Least-squares estimation in phase-measurement interferometry

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It is shown that phase-extraction algorithms in phase-measurement interferometry can be developed from the principle of least-squares estimation. It is then demonstrated that this view can be taken to develop algorithms that estimate the phase in the presence of external perturbations. To illustrate, an algorithm is developed that extracts the phase in the presence of a linear time-dependent drift.

Several authors[1–3] have discussed the use of synchronous sampling of interferometric data for the purpose of determining the phase of an unknown wave front. Each has proposed a phase-extraction algorithm appropriate for his particular data-acquisition scheme. Bruning[1] presented an arctangent algorithm suitable for instantaneous detection of the intensity field. Wyant[3] presented an algorithm to be used with detectors that integrated the intensity over a finite time interval. This was intended primarily for use with charge-coupled devices. The purpose of this discussion is to demonstrate that both algorithms are derivable from the principle of least-squares estimation. This in itself does not provide any new information about the phase data when interferometer measurements are obtained under ideal conditions. However, it does suggest a rational method of reconstructing such data from interferometer measurements taken under nonideal circumstances. Since the least-squares approach also allows one to estimate the perturbations, it may be applicable in an active phase-measurement interferometry (PMI) sense, in which perturbations estimated during one sampling period are used to make hardware adjustments before the next set of measurements is taken. In the final example discussed below, we develop a least-squares algorithm that estimates both the phase and the perturbation for the case of a linear time-dependent drift.

It is known that, when a function is represented by a Fourier series, the Fourier coefficients are determined by the requirement that the residual error between the function and the series be a minimum in the least-squares sense. Since the above algorithms were derived by using properties of Fourier series and their orthogonality relationships,[1] it is not surprising that these results could be obtained directly by least-squares estimation.

To begin, the Bruning–Herriott algorithm will be derived by using least-squares estimation. Assume that we are using a two-beam interferometer, such as the Twyman–Green interferometer shown in Fig. 1. Light of wavelength λ returning from the test and the reference surfaces recombines at the beam splitter and interferes. A piezoelectric driver is mounted to the reference optic so that repeated intensity samples can be taken while the optical path length ΔL varies linearly in time over a one-wavelength range. The object of these measurements is to determine the unknown surface $W(X, Y)$ of the test optic. The normalized intensity distribution in the interference pattern can be represented mathematically:

$$I(X, Y, t) = 1 + \gamma \cos \left\{\frac{4\pi}{\lambda} \left[W(X, Y) - \Delta L\right]\right\}, \quad (1)$$

where γ is the fringe visibility. As the reference optic translates through its half-wavelength range, the intensity at each point of space is sampled at N times, t_j. Let the measured intensity samples taken at times t_j be given by $I_j{*}$. Then, by using Eq. (1) and the measurements $I_j{*}$, the following least-squares penalty function can be formed:

$$Q = \sum_{j=1}^{N} \left\{I_j{*} - 1 - \gamma \cos \left[\frac{4\pi W(X, Y)}{\lambda} - \frac{2\pi}{N} j\right]\right\}^2 . \quad (2)$$

The best estimate of the surface W is that which minimizes Q. Therefore equating to zero the partial derivative of Q with respect to W gives

$$\sum_{j=1}^{N} \left\{I_j{*} - 1 - \gamma \cos \left[\frac{4\pi W(X, Y)}{\lambda} - \frac{2\pi}{N} j\right]\right\} \times \sin \left[\frac{4\pi W(X, Y)}{\lambda} - \frac{2\pi}{N} j\right] = 0. \quad (3)$$

Then, by expanding the sine and cosine functions with the trigonometric addition formulas and making use of the orthogonality relations[4]

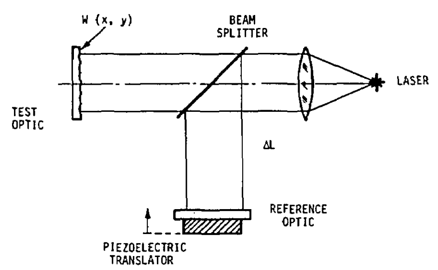

Fig. 1. Twyman–Green interferometer with piezoelectric translator in reference arm.

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$$\sum_{j=1}^{N} \sin\left(\frac{2\pi}{N} j\right) \cos\left(\frac{2\pi}{N} j\right) = 0 \tag{4}$$

and

$$\sum_{j=1}^{N} \sin^2\left(\frac{2\pi}{N} j\right) = \sum_{j=1}^{N} \cos^2\left(\frac{2\pi}{N} j\right) . \tag{5}$$

Equation (3) can be solved for $W(X, Y)$ to give

$$W(X, Y) = \frac{\lambda}{4\pi} \tan^{-1} \left[\sum_{j=1}^{N} I_j^* \sin\left(\frac{2\pi}{N} j\right) \Big/ \sum_{j=1}^{N} I_j^* \cos\left(\frac{2\pi}{N} j\right)\right] . \tag{6}$$

Equation (6) is the algorithm used by Bruning *et al.*

With only slight modifications, the above argument can be applied to derive an algorithm that is suitable for use with charge-coupled devices. Assume that N measurements M_j^* are made during the data-acquisition period T. If each detector integrates the two-beam intensity $I(t)$ over an interval $\tau = T/N$, the detected signal can be expressed mathematically as

$$M(\tau) = \int_{T_0}^{T_0+\tau} dt' \left[1 + \gamma \cos\left(\frac{2\pi}{N\tau} t' + \phi\right)\right] , \tag{7}$$

where γ is defined as in Eq. (1), ϕ is the unknown phase, and T_0 is the starting time for the detector measurement. Each of the N measurements made during the period T can be represented by Eq. (7) if T_0 is allowed to take on a set of N values separated from one another by the interval τ. Furthermore, we will choose the T_0 values so that the integration limits in Eq. (7) will coincide, for the case of $N = 4$, with those selected by Wyant. Since his four measurement intervals were

$$\left(-\frac{T}{8}, \frac{T}{8}\right), \quad \left(\frac{T}{8}, \frac{3T}{8}\right), \quad \left(\frac{3T}{8}, \frac{5T}{8}\right), \quad \left(\frac{5T}{8}, \frac{7T}{8}\right),$$

we will define T_0 as

$$T_0 = j\tau - \frac{\tau}{2} , \tag{8}$$

where $j = 0, 1, \ldots, N-1$. Therefore, inserting Eq. (8) into Eq. (7) and carrying out the integration, we obtain for the jth detected signal

$$M_j = \tau + \frac{N\gamma\tau}{\pi} \sin\left(\frac{\pi}{N}\right) \cos\left(\phi + \frac{2\pi}{N} j\right). \tag{9}$$

Next, M_j^* and M_j are used to form the penalty function, as was done in Eq. (2). From this point the steps are identical with those leading to Eq. (6). Therefore, minimizing the penalty function and simplifying, we obtain

$$\phi = \tan^{-1} \left[\frac{-\sum_{j=0}^{N-1} M_j^* \sin\left(\frac{2\pi}{N} j\right)}{\sum_{j=0}^{N-1} M_j^* \cos\left(\frac{2\pi}{N} j\right)}\right] . \tag{10}$$

Setting $N = 4$ in Eq. (10) gives

$$\phi = \tan^{-1} \left[\frac{-(M_1^* - M_3^*)}{(M_0^* - M_2^*)}\right] \tag{11}$$

$$= \frac{\pi}{2} + \tan^{-1} \left(\frac{M_0^* - M_2^*}{M_1^* - M_3^*}\right) . \tag{12}$$

Equation (12) is identical with Wyant's result except for the $\pi/2$ phase shift. This difference is due to the fact that we represented the interferometer intensity with a cosine function, whereas Wyant used a sine function.

We have demonstrated the utility of least-squares estimation as a technique for deriving phase algorithms for a variety of detection schemes under ideal conditions. We are now in a position to extend this approach to develop new algorithms that will estimate the phase in the presence of external perturbations. One approach to this end is to make some assumptions about the mathematical form of the time-dependent perturbations. This form will contain certain unspecified parameters that will be determined by the least-squares criterion. In general, there will be one equation for each parameter to be estimated. Furthermore, these equations will be coupled and nonlinear and will have to be solved by iteration.

As a simple but practical example, we will consider phase estimation in the presence of an unknown linear drift. We further assume a detection scheme similar to the one used by Bruning *et al.* In practice, linear drift can result from an error in the range of the piezoelectric drive that moves the reference mirror of the interferometer. For the purpose of this discussion, linear drift will be defined in terms of the percentage overshoot of the optical path beyond one wave at the end of the sampling period. Let this percentage be given by F. Then, denoting the measured intensity at time t_j by I_j^*, we find that the penalty function becomes

$$Q = \sum_{j=1}^{N} [I_j^* - 1 - \gamma \cos(\theta_j)]^2, \tag{13}$$

where θ_j is given by

$$\theta_j = \frac{4\pi W(X, Y)}{\lambda} - \frac{2\pi}{N} (1 + F) j. \tag{14}$$

We want to find the (W, F) pair that minimizes Eq. (13). Equating to zero the partial derivatives of Q with respect to W and F, respectively, we obtain two equations in two unknowns:

$$\sum_{j=1}^{N} I_j^* \sin(\theta_j) = \sum_{j=1}^{N} [\sin(\theta_j) + \tfrac{1}{2}\gamma \sin(2\theta_j)], \tag{15}$$

$$\sum_{j=1}^{N} j I_j^* \sin(\theta_j) = \sum_{j=1}^{N} j[\sin(\theta_j) + \tfrac{1}{2}\gamma \sin(2\theta_j)]. \tag{16}$$

The presence of the drift parameter F in Eqs. (15) and (16) has destroyed the sine–cosine orthogonality that helped to simplify the earlier equations. Nevertheless, the equations can be solved iteratively.

To test the algorithm, values of I_j^* were simulated on a computer, using known values of W and F. For this test the fringe visibility γ was assumed to have a constant value of 0.8. These intensities were then inserted into Eqs. (15) and (16), and a two-dimensional Newton–Raphson procedure was applied. For the

Table 1. Iterative Solutions

True F	Initial Guess for W (waves)[a]	Values Returned by Newton and Raphson	
		W (waves)	F
0.01	0.1005	0.09983	0.0094125
		0.09999	0.0099
		0.100	0.010
0.03	0.1015	0.0985	0.0248
		0.0999	0.0299
		0.100	0.030
0.05	0.1022	0.0957	0.0359
		0.0998	0.0494
		0.0999	0.0499
		0.100	0.050
0.07	0.1028	0.0916	0.0428
		0.0992	0.0677
		0.0999	0.0699
		0.100	0.070
0.09	0.1032	0.0858	0.0454
		0.0979	0.0838
		0.0999	0.0898
0.11	0.1034	0.0785	0.0437
		0.0957	0.0968
		0.0998	0.1093
		0.0999	0.1099
		0.100	0.1100
0.13	0.1032	0.0696	0.0380
		0.0925	0.1065
		0.0993	0.1277
		0.0999	0.1299
		0.100	0.130
0.15	0.1028	0.0592	0.0281
		0.0884	0.1132
		0.0984	0.1448
		0.0999	0.1499
		0.100	0.150

[a] Initial guess for W obtained from Eq. (6) assuming that $F = 0$.

purpose of illustration, Table 1 gives the solutions of Eqs. (15) and (16) for the case of $W = 0.1\lambda$, $N = 4$, and for values of F ranging from 0.01 to 0.15. To obtain an initial guess for the iteration, F was set to zero and W was computed by using the conventional unperturbed algorithm given in Eq. (6). The resulting W values are shown in the second column of Table 1. The last two columns of the table give the results of successive iterations for the W and F values that were returned by the Newton–Raphson calculation. For each case considered below, the iteration process converged to the correct values for W and F.

In conclusion, algorithms traditionally used in PMI can be derived in a straightforward manner by using the principle of least squares. Furthermore, this approach can be used to estimate the phase in a perturbed environment, provided that one can make a reasonable guess about the mathematical form of the perturbation.

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