Phase-shift extraction and wave-front reconstruction in phase-shifting interferometry with arbitrary phase steps

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Received May 8, 2003

A new approach to reconstructing the object wave front in phase-shifting interferometry with arbitrary unknown phase steps is proposed. With this method the actual phase steps are first determined from measured intensities with an algorithm based on the statistic property of the object phase distribution in the recording plane. Then the original object field is calculated digitally with a derived formula. This method is simple, accurate, and capable of retrieving the original object field, including its amplitude and phase distributions simultaneously, with arbitrary and unequal phase steps in a three- or four-frame method. The effectiveness and correctness of this approach are verified by a series of computer simulations for both smooth and diffusing surfaces. © 2003 Optical Society of America

OCIS codes: 090.2880, 100.3010, 100.2000, 120.5050.

The standard phase-shifting interferometry[1–5] (PSI) requires a special and constant phase shift, $2\pi/N$, where N is an integer equal to or larger than 3. However, this requirement is often difficult to meet exactly due to many practical factors, so special efforts must be made to calibrate the phase shifter.[6,7] To eliminate this inconvenience, a generalized method dealing with arbitrary phase shifts was developed by Greivenkamp,[8] where the amount of phase shifts must still be known precisely but does not necessarily have special values. Stoilov and Dragastinov[9] suggested a five-frame approach in which the phase shifts may be arbitrary but still equal. On the other hand, some algorithms of phase retrieval with unknown phase shifts have also been developed.[10–13] Usually these methods have substantial computation loads, and some of them need as many as 15 interferograms[13] or an additional optical device.[12] Recently, an approach to calculating the unknown phase shifts by use of measured object and reference wave intensities was suggested,[14] but it can be used only for pure phase objects, and the approximate values of the phase steps are still required.

In this Letter we introduce a method that can be used to calculate the object field, including both its amplitude and its phase distribution, from three or four interferograms in PSI with arbitrary unknown phase shifts. In this method the actual phase steps are first extracted from the measured intensity based on the statistic property of the object phase distribution. Then the object field can be calculated analytically by the formulas that we derived. It seems to be, to our knowledge, the simplest method of this kind. We first discuss its principles, then give its verification by computer simulations.

Assuming that an object is a certain distance from the recording plane, P_H, the complex amplitude distribution of the object wave in plane P_H is $O(x,y) = A_0(x,y)\exp[i\varphi_0(x,y)]$, and an on-axis reference wave in this plane at the jth exposure is $R_j(x,y) = A_r \exp(i\delta_j)$, we can write the intensity distribution of the jth interferogram as

$$I_j(x,y) = A_0^2(x,y) + A_r^2 + 2A_0(x,y)A_r \cos[\varphi_0(x,y) - \delta_j]. \tag{1}$$

Consequently, we have

$$\begin{aligned} I_{j+1} - I_j &= 2A_0(x,y)A_r\{\cos[\varphi_0(x,y) - \delta_{j+1}] \\ &\quad - \cos[\varphi_0(x,y) - \delta_j]\} \\ &= 4A_0(x,y)A_r \sin[\varphi_0(x,y) \\ &\quad - (\delta_j + \delta_{j+1})/2]\sin(\alpha_j/2), \end{aligned} \tag{2}$$

where $\alpha_j = \delta_{j+1} - \delta_j$ is the phase step between the jth and $(j+1)$th frames, and we assume that $0 < \alpha_j < \pi$ to avoid possible ambiguity in the following calculation. Since the intensities $I_0 = A_0^2$ and $I_r = A_r^2$ can be measured, we can rewrite Eq. (2) as

$$\frac{I_{j+1} - I_j}{4\sqrt{I_0 I_r}} = \sin\left[\varphi_0(x,y) - \frac{\delta_j + \delta_{j+1}}{2}\right]\sin\frac{\alpha_j}{2}, \tag{3}$$

where the coordinate dependence of I_j and I_0 is omitted for conciseness. Because $\varphi_0(x, y)$ is generally a spatially random distribution because of Fresnel diffraction, we can reasonably expect

$$\langle|\sin[\varphi_0(x,y) - (\delta_j + \delta_{j+1})/2]|\rangle = \langle|\sin\varphi_0(x,y)|\rangle = 2/\pi, \tag{4}$$

regardless of the values of the constants δ_j and δ_{j+1}, where $\langle\ \rangle$ mean average over the whole frame. With this result we can introduce a parameter

$$p_j = \left\langle\left|\frac{I_{j+1} - I_j}{4\sqrt{I_0 I_r}}\right|\right\rangle = \frac{2}{\pi}\sin\frac{\alpha_j}{2}, \tag{5}$$

and then we have

$$\alpha_j = 2 \sin^{-1}(\pi p_j/2). \tag{6}$$

Therefore all the phase steps α_j ($j = 1, 2, \ldots, N - 1$) can be calculated from p_j, and all the δ_j ($j = 1, 2, \ldots, N$) can be further determined from α_j easily.

When all the δ_j are known, an analytic expression of $O(x, y)$ can be derived from Eq. (1). For example, in the three-frame case we have

$$I_1 - I_3 = 4A_0A_r \sin[(\delta_3 - \delta_1)/2] \\ - \sin \varphi_0 \cos[(\delta_1 + \delta_3)/2]\}. \tag{7}$$

$$I_1 - I_3 = 4A_0A_r \sin[(\delta_3 - \delta_1)/2] \\ \times \{\cos \varphi_0 \sin[(\delta_1 + \delta_3)/2] \\ - \sin \varphi_0 \cos[(\delta_1 + \delta_3)/2]\}. \tag{8}$$

Solving these equations for $\cos \varphi_0$ and $\sin \varphi_0$ yields

$$O(x, y) = A_0(\cos \varphi_0 + i \sin \varphi_0) \\ = \frac{1}{4A_r \sin[(\delta_3 - \delta_2)/2]} \left\{ \frac{\exp[i(\delta_1 + \delta_2)/2]}{\sin[(\delta_3 - \delta_1)/2]} \right. \\ \left. \times (I_1 - I_3) - \frac{\exp[i(\delta_1 + \delta_3)/2]}{\sin[(\delta_2 - \delta_1)/2]} (I_1 - I_2) \right\}. \tag{9}$$

Similarly, in the four-frame case we have

$$O(x, y) = \frac{1}{4A_r \sin[(\delta_2 + \delta_4 - \delta_1 - \delta_3)/2]} \\ \times \left\{ \frac{\exp[i(\delta_1 + \delta_3)/2]}{\sin[(\delta_4 - \delta_2)/2]} (I_2 - I_4) \right. \\ \left. - \frac{\exp[i(\delta_2 + \delta_4)/2]}{\sin[(\delta_3 - \delta_1)/2]} (I_1 - I_3) \right\}. \tag{10}$$

We may easily verify that Eqs. (9) and (10) will be simplified as standard formulas when $\alpha_j = \pi/2$.

After determining all the δ_j, we can calculate the object field $O(x, y)$ in plane P_H by Eq. (9) or (10); thus both $A_0(x, y)$ and $\varphi_0(x, y)$ can be obtained. To further improve the accuracy of this method, we compute

$$C_j = \langle |\sin[\varphi_0(x, y) - (\delta_j + \delta_{j+1})/2]| \rangle, \tag{11}$$

with retrieved $\varphi_0(x, y)$ and δ_j, and find a new value

$$\alpha_j = 2 \sin^{-1}(p_j/C_j). \tag{12}$$

In turn, the new set of α_j yields a new set of δ_j and a new $\varphi_0(x, y)$, which gives rise to the third set of α_j. This iteration process may go on until the difference of α_j between two adjacent steps is less than a small tolerance.

When the actual phase shifts are extracted and the object field $O(x, y)$ in plane P_H is found this way, the optical field in the actual object plane, $O'(x', y')$, can be computed with inverse Fresnel diffraction.[4]

Since the exact expression of an actual object surface is hard to know due to many practical factors, we carried out a series of computer simulations to verify the effectiveness of the proposed method. In these simulations spherical surfaces with different radii R were used as test objects. An axial plane wave of $\lambda = 532$ nm illuminates the surface and is then reflected back onto a CCD plane of 512×512 pixels with 15-μm pitch and $z = 216.5$ mm (all the parameters were chosen according to sampling theory). With paraxial approximation we can write the phase distribution in the plane P_0, which is tangential to the surface vertex, as $4\pi h(x, y)/\lambda$, where h is the depth of the surface determined by R. To account for the possible irradiance variation of a real object, we assume a Guassian intensity distribution of the object wave in plane P_0, gradually decreasing from 1 at the center to 1/4 at the edge. Both smooth and diffusing surfaces with a certain Gaussian fluctuation Δh added to h are considered. Two-dimensional Fresnel diffraction of the complex amplitude of plane P_0 leads to the object wave, $O(x, y)$, in recording plane P_H. Different interferograms are computer generated for different nonstandard phase shifts. Both the three- and four-frame cases with a large range of R, from 0.2 m to 20 m, are tested. Here we give only a few results of the four-frame case as examples for the sake of page limits.

First, we present the ability of our proposed algorithm to retrieve the unknown arbitrary phase steps. Table 1 gives an example with assumed parameters $R = 2$ m, $\delta_1 = 0$, $\alpha_1 = 1.3963$ (80°), $\alpha_2 = 1.4312$ (82°), and $\alpha_3 = 1.4137$ (81°), where the left and right halves correspond to the case without and with phase noise of the object surface (zero mean and $\lambda/12$ standard deviation of h). We can see the excellent performance of this algorithm for phase-step extraction. In both cases even the phase steps directly calculated from Eq. (6) are accurate; the maximum relative error is less than 3.6×10^{-3}. The first iteration decreases this error greatly to 5.7×10^{-4}. After three or four iterations, the difference between two sequential values continues to be 0 or 0.0001, and this holds true until all the phase steps reach their steady values after five or six iterations. This means that the maximum relative error is less than 7×10^{-5}. The tests with other parameters and the three-frame method show similar behavior.

Second, we show the ability of this algorithm to retrieve the object phase distribution in Fig. 1, where the left and right columns correspond to the two cases (without and with phase noise) mentioned above; all the phase shifts remain unchanged, but $R = 10$ m here for the clarity of phase display. In each column we can see (from the top to the bottom) the original object phase along the central line in Fig. 1(a) [wrapped, as is the case in Figs. 1(b) and 1(c)], the retrieved object phase from calculated phase shifts with our algorithm and Eq. (10) and inverse Fresnel diffraction

Table 1. Phase-Step Retrieval Results for a Smooth Surface and a Diffusing Surface

	α_1 (Smooth surface)	α_2	α_3	α_1 (Diffusing surface)	α_2	α_3
Assumed value	1.3963	1.4312	1.4137	1.3963	1.4312	1.4137
Extracted by Eq. (6)	1.3913	1.4319	1.4171	1.3948	1.4330	1.4125
First iteration	1.3957	1.4317	1.4134	1.3955	1.4319	1.4130
Second iteration	1.3960	1.4314	1.4135	1.3959	1.4315	1.4134
Third iteration	1.3962	1.4313	1.4137	1.3961	1.4313	1.4136
Fourth iteration	1.3962	1.4312	1.4137	1.3962	1.4312	1.4137
Fifth iteration	1.3963	1.4311	1.4138	1.3962	1.4312	1.4137
Sixth iteration				1.3963	1.4311	1.4138

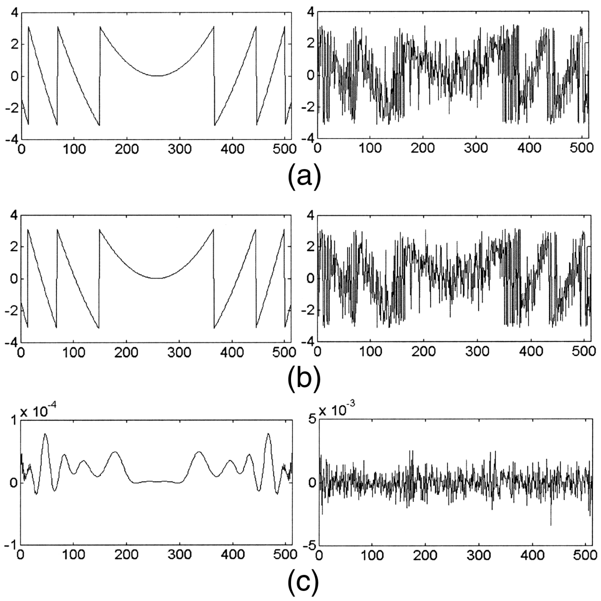

Fig. 1. Simulation results of object phase reconstruction with the four-frame method with retrieved nonstandard phase shifts. The left and right columns correspond to the smooth and the diffusing surface. (a) Original object phase. (b) Retrieved object phase. (c) Their difference.

in Fig. 1(b), and the difference between the two in Fig. 1(c). The maximum phase error is less than 1×10^{-4} rad for the smooth surface and 3×10^{-3} rad for the diffusing surface. Other simulations with $R =$ 0.2 m, 2 m, and the three-step method yield similar results. But for relatively small R the phase change in the object plane varies too rapidly to be displayed directly, and a virtual reference spherical surface may be digitally introduced to show the contour map.[15]

Finally, we want to mention two previous works by Kadono and Toyoka[16] and Kadono *et al.*[17] who proposed statistical interferometry based on a fully developed speckle field. In comparing their work with ours, we note that the basic idea of using the statistical property of optical fields is similar, but the calculation algorithms are totally different. The calculation here is much simpler and more direct, and it yields accurate results even for smooth object surfaces, whereas two diffusers had to be employed in the previous approaches.

In summary, we have proposed a new method for retrieving actual phase steps in PSI with arbitrary and unequal phase shifts and then analytically reconstructing the original object, including its amplitude and phase distributions. The effectiveness of this algorithm is verified satisfactorily by a series of computer simulations. This algorithm is simple compared with other methods of this kind and can be used for both smooth and diffusing object surfaces with high accuracy. Naturally, in experiments practical factors may introduce new errors, but the principles of this approach have been proved to be correct, and further investigation of its practical applications will be our next task.

This work was supported by the National Natural Science Foundation (grant 60177002) and the Ph.D. Training Foundation of the Education Ministry of China. L. Z. Cai's e-mail address is lzcai@sdu.edu.cn.

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