Supplemental Material: Proof for Lemma 1

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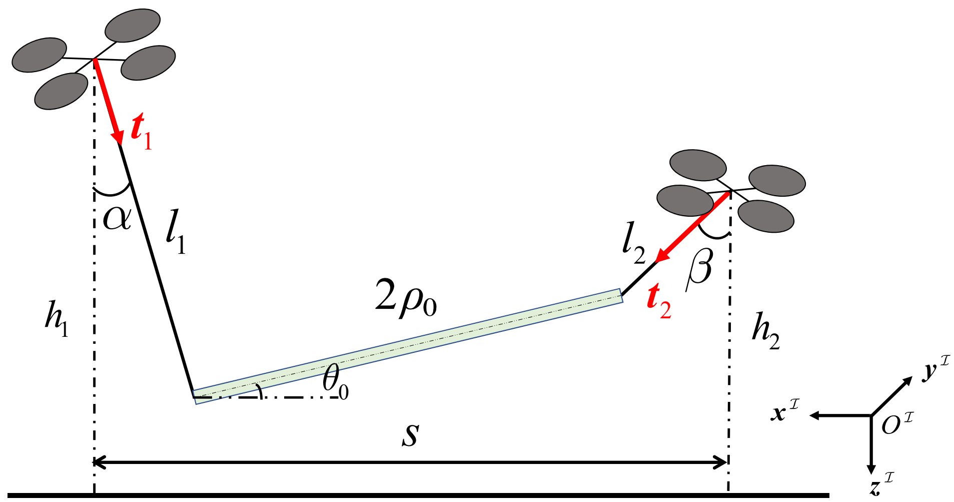

Fig. 1. Regulation of the height of quadrotor 2.

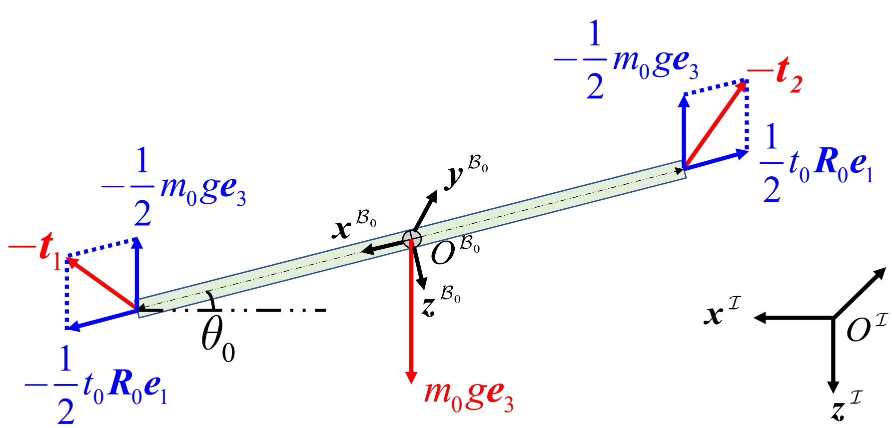

Fig. 2. Force analysis for the pipe.

According to [1], the stable attitude and position of the pipe are unique when the positions of the two quadrotors or two suspension points are fixed, and so is the vertical cable force error

$t_0 \sin\theta_0$. Hence, when quadrotor 2 only regulates its height and quadrotor 1 keeps its position, the vertical cable force error $t_0 \sin\theta_0$ can always be treated as the function of the single variable $\tilde{p}_{2,z}$. In this supplemental material, we attempt to prove that $t_0 \sin\theta_0$ is negatively correlated with $\tilde{p}_{2,z}$ in the local sense and its derivative with respect to $\tilde{p}_{2,z}$ is bounded, which is summarized in the following Lemma 1.

Lemma 1. *$t_0 \sin\theta_0$ and $\tilde{p}_{2,z}$ are negatively correlated and satisfy the following equation*

$$t_0 \sin\theta_0 = -\sigma\left(\tilde{p}_{2,z}\right) \tag{1}$$

where $\sigma(x)$ is a strictly increasing function with $\sigma(0) = 0$. Moreover, the slope of $\sigma(x)$ satisfies

$$0 < \underline{\sigma} < \frac{d\sigma(x)}{dx} < \overline{\sigma} \tag{2}$$

where $\underline{\sigma} \in \mathbb{R}^+$ and $\overline{\sigma} \in \mathbb{R}^+$ are the constant lower and upper bounds.

Proof. As shown in Fig 1, the height of the two quadrotors are h_1 and h_2, which satisfies

$$h_1 = -p_{1,z}, \quad h_2 = -p_{2,z} = -\tilde{p}_{2,z} - p_{2,z}^{des} \tag{3}$$

where $p_{2,z}^{des}$ and $\tilde{p}_{2,z}$ are the desired vertical position and vertical tracking error of quadrotor 2, respectively. Only the scene where the internal force $t_0 > 0$ in Fig. 1 is considered, where the cable angles α, β and the load angle θ_0 satisfy

$$0 < \alpha < \frac{\pi}{2}, \quad 0 < \beta < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \theta_0 < \frac{\pi}{2}. \tag{4}$$

As the payload stays in the XZ plane of NED frame shown in Fig. 2 under the quasi-static condition, the cable forces can be computed as [2]

$$\begin{aligned} \boldsymbol{t}_1 &= \frac{1}{2}m_0 g\boldsymbol{e}_3 + \frac{1}{2}t_0\boldsymbol{R}_0\boldsymbol{e}_1 = \left[\frac{t_0\cos\theta_0}{2}, 0, \frac{-t_0\sin\theta_0 + m_0 g}{2}\right]^\top, \\ \boldsymbol{t}_2 &= \frac{1}{2}m_0 g\boldsymbol{e}_3 - \frac{1}{2}t_0\boldsymbol{R}_0\boldsymbol{e}_1 = \left[-\frac{t_0\cos\theta_0}{2}, 0, \frac{t_0\sin\theta_0 + m_0 g}{2}\right]^\top \end{aligned} \tag{5}$$

where θ_0 is the pitch angle of the pipe notated in Fig. 1.

From (5), trigonometric functions of α and β are computed as

$$\begin{aligned} \cos\alpha &= \frac{-t_0\sin\theta_0 + m_0 g}{\sqrt{(t_0\cos\theta_0)^2 + (-t_0\sin\theta_0 + m_0 g)^2}} &&= \frac{k - \sin\theta_0}{g_1} \\ \cos\beta &= \frac{t_0\sin\theta_0 + m_0 g}{\sqrt{(t_0\cos\theta_0)^2 + (t_0\sin\theta_0 + m_0 g)^2}} &&= \frac{k + \sin\theta_0}{g_2} \\ \sin\alpha &= \frac{t_0\cos\theta_0}{\sqrt{(t_0\cos\theta_0)^2 + (-t_0\sin\theta_0 + m_0 g)^2}} &&= \frac{\cos\theta_0}{g_1} \\ \sin\beta &= \frac{t_0\cos\theta_0}{\sqrt{(t_0\cos\theta_0)^2 + (t_0\sin\theta_0 + m_0 g)^2}} &&= \frac{\cos\theta_0}{g_2} \end{aligned} \tag{6}$$

where $k = \frac{m_0 g}{t_0}$, $g_1 = \sqrt{k^2 - 2k\sin\theta_0 + 1}$, and $g_2 = \sqrt{k^2 + 2k\sin\theta_0 + 1}$ are used for substitutions. Here the derivatives of $g_1(k,\theta_0)$ and $g_2(k,\theta_0)$ with respect to θ_0 are computed as

$$\frac{dg_1}{d\theta_0} = \frac{1}{g_1}\left[(k - \sin\theta_0)\frac{dk}{d\theta_0} - k\cos\theta_0\right], \quad \frac{dg_2}{d\theta_0} = \frac{1}{g_2}\left[(k + \sin\theta_0)\frac{dk}{d\theta_0} + k\cos\theta_0\right]. \tag{7}$$

From here the dependency of g_1 and g_2 on the variables k and θ_0 is sometimes omitted for notation simplicity.

When the desired formation shape $\boldsymbol{p}_{12,xy}^{des}$ has been achieved, the following constraint equation is obtained

$$l_1\sin\alpha + l_2\sin\beta + 2\rho_0\cos\theta_0 = s. \tag{8}$$

Using the substitutions in (6) yields

$$l_1\frac{\cos\theta_0}{g_1(k,\theta_0)} + l_2\frac{\cos\theta_0}{g_2(k,\theta_0)} + 2\rho_0\cos\theta_0 = s. \tag{9}$$

Although attempt is made to configure out the stable configuration of the payload by convex formulation in [3], the constraint equation (9) is actually a high-order equation with respect to k, from which multiple solutions are provided [1], so it is quite hard to express the variable k as a specified analytic function of θ_0 in this scene. In this material, we try to obtain the relationship between them by analyzing the derivative of the variable k with respect to θ_0. Differentiating (9) with respect to θ_0 yields

$$\frac{l_1}{g_1^3}[(k - \sin\theta_0)\frac{dk}{d\theta_0} - k\cos\theta_0] + \frac{l_2}{g_2^3}[(k + \sin\theta_0)\frac{dk}{d\theta_0} + k\cos\theta_0] = -\frac{s\cdot\sin\theta_0}{\cos^2\theta_0}. \tag{10}$$

Combining (9), the derivative of k with respect to θ_0 is computed as

$$\frac{dk}{d\theta_0} = -\frac{\frac{l_1}{g_1^3}(k - \sin\theta_0)(k\sin\theta_0 - 1) + \frac{l_2}{g_2^3}(k + \sin\theta_0)(k\sin\theta_0 + 1) + 2\rho_0\sin\theta_0}{\left[\frac{l_1(k-\sin\theta_0)}{g_1^3} + \frac{l_2(k+\sin\theta_0)}{g_2^3}\right]\cos\theta_0} \tag{11}$$

Next, the height of quadrotor 2 is calculated as

$$\begin{aligned} h_2 &= l_2 \cos\beta + 2\rho_0 \sin\theta_0 - l_1 \cos\alpha + h_1 \\ &= l_2 \frac{k+\sin\theta_0}{g_2(k,\theta_0)} + 2\rho_0 \sin\theta_0 - l_1 \frac{k-\sin\theta_0}{g_1(k,\theta_0)} + h_1 \end{aligned} \tag{12}$$

where $h_1 = -p_{1,z}$ is assumed to be static at the steady state. The lengths of cables l_1, l_2 and the length of pipe $2\rho_0$ are also fixed. Then h_2 can be seen as a continuous function of θ_0. Differentiating h_2 with respect to θ_0 yields

$$\begin{aligned} \frac{dh_2}{d\theta_0} &= l_2 \frac{(\frac{dk}{d\theta_0} + \cos\theta_0) g_2 - (k+\sin\theta_0)\frac{dg_2}{d\theta_0}}{g_2^2} - l_1 \frac{(\frac{dk}{d\theta_0} - \cos\theta_0) g_1 - (k-\sin\theta_0)\frac{dg_1}{d\theta_0}}{g_1^2} + 2\rho_0 \cos\theta_0 \\ &= (\frac{l_2}{g_2^3} - \frac{l_1}{g_1^3}) \frac{dk}{d\theta_0} \cos^2\theta_0 + \frac{l_2 \cos\theta_0 (k\sin\theta_0 + 1)}{g_2^3} + \frac{l_1 \cos\theta_0 (1 - k\sin\theta_0)}{g_1^3} + 2\rho_0 \cos\theta_0 \\ &= k\cos\theta_0 \frac{\frac{4l_1 l_2}{g_1^3 g_2^3} + 2\rho_0 (\frac{l_2}{g_2^3} + \frac{l_1}{g_1^3})}{\frac{l_1(k-\sin\theta_0)}{g_1^3} + \frac{l_2(k+\sin\theta_0)}{g_2^3}}. \end{aligned} \tag{13}$$

Since $p_{1,z}^{des}$ and p_{12}^{des} are fixed, $p_{2,z}^{des}$ is also fixed according to the equilibrium analysis in the article, i.e., $\dot{p}_{2,z}^{des} = 0$. Therefore, the following equation is obtained

$$\frac{d\tilde{p}_{2,z}}{d\theta_0} = \frac{d\left(-h_2 - p_{2,z}^{des}\right)}{d\theta_0} = -\frac{dh_2}{d\theta_0} \tag{14}$$

where $p_{2,z} = -h_2$ is used. Then combining (13) the derivative of the inverse function satisfies

$$\frac{d\theta_0}{d\tilde{p}_{2,z}} = -\frac{d\theta_0}{dh_2} = -\frac{\frac{l_1(k-\sin\theta_0)}{g_1^3} + \frac{l_2(k+\sin\theta_0)}{g_2^3}}{k\cos\theta_0 \left[\frac{4l_1 l_2}{g_1^3 g_2^3} + 2\rho_0(\frac{l_2}{g_2^3} + \frac{l_1}{g_1^3})\right]}. \tag{15}$$

Finally, differentiating $t_0 \sin\theta_0$ with respect to $\tilde{p}_{2,z}$ yields

$$\begin{aligned} \frac{d(t_0 \sin\theta_0)}{d\tilde{p}_{2,z}} &= \frac{dt_0}{d\theta_0} \frac{d\theta_0}{d\tilde{p}_{2,z}} \sin\theta_0 + \frac{d(\sin\theta_0)}{d\tilde{p}_{2,z}} t_0 \\ &= -\frac{dk}{d\theta_0} \frac{d\theta_0}{d\tilde{p}_{2,z}} \frac{m_0 g \sin\theta_0}{k^2} + \frac{d\theta_0}{d\tilde{p}_{2,z}} \frac{m_0 g \cos\theta_0}{k} \\ &= -\frac{m_0 g}{k^3 \cos^2\theta_0} \frac{\frac{l_1}{g_1^3}(k-\sin\theta_0)^2 + \frac{l_2}{g_2^3}(k+\sin\theta_0)^2 + 2\rho_0 \sin^2\theta_0}{\frac{4l_1 l_2}{g_1^3 g_2^3} + 2\rho_0(\frac{l_1}{g_1^3} + \frac{l_2}{g_2^3})} < 0, \end{aligned} \tag{16}$$

from which the function $t_0 \sin\theta_0$ is proved to be strictly decreasing with respect to the variable $\tilde{p}_{2,z}$ in the local sense. Based on the constraint function (9), the following inequality can be deduced

$$\begin{aligned}\frac{s}{\cos\theta_0}-2\rho_0 &= \frac{l_1}{\sqrt{(k-\sin\theta_0)^2+\cos^2\theta_0}}+\frac{l_2}{\sqrt{(k+\sin\theta_0)^2+\cos^2\theta_0}}\\ &\geq \frac{2\sqrt{2l_1l_2}}{\sqrt{(k-\sin\theta_0)^2+(k+\sin\theta_0)^2+2\cos^2\theta_0}}\\ &= \frac{2\sqrt{l_1l_2}}{\sqrt{k^2+1}}\end{aligned} \tag{17}$$

which implies

$$k^2 \geq \frac{4l_1l_2}{\left(\frac{s}{\cos\theta_0}-2\rho_0\right)^2}-1. \tag{18}$$

Assumption 1. *Consider the pipe suspended by two quadrotors by cables in the XZ plane of NED frame shown in Fig 1, the pitch angle of the pipe θ_0 is assumed to satisfy the following bounded condition*

$$-\frac{\pi}{2} < -\theta^* \leq \theta_0 \leq \theta^* < \frac{\pi}{2} \tag{19}$$

and the cable angles α and β are upper bounded by κ, i.e.,

$$0 < \alpha \leq \kappa < \frac{\pi}{2}, \qquad 0 < \beta \leq \kappa < \frac{\pi}{2}, \tag{20}$$

where θ^* and κ are positive constants.

According to the inequality (18), the horizontal distance s can be adjusted to set the lower bound for k, i.e.,

$$k \geq \underline{k} = \sqrt{\frac{4l_1l_2}{\left(\frac{s}{\cos\theta^*}-2\rho_0\right)^2}-1} > 0 \tag{21}$$

where the lower bound $\underline{k}$ corresponds to the upper bound of the internal force $\bar{t}_0$.

For the upper bound $\bar{\sigma}$ of $\frac{d\sigma(x)}{dx}$,

$$\begin{aligned}\frac{d\sigma(x)}{dx} = -\frac{d(t_0\sin\theta_0)}{d\tilde{p}_{2,z}} &= \frac{m_0g}{k^3\cos^3\theta_0}\frac{l_1\sin\alpha\cos^2\alpha+l_2\sin\beta\cos^2\beta+2\rho_0\cos\theta_0\sin^2\theta_0}{\frac{4l_1l_2}{g_1^3g_2^3}+2\rho_0\left(\frac{l_1}{g_1^3}+\frac{l_2}{g_2^3}\right)}\\ &\leq \frac{m_0g}{k^3\cos^2\theta_0}\frac{(l_1\sin\alpha+l_2\sin\beta+2\rho_0\cos\theta_0)\cdot\max\{\cos^2\alpha,\cos^2\beta,\sin^2\theta_0\}}{2\rho_0\left(\frac{l_1\sin\alpha}{g_1^2}+\frac{l_2\sin\beta}{g_2^2}\right)}\\ &\leq \frac{m_0g}{k^3\cos^2\theta_0}\frac{s}{2\rho_0\cdot\frac{l_1\sin\alpha+l_2\sin\beta}{k^2+2k+1}}\\ &\leq \frac{m_0g}{k}\left(1+\frac{1}{k}\right)^2\frac{\frac{s}{\cos^2\theta_0}}{2\rho_0(s-2\rho_0\cos\theta_0)}\\ &\leq \frac{m_0g}{\underline{k}}\left(1+\frac{1}{\underline{k}}\right)^2\frac{s}{2\rho_0\cos^2\theta^*(s-2\rho_0)} = \bar{\sigma}\end{aligned} \tag{22}$$

where $g_1^2 \leq k^2 + 2k + 1$ and $g_2^2 \leq k^2 + 2k + 1$ are used.

For the lower bound $\underline{\sigma}$ of $\frac{d\sigma(x)}{dx}$,

$$\begin{aligned} \frac{d\sigma(x)}{dx} = -\frac{d(t_0 \sin\theta_0)}{d\tilde{p}_{2,z}} = \frac{m_0 g}{k^3 \cos^3\theta_0} & \frac{l_1 \sin\alpha \cos^2\alpha + l_2 \sin\beta \cos^2\beta + 2\rho_0 \cos\theta_0 \sin^2\theta_0}{\frac{4 l_1 l_2}{g_1^3 g_2^3} + 2\rho_0 \left(\frac{l_1}{g_1^3} + \frac{l_2}{g_2^3}\right)} \\ \geq m_0 g & \frac{l_1 \sin\alpha \cos^2\alpha + l_2 \sin\beta \cos^2\beta}{\frac{4 l_1 l_2 k^3 \cos^3\theta_0}{g_1^3 g_2^3} + 2\rho_0 \left(\frac{l_1 k^3 \cos^3\theta_0}{g_1^3} + \frac{l_2 k^3 \cos^3\theta_0}{g_2^3}\right)} \\ \geq m_0 g & \frac{(l_1 \sin\alpha + l_2 \sin\beta) \cos^2 \max\{\alpha, \beta\}}{\frac{4 l_1 l_2}{\cos^3\theta_0} + 2\rho_0 l_1 + 2\rho_0 l_2} \\ \geq m_0 g & \frac{(s - 2\rho_0) \cos^2\kappa}{\frac{4 l_1 l_2}{\cos^3\theta^*} + 2\rho_0 l_1 + 2\rho_0 l_2} = \underline{\sigma} > 0 \end{aligned} \tag{23}$$

where $g_1 \geq k \cos\theta_0$ and $g_2 \geq k \cos\theta_0$ are used.

In this manner, $t_0 \sin\theta_0$ can be described as a monotonic function of $\tilde{p}_{2,z}$ in the local sense, i.e.,

$$t_0 \sin\theta_0 = -\sigma\left(\tilde{p}_{2,z}\right) \tag{24}$$

where $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $\sigma(0) = 0$ and satisfies

$$0 < \underline{\sigma} \leq \frac{d\sigma(x)}{dx} \leq \overline{\sigma}. \tag{25}$$

Here $\underline{\sigma} = m_0 g \frac{(s-2\rho_0)\cos^2\kappa}{\frac{4 l_1 l_2}{\cos^3\theta^*} + 2\rho_0 l_1 + 2\rho_0 l_2}$ and $\overline{\sigma} = \frac{m_0 g}{\underline{k}} \left(1 + \frac{1}{\underline{k}}\right)^2 \frac{s}{2\rho_0 \cos^2\theta^* (s - 2\rho_0)}$. $\square$

References

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