(t)-X^{+}(t)=\mathbf{0}$. Then, a matrix-valued error function can be defined as $E(t)=A(t)-X^{+}(t)$. By adopting the ZNN design formula $\dot{E}(t)=-\gamma E(t)$, we obtain $\dot{A}(t)-\dot{X}^{+}(t)=-\gamma(A(t)-X^{+}(t))$, where $\gamma>0$ is used to scale the convergence rate of the neural network and should be set as large as the hardware would permit [18]. By employing Lemma 3 in [19], the above equation can be further modified as $X^{+}(t)\dot{X}(t)X^{+}(t)=-\dot{A}(t)-\gamma(A(t)-X^{+}(t))$. Reformulating the above equation, we have

$$\dot{X}(t) = -X(t)\dot{A}(t)X(t) - \gamma(X(t)A(t)X(t)-X(t)), \tag{3}$$

of which the exponential convergence was proved in [19]. Note that Eq. (3) is the fifth ZNN model with linear activation function in [19], which is exactly the G-M dynamic system for the online solution of the time-varying matrix pseudoinverse. It is also worth pointing out that there are five continuous-time ZNN models in [19], of which the first four are depicted in implicit dynamics that are difficult to be discretized directly (e.g., $A^{\mathrm{T}}(t)A(t)\dot{X}(t)=\cdots$). On the contrary, the fifth ZNN model (i.e., Eq. (3)) is depicted in explicit dynamics, i.e., $\dot{X}(t)=\cdots$, which can be discretized directly and the corresponding discrete-time models can be readily implemented by digital hardware.

Remark 1. Before constructing specific discrete-time neural networks from (3) for solving (2), the main characteristics, difficulties and even challenges of online time-varying matrix pseudoinversion, which are evidently different from static case, are discussed here. (1) In view of the time variation of coefficient matrix $A(t)$, how to develop the discrete-time model(s) without using future data is a key point for the time-varying matrix pseudoinversion. In other words, at the present time instant t_k, we have no mathematical function to express deterministically the future such as A_{k+1}, but we have to find in advance an optimal solution to it. (2) Computation consumes time inevitably; and time is precious especially for the time-varying pseudoinversion. Thus, how to design a simple discrete-time model with higher accuracy is important. In other words, the discrete-time model should satisfy the requirement of real-time computation.

3. Taylor-type ZNN discretization

A Taylor-type numerical differentiation formula is presented in this section, which is used to discretize the continuous-time ZNN model (3). Two Taylor-type discrete-time ZNN models are thus proposed for discrete-time matrix pseudoinversion (2), of which the stability and convergence are proved theoretically as well.

3.1. New effective $O(\tau^2)$ formula

A Taylor series is an infinite sum of terms that are calculated from the values of derivatives of a function at a single point. In scientific and engineering fields, the partial sums can be accumulated until an approximation to the function is obtained that achieves the specified accuracy. Consequently, a new Taylor-type numerical differentiation formula is constructed in this subsection for the first-order derivative approximation by eliminating the second-order derivative, which can achieve higher computational accuracy in the application of ZNN discretization. In what follows, the notation $C^n[a,b]$ stands for the set of all functions f such that f and its first n derivatives are continuous on $[a,b]$.

Theorem 1. *Assume that $f\in C^3[a,b]$ and that t_{k-2}, t_{k-1}, t_k, $t_{k+1} \in [a,b]$. Then, the new Taylor-type numerical differentiation formula is formulated as*

$$f'(t_k) \approx \frac{2f(t_{k+1})-3f(t_k)+2f(t_{k-1})-f(t_{k-2})}{2\tau}, \tag{4}$$

which has a truncation error of $O(\tau^2)$, that is

$$f'(t_k) = \frac{2f(t_{k+1})-3f(t_k)+2f(t_{k-1})-f(t_{k-2})}{2\tau} + O(\tau^2). \tag{5}$$

Proof. See the Appendix for details.

Based on Theorem 1, a new effective $O(\tau^2)$ formula is obtained for first-order derivative approximation, which is expected to be applied for ZNN discretization for a higher accuracy in comparison with the Euler-type discrete-time ZNN models and Newton iteration.

3.2. T-ZNN-K and T-ZNN-U models

Based on the presented Taylor-type numerical differentiation formula (4), two new discrete-time ZNN models are developed and investigated in this subsection. Specifically, it follows from (3) that

$$X_{k+1} = -\tau X_k\dot{A}_kX_k - h(X_kA_kX_k - X_k) + \tfrac{3}{2}X_k - X_{k-1} + \tfrac{1}{2}X_{k-2}, \tag{6}$$

where step-size $h=\tau\gamma>0$. For presentation convenience, Eq. (6) is called the Taylor-type discrete-time ZNN model with $\dot{A}(t)$ known, i.e., the T-ZNN-K model.

As we know, it may be difficult to know or obtain the value of $\dot{A}(t)$ directly in certain real-world applications. Thus, it is worth investigating the Taylor-type discrete-time ZNN model with $\dot{A}(t)$ unknown. In this situation, $\dot{A}(t)$ can be estimated from $A(t)$ by employing the backward-difference rule of first derivative with a third-order accuracy [20]:

$$\dot{A}_k = \frac{11A_k - 18A_{k-1} + 9A_{k-2} - 2A_{k-3}}{6\tau} + \mathbf{O}(\tau^3), \tag{7}$$

where $\mathbf{O}(\tau^3)$ denotes a matrix with each element being $O(\tau^3)$. Thus, the Taylor-type discrete-time ZNN model with $\dot{A}(t)$ unknown, i.e., the T-ZNN-U model can be formulated as

$$\begin{aligned} X_{k+1} = {} & -X_k\left(\tfrac{11}{6}A_k - 3A_{k-1} + \tfrac{3}{2}A_{k-2} - \tfrac{1}{3}A_{k-3}\right)X_k \\ & - h(X_kA_kX_k - X_k) + \tfrac{3}{2}X_k - X_{k-1} + \tfrac{1}{2}X_{k-2}. \end{aligned} \tag{8}$$

3.3. Theoretical analyses and results

Before investigating the performance of T-ZNN-K model (6) and T-ZNN-U model (8), the following definitions are provided as a basis for further discussion [22].

Definition 2. An N step method/model $\sum_{j=0}^{N}\alpha_j x_{k+j}=\tau\sum_{j=0}^{N}\beta_j g_{k+j}$ can be checked 0-stability by determining the roots of the characteristic polynomial $P_N(q)=\sum_{j=0}^{N}\alpha_j q^j$. If the roots of $P_N(q)=0$ are such that $|q|\le 1$ and those for which $|q|=1$ are simple, then, the N step method is 0-stable. In addition, the 0-stability is sometimes called Dahlquist stability or root stability.

Definition 3. An N step method is said to be consistent of order p if its truncation error is $O(\tau^p)$ with $p>0$ for the smooth exact solution.

Definition 4. An N step method is convergent, i.e., $x_{[(t-t_0)/\tau]}\to x^*(t)$, for all $t\in[t_0,t_\mathrm{f}]$, as $\tau\to 0$, if and only if the method is consistent and 0-stable. In other words, consistency plus 0-stability leads to convergence. In particular, a 0-stable consistent method converges with the order of its truncation error.

Based on the above three definitions, we have the following theoretical results about T-ZNN-K model (6) and T-ZNN-U model (8).

Theorem 2. *T-ZNN-K model (6) is 0-stable.*

Proof. According to Definition 2, the characteristic polynomial of T-ZNN-K model (6) can be derived as

$$P_1(q)=q^3-1.5q^2+q-0.5=0,$$

which has three roots on or in the unit disk, i.e., $q_1=1$, $q_2=0.25+i0.6614$ and $q_3=0.25-i0.6614$, with i denoting the imaginary unit. Therefore, T-ZNN-K model (6) is 0-stable. The proof is thus completed. □

Theorem 3. *T-ZNN-K model (6) is consistent and convergent, which converges with the order of truncation error being $\mathbf{O}(\tau^3)$ for all $t_k\in[t_0,t_\mathrm{f}]$.*

Proof. In view of (5), we have the following equation:

$$X_{k+1}=-\tau X_k\dot{A}_kX_k-hX_k(A_kX_k-I)+\tfrac{3}{2}X_k-X_{k-1}+\tfrac{1}{2}X_{k-2}+\mathbf{O}(\tau^3). \quad (9)$$

Note that, dropping $\mathbf{O}(\tau^3)$ of (9) yields exactly T-ZNN-K model (6), and thus the truncation error of T-ZNN-K model (6) is $\mathbf{O}(\tau^3)$. Therefore, according to Definition 3, T-ZNN-K model (6) is consistent of order 3. In view of Theorem 2, T-ZNN-K model (6) is both 0-stable and consistent. Finally, based on Definition 4, it can be concluded that T-ZNN-K model (6) is consistent and convergent, which converges with the order of truncation error being $\mathbf{O}(\tau^3)$ for all $t_k\in[t_0,t_\mathrm{f}]$. The proof is thus completed. □

Theorem 4. *Consider online discrete-time varying matrix pseudoinversion problem (2). The steady-state residual error $\lim_{k\to\infty}\|X_{k+1}A_{k+1}A_{k+1}^{\mathrm{T}}-A_{k+1}^{\mathrm{T}}\|_\mathrm{F}$ of T-ZNN-K model (6) is $O(\tau^3)$.*

Proof. In view of Definition 3, Theorems 2 and 3, it can be concluded that $A_{k+1}^{+}=X_{k+1}+\mathbf{O}(\tau^3)$ with k large enough. Then, we can obtain

$$\|X_{k+1}A_{k+1}A_{k+1}^{\mathrm{T}}-A_{k+1}^{\mathrm{T}}\|_\mathrm{F}=\|A_{k+1}^{+}A_{k+1}A_{k+1}^{\mathrm{T}}-A_{k+1}^{\mathrm{T}}+\mathbf{O}(\tau^3)A_{k+1}A_{k+1}^{\mathrm{T}}\|_\mathrm{F}. \quad (10)$$

From Lemma 1, we get $A_{k+1}^{+}A_{k+1}A_{k+1}^{\mathrm{T}}-A_{k+1}^{\mathrm{T}}=\mathbf{0}$. Note that, with $A_{k+1}A_{k+1}^{\mathrm{T}}$ being a constant matrix at a certain time instant, the term $\mathbf{O}(\tau^3)A_{k+1}A_{k+1}^{\mathrm{T}}$ changes in an $\mathbf{O}(\tau^3)$ pattern; i.e., with τ decreasing to one tenth of its original value, $\|\mathbf{O}(\tau^3)A_{k+1}A_{k+1}^{T}\|_\mathrm{F}$ decreases to one thousandth of its original value. Therefore, we have $\|\mathbf{O}(\tau^3)A_{k+1}A_{k+1}^{\mathrm{T}}\|_\mathrm{F}=\|\mathbf{O}(\tau^3)\|_\mathrm{F}$ and further have

$$\|X_{k+1}A_{k+1}A_{k+1}^{\mathrm{T}}-A_{k+1}^{\mathrm{T}}\|_\mathrm{F}=\|\mathbf{O}(\tau^3)A_{k+1}A_{k+1}^{\mathrm{T}}\|_\mathrm{F}=\|\mathbf{O}(\tau^3)\|_\mathrm{F}=O(\tau^3),$$

which now completes the proof. □

Theorem 5. *Consider online discrete-time varying matrix pseudoinversion problem (2). The steady-state residual error $\lim_{k\to\infty}\|X_{k+1}A_{k+1}A_{k+1}^{\mathrm{T}}-A_{k+1}^{\mathrm{T}}\|_\mathrm{F}$ of T-ZNN-U model (8) is $O(\tau^3)$.*

Proof. In view of (7) and (10), we can have the following equation for T-ZNN-U model (8):

$$\begin{aligned}X_{k+1}&=-\tau X_k(\dot{A}_k+\mathbf{O}(\tau^3))X_k-hX_k(A_kX_k-I)\\&\quad+\tfrac{3}{2}X_k-X_{k-1}+\tfrac{1}{2}X_{k-2}+\mathbf{O}(\tau^3)\\&=-\tau X_k\dot{A}_kX_k-hX_k(A_kX_k-I)\\&\quad+\tfrac{3}{2}X_k-X_{k-1}+\tfrac{1}{2}X_{k-2}-\tau X_k(\mathbf{O}(\tau^3))X_k+\mathbf{O}(\tau^3)\\&=-\tau X_k\dot{A}_kX_k-hX_k(A_kX_k-I)+\tfrac{3}{2}X_k-X_{k-1}+\tfrac{1}{2}X_{k-2}+\mathbf{O}(\tau^3).\end{aligned} \quad (11)$$

Note that, dropping $\mathbf{O}(\tau^3)$ of (11) yields exactly T-ZNN-K model (6), and thus the truncation error of T-ZNN-U model (8) is $\mathbf{O}(\tau^3)$. Therefore, according to Definition 3, the T-ZNN-U model (8) is consistent of order 3. By following Theorems 3 and 4, it can be readily generalized and similarly proved that the steady-state residual error $\lim_{k\to\infty}\|X_{k+1}A_{k+1}A_{k+1}^{\mathrm{T}}-A_{k+1}^{\mathrm{T}}\|_\mathrm{F}$ of T-ZNN-U model (8) is $O(\tau^3)$. The proof is thus completed. □

4. Euler-type ZNN discretization and Newton iteration

In this section, for comparison purposes, Euler-type ZNN models and Newton iteration are presented and analyzed to solve discrete-time varying matrix pseudoinverse problem (2).

4.1. E-ZNN-K and E-ZNN-U models

By following the similar steps given in Section 3.2, the Euler-type ZNN model via Euler forward-difference rule (i.e., $\dot{X}_k=(X_{k+1}-X_k)/\tau+\mathbf{O}(\tau)$) with $\dot{A}(t)$ known (i.e., E-ZNN-K model) is directly given as

$$X_{k+1}=X_k-\tau X_k\dot{A}_kX_k-h(X_kA_kX_k-X_k). \quad (12)$$

By discretizing $\dot{A}(t)$ in (12) with Euler backward-difference rule (i.e., $\dot{A}_k=(A_k-A_{k-1})/\tau+\mathbf{O}(\tau)$), the Euler-type ZNN model with $\dot{A}(t)$ unknown (i.e., E-ZNN-U model) is obtained as

$$X_{k+1}=X_k-X_k(A_k-A_{k-1})X_k-h(X_kA_kX_k-X_k). \quad (13)$$

Theorem 6. *E-ZNN-K model (12) is 0-stable.*

Proof. According to Definition 2, the characteristic polynomial of E-ZNN-K model (12) can be derived as

$$P_1(q)=q-1,$$

which has only one root (i.e., $q=1$) on the unit disk. Hence, E-ZNN-K model (12) is 0-stable. This completes the proof. □

Theorem 7. *Consider online discrete-time varying matrix pseudoinversion problem (2). Both the steady-state residual errors $\lim_{k\to\infty}\|X_{k+1}A_{k+1}A_{k+1}^{\mathrm{T}}-A_{k+1}^{\mathrm{T}}\|_\mathrm{F}$ of E-ZNN-K model (12) and E-ZNN-U model (13) are $O(\tau^2)$.*

Proof. Based on Theorem 6, by following Theorems 3–5, it can be readily generalized and similarly proved that both the steady-state residual errors $\lim_{k\to\infty}\|X_{k+1}A_{k+1}A_{k+1}^{\mathrm{T}}-A_{k+1}^{\mathrm{T}}\|_\mathrm{F}$ of E-ZNN-K model (12) and E-ZNN-U model (13) are $O(\tau^2)$. The proof is thus completed. □

4.2. Newton iteration

The classical Newton iteration is generalized and developed to solve (2), which is formulated as [6]

$$X_{k+1} = X_k - (X_k A_k X_k - X_k). \tag{14}$$

Evidently, the above Newton iteration is actually a special case of the presented E-ZNN-K model (12) by taking step-size $h=1$ and omitting the time-derivative matrix $\dot{A}_k$. In other words, in addition to differences, we discover the connections between E-ZNN-K model (12) and Newton iteration. That is, a more general form of Newton iteration (14) for (2) is E-ZNN-K model (12); a simplified form of E-ZNN-K model (12) is Newton iteration (14); and the methods of discrete-time ZNN and Newton iteration are closely related. This discovery is also a contribution of this paper. By following the similar steps, one can prove that the steady-state residual error of Newton iteration (14) is theoretically $O(\tau)$. In addition, we have the following theorem to reveal that the steady-state residual error of any method designed intrinsically for static matrix pseudoinversion when employed for online discrete-time varying one is $O(\tau)$.

Theorem 8. *Suppose that a method designed intrinsically for static matrix pseudoinversion converges to the optimal solution to a static matrix pseudoinversion within computational time τ. If the method is employed for online discrete-time varying matrix pseudoinversion (2), then the steady-state residual error of the method is $O(\tau)$.*

Proof. As assumed, the time derivative of $X(t)$ exists, i.e., $\mathrm{d}X_k^{ij}/\mathrm{d}t = p^{ij}$ at time instant $t=k\tau$ with $X^{ij}(t)$ being the ijth element of matrix $X(t)$ and p^{ij} being a constant. Then, it can be readily derived that $\lim_{\tau\to 0}\Delta X_k^{ij}/\tau = \mathrm{d}X_k^{ij}/\mathrm{d}t = p^{ij}$ and $\Delta X_k^{ij} \approx p^{ij}\tau$. Therefore, ΔX_k^{ij} changes in an $O(\tau)$ pattern, i.e., $\Delta X_k^{ij} = O(\tau)$ and $\Delta X_k = \mathbf{O}(\tau)$. Note that, at computational time interval $[k\tau,(k+1)\tau)$, the method converges to the optimal solution X_k^* to the time-varying matrix pseudoinversion problem at time instant $t=k\tau$ and $X_{k+1}^* = X_k^* + \Delta X_k$. Thus, at time instant $t=(k+1)\tau$, the difference between the solution generated by the method and the optimal solution is ΔX_k, i.e., $X_{k+1}^* = X_k^* + \mathbf{O}(\tau) = X_{k+1} + \mathbf{O}(\tau)$. Then, we can obtain

$$\|X_{k+1}A_{k+1}A_{k+1}^{\mathrm{T}} - A_{k+1}^{\mathrm{T}}\|_{\mathrm{F}} = \|A_{k+1}^{+}A_{k+1}A_{k+1}^{\mathrm{T}} - A_{k+1}^{\mathrm{T}} + \mathbf{O}(\tau)A_{k+1}A_{k+1}^{\mathrm{T}}\|_{\mathrm{F}}.$$

From Lemma 1, we get $A_{k+1}^{+}A_{k+1}A_{k+1}^{\mathrm{T}} - A_{k+1}^{\mathrm{T}} = \mathbf{0}$. Then, we further have

$$\|X_{k+1}A_{k+1}A_{k+1}^{\mathrm{T}} - A_{k+1}^{\mathrm{T}}\|_{\mathrm{F}} = \|\mathbf{O}(\tau^3)A_{k+1}A_{k+1}^{\mathrm{T}}\|_{\mathrm{F}} = \|\mathbf{O}(\tau)\|_{\mathrm{F}} = O(\tau).$$

The proof is thus completed. □

Thus, supported by Theorem 8, it can be concluded that the steady-state residual errors of the methods in [5–8] have an $O(\tau)$ pattern for online discrete-time varying matrix pseudoinversion (2).

In conclusion, we have constructed four discrete-time ZNN models [i.e., T-ZNN-K model (6), T-ZNN-U model (8), E-ZNN-K model (12) and E-ZNN-U model (13)] and Newton iteration (14) for solving (2). For readers' convenience and also for comparison purposes, we list comparatively these discrete-time ZNN models and Newton iteration in Table 1.

It is noted that, three initial states X_0, X_1 and X_2 are needed for the initialization of T-ZNN-K model (6). Besides, from (7), we can not obtain $\dot{A}_0$ since t starts from 0 s and thus A_{-1} is undefined. Therefore, in the ensuing numerical experiments as well as the application to motion generation, in view of Newton iteration has the simplest structure, we exploit Newton iteration (14) for the initializations of T-ZNN-K model (6), T-ZNN-U model (8) and E-ZNN-U model (13). Besides, in the ensuing section, the maximal steady-state residual error (MSSRE) is based on the following criterion: $(\|E(k+1)\|_{\mathrm{F}} - \|E(k)\|_{\mathrm{F}})/\|E(k)\|_{\mathrm{F}} < 0.01$. When the criterion is satisfied for the first time during the solving process, we have $k_{\mathrm{m}} = k+1$ and $\mathrm{MSSRE} = \max\{\|E(j)\|_{\mathrm{F}}\}$ with $j = k_{\mathrm{m}}, k_{\mathrm{m}}+1, \ldots$.

Table 1
Different discrete-time ZNN models and Newton iteration for solving (2), where TK, TU, EK, EU and NI denote T-ZNN-K model (6), T-ZNN-U model (8), E-ZNN-K model (12), E-ZNN-U model (13) and Newton iteration (14), respectively.

TK	$X_{k+1} = -\tau X_k \dot{A}_k X_k - h(X_k A_k X_k - X_k) + 3X_k/2 - X_{k-1} + X_{k-2}/2$
TU	$X_{k+1} = -X_k(11A_k/6 - 3A_{k-1} + 3A_{k-2}/2 - A_{k-3}/3)X_k$ $- h(X_k A_k X_k - X_k) + 3X_k/2 - X_{k-1} + X_{k-2}/2$
EK	$X_{k+1} = X_k - \tau X_k \dot{A}_k X_k - h(X_k A_k X_k - X_k)$
EU	$X_{k+1} = X_k - X_k(A_k - A_{k-1})X_k - h(X_k A_k X_k - X_k)$
NI	$X_{k+1} = X_k - (X_k A_k X_k - X_k)$

5. Numerical experiments and verifications

In this section, numerical experiments based on two time-varying matrices are provided to verify the efficacy of the proposed discrete-time ZNN models for the online discrete-time varying matrix pseudoinversion.

Example 1. Let us consider the following discrete-time varying matrix pseudoinversion problem with X_{k+1} to be computed at each computational time interval $[k\tau,(k+1)\tau) \subseteq [0,10]$:

$$A_k = \begin{bmatrix} \sin(t_k) & \cos(t_k) & -\sin(t_k) \\ -\cos(t_k) & \sin(t_k) & \cos(t_k) \end{bmatrix}. \tag{15}$$

To check the solution correctness of the proposed discrete-time ZNN models, from (1), we can obtain the theoretical time-varying pseudoinverse of matrix A_k as

$$A_k^{+} = \begin{bmatrix} 0.5\sin(t_k) & -0.5\cos(t_k) \\ \cos(t_k) & \sin(t_k) \\ -0.5\sin(t_k) & 0.5\cos(t_k) \end{bmatrix}.$$

Starting with $X(0) = [0, -2/3; 2/3, 0; 0, 2/3]$, the corresponding numerical experimental results are shown in Fig. 1. Specifically, the element trajectories of the state X_k are shown in Fig. 1(a), from which we could observe that the solution of T-ZNN-K model (6) converges to the theoretical time-varying pseudoinverse. In addition, the residual error (i.e., $\|E(k)\|_{\mathrm{F}} = \|X_k A_k A_k^{\mathrm{T}} - A_k^{\mathrm{T}}\|_{\mathrm{F}}$) synthesized by T-ZNN-K model (6) converges to near zero rapidly. Therefore, the efficacy of the proposed T-ZNN-K model (6) for discrete-time varying matrix pseudoinversion (2) is illustrated primarily. Note that the figures generated by other ZNN models and Newton iteration are similar to Fig. 1, and are thus omitted.

To clearly compare T-ZNN-K model (6) and T-ZNN-U model (8) with E-ZNN-K model (12), E-ZNN-U model (13) and Newton iteration (14) for solving discrete-time varying matrix pseudoinverse, the further numerical experimental results are visualized in Fig. 2. As seen from Fig. 2(a), with the same initial state $X(0) = [0, -2/3; 2/3, 0; 0, 2/3]$, $h=0.3$ and $\tau=0.1$, the residual errors of ZNN models converge to near zero rapidly, whereas the residual error synthesized by Newton iteration has an obvious lagging error. As shown in Fig. 2(b), the MSSRE synthesized by T-ZNN-K model (6) and T-ZNN-U model (8) are of order 10^{-3} and the MSSREs synthesized by E-ZNN-K model (12) and E-ZNN-U model (13) are of order 10^{-2}. By contrast, the MSSREs synthesized by Newton iteration (14) are of order 10^{-1}. Besides, the residual errors of these five discrete-time models with sampling gap τ being 0.01 and 0.001 are displayed in Figs. 2(c) and (d), respectively. These comparative results substantiate the higher accuracy of the proposed T-ZNN-K model (6) and T-ZNN-U model (8) [compared with E-ZNN-K model (12), E-ZNN-U model (13)

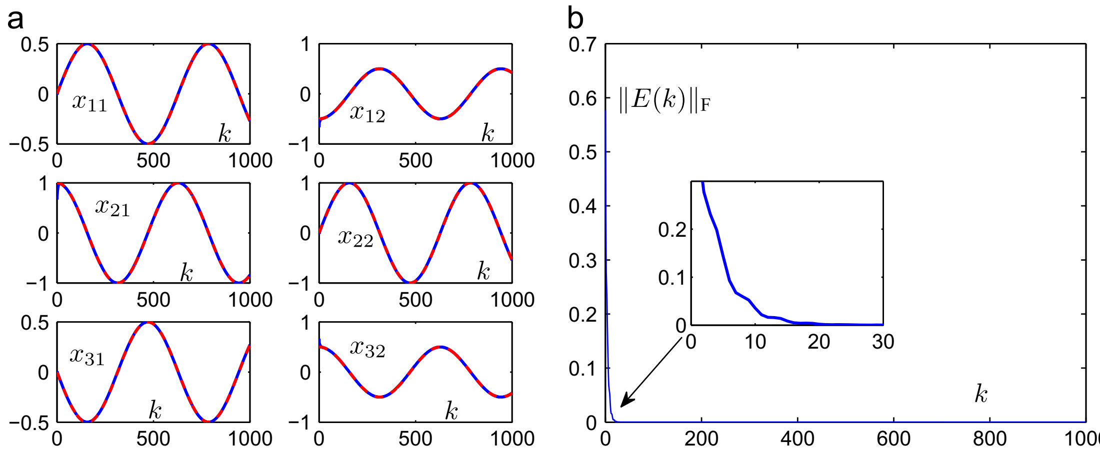

Fig. 1. Convergence performance of T-ZNN-K model (6) for discrete-time varying matrix pseudoinversion (15). In subfigure (a), the red solid curves correspond to the theoretical solution and the blue dash-dotted curves correspond to the neural-network solutions: (a) state matrix X_k of (6) and (b) residual error $\|E(k)\|_F$ of (6). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

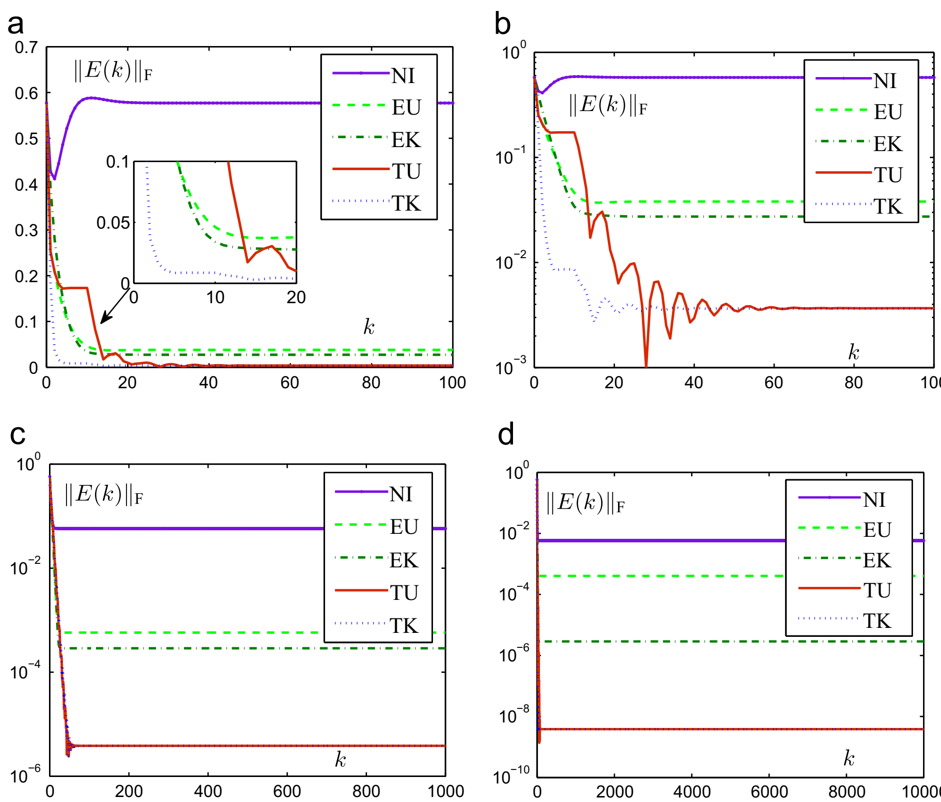

Fig. 2. Residual errors $\|E(k)\|_F = \|X_kA_kA_k^T - A_k^T\|_F$ of five models for discrete-time varying matrix pseudoinversion (15): (a) $\|E(k)\|_F$ with $\tau = 0.1$; (b) order of $\|E(k)\|_F$ with τ=0.1; (c) order of $\|E(k)\|_F$ with τ=0.01; (d) order of $\|E(k)\|_F$ with τ=0.001.

and Newton iteration (14)] as well as the important role of the time-derivative information in obtaining the higher solution accuracy.

Example 2. In this example, we consider a more complicated situation of the discrete-time varying matrix pseudoinversion problem with X_{k+1} to be computed at each computational time interval $[k\tau, (k+1)\tau) \subseteq [0, 10]$:

$$A_k = \begin{bmatrix} c_{11}(t_k) & c_{12}(t_k) & c_{13}(t_k) & \cdots & c_{1n}(t_k) \\ c_{21}(t_k) & c_{22}(t_k) & c_{23}(t_k) & \cdots & c_{2n}(t_k) \\ c_{31}(t_k) & c_{32}(t_k) & c_{33}(t_k) & \cdots & c_{3n}(t_k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1}(t_k) & c_{m2}(t_k) & c_{m3}(t_k) & \cdots & c_{mn}(t_k) \end{bmatrix} \in \mathbb{R}^{m\times n}, \tag{16}$$

where $m < n$. Thereinto,

$$c_{mn}(t_k) = \begin{cases} m + \sin(5t_k), & m = n, \\ \cos(5t_k)/(m-n), & m > n, \\ \sin(5t_k)/(n-m), & m < n. \end{cases}$$

Due to the complexity of matrix (16) (with large dimensions, i.e., m=8 and n=9), the analytical theoretical pseudoinverse solution is difficult to be obtained. Therefore, we only present the residual errors $\|E(k)\|_F = \|X_kA_kA_k^T - A_k^T\|_F$ synthesized by five discrete-time models, which are shown in Fig. 3. Besides, more detailed MSSRE data and the average computing time per updating (ACTPU) of five models are shown in Table 2 with respect to different values of

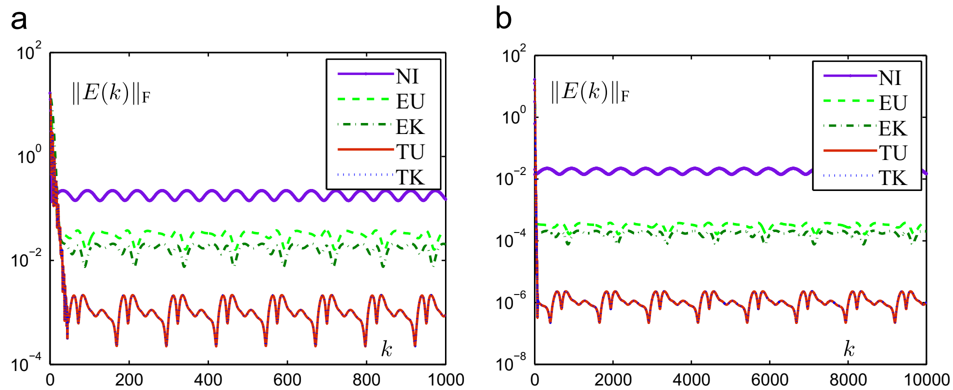

Fig. 3. Residual errors $\|E(k)\|_{\rm F} = \|X_k A_k A_k^{\rm T} - A_k^{\rm T}\|_{\rm F}$ of five models for discrete-time varying matrix pseudoinversion (16): (a) order of $\|E(k)\|_{\rm F}$ with τ=0.01 and (b) order of $\|E(k)\|_{\rm F}$ with τ=0.001.

Table 2
Comparisons among five discrete-time models in terms of MSSRE and ACTPU with h=0.3 for discrete-time varying matrix pseudoinversion (16).

τ (s)	Model	MSSRE	ACTPU (s)
0.1	T-ZNN-K model (6)	2.7350	4.6212×10^{-4}
	T-ZNN-U model (8)	2.6852	4.6151×10^{-4}
	E-ZNN-K model (12)	2.1311	4.4386×10^{-4}
	E-ZNN-U model (13)	2.1461	4.3773×10^{-4}
	Newton iteration (14)	2.2268	3.9950×10^{-4}
0.01	T-ZNN-K model (6)	2.1465×10^{-3}	4.6599×10^{-4}
	T-ZNN-U model (8)	2.1616×10^{-3}	4.6342×10^{-4}
	E-ZNN-K model (12)	2.0953×10^{-2}	4.6054×10^{-4}
	E-ZNN-U model (13)	3.7839×10^{-2}	4.5768×10^{-4}
	Newton iteration (14)	2.2107×10^{-1}	4.0379×10^{-4}
0.001	T-ZNN-K model (6)	2.3170×10^{-6}	4.6634×10^{-4}
	T-ZNN-U model (8)	2.3185×10^{-6}	4.6183×10^{-4}
	E-ZNN-K model (12)	2.1273×10^{-4}	4.5747×10^{-4}
	E-ZNN-U model (13)	3.8356×10^{-4}	4.5922×10^{-4}
	Newton iteration (14)	2.2120×10^{-2}	4.1326×10^{-4}

sampling gap τ. From Fig. 3 as well as Table 2, the following important facts are summarized, which further substantiate the theoretical analyses very well.

1. The MSSRE of Newton iteration (14) without utilizing time derivative of time-varying coefficient changes in an $O(\tau)$ pattern. For example, it follows from Table 2 that the MSSREs of Newton iteration (14) are of order 10^0, 10^{-1} or 10^{-2}, corresponding to τ=0.1, 0.01 or 0.001, respectively.
2. The MSSREs of E-ZNN-K model (12) and E-ZNN-U model (13), which utilize time derivative of time-varying coefficient and are discretized via Euler difference rule, change in an $O(\tau^2)$ pattern. This fact is clearly shown in Table 2.
3. The MSSREs of T-ZNN-K model (6) and T-ZNN-U model (8), which utilize time derivative of time-varying coefficient and are discretized via the proposed Taylor-type formula (4), change in an $O(\tau^3)$ pattern. For example, as displayed in Table 2, the MSSREs of T-ZNN-K model (6) and T-ZNN-U model (8) are of order 10^0, 10^{-3} or 10^{-6}, corresponding to τ=0.1, 0.01 or 0.001, respectively.
4. Even for a small sampling gap τ, such as 0.001 s (i.e., 1 ms), the five discrete-time models all have a small ACTPU (i.e., around of order 10^{-4}), thereby satisfying well the requirement of real-time computation.

6. Application to motion generation

In this section, T-ZNN-U model (8) and Newton iteration (14) are applied to the motion generation of a five-link planar robot manipulator. For such a robot operating with task duration T, at time instant $t\in[0,T]$, the relation between the end-effector Cartesian position vector $r\in\mathbb{R}^2$ and the joint-angle vector $\theta\in\mathbb{R}^5$ is [16,17]

$$\phi(\theta(t)) = r(t), \tag{17}$$

where $\phi(\cdot)$ denotes the continuous forward-kinematics mapping function with known structure and parameters for a given manipulator. The inverse-kinematic problem [i.e., given $r(t)$, to solve for $\theta(t)$] is usually solved at the joint-velocity level by differentiating (17) with respect to time t. Then, the following relation between the end-effector and Cartesian velocity $\dot r(t)$ and the joint velocity $\dot\theta(t)$ can be derived:

$$J(\theta(t))\dot\theta(t) = \dot r(t), \tag{18}$$

where Jacobian matrix $J(\theta(t)) = \partial\phi(\theta(t))/\partial\theta(t)$. Evidently, the analytical solution to the inverse-kinematic problem is $\dot\theta(t) = J^+(\theta(t))\dot r(t)$, where $J^+(\theta(t))$ is the pseudoinverse of time-varying Jacobian matrix. We need to obtain $J^+(\theta(t))$ in real time t for the control of the robot. Defining $A_k = J(\theta(k\tau))$, we could exploit the aforementioned ZNN models to solve $J^+(\theta(t))$ as well as to expedite the computation process and achieve better control precision. For simplicity and illustration, with each link length being 1 m, the five-link planar robot manipulator is investigated to track a square path with the side length being 2.4 m, where T=20 s and initial joint state $\theta(0) = [\pi/4, \pi/12, \pi/4, \pi/12, \pi/4]^{\rm T}$ rad. Observing that the ACTPU is smaller than 1 ms in Example 2, we choose sampling gap $\tau = 1$ ms and step-size h=0.3.

Numerical results synthesized by T-ZNN-U model (8) and Newton iteration (14) are shown in Figs. 4 and 5. Note that, similar to Fig. 4 synthesized by T-ZNN-U model (8), the joint-angle, joint-velocity and motion trajectories generated by Newton iteration (14) are omitted due to space limitation.

As seen from Fig. 5(a), the maximum position error synthesized by T-ZNN-U model (8) is of order 10^{-5} m, which is roughly 10 times smaller than that by Newton iteration (14) in Fig. 5(b). This application to the motion generation of the five-link planar robot manipulator further illustrates the superiority of the proposed T-ZNN-U model (8) for discrete-time varying pseudoinverse solving.

7. Conclusions

In this paper, in order to achieve higher computational accuracy in approximating the first-order derivative and discretize effectively the continuous-time ZNN model, a Taylor-type numerical differentiation formula (4) has been first presented and investigated. Afterwards, based on the formula, T-ZNN-K model (6) and T-ZNN-U

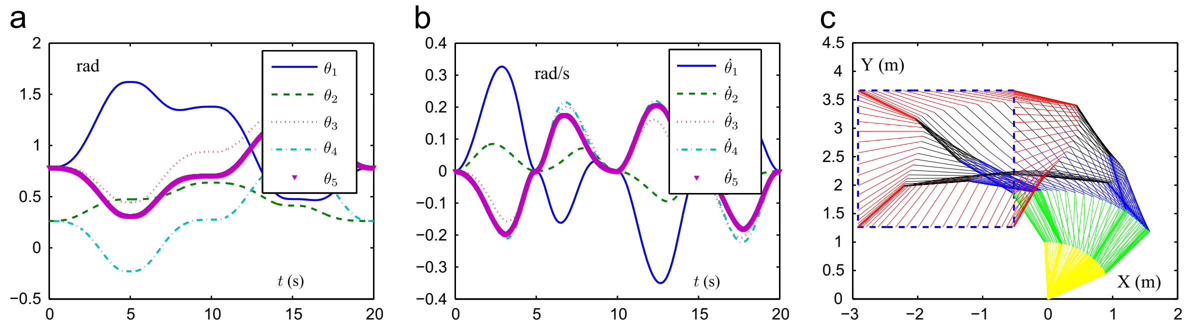

Fig. 4. Motion generation of the five-link planar robot manipulator with its end-effector tracking a square path: (a) θ by T-ZNN-U (8); (b) $\dot{\theta}$ by T-ZNN-U (8); and (c) motion trajectories.

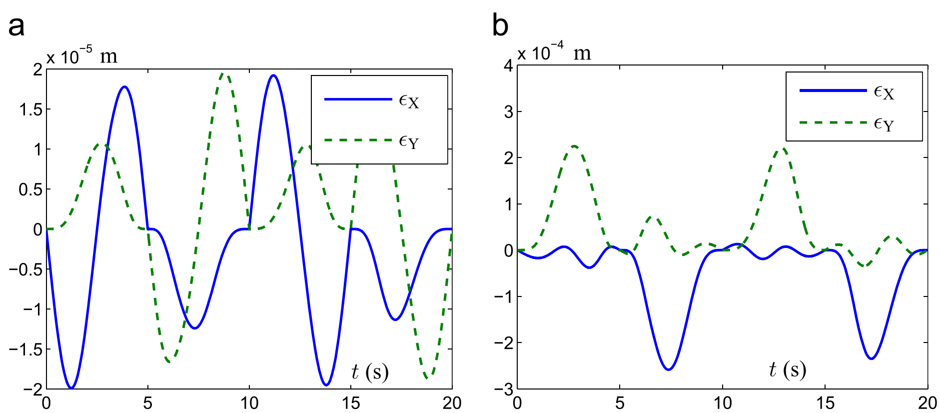

Fig. 5. Position error ϵ of the robot end-effector when tracking the square path as synthesized by T-ZNN-U model (8) and Newton iteration (14): (a) ϵ by T-ZNN-U (8) and (b) ϵ by Newton iteration (14).

model (8) have been proposed for solving discrete-time varying pseudoinverse. For comparison, E-ZNN-K model (12), E-ZNN-U model (13) and Newton iteration (14) have been presented for the same solution task. Meanwhile, interesting connections between E-ZNN-K model (12) and Newton iteration (14) have been discovered. Moreover, the stability and convergence of the five discrete-time models have been investigated in detail. Theoretical analyses have shown that MSSREs of T-ZNN-K model (6) and T-ZNN-U model (8) have an $O(\tau^3)$ pattern; E-ZNN-K model (12) and E-ZNN-U model (13) have an $O(\tau^2)$ pattern; and the Newton iteration (14) has an $O(\tau)$ pattern. Numerical results of three illustrative examples (including one application example) have further illustrated the efficacy and superiority of the proposed T-ZNN-K model (6) and T-ZNN-U model (8) (compared with E-ZNN-K model (12), E-ZNN-U model (13) and Newton iteration (14)).

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Appendix A

A.1. Proof of Theorem 1

Based on Taylor expansion theorem [20], we have the following equation:

$$f(t_{k+1})=f(t_k+\tau)=f(t_k)+\tau f'(t_k)+\frac{\tau^2}{2!}f''(t_k)+\frac{\tau^3}{3!}f'''(c_1),$$

where $n!$ denotes the factorial of n, and c_1 lies between t_k and t_{k+1}. It can be rewritten as

$$f'(t_k)=\frac{f(t_{k+1})-f(t_k)}{\tau}-\frac{\tau}{2!}f''(t_k)-\frac{\tau^2}{3!}f'''(c_1). \tag{19}$$

In a similar way, one can obtain

$$f(t_{k-1})=f(t_k-\tau)=f(t_k)-\tau f'(t_k)+\frac{\tau^2}{2!}f''(t_k)-\frac{\tau^3}{3!}f'''(c_2),$$

and

$$f(t_{k-2})=f(t_k)-2\tau f'(t_k)+\frac{4\tau^2}{2!}f''(t_k)-\frac{8\tau^3}{3!}f'''(c_3),$$

with c_2 and c_3 lying in (t_{k-1},t_k) and (t_{k-2},t_k), respectively. Then, the above two equations can be rewritten as

$$f'(t_k)=\frac{f(t_k)-f(t_{k-1})}{\tau}+\frac{\tau}{2!}f''(t_k)-\frac{\tau^2}{3!}f'''(c_2), \tag{20}$$

and

$$f'(t_k)=\frac{f(t_k)-f(t_{k-2})}{2\tau}+\tau f''(t_k)-\frac{2\tau^2}{3}f'''(c_3). \tag{21}$$

Let (19) plus (21), then subtract (20). Thus, we obtain the following difference formula:

$$f'(t_k)=\frac{2f(t_{k+1})-3f(t_k)+2f(t_{k-1})-f(t_{k-2})}{2\tau}+\tau^2\left(-\frac{1}{3!}f'''(c_1)+\frac{1}{3!}f'''(c_2)-\frac{2}{3}f'''(c_3)\right). \tag{22}$$

Since $f'''(t)$ is continuous, the above equation can be further rewritten as Eq. (5), i.e.,

$$f'(t_k)=\frac{2f(t_{k+1})-3f(t_k)+2f(t_{k-1})-f(t_{k-2})}{2\tau}+O(\tau^2).$$

Dropping $O(\tau^2)$ of (5) yields exactly new Taylor-type differentiation formula as below:

$$f'(t_k) \approx \frac{2f(t_{k+1}) - 3f(t_k) + 2f(t_{k-1}) - f(t_{k-2})}{2\tau}.$$

The proof is thus completed.

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