Communications are short papers. Appropriate material for this section includes reports of incidental research results, comments on papers previously published, and short descriptions of theoretical and experimental techniques. Communications are handled much the same as regular papers. Proofs are provided.

Phase-shifting algorithms for nonlinear and spatially nonuniform phase shifts: comment

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In the framework of phase-shifting interferometry, the characteristic polynomial theory is extended to deal with nonlinear and nonuniform phase-shift miscalibration. A general procedure for designing algorithms that are insensitive to those errors is presented. It is also shown how analytical expressions for the residual phase errors can be obtained. Finally, it is demonstrated that the coefficients of any algorithm can be given a Hermitian symmetry. © 1998 Optical Society of America [S0740-3232(98)01605-6]

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1. INTRODUCTION

In a recent paper Hibino *et al.*[1] investigated two specific aspects of digital phase detection: (i) the influence of nonlinear phase shifts and (ii) their possible spatial nonuniformity. The authors also propose a Hermitian symmetry condition over the coefficient of an algorithm, thus ensuring a minimal number of samples. In this comment we indicate how the first issue can be addressed in the framework of our characteristic polynomial theory[2] and that the corresponding design procedure is very similar to the one in the linear case. Then we give a general solution of the second problem, but we suggest that the algorithmic approach may not be the best. Finally, we show how the coefficients of almost any published algorithm can be given the Hermitian symmetry mentioned by the authors.

2. NONLINEAR PHASE SHIFT

In the framework of the characteristic polynomials theory, rule 3 in Subsection 2.G of a preceding paper[2] states that the harmonic m in the intensity signal has no effect in the presence of a phase-shift miscalibration ϵ if the complex number $\exp(im\delta)$ is a root of order $k+1$ of the characteristic polynomial. If this condition is true, the detected phase will contain no term in ϵ^l, $l \leqslant k$.

However, this rule is valid in the case of a linear miscalibration, that is, if the kth phase shift is $\delta_k = k\delta(1+\epsilon)$. We show here that a very similar result is obtained in the case of a nonlinear phase shift:

$$\delta_k = k\delta(1+\epsilon_1+k\epsilon_2+k^2\epsilon_3+\ldots). \tag{1}$$

As reported in Ref. 2, the phase detection can be explained in terms of the computation of a complex linear combination $S(\phi)=\sum_{k=0}^{M-1} c_k I(\phi+\delta_k)$, where the numbers $c_k = a_k + ib_k$ are related to the coefficients in the numerator (b_k) and denominator (a_k) of the algorithm expressed in its arctangent form. If the intensity signal contains a harmonic $\alpha_m \exp(im\phi)$, the sum $S(\phi)$ will contain the corresponding term

$$S_m(\phi) = \alpha_m \sum_{k=0}^{M-1} c_k \exp[im(\phi+\delta_k)] = \alpha_m \exp(im\phi)\sum_{k=0}^{M-1} c_k \exp(im\delta_k). \tag{2}$$

In the case of the nonlinear phase shift expressed by Eq. (1), the sum in Eq. (2) is written as

$$\begin{aligned}\sum_{k=0}^{M-1} c_k \exp(im\delta_k) &= \sum_{k=0}^{M-1} c_k \exp[imk\delta(1+\epsilon_1+k\epsilon_2+k^2\epsilon_3+\ldots)]\\ &= \sum_{k=0}^{M-1} c_k[\exp(i\delta)^m]^k\Bigg[1+imk\delta\epsilon_1+imk^2\delta\epsilon_2\\ &\quad + imk^3\delta\epsilon_3+\ldots+\frac{(imk\delta\epsilon_1)^2}{2!}+\frac{(imk^2\delta\epsilon_2)^2}{2!}\\ &\quad + \frac{(imk^3\delta\epsilon_3)^2}{2!}+\ldots+\frac{2(imk\delta\epsilon_1)(imk^2\delta\epsilon_2)}{2!}\\ &\quad + \frac{2(imk\delta\epsilon_1)(imk^3\delta\epsilon_3)}{2!}\\ &\quad + \frac{2(imk^2\delta\epsilon_2)(imk^3\delta\epsilon_3)}{2!}+\ldots\Bigg].\end{aligned} \tag{3}$$

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Stipulating $\zeta = \exp(i\delta)$, we can see that the preceding expression involves many different polynomials of ζ^m, with coefficients $\{c_k\}$, $\{kc_k\}$, $\{k^2c_k\}$, etc. In terms of a dummy variable x, the first one is the characteristic polynomial $P(x)$. The others are deduced from $P(x)$ by iterative application of the operator $\mathbf{D} = x \cdot (d/dx)$ already introduced in Ref. 2. So, taking into account Eqs. (2) and (3), we can, in the case of a nonlinear phase shift, write the supplementary term in the sum $S(\phi)$ related to the mth harmonic as

$$\begin{aligned} S_m(\phi) = \alpha_m \exp(im\phi)\Bigg[& P(\zeta^m) + im\delta\epsilon_1\mathbf{D}P(\zeta^m) \\ & + im\delta\epsilon_2\mathbf{D}^2P(\zeta^m) + im\delta\epsilon_3\mathbf{D}^3P(\zeta^m) + \ldots \\ & + \frac{(im\delta\epsilon_1)^2}{2!}\mathbf{D}^2P(\zeta^m) + \frac{(im\delta\epsilon_2)^2}{2!}\mathbf{D}^4P(\zeta^m) \\ & + \frac{(im\delta\epsilon_3)^2}{2!}\mathbf{D}^6P(\zeta^m) + \ldots \\ & + \frac{2(im\delta\epsilon_1)(im\delta\epsilon_2)}{2!}\mathbf{D}^3P(\zeta^m) \\ & + \frac{2(im\delta\epsilon_1)(im\delta\epsilon_3)}{2!}\mathbf{D}^4P(\zeta^m) \\ & + \frac{2(im\delta\epsilon_2)(im\delta\epsilon_3)}{2!}\mathbf{D}^5P(\zeta^m) + \ldots\Bigg]. \end{aligned} \tag{4}$$

The conditions driving the insentivity to a nonlinear phase shift clearly appear in this equation and consist in canceling the adequate terms in $\epsilon_{r_1}{}^{s_1}\epsilon_{r_2}{}^{s_2}\ldots\epsilon_{r_t}{}^{s_t}$, which requires $\mathbf{D}^{r_1s_1 + r_2s_2 + \ldots + r_ts_t)}P(\zeta^m)$ to be canceled. As shown in Ref. 2, $\mathbf{D}^nP(\zeta^m)$ is canceled if ζ^m is a root of order $n + 1$ of the characteristic polynomial. For example, the quadratic nonlinearity has no incidence up to the first order in ϵ_2 if $\mathbf{D}^2P(\zeta^m) = 0$, that is if ζ^m is a triple root of the characteristic polynomial. This condition is the same as the one that provides an improved insensitivity to a linear miscalibration; that is, it cancels the term in $\epsilon_1{}^2$ as well.

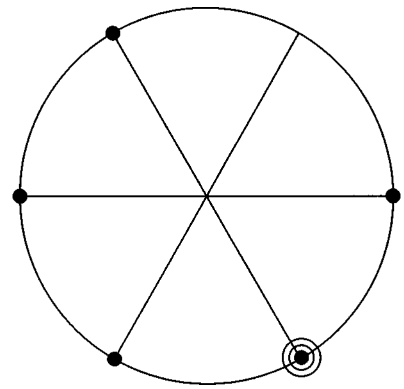

Fig. 1. Characteristic diagram canceling the harmonic $m = -1$ in the presence of a quadratic miscalibration. The diagram is shown for the special case of $N = 6$.

A simple monomial product allows us to find the proper algorithm. When

$$\begin{aligned} P_N(x) &= 1 + \zeta^{-1}x + \zeta^{-2}x^2 + \ldots + \zeta^{-(N-1)}x^{N-1} \\ &= \zeta \prod_{l=0,l\neq 1}^{N-1} (x - \zeta^l) \end{aligned} \tag{5}$$

denotes the characteristic polynomial of the N-bucket algorithm, the following polynomial corresponds to the diagram in Fig. 1 and will yield the algorithm coefficients

$$P(x) = \zeta^{1/2}P_N(x)\left(\frac{x - \zeta^{-1}}{\zeta - \zeta^{-1}}\right)^2. \tag{6}$$

The constant coefficients $\zeta^{1/2}$ and $(\zeta - \zeta^{-1})^2$ are introduced to allow us to obtain the algorithm in a form directly comparable with the result presented in Ref. 1. An elementary computation shows that

$$\begin{aligned} P(x) = {} & \frac{\zeta^{-3/2} - \zeta^{3/2}x - \zeta^{-3/2}x^N + \zeta^{3/2}x^{N+1}}{(\zeta - \zeta^{-1})^2} \\ & + \sum_{r=1}^{N} \zeta^{-(r-1/2)}x^r. \end{aligned} \tag{7}$$

From this expression, and using $(\zeta - \zeta^{-1})^2 = -4\sin^2(2\pi/N)$, we can then write the algorithm straightforwardly as

$$\phi = \arctan\left\{\frac{\frac{1}{4}(I_0 + I_1 - I_N - I_{N+1})\dfrac{\sin(3\pi/N)}{\sin^2(2\pi/N)} - \displaystyle\sum_{r=1}^{N} I_r \sin[2\pi(r - \tfrac{1}{2})/N]}{\frac{1}{4}(-I_0 + I_1 + I_N - I_{N+1})\dfrac{\cos(3\pi/N)}{\sin^2(2\pi/N)} + \displaystyle\sum_{r=1}^{N} I_r \cos[2\pi(r - \tfrac{1}{2})/N]}\right\}. \tag{8}$$

We consider as an example, the problem addressed by Hibino *et al.*[1] in their paper about the generic algorithm that is insensitive to a quadratic phase shift and with harmonics eliminated up to the order j. In this case the root corresponding to $m = -1$ has to be triple. This is described by the characteristic diagram shown in Fig. 1. To obtain this algorithm in the exact form given in Eq. (56) of Ref. 1, apart from the sign of ϕ, we have to change the samples indexing scheme so that it starts from 1 instead of 0, introduce $-\pi$ into the trigonometric lines of the right-hand term of the numerator and denominator, and use the relation $N = j + 2$. Strictly speaking, the miscalibration (linear or quadratic) is eliminated only for the first harmonic ($m = -1$).

3. RESIDUAL ERRORS

The result expressed as Eq. (4) allows us to easily evaluate any residual error on the measured phase. We consider as an example the residuals related to harmonics 1 and -1; that is, we assume that the signal is sinusoidal. Without loss of generality, we also assume that $\alpha_1 = \alpha_{-1} = 1$. Also, we restrict our evaluation to a quadratic miscalibration; i.e., we assume $\epsilon_r = 0$ for $r \geqslant 3$. So the sum $S(\phi)$ is written as

$$\begin{aligned} S(\phi) &= S_1(\phi) + S_{-1}(\phi) \\ &= \exp(i\phi)P(\zeta)\left\{1 + i\delta\epsilon_1 \frac{\mathbf{D}P(\zeta)}{P(\zeta)} + i\delta\epsilon_2 \frac{\mathbf{D}^2P(\zeta)}{P(\zeta)}\right. \\ &\quad + \exp(-2i\phi)\left[-i\delta\epsilon_1 \frac{\mathbf{D}P(\zeta^{-1})}{P(\zeta)}\right. \\ &\quad \left.\left. - i\delta\epsilon_2 \frac{\mathbf{D}^2P(\zeta^{-1})}{P(\zeta)}\right]\right\}, \end{aligned} \tag{9}$$

where it is considered that $P(\zeta^{-1}) = 0$. Thus the measured phase ϕ^* is written as

$$\phi^* = \phi + \arg[P(\zeta)] + \Delta\phi, \tag{10}$$

with

$$\begin{aligned} \Delta\phi = \delta&\left\{\epsilon_1 \Re\left[\frac{\mathbf{D}P(\zeta)}{P(\zeta)}\right] + \epsilon_2 \Re\left[\frac{\mathbf{D}^2P(\zeta)}{P(\zeta)}\right]\right. \\ &\left. + \left[\left|\epsilon_1 \frac{\mathbf{D}P(\zeta^{-1})}{P(\zeta)} + \epsilon_2 \frac{\mathbf{D}^2P(\zeta^{-1})}{P(\zeta)}\right| \sin(2\phi + \psi)\right]\right\}, \end{aligned} \tag{11}$$

where $\Re$ is the real part operator and ψ is some unessential phase lag.

If the terms linear in ϵ_1 or ϵ_2 cancel, or if other harmonics are considered, the computation can be extended in an obvious way, starting from Eq. (4).

4. SPATIALLY NONUNIFORM PHASE SHIFT

It can be seen from Eq. (11) that, besides the usual error at twice the signal frequency, there is a static error governed by terms of the form $\mathbf{D}^r P(\zeta)$. Until the publication of Ref. 1, little attention seems to have been payed to this static term. In that paper the notion of spatial nonuniformity of the miscalibration is pointed out, which involves a variation of ϵ_1 or ϵ_2 over the field of view. Equation (11) shows the way to design algorithms that eliminate this static error. However, the problem happens to be quite different when one deals with a nonuniform linear or with a nonuniform quadratic miscalibration.

Before addressing this point, we give some identities that will turn out to be useful in the sequel. All of these results can be demonstrated very easily. First, if $P_1, P_2 \ldots P_n$ are polynomials, then

$$\frac{\mathbf{D}(P_1P_2\ldots P_n)}{P_1P_2\ldots P_n} = \sum_k \frac{\mathbf{D}P_k}{P_k}, \tag{12}$$

$$\frac{\mathbf{D}^2(P_1P_2\ldots P_n)}{P_1P_2\ldots P_n} = \sum_k \frac{\mathbf{D}^2P_k}{P_k} + 2\prod_{k<l} \frac{\mathbf{D}P_k}{P_k}\frac{\mathbf{D}P_l}{P_l}. \tag{13}$$

Other useful results are

$$P(x) = x^p \Rightarrow \frac{\mathbf{D}^nP(x)}{P(x)} = p^n, \tag{14}$$

$$P(x) = x - \xi \Rightarrow \forall n, \quad \frac{\mathbf{D}^nP(x)}{P(x)} = \frac{x}{x - \xi}. \tag{15}$$

Finally, if $\xi = \zeta \exp(i\psi)$, then

$$\frac{\zeta}{\zeta - \xi} = \frac{1}{1 - \exp(i\psi)} = \frac{1}{2}[1 + i\,\mathrm{cotan}(\psi/2)]. \tag{16}$$

A. Nonuniform Linear Miscalibration

A nonuniform linear miscalibration ($\epsilon_1 \neq 0$, $\epsilon_2 = 0$) may correspond, for example, to a slight parasitic tilt of the moving mirror as it moves. In this case the condition we must verify to eliminate the static error is

$$\Re\left[\frac{\mathbf{D}P(\zeta)}{P(\zeta)}\right] = 0. \tag{17}$$

There is a simple way to achieve this without any supplementary sample, merely by a change of the reference phase. Simply consider the polynomial $Q(x) = x^{-p}P(x) = \sum_{k=0}^{M-1} c_k x^{k-p}$ (actually the term polynomial may be improper as some powers of x are negative if $p > 0$). The number of coefficients is the same as in $P(x)$, so the corresponding algorithm requires the same number of samples. From Eq. (2), it can be seen that this corresponds simply to changing δ_k to

$$\begin{aligned} \delta_{k-p} = (k - p)\delta(1 &+ \epsilon_1 + (k - p)\epsilon_2 \\ &+ (k - p)^2\epsilon_3 + \ldots). \end{aligned} \tag{18}$$

This kind of phase-shift renumbering has already been introduced by some authors.[1,3] It consists of considering that the reference position of the phase shifter, with respect to which the algorithm will calculate the phase, is the pth rather than the first.

From Eqs. (14) and (12), it is easy to obtain

$$\frac{\mathbf{D}Q(\zeta)}{Q(\zeta)} = \frac{\mathbf{D}P(\zeta)}{P(\zeta)} - p, \tag{19}$$

so the condition stated in Eq. (17) is verified for $Q(x)$ when

$$p = \Re\left[\frac{\mathbf{D}P(\zeta)}{P(\zeta)}\right]. \tag{20}$$

Most often, all the roots of the characteristic polynomial $P(x)$ of an M-sample algorithm lie on the complex unit circle; that is

$$P(x) = \alpha \prod_{r=1}^{M-1} (x - \xi_r), \tag{21}$$

with $\xi_r = \zeta \exp(i\psi_r)$. From Eqs. (12), (15), and (16), we see that, in this case

$$\Re\left[\frac{\mathbf{D}P(\zeta)}{P(\zeta)}\right] = \frac{M-1}{2}. \tag{22}$$

So the general solution is to take $p = (M-1)/2$, that is, to choose as the reference phase-shifter position the one in the middle of the shift range.

As an example, we take the seven-sample algorithm proposed by de Groot[3] ($\delta = \pi/2$, $\zeta = i$):

$$\phi = \arctan\left[\frac{I_0 - I_6 - 7(I_2 - I_4)}{4(I_1 + I_5) - 8I_3}\right]. \tag{23}$$

Here $M = 7$, so $p = 3$, and there is no static error proportional to ϵ_1 with the phase shifts δ_{k-3}, as indicated in Ref. 3.

However, all this discussion shows that what is involved is the phase-reference definition, which is basically arbitrary.

B. Nonuniform Quadratic Miscalibration

In the case of a nonuniform quadratic miscalibration, it can be seen from Eq. (11) that both of the following conditions must be satisfied:

$$\Re\left[\frac{\mathbf{D}P(\zeta)}{P(\zeta)}\right] = 0, \tag{24}$$

$$\Re\left[\frac{\mathbf{D}^2P(\zeta)}{P(\zeta)}\right] = 0. \tag{25}$$

As shown in the preceding subsection, the first condition is satisfied with the phase-shift renumbering $k \to k - p$, with $p = (M-1)/2$, where M is the total number of samples.

Let us start from an algorithm described by the characteristic polynomial $P(x)$, whose roots are chosen on the unit complex circle to ensure some required properties about intensity to miscalibration, elimination of harmonics, etc. We now consider the polynomial

$$Q(x) = [P(x)x^p](x - \xi) = R(x)(x - \xi) \tag{26}$$

corresponding to an algorithm with one more sample. We designate M this final number of samples. The problem now is to choose ξ so that Eq. (25) is satisfied. However, ξ is a complex number and so depends on two parameters, and there is only one equation, so we will impose one more condition on ξ, namely, that it lie on the complex unit circle; that is,

$$\xi = \zeta \exp(i\psi). \tag{27}$$

Using Eqs. (12), (16), and (26), we can rewrite the first condition described by Eq. (24) as

$$\Re\left[\frac{\mathbf{D}Q(\zeta)}{Q(\zeta)}\right] = \Re\left[\frac{\mathbf{D}R(\zeta)}{R(\zeta)}\right] + \frac{1}{2} = 0. \tag{28}$$

Using Eq. (13), we obtain

$$\frac{\mathbf{D}^2Q(\zeta)}{Q(\zeta)} = \frac{\mathbf{D}^2R(\zeta)}{R(\zeta)} + 2\frac{\mathbf{D}R(\zeta)}{R(\zeta)}\frac{1}{2}[1 + i\ \mathrm{cotan}(\psi/2)] + \frac{1}{2}[1 + i\ \mathrm{cotan}(\psi/2)], \tag{29}$$

so that, taking into account Eq. (28), the second condition described by Eq. (25) can be written as

$$\Re\left[\frac{\mathbf{D}^2Q(\zeta)}{Q(\zeta)}\right] = \Re\left[\frac{\mathbf{D}^2R(\zeta)}{R(\zeta)}\right] + 2\left[-\frac{1}{2}\right]\frac{1}{2} - 2\Im\left[\frac{\mathbf{D}R(\zeta)}{R(\zeta)}\right]\frac{\mathrm{cotan}(\psi/2)}{2} + \frac{1}{2} = 0, \tag{30}$$

which gives

$$\tan(\psi/2) = \frac{\Im[\mathbf{D}R(\zeta)/R(\zeta)]}{\Re[\mathbf{D}^2R(\zeta)/R(\zeta)]}. \tag{31}$$

So if we denote

$$A = \Re\left[\frac{\mathbf{D}^2R(\zeta)}{R(\zeta)}\right] + i\Im\left[\frac{\mathbf{D}R(\zeta)}{R(\zeta)}\right], \tag{32}$$

the final solution is given by

$$\xi = \zeta\frac{A^2}{|A|^2}. \tag{33}$$

As an example, we follow this procedure to investigate the case, considered by Hibino *et al.*[1] of an algorithm that compensates for a quadratic and spatially nonuniform phase shift when the signal is sinusoidal, with a phase shift $\delta = \pi/3$. Following the results of Section 2, we start with the polynomial $P(x) = (x-1)(x-\zeta^{-1})^3$, with $\zeta = \exp(i\pi/3)$. The corresponding number of samples is 5, so there will be $M = 6$ samples in the final algorithm, and so $p = 5/2$. It is easy to find

$$\frac{\mathbf{D}P(\zeta)}{P(\zeta)} = 2 - i\sqrt{3},$$

$$\frac{\mathbf{D}^2P(\zeta)}{P(\zeta)} = 3 - 4i\sqrt{3},$$

$$\frac{\mathbf{D}R(\zeta)}{R(\zeta)} = \frac{\mathbf{D}P(\zeta)}{P(\zeta)} - p = -\frac{1}{2} - i\sqrt{3},$$

$$\frac{\mathbf{D}^2R(\zeta)}{R(\zeta)} = \frac{\mathbf{D}^2P(\zeta)}{P(\zeta)} - 2p\frac{\mathbf{D}P(\zeta)}{P(\zeta)} + p^2 = -\frac{3}{4} + i\sqrt{3}, \tag{34}$$

which yields $A = -3/4 - i\sqrt{3}$ and $\xi = -(37 + 5i\sqrt{3})/38$. The final polynomial (without the term x^{-p}, which corresponds only to a phase-shift renumbering) can be written as

$$\frac{Q(x)}{x^{-5/2}} = (1 - 5i\sqrt{3})(x-1)(x-\zeta^{-1})^3 \times \left(x + \frac{37}{38} + \frac{5i\sqrt{3}}{38}\right), \tag{35}$$

where the multiplying factor $1 - 5i\sqrt{3}$ is there just to provide a Hermitian symmetry to the coefficients, as shown in the algorithm arctangent form

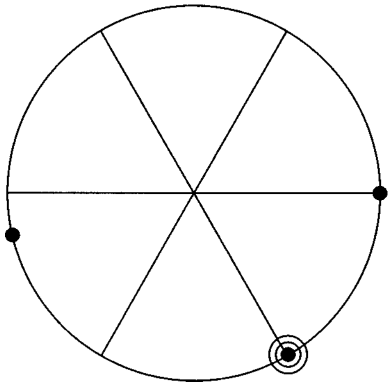

Fig. 2. Characteristic diagram of the six-sample algorithm described by Eq. (36).

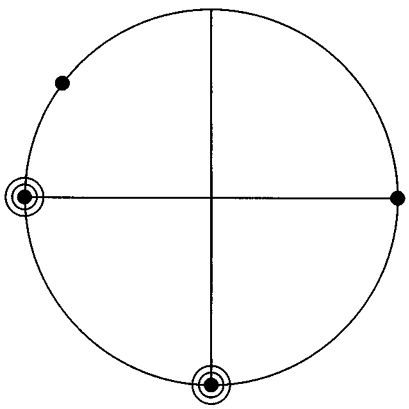

Fig. 3. Characteristic diagram of the nine-sample algorithm described by Eq. (42).

$$\phi = \arctan \times \left\{ \frac{\sqrt{3}[-5(I_0 - I_5) + 6(I_1 - I_4) + 17(I_2 - I_3)]}{I_0 + I_5 - 26(I_1 + I_4) + 25(I_2 + I_3)} \right\}, \tag{36}$$

which is (except for the sign) the solution obtained by Hibino *et al.*[1] Note that the phase shifts α_r that these authors consider in their paper are expressed as a function of $r - 7/2$, but their numbering scheme for r starts from 1, so they are equal to our phase shifts δ_{k-p} with $p = 5/2$. The characteristic diagram of this algorithm is shown in Fig. 2.

However, a drawback of this algorithm must be immediately emphasized: It has poor efficiency in terms of signal strength. In Ref. 4 the parameter η that governs the signal-to-noise ratio (SNR) on the sum $S(\phi)$ was defined. This parameter varies between 0 (SNR = 0) and 1, the optimum value corresponding to the N-bucket algorithm where all the samples have equal weight. It is shown in Ref. 4 that this parameter is seldom below 0.8 for almost every published algorithm, but it is equal to 0.495 for the one described by Eq. (36).

From Eq. (4), and using the same type of computation as the one used in Section 3, we can easily evaluate the residual phase errors. The static one is

$$-\delta^2 \epsilon_1 \epsilon_2 \Im\left[\frac{\mathbf{D}^3 Q(\zeta)}{Q(\zeta)}\right] = -\delta^2 \epsilon_1 \epsilon_2 \Im\left(\frac{-265 i \sqrt{3}}{96}\right), \tag{37}$$

and the error at twice the signal frequency is the imaginary part of

$$\left[-\frac{\delta^2 {\epsilon_2}^2}{2} \frac{\mathbf{D}^4 Q(\zeta^{-1})}{Q(\zeta)} - \delta^2 \epsilon_1 \epsilon_2 \frac{\mathbf{D}^3 Q(\zeta^{-1})}{Q(\zeta)}\right] \exp(-2i\phi) = -\delta^2 \epsilon_2 \left(\epsilon_2 - \frac{7 i \sqrt{3}}{12} \epsilon_1\right) \exp(-2i\phi), \tag{38}$$

and so the phase error is written as

$$\delta\phi = \frac{\pi^2 \epsilon_2}{9} \left[\frac{265\sqrt{3}}{96} \epsilon_1 + \epsilon_2 \sin(2\phi) + \frac{7\sqrt{3}}{12} \epsilon_1 \cos(2\phi)\right], \tag{39}$$

which has been checked by a numerical simulation and is almost the same expression as the one present in Ref. 1 (where ϵ_2 has a different definition).

The second example considered by Hibino *et al.*[1] deals with the elimination of the nonuniform quadratic miscalibration for the second harmonic, with $\delta = \pi/2$, $\zeta = i$. Here the basic polynomial to start from is $P(x) = (x - 1)(x + i)^3(x + 1)^3$. The final number of samples will be 9, so $p = 4$. A straightforward calculation gives

$$\begin{aligned}
\frac{\mathbf{D}P(\zeta)}{P(\zeta)} &= \frac{7}{2} + i, \\
\frac{\mathbf{D}^2 P(\zeta)}{P(\zeta)} &= 14 + 7i, \\
\frac{\mathbf{D}R(\zeta)}{R(\zeta)} &= \frac{\mathbf{D}P(\zeta)}{P(\zeta)} - p = -\frac{1}{2} + i, \\
\frac{\mathbf{D}^2 R(\zeta)}{R(\zeta)} &= \frac{\mathbf{D}^2 P(\zeta)}{P(\zeta)} - 2p \frac{\mathbf{D}P(\zeta)}{P(\zeta)} + p^2 = 2 - i,
\end{aligned} \tag{40}$$

which gives $A = 2 + i$ and $\xi = -4/5 + 3i/5$. The final polynomial can be written as

$$\frac{Q(x)}{x^p} = \left(-1 + \frac{i}{2}\right)(x - 1)(x + i)^3(x + 1)^3 \times \left(x + \frac{4}{5} - \frac{3i}{5}\right), \tag{41}$$

which gives the same algorithm as in Ref. 1:

$$\phi = -\arctan\left[\frac{(1/2)(I_0 - I_8) - (I_1 - I_7) - 7(I_2 - I_6) - 9(I_3 - I_5)}{-(I_0 + I_8) - 4(I_1 + I_7) - 4(I_2 + I_6) + 4(I_3 + I_5) + 10 I_4}\right]. \tag{42}$$

The corresponding diagram is plotted in Fig. 3. The parameter η indicating its efficiency in terms of SNR is 0.497.

C. Discussion

From the preceding subsections, it appears that perhaps the spatial miscalibration nonuniformity is not a problem to address from the algorithmic point of view.

Probably the main concern in algorithm design is to eliminate as much as possible residual oscillations in the detected phase. With regard to the nonuniform static error, a possible approach may be to perform an appropriate calibration of the phase detection setup on a perfect flat. The important point is that this static error does not depend on ϕ.

A striking point is that the existence of a nonuniform nonlinear miscalibration means that the mirror surface becomes distorted as it moves. This shows that the definition of a phase reference surface is not unequivocal. This could explain why a purely algorithmic approach may not be appropriate in this case.

5. COEFFICIENT SYMMETRY

We show in this section that the coefficients $\{c_k\}$ of any polynomial

$$P(x) = \sum_{k=0}^{M-1} c_k x^k = c_{M-1} \prod_{r=1}^{M-1} (x - \xi_r)$$

whose roots $\{\xi_r\}$ are located on the complex unit circle can be given the following Hermitian symmetry:

$$c_k{}^* = c_{M-k-1}, \tag{43}$$

where the asterisk stands for complex conjugation. Indeed, owing to

$$\begin{aligned}(x^*)^{M-1} P\left(\frac{1}{x^*}\right) &= \sum_{k=0}^{M-1} c_k (x^*)^{M-k-1} \\ &= \sum_{k=0}^{M-1} c_{M-k-1} (x^*)^k,\end{aligned} \tag{44}$$

the symmetry described by Eq. (43) is equivalent to

$$(x^*)^{M-1} P\left(\frac{1}{x^*}\right) = [P(x)]^*. \tag{45}$$

Using $\xi_r \xi_r{}^* = 1$, we can write

$$\begin{aligned}(x^*)^{M-1} P\left(\frac{1}{x^*}\right) &= c_{M-1} \prod_{r=1}^{M-1} (1 - x^* \xi_r) \\ &= (-1)^{M-1} \frac{c_{M-1}}{c_{M-1}^*} \left(\prod_{r=1}^{M-1} \xi_r \right) \\ &\quad \times \left[c_{M-1}^* \prod_{r=1}^{M-1} (x^* - \xi_r{}^*) \right] \\ &= (-1)^{M-1} \frac{c_{M-1}}{c_{M-1}^*} \left(\prod_{r=1}^{M-1} \xi_r \right) [P(x)]^*.\end{aligned} \tag{46}$$

The condition described by Eq. (45) is verified when the coefficient c_{M-1} is chosen so that

$$(-1)^{M-1} \frac{c_{M-1}}{c_{M-1}^*} \prod_{r=1}^{M-1} \xi_r = 1. \tag{47}$$

With ψ denoting the argument of c_{M-1}, Eq. (47) is equivalent to

$$\exp(-2i\psi) = (-1)^{M-1} \prod_{r=1}^{M-1} \xi_r. \tag{48}$$

Taking into account the fact that $1/[1 + \exp(-2i\psi)] = (1 + i \tan \psi)/2$ has its argument equal to ψ, we find that the polynomial coefficients have a Hermitian symmetry when

$$c_{M-1} = \frac{a}{1 + (-1)^{M-1} \prod_{r=1}^{M-1} \xi_r}, \tag{49}$$

where a is any real number. This technique was used to calculate the leading coefficients in Eqs. (35) and (41).

So this Hermitian symmetry of the polynomial coefficients cannot be a necessary condition for any specific property of the related algorithm. However, this symmetry is useful for its computer implementation, as it reduces the required number of addition and multiplication operations. Indeed, this symmetry allows us to group intensity terms in pairs, as in Eqs. (36) and (42).

6. CONCLUSION

In this comment we demonstrate that the characteristic polynomial theory can be extended to address the problem of nonuniform and nonlinear phase-shift miscalibration. It is shown that the rules for designing algorithms that are insensitive to linear and nonlinear miscalibration are very similar and involve introducing multiple roots of the characteristic polynomial.

A systematic procedure for designing algorithms that eliminates errors due to the possible miscalibration nonuniformity is presented. In the case of a nonuniform linear miscalibration, it is shown that a mere renumbering of the phase shifts starting from the middle of the phase-shift range makes the static phase error disappear. In the case of a nonuniform quadratic miscalibration, one extra sample is necessary, but the algorithms obtained have a poor efficiency in terms of their SNR. It is suggested that this problem may be better dealt with through recording the static phase error related to this nonuniformity by use of reference flat, so that this error is numerically subtracted from measurements.

Also, it is shown in this paper that the characteristic polynomial theory allows us to give very quickly an analytical expression for the residual phase errors. Finally, it is demonstrated how the coefficients of almost any algorithm can be given a Hermitian symmetry that will allow an efficient computer implementation.

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REFERENCES

1. K. Hibino, B. Oreb, D. Farrant, and K. Larkin, "Phase-shifting algorithms for nonlinear and spatially nonuniform phase shifts," J. Opt. Soc. Am. A **14**, 918–930 (1997).
2. Y. Surrel, "Design of algorithms for phase measurements by the use of phase stepping," Appl. Opt. **35**, 51–60 (1996).
3. P. de Groot, "Derivation of algorithms for phase-shifting interferometry using the concept of data-sampling window," Appl. Opt. **34**, 4723–4730 (1995).
4. Y. Surrel, "Additive noise effect in digital phase detection," Appl. Opt. **36**, 271–276 (1997).