Advanced iterative algorithm for phase extraction of randomly phase-shifted interferograms

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Received March 1, 2004

An advanced random phase-shifting algorithm to extract phase distributions from randomly phase-shifted interferograms is proposed. The algorithm is based on a least-squares iterative procedure, but it copes with the limitation of the existing iterative algorithms by separating a frame-to-frame iteration from a pixel-to-pixel iteration. The algorithm provides stable convergence and accurate phase extraction with as few as three interferograms, even when the phase shifts are completely random. The algorithm is simple, fast, and fully automatic. A computer simulation is conducted to prove the concept.

OCIS codes: 120.5050, 120.3180, 100.2650, 100.2000.

Since the phase-shifting algorithm was introduced in the 1960s, numerous self-calibrating algorithms to cope with the phase-shift errors caused by imperfect phase-shifting mechanisms have been proposed. Among them, an overdeterministic approach that uses least-squares algorithms has been studied extensively for randomly shifted interferograms. Okada *et al.*[1] first proposed a least-squares-based iterative algorithm. Their research was a substantial extension of that previously proposed by Greivenkamp.[2] In the Okada algorithm, one solves the approximate linear equations iteratively to determine phase-shift amounts and phase distributions simultaneously. Similar methods were proposed by Lassahn *et al.*,[3] Han and Kim,[4] Kim *et al.*,[5] and Wei *et al.*[6]

For all the iterative methods cited above it was claimed that four frames should be sufficient for accurate extraction of phase information. As indicated by Larkin,[7] however, no published results of algorithms that used fewer than five frames can be found in the literature. In fact, it was found from an extensive study conducted by the present authors that the existing algorithms do not always provide accurate phase information unless a large number of frames are utilized (usually more than 15 frames). If a practically feasible number of frames (say, ≤ 5) is employed, the existing algorithms produce significant phase-shift errors unless the shift intervals are reasonably uniform and the initial estimates of shift amounts are within a few degrees of the actual shift amounts. These stringent requirements limit the applications of the existing iterative algorithms in practice.

In this Letter an improved iterative algorithm to cope with the above problem is proposed. The proposed algorithm makes the least squares converge accurately and rapidly, which substantially relaxes the required amount of input data; only three randomly shifted interferograms are sufficient for accurate extraction of phase information. The algorithm does not require accurate initial estimation of phase-shift amounts. The phase shifts can be completely random as long as at least three frames have different phase-shift amounts. The details of the proposed algorithm are described below.

Step 1. Pixel-by-pixel iteration to determine phase distribution: The intensity of an interferogram can be expressed as

$$I_{ij}{}^{t} = A_{ij} + B_{ij} \cos(\phi_j + \delta_i), \tag{1}$$

where the superscript t denotes the theoretical value, the subscript i denotes the ith phase-shifted image ($i = 1, 2, \ldots, M$), and j denotes the individual pixel locations in each image ($j = 1, 2, \ldots, N$). In Eq. (1), A_{ij} is the background or mean intensity, B_{ij} is the modulation amplitude, ϕ_j is the angular phase information, δ_i is the phase-shift amount of each of M frames ($M \geq 3$), and N is the total number of pixels in each frame.

As in the conventional phase-shifting algorithms, in step 1 it is assumed that the background intensity and the modulation amplitude do not have frame-to-frame variation; i.e., they are functions of pixels only. Under that assumption, A_{ij} and B_{ij} become single-order tensors, i.e., $A_{1j} = A_{2j} = \ldots = A_{Mj}$ and $B_{1j} = B_{2j} = \ldots = B_{Mj}$. Defining a new set of variables as $a_j = A_{ij}$, $b_j = B_{ij} \cos \phi_j$, and $c_j = -B_{ij} \sin \phi_j$, we can express Eq. (1) as

$$I_{ij}{}^{t} = a_j + b_j \cos \delta_i + c_j \sin \delta_i . \tag{2}$$

If δ_i is known, there are $(3N)$ unknowns and (MN) equations. The unknowns can be solved by use of the overdetermined least-squares method. An expression of the least-squares error S_j accumulated from all the images described by Eqs. (1) and (2) can be written as

$$\begin{aligned} S_j &= \sum_{i=1}^{M} (I_{ij}{}^{t} - I_{ij})^2 \\ &= \sum_{i=1}^{M} (a_j + b_j \cos \delta_i + c_j \sin \delta_i - I_{ij})^2, \end{aligned} \tag{3}$$

where I_{ij} is the experimentally measured intensity of the interferogram.

For the known δ_i, the least-squares criteria require that

$$\partial S_j / \partial a_j = 0 , \qquad \partial S_j / \partial b_j = 0 , \qquad \partial S_j / \partial c_j = 0 . \tag{4}$$

0146-9592/04/141671-03$15.00/0

Equations (4) yield

$$\{X_j\} = [A]^{-1}\{B_j\}, \tag{5}$$

where

$$[A] = \begin{bmatrix} M & \sum_{i=1}^{M} \cos \delta_i & \sum_{i=1}^{M} \sin \delta_i \\ \sum_{i=1}^{M} \cos \delta_i & \sum_{i=1}^{M} \cos^2 \delta_i & \sum_{i=1}^{M} \cos \delta_i \sin \delta_i \\ \sum_{i=1}^{M} \sin \delta_i & \sum_{i=1}^{M} \sin \delta_i \cos \delta_i & \sum_{i=1}^{M} \sin^2 \delta_i \end{bmatrix}, \tag{6}$$

$$\{X_j\} = \{a_j \quad b_j \quad c_j\}^T, \tag{7}$$

$$\{B_j\} = \left\{ \sum_{i=1}^{M} I_{ij} \quad \sum_{i=1}^{M} I_{ij} \cos \delta_i \quad \sum_{i=1}^{M} I_{ij} \sin \delta_i \right\}^T. \tag{8}$$

Equation (5) requires at least three different phase shifts δ_i to ensure that $[A]$ is nonsingular. From Eqs. (5)–(8) the unknowns a_j, b_j, and c_j can be solved and the phase ϕ_j can be determined from

$$\phi_j = \tan^{-1}(-c_j/b_j). \tag{9}$$

It is worth noting that $[A]^{-1}$ has to be assembled only once because its components depend only on δ_i. Therefore this step requires the same order of calculation time as that of the conventional phase-shifting algorithms.[8]

Step 2. Frame-by-frame iteration to determine phase shifts: In the second step it is assumed that the background intensity and the modulation amplitude do not have pixel-to-pixel variation, i.e., that they are functions of frames only. Under that assumption, A_{ij} and B_{ij} become single-order tensors; i.e., $A_{i1} = A_{i2} = \ldots = A_{iN}$ and $B_{i1} = B_{i2} = \ldots = B_{iN}$. Defining another set of variables for each frame as $a_i' = A_{ij}$, $b_i' = B_{ij} \cos \delta_i$, and $c_i' = -B_{ij} \sin \delta_i$, we can express intensity equation (1) as

$$I_{ij}{}^t = a_i' + b_i' \cos \phi_j + c_i' \sin \phi_j. \tag{10}$$

If ϕ_j is known (as obtained from step 1), there are $(3M)$ unknowns and (MN) equations. The unknowns can be solved again by use of the overdetermined least-squares method. An expression for the error S_i' accumulated from all the pixels in the ith image can be expressed as

$$S_i' = \sum_{j=1}^{N} (I_{ij}{}^t + I_{ij})^2 = \sum_{j=1}^{N} (a_i' + b_i' \cos \phi_j + c_i' \sin \phi_j - I_{ij})^2. \tag{11}$$

For the known ϕ_j, the least-squares criterion yields

$$\{X_i'\} = [A']^{-1}\{B_i'\}, \tag{12}$$

where

$$[A'] = \begin{bmatrix} N & \sum_{j=1}^{N} \cos \phi_j & \sum_{j=1}^{N} \sin \phi_j \\ \sum_{j=1}^{N} \cos \phi_j & \sum_{j=1}^{N} \cos^2 \phi_j & \sum_{j=1}^{N} \cos \phi_j \sin \phi_j \\ \sum_{j=1}^{N} \sin \phi_j & \sum_{j=1}^{N} \sin \phi_j \cos \phi_j & \sum_{j=1}^{N} \sin^2 \phi_j \end{bmatrix}, \tag{13}$$

$$\{X_i'\} = \{a_i' \quad b_i' \quad c_i'\}^T, \tag{14}$$

$$\{B_i'\} = \left\{ \sum_{j=1}^{N} I_{ij} \quad \sum_{j=1}^{N} I_{ij} \cos \phi_j \quad \sum_{j=1}^{N} I_{ij} \sin \phi_j \right\}^T. \tag{15}$$

The unknowns a_i', b_i', and c_i' can be solved from Eqs. (12)–(15). Then the amount of phase shift in each frame can be determined from

$$\delta_i = \tan^{-1}(-c_i'/b_i'). \tag{16}$$

Step 3. Convergence limit: The algorithm repeats steps 1 and 2 until the phase-shift values converge. The convergence criteria for relative phase-shift amounts can be expressed as

$$|(\delta_i{}^k - \delta_1{}^k) - (\delta_i{}^{k-1} - \delta_1{}^{k-1})| < \epsilon, \tag{17}$$

where k represents the number of iterations and ϵ is a predefined accuracy requirement, e.g., 10^{-4}. When the convergence criteria are satisfied, the correct phase distributions are determined from step 1.

A computer simulation was conducted to prove the concept. The base phase map used in the simulation is shown in Fig. 1, which represents a monotonically increasing phase map with random noise ranging from $-\pi/4$ to $\pi/4$. The white curve represents an

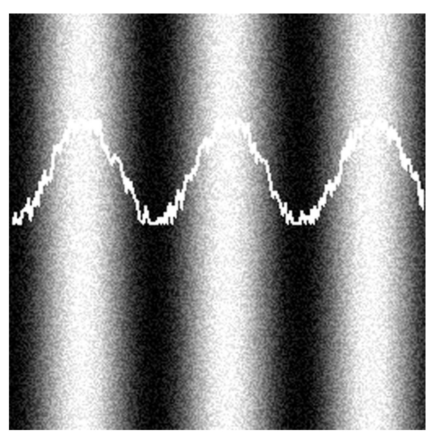

Fig. 1. Computer-generated phase map with random noise ranging from $-\pi/4$ to $\pi/4$.

Table 1. Comparison of Simulation Results of the Proposed Advanced Iterative Algorithm (AIA) and Okada's Algorithm for Four-Frame Phase Shifting[a]

	Case 1			Case 2			Case 3		
	Real	Okada	AIA	Real	Okada	AIA	Real	Okada	AIA
δ_1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
δ_2	4.3497	3.8823	4.3486	1.5977	1.8279	1.6037	5.3618	3.3496	5.3696
δ_3	1.9003	1.6869	1.8999	4.2415	5.2885	4.2479	0.4359	0.6033	0.4679
δ_4	2.1561	1.9755	2.1561	1.9825	2.0418	1.9882	0.5755	0.9066	0.6184
e^{ave}		0.2514	0.0007		0.6782	0.0023		1.4200	0.0143
n^a		96	9		206	11		113	22

[a] n is the number of iteration cycles to satisfy the convergence limit (10^{-4}).

Table 2. Simulation Results of the Proposed Advanced Iterative Algorithm (AIA) for Three-Frame Phase Shifting

	Case 1		Case 2		Case 3	
	Real	AIA	Real	AIA	Real	AIA
δ_1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
δ_2	2.1501	2.1493	2.9473	2.9408	2.6134	2.5871
δ_3	4.2646	4.2625	1.5221	1.5126	2.5766	2.5471
e^{ave}		0.0011		0.0047		0.0147
n^a		6		9		15

[a] n is the number of iteration cycles to satisfy the convergence limit (10^{-4}).

intensity distribution along a horizontal center line. A total of 100 sets of randomly phase-shifted phase maps (four frames in each set) was produced by use of the base phase map, and the phase shifts were calculated by the proposed algorithm as well as by Okada's algorithm. Although they could be chosen randomly, we used initial phase shifts of 0, $\pi/2$, π, and $3\pi/2$ in all calculations to compare the two algorithms directly. In addition, the first phase-shift amount was subtracted from each phase-shift amount to present the relative phase shift among the phase maps because the absolute phase shift was not meaningful.

Table 1 summarizes the results from three representative cases for which the average phase-shift error is defined as $e^{\mathrm{ave}} = (1/M - 1)\sum_{i=1}^{M-1} |(\delta_{i+1} - \delta_i) - ({\delta_{i+1}}^t - {\delta_i}^t)|$. The results clearly indicate that the proposed algorithm provides excellent results, whereas Okada's algorithm produces significant errors. The average phase-shift errors of the proposed algorithm vary from 0.0 to 0.0143, whereas those of Okada's algorithm vary significantly, from 0.0105 to 1.420. It is also important to note that the proposed algorithm provides much faster convergence. Although only three representative cases are shown here, the trend in accuracy and convergence prevailed in all 100 cases.

It is also important to note that Okada's algorithm requires at least four frames to ensure a proper iteration procedure, whereas the minimum number of frames required for the proposed algorithm is three. Table 2 shows simulation results from three-frame phase shifting.

The most important concept of the proposed advanced iterative algorithm (AIA) lies in step 2. The existing algorithms use ϕ_j, A_{Mj}, and B_{Mj} obtained from step 1 (a total of $3N$ values) in step 2. If the initial phase-shift estimates in step 1 deviate from the actual shift amounts, the values of ϕ_j, A_{Mj}, and B_{Mj} obtained from the first iteration can be significantly different from the actual values. Consequently, numerous sets of phase shifts in step 2 exist to satisfy the convergence criteria unless a large number of frames are used. This is precisely the reason why the existing iterative algorithms are unstable when only a few frames are employed.

The proposed algorithm does not use A_{Mj} and B_{Mj} in step 2. Instead, it introduces a new set of variables (A_{iN} and B_{iN}) in step 2, and they are determined simultaneously while δ_i is determined; that is, only ϕ_j and δ_i are updated after each iteration, whereas (A_{Mj} and B_{Mj}) and (A_{iN} and B_{iN}) are simply recalculated in each iteration. This procedure results in rapid but accurate convergence with only a few randomly phase-shifted interferograms.

In summary, we have proposed using an advanced random phase-shifting algorithm to determine phase-shift amounts and phase distributions simultaneously. The proposed algorithm is based on the least-squares iterative algorithm, but it provides extremely stable and correct convergence. The proposed algorithm has been proved effective for any three or more images with completely random phase shifts. Applications to various real problems are anticipated.

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