Phase shifting for nonsinusoidal waveforms with phase-shift errors

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In phase measurement systems that use phase-shifting techniques, phase errors that are due to nonsinusoidal waveforms can be minimized by applying synchronous phase-shifting algorithms with more than four samples. However, when the phase-shift calibration is inaccurate, these algorithms cannot eliminate the effects of nonsinusoidal characteristics. It is shown that, when a number of samples beyond one period of a waveform such as a fringe pattern are taken, phase errors that are due to the harmonic components of the waveform can be eliminated, even when there exists a constant error in the phase-shift interval. A general procedure for constructing phase-shifting algorithms that eliminate these errors is derived. It is shown that $2j + 3$ samples are necessary for the elimination of the effects of higher harmonic components up to the jth order. As examples, three algorithms are derived, in which the effects of harmonic components of low orders can be eliminated in the presence of a constant error in the phase-shift interval.

Key words: phase measurement, phase-shifting algorithm, phase-shift error, nonsinusoidal signals, interferometry.

1. INTRODUCTION

With phase shifting[1] the phase difference between an object wave front and a reference wave front is changed either stepwise or linearly, and the resulting irradiance distribution is stored at each step or bucket[2] in the frame memory of a computer. The object phase can be obtained from the arctangent of the ratio between two combinations of the observed irradiances, according to the phase-shifting algorithm used.

Inaccurate calibration of phase shifters is one of the most significant sources of error in high-precision optical interferometers.[2] Many phase-shifting algorithms have been developed that minimize absolute phase errors that are due to the phase-shift error.[3–8] Carre[3] developed a four-sample algorithm from which the object phase can be determined as long as the interval of phase shift remains constant. However, these algorithms were designed for sinusoidal waveforms.

In practical measurements observed waveforms such as fringes in an interferogram often become nonsinusoidal because of, for example, the nonlinearity of the detector or multiple-beam interference. Stetson and Brohinsky[9] showed that the n-sample synchronous phase-shifting algorithm eliminates the effect of harmonic components up to the $(n - 2)$th order. However, the synchronous phase-shifting algorithm is not insensitive to phase-shift errors.

Larkin and Oreb[10] derived an $(n + 1)$-sample symmetrical phase-shifting algorithm based on Fourier representation[11] and discussed the effects of nonsinusoidal waveforms and a constant phase-shift error. They also modified the symmetrical phase-shifting algorithm so that the phase errors could be minimized in the presence of a constant phase-shift error. However, the modified symmetrical algorithm was designed to eliminate the effect of a constant phase-shift error for sinusoidal fringes, so the algorithm still gives residual errors in calculated phase that are due to the higher harmonic components when a phase-shift error exists.

To the best of our knowledge, phase-shifting algorithms that eliminate the effects of higher harmonic components in the presence of a phase shift error have not been reported.

In this paper we discuss the deviations of the arctangent function of a phase-shifting algorithm that are due to the effects of harmonic components of the signal and a constant phase-shift error. It is shown that, by taking a number of samples beyond one period of the sampled waveform, one can simultaneously eliminate, to first order, phase errors that are due to these two factors. A general procedure for constructing phase-shifting algorithms with extended symmetry that eliminates these errors is derived in terms of linear simultaneous equations in the sampling amplitudes. It is shown that $2j + 3$ samples are necessary for the elimination of the effects of higher harmonic components up to the jth order in the presence of a constant phase-shift error, where the phase-shift interval is set to $2\pi/(j + 2)$ rad.

As examples, 5-sample, 7-sample, and 11-sample algorithms are derived. In the 7-sample and 11-sample algorithms the effects of higher harmonic components up to the second and fourth orders, respectively, are eliminated in the presence of a constant phase-shift error.

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2. DERIVATION OF PHASE-SHIFTING ALGORITHMS

A. Suppression of Harmonic Components

In order to construct the phase-shifting algorithm, we first consider the case in which the signal is nonsinusoidal but the measurement system has no phase-shift error. We will derive the conditions that the sampling amplitudes must satisfy to eliminate errors that are due to the effects of harmonic components of the signal. In Subsection 2.B below we will discuss the case of a constant phase-shift error.

Let $I(x, y, \alpha)$ be the signal irradiance (of the waveform whose phase is to be measured) at a designated point (x, y), where α is the phase-shift parameter. Since signal $I(x, y, \alpha)$ is a periodic function of parameter α with period 2π, it can be expanded into a Fourier series as

$$I(x, y, \alpha) = \sum_{k=0}^{\infty} s_k(x, y)\cos[k\alpha - \phi_k(x, y)], \tag{1}$$

where $s_k(x, y)$ and $\phi_k(x, y)$ are the amplitude and the phase of the kth-order harmonic component at point (x, y) and $\phi_0(x, y)$ is defined as zero for any point. The phase distribution $\phi_1(x, y)$ of the fundamental order is the object phase to be measured. To simplify the mathematical notation, we will henceforth confine the analysis to a single point, although it is applicable to all other points of interest. Therefore the (x, y) dependence of I, s_k, and ϕ_k will not be highlighted.

Let us consider an m-sample phase-shifting algorithm in which the reference phases are separated by $m - 1$ equal intervals of $2\pi/n$ rad, where n is a positive integer. When there is no phase-shift error, parameter α is always an exact multiple of $2\pi/n$ rad. Here we confine our discussion to an algorithm in which the denominator and the numerator of the arctangent function are given by a linear combination of the sampled irradiances.

A general expression for m-sample phase-shifting algorithms is

$$\phi = \arctan \frac{\sum_{i=1}^{m} b_i I_i}{\sum_{i=1}^{m} a_i I_i}, \tag{2}$$

where ϕ is the calculated phase at point (x, y), a_i and b_i are the sampling amplitudes of the ith sample, and I_i is the ith sampled irradiance defined by

$$I_i = I(\alpha_i), \tag{3}$$

where $\alpha_i = 2\pi(i - l)/n$ and l is an integer defining the initial value of the reference phase, which is introduced for convenience of notation. The choice of integer l affects the calculated phase only as a spatially uniform constant bias, which is usually irrelevant to the measurements. Here we define integer l as

$$l = \begin{cases} m/2 & \text{for even } m \\ (m+1)/2 & \text{for odd } m \end{cases}. \tag{4}$$

It can be seen from Eq. (2) that, if one is to obtain the correct object phase ϕ_1 from the calculated phase ϕ [see Eq. (1)], the denominator and the numerator of the right-hand side of Eq. (2) should give values that are proportional to the cosine and the sine of the object phase, respectively. For convenience, we choose amplitude s_1 as the proportionality constant. Then the sampling amplitudes a_i and b_i satisfy

$$\sum_{i=1}^{m} a_i I_i = s_1 \cos \phi_1, \tag{5}$$

$$\sum_{i=1}^{m} b_i I_i = s_1 \sin \phi_1. \tag{6}$$

Substituting the expression for I_i of Eqs. (1) and (3) into the left-hand side of Eq. (5) and comparing both sides of Eq. (5), we find that the sampling amplitudes a_i satisfy

$$\sum_{i=1}^{m} a_i \sin(k\alpha_i) = 0, \tag{7}$$

$$\sum_{i=1}^{m} a_i \cos(k\alpha_i) = \delta(k, 1), \tag{8}$$

where the integer k is the order of the harmonic component of the signal and takes all nonnegative integer values and $\delta(k, 1)$ is the Kronecker delta function. Similarly, we find from Eq. (6) that the sampling amplitudes b_i satisfy

$$\sum_{i=1}^{m} b_i \sin(k\alpha_i) = \delta(k, 1), \tag{9}$$

$$\sum_{i=1}^{m} b_i \cos(k\alpha_i) = 0. \tag{10}$$

It can be seen that Eqs. (7)–(10) are the necessary and sufficient conditions for the sampling amplitudes to satisfy Eqs. (5) and (6) and therefore to eliminate errors in ϕ that are due to the kth-order harmonic component of the signal.

Since the number of samples m is finite, it is impossible to satisfy Eqs. (7)–(10) for all nonnegative integer values of k. However, it is generally possible to choose sampling amplitudes a_i and b_i such that Eqs. (7)–(10) hold true for a set of specific integers k, including $k = 0$ and 1. When the sampling amplitudes are chosen so that they satisfy the simultaneous equations (7)–(10) for $k = 0, 1, \ldots, j$, the algorithm becomes insensitive to phase errors that are due to the effects of harmonic components up to the jth order. In this case, only the terms involving amplitude s_k of higher than the jth order $(k > j)$ would affect the calculated phase as residual errors.

It will now be shown that, in order to satisfy Eqs. (7)–(10) for $k = 0, 1, \ldots, j$, one should set the phase-shift interval equal to or smaller than $2\pi/(j + 2)$ rad. This means that the sampling divisor n satisfies

$$n \geq j + 2. \tag{11}$$

If $n \leq j + 1$, the left-hand side of Eq. (8) for $k = n - 1$ can be rewritten [where we note that $\alpha_i = (2\pi/n)(i - l)$] as

$$\sum_{i=1}^{m} a_i \cos(n-1)\alpha_i = \sum_{i=1}^{m} a_i \cos \alpha_i. \tag{12}$$

From Eq. (12) it can be seen that the left-hand side of Eq. (8) for $k = n - 1$ is identical with that of Eq. (8) for

$k = 1$. Since the left-hand side of Eq. (8) for $k = 1$ should be equal to unity, Eq. (8) for $k = n - 1$ cannot become zero. Thus Eq. (8) for $k = 1$ and $k = n - 1$ $(\leq j)$ cannot be satisfied simultaneously when $n \leq j + 1$. Therefore it can be concluded that, in order to suppress the effects of harmonic components up to the jth order, we must set the sampling interval smaller than or equal to $2\pi/(j + 2)$ rad.

For the convenience of the discussion below, here we estimate the minimum number of samples necessary to construct an algorithm that is insensitive to phase errors that are due to the effects of harmonic components up to the jth order. We therefore must determine the minimum number of sampling amplitudes necessary to satisfy the simultaneous equations (7)–(10) for $k = 0, 1, \ldots, j$. We note that Eqs. (7)–(10) for $k = 0, 1, \ldots, j$ are generally not independent. For the simplest case of $j = 1$ it can be seen that Eqs. (7)–(10) for $k = 0, 1$ give three independent equations for each sampling amplitude a_i and b_i. In order to satisfy three independent linear equations, we need at least three variables. Therefore, when the algorithm concerns only the signal of the fundamental order (while neglecting all higher harmonic components), the minimum number of samples necessary to determine the object phase is 3, which is well known in phase-shifting interferometry.

For integer j larger than unity the number of independent equations depends on the value of the phase-shift interval. Here we consider the interesting case in which the phase-shift interval equals $2\pi/(j + 2)$ rad.

It is shown in Appendix A below that Eqs. (7)–(10) for $k = 0, 1, \ldots, j$ reduce to $j + 2$ independent equations for each amplitude a_i and b_i. Therefore the minimum number of samples necessary to eliminate the effect of harmonic components up to the jth order is $j + 2$ when the phase-shift interval is set to $2\pi/(j + 2)$ rad.

One of the solutions for this case has already been given. It was shown[9] that the $(j + 2)$-sample synchronous phase-shifting algorithm is insensitive to the effects of harmonic components up to the jth order. The sampling amplitudes of the algorithm are given by

$$a_i = [2/(j + 2)]\cos[2\pi(i - l)/(j + 2)], \quad i = 1, \ldots, j + 2, \tag{13}$$

$$b_i = [2/(j + 2)]\sin[2\pi(i - l)/(j + 2)], \quad i = 1, \ldots, j + 2. \tag{14}$$

An elementary calculation can verify that these amplitudes satisfy Eqs. (7)–(10) for $k = 0, 1, \ldots, j$.

B. Suppression of Harmonic Components in the Presence of a Phase-Shift Error

We consider here the case in which a measurement system has a constant error in phase-shift interval and detects the phase of a signal consisting of harmonic components.

When a constant phase-shift error is present, the phase-shift interval is changed from the correct value of $2\pi/n$ rad to $2\pi(1 + \epsilon)/n$ rad, where error ϵ is a real number assumed to be much smaller than unity. From Eq. (3) the ith sampled intensity I_i is now given by

$$I_i = I[(1 + \epsilon)\alpha_i], \tag{15}$$

where the signal irradiance $I(\alpha)$ is given by Eq. (1).

We now derive the conditions for simultaneous elimination of the effects of harmonic components up to the jth order and a constant phase-shift error ϵ, to first order in ϵ. If we substitute Eqs. (1) and (15) into the algorithm of Eq. (2) and expand the denominator and the numerator of the right-hand side of Eq. (2) into powers of error ϵ, the calculated phase is given by

$$\phi = \arctan\left\{\frac{\sum_{k=0}^{\infty}\sum_{i=1}^{m} b_i s_k[\cos(k\alpha_i - \phi_k) - \epsilon k\alpha_i \sin(k\alpha_i - \phi_k) + o(\epsilon^2)]}{\sum_{k=0}^{\infty}\sum_{i=1}^{m} a_i s_k[\cos(k\alpha_i - \phi_k) - \epsilon k\alpha_i \sin(k\alpha_i - \phi_k) + o(\epsilon^2)]}\right\}, \tag{16}$$

where $o(\epsilon^2)$ are the terms involving the powers of error ϵ higher than the first.

The algorithm of Eq. (16) should give the correct value of object phase ϕ_1 for arbitrary small values of ϵ. When $\epsilon = 0$, the algorithm is reduced to one that deals with only the harmonic components of the signal, as we discussed in Subsection 2.A above. It has already been shown in Subsection 2.A that without the phase-shift error the sampling amplitudes satisfy Eqs. (7)–(10) for $k = 0, 1, \ldots, j$ to eliminate the effects of harmonic components up to the jth order. Similarly, in the presence of a phase-shift error it is necessary for the sampling amplitudes to satisfy Eqs. (7)–(10) for $k = 0, 1, \ldots, j$.

When we use Eqs. (7)–(10) and neglect the terms involving the harmonic components of higher than the jth order and those of $o(\epsilon^2)$, Eq. (16) is reduced to

$$\begin{aligned}\phi &= \arctan\left[\frac{s_1 \sin\phi_1 - \epsilon \sum_{k=1}^{j}\sum_{i=1}^{m} b_i\alpha_i k s_k \sin(k\alpha_i - \phi_k)}{s_1 \cos\phi_1 - \epsilon \sum_{k=1}^{j}\sum_{i=1}^{m} a_i\alpha_i k s_k \sin(k\alpha_i - \phi_k)}\right] \\ &= \phi_1 + \frac{\epsilon}{2}\sum_{k=1}^{j}\sum_{i=1}^{m} k s_k \alpha_i \\ &\quad \times \left[\frac{a_i \sin(k\alpha_i - \phi_k)}{\cos\phi_1} - \frac{b_i \sin(k\alpha_i - \phi_k)}{\sin\phi_1}\right]\sin(2\phi_1) \\ &\quad + o(\epsilon^2),\end{aligned} \tag{17}$$

where the approximation $\arctan(1 + \delta)\tan\phi = \phi + (\delta/2)\sin(2\phi) + o(\delta^2)$ for $|\delta| \ll 1$ was used.

The second terms on the right-hand side of Eq. (17) are the leading terms of phase error that are due to a constant phase-shift error. If we are to obtain the correct object phase ϕ_1 in the linear approximation, the second terms of Eq. (17) should remain constant and equal to zero for arbitrary values of error ϵ, amplitude s_k, and phase ϕ_k

for $k = 1, \ldots, j$. This requirement can be fulfilled if the balancing relation

$$\frac{\sum_{i=1}^{m} b_i \alpha_i \sin(k\alpha_i - \phi_k)}{\sin \phi_1} = \frac{\sum_{i=1}^{m} a_i \alpha_i \sin(k\alpha_i - \phi_k)}{\cos \phi_1} \tag{18}$$

is satisfied for arbitrary values of ϕ_k.

Expanding the trigonometric functions on both sides and noting that $\alpha_i = 2\pi(i - l)/n$, we can rewrite Eq. (18) as

$$\begin{aligned}\sum_{i=1}^{m} b_i(i - l)(\cos \phi_1)[\sin(k\alpha_i)\cos \phi_k - \cos(k\alpha_i)\sin \phi_k] \\ = \sum_{i=1}^{m} a_i(i - l)(\sin \phi_1)[\sin(k\alpha_i)\cos \phi_k \\ - \cos(k\alpha_i)\sin \phi_k].\end{aligned} \tag{19}$$

Comparing each term involving $\cos \phi_k$ or $\sin \phi_k$ for each k ($k = 1, \ldots, j$) on both sides of Eq. (19), we find that the necessary and sufficient conditions for Eq. (19) to hold are

$$\sum_{i=1}^{m} a_i(i - l)\sin(k\alpha_i) = 0, \qquad k = 2, \ldots, j, \tag{20}$$

$$\sum_{i=1}^{m} a_i(i - l)\cos(k\alpha_i) = 0, \qquad k = 1, \ldots, j, \tag{21}$$

$$\sum_{i=1}^{m} b_i(i - l)\sin(k\alpha_i) = 0, \qquad k = 1, \ldots, j, \tag{22}$$

$$\sum_{i=1}^{m} b_i(i - l)\cos(k\alpha_i) = 0, \qquad k = 2, \ldots, j, \tag{23}$$

$$\sum_{i=1}^{m} a_i(i - l)\sin \alpha_i = -\sum_{i=1}^{m} b_i(i - l)\cos \alpha_i, \tag{24}$$

where Eqs. (20) and (23) are necessary only when integer j is larger than unity.

When the sampling amplitudes a_i and b_i satisfy Eqs. (20)–(24) as well as Eqs. (7)–(10) for $k = 0, \ldots, j$, the algorithm of Eq. (17) gives the correct object phase ϕ_1 within the approximation that neglects the terms involving harmonic amplitudes higher than the jth order and terms of $o(\epsilon^2)$. Therefore the simultaneous equations consisting of the nine equations (7)–(10) and (20)–(24) are the necessary and sufficient conditions for a phase-shifting algorithm to eliminate, to a first-order approximation, phase errors that are due to the effects of higher harmonic components up to the jth order and a constant phase-shift error.

In order to find the minimum number of samples necessary to construct the algorithm, we need to calculate the number of independent equations among the simultaneous equations (7)–(10) and (20)–(24).

When the integer j equals unity, it can be seen that Eqs. (7)–(10) for $k = 0$, 1 and Eqs. (20)–(23) give four independent equations for each amplitude a_i and b_i and Eq. (24) gives one independent equation for both amplitudes. Therefore, when the algorithm concerns only the signal of the fundamental order ($j = 1$), five samples at least are necessary.

For integer j larger than unity the minimum number of samples depends on the value of the phase-shift interval. It has already been shown in Subsection 2.A above that, for elimination of the effects of harmonic components up to the jth order, the divisor n of the phase-shift interval should satisfy $n \geq j + 2$. Here we confine our discussion to the interesting case in which n equals $j + 2$ or, in other words, the phase-shift interval is set to $2\pi/(j + 2)$ rad, since the number of samples is then as small as possible for a given j. It was shown in Subsection 2.A that Eqs. (7)–(10) for $k = 0, \ldots, j$ reduce to $j + 2$ independent equations for each amplitude a_i and b_i. It is shown in Appendix A below that, in addition to these $j + 2$ equations, Eqs. (20)–(23) give j independent equations for each amplitude a_i and b_i and Eq. (24) gives an additional independent equation involving both amplitudes.

Therefore the minimum number of samples necessary to eliminate the effects of harmonic components up to the jth order in the presence of a phase-shift error is $2j + 3$, where, except for the case $j = 1$, the phase-shift interval is set to $2\pi/(j + 2)$rad.

Table 1 shows the minimum number of samples and typical phase-shift intervals necessary to eliminate both errors. We can see from Table 1 that the minimum number of samples necessary to eliminate the effects of harmonic components up to the second, third, and fourth orders are 7, 9, and 11, respectively. Note that for the case $j = 1$ the minimum number of samples does not depend on the value of the phase-shift interval, and n is an arbitrary integer ≥ 3.

It is also worth noting that, in the case $j = 1$ and a constant phase-shift error, Carre[3] showed that four samples are enough to obtain the correct object phase. The result by Carre seems at first glance to contradict our present result. However, the discrepancy can be understood if we note that our present discussion is confined to an algorithm in which the tangent value of the calculated phase is linear in the sampled intensities. In contrast, the tangent value is nonlinear in the formula by Carre. Therefore the number of samples is not strictly optimized in our present algorithms. However, Carre's approach becomes complicated and inefficient when it deals with the higher harmonic components. On the contrary, as we will see in Section 3 below, our procedure for constructing the algorithms can be easily applied to any order of harmonic components. In Section 3 we derive as examples the specific forms of the 5-sample ($j = 1$), 7-sample ($j = 2$), and 11-sample ($j = 4$) algorithms.

Table 1. Minimum Number of Samples and Typical Phase-Shift Intervals Necessary to Eliminate the Effects of Harmonic Components up to the jth Order in the Presence of a Constant Phase-Shift Error[a]

Harmonic Orders j	Number of Samples $2j + 3$	Phase-Shift Interval $2\pi/(j + 2)$
1	5	$2\pi/n$
2	7	$2\pi/4$
3	9	$2\pi/5$
4	11	$2\pi/6$
5	13	$2\pi/7$
6	15	$2\pi/8$

[a]For $j = 1$, n is an arbitrary integer ≥ 3.

3. EXAMPLES OF PHASE-SHIFTING ALGORITHMS

The simplest example is the case $j = 1$ (see Table 1), which gives a five-sample algorithm. From Table 1 the five-sample algorithm can be constructed with any phase-shift interval of $2\pi/n$ rad for $n = 3, 4, \ldots$. When the phase-shift interval is $\pi/2$, the algorithm leads to the well-known five-bucket algorithm[5,8,10,12]:

$$\phi = \arctan\left[\frac{2(I_4 - I_2)}{-I_1 + 2I_3 - I_5}\right], \tag{25}$$

where the sampled irradiances I_i are defined in terms of signal irradiance $I(\alpha_i)$ and phase-shift error ϵ as $I_i = I[\pi(i - 3)(1 + \epsilon)/2]$ for $i = 1, \ldots, 5$. Since the procedure for deriving this algorithm is similar to those of the following examples, it is omitted here.

The self-calibrating feature of the algorithm for a phase-shift error has already been pointed out by several authors.[5,8,10,12] When the signal irradiance has a sinusoidal waveform, the algorithm gives phase errors of order $o(\epsilon^2)$.

A. Seven-Sample Algorithm ($j = 2$)

The second example is the case $j = 2$ (see Table 1), which gives a seven-sample algorithm. The algorithm eliminates phase errors that are due to the effect of the second harmonic component of a signal in the presence of a constant phase-shift error.

The sampling amplitudes of the algorithm should satisfy the simultaneous equations (7)–(10) for integer $k = 0, 1, 2$ and Eqs. (20)–(24) for $j = 2$. From Table 1 the algorithm can be constructed with a phase-shift interval of $\pi/2$ rad. The parameter values in these equations are thus $m = 7$, $n = 4$, and $l = 4$.

Substituting these parameter values into Eqs. (7)–(10) and (20)–(24) shows that the simultaneous equations consist of a total of 13 independent equations involving amplitudes a_i and b_i. Since the seven-sample algorithm has 14 independent sampling amplitudes a_i and b_i, the solution cannot be determined uniquely. Any of the sampling amplitudes can be set to any value. Let us assume that $a_1 = 0$, so that the number of unknowns matches the number of equations. The solution can then be found as

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (0, -1/4, 0, 1/2, 0, -1/4, 0), \tag{26}$$

$$(b_1, b_2, b_3, b_4, b_5, b_6, b_7) = (1/8, 0, -3/8, 0, 3/8, 0, -1/8). \tag{27}$$

The explicit form of the algorithm is obtained by substituting Eqs. (26) and (27) into Eq. (2), which results in

$$\phi = \arctan\left[\frac{I_1 - 3I_3 + 3I_5 - I_7}{2(-I_2 + 2I_4 - I_6)}\right], \tag{28}$$

where the sampled irradiances are defined as $I_i = I[\pi(1 + \epsilon)(i - 4)/2]$ for $i = 1, \ldots, 7$.

The seven-sample algorithm of Eq. (28) eliminates phase errors that are due to the second-harmonic component in the presence of a constant phase-shift error. In order to verify this, we input a signal with a second-harmonic component:

$$I(\alpha) = s_0 + s_1 \cos(\alpha - \phi_1) + s_2 \cos(2\alpha - \phi_2). \tag{29}$$

Substituting Eq. (29) into Eq. (28) and expanding the cosines into powers of error ϵ give a phase error of

$$\begin{aligned}\Delta\phi &= \phi - \phi_1 \\ &= (\pi\epsilon)^2\left[\frac{s_2}{s_1}\sin\phi_1\cos\phi_2 - \frac{1}{16}\sin(2\phi_1)\right] + o(\epsilon^3),\end{aligned} \tag{30}$$

where the approximation $\arctan(1 + \delta)\tan\phi = \phi + (\delta/2)\sin(2\phi) + o(\delta^2)$ for $|\delta| \ll 1$ was used. From Eq. (30) it can be seen that the algorithm eliminates the error terms linear in phase-shift error ϵ and linear in amplitude s_2. Therefore, to a first-order approximation, the algorithm gives the correct object phase ϕ_1.

Numerical calculations show that, when a system with 5% error in phase-shift interval ($\epsilon = 0.05$) measures the phase of a signal with a second-harmonic component with amplitude 30% of that of the fundamental component ($s_2/s_1 = 0.3$), the algorithm gives a peak-to-valley phase error of $\pi/200$ rad, and the five-bucket algorithm[5,10,12] [defined by Eq. (25)], the seven-sample synchronous algorithm [see Eqs. (13) and (14)], and the seven-sample symmetrical algorithm[10] give errors of $\pi/33$, $\pi/26$, and $\pi/37$ rad, respectively. We can see that the phase error that is due to the second-harmonic component and a constant phase-shift error are $o(\epsilon^2)$ in the present algorithm.

B. Eleven-Sample Algorithm ($j = 4$)

From Table 1 we can derive an 11-sample $\pi/3$-interval algorithm for the case $j = 4$. The 11-sample algorithm eliminates the effects of harmonic components up to the fourth order in the presence of a constant phase-shift error. Substituting the parameter values of $m = 11$, $n = 6$, and $l = 6$ into Eqs. (7)–(10) for $k = 0, 1, \ldots, 4$ and Eqs. (20)–(24) for $j = 4$, we obtain simultaneous equations consisting of a total of 21 independent equations for the sampling amplitudes. Solving the equations results in the 11-sample algorithm given by

$$\phi = \arctan\left[\frac{\sqrt{3}(-I_2 - 4I_3 - 7I_4 - 6I_5 + 6I_7 + 7I_8 + 4I_9 + I_{10})}{-2I_1 - 5I_2 - 6I_3 - I_4 + 8I_5 + 12I_6 + 8I_7 - I_8 - 6I_9 - 5I_{10} - 2I_{11}}\right], \tag{31}$$

where the sampling irradiances are defined as $I_i = I[\pi(1 + \epsilon)(i - 6)/3]$ for $i = 1, \ldots, 11$.

When a system measures the phase of a signal with harmonic components,

$$I(\alpha) = s_0 + \sum_{k=1}^{4} s_k \cos(k\alpha - \phi_k), \tag{32}$$

the algorithm gives a phase error of

$$\Delta\phi = \phi - \phi_1 = \frac{(\pi\epsilon)^2}{9}\left[\frac{2}{3}\sin(2\phi_1) + \frac{s_2}{s_1}\sin(\phi_1 + \phi_2) + \frac{3s_3}{s_1}\sin\phi_1\cos\phi_3 + \frac{4s_4}{s_1}\sin(\phi_1 - \phi_4)\right] + o(\epsilon^3). \quad (33)$$

Consider a system with 5% error in phase-shift interval ($\epsilon = 0.05$) that measures the phase of a signal with the second-, third-, and fourth-harmonic components with relative intensities that are respectively 30%, 15%, and 7% of that of the fundamental component ($s_2/s_1 = 0.3$, $s_3/s_1 = 0.15$, and $s_4/s_1 = 0.07$). Numerical calculations show that the 11-sample algorithm gives a peak-to-valley phase error of $\pi/340$ rad, and the 11-sample synchronous phase-shifting algorithm [see Eqs. (13) and (14)] gives an error of $\pi/20$ rad.

We can see that the present algorithm is well compensated for phase errors that are due to effects of harmonic components in the presence of a phase-shift error. The algorithm can be applied to high-precision phase-measuring interferometers such as those in phase-shifting Ronchi tests and multiple-beam interferometers, which have high harmonic content.

4. FOURIER CHARACTERISTICS OF THE SAMPLING AMPLITUDES

In Section 3 we have derived two phase-shifting algorithms that are insensitive to the effects of certain harmonic components of a signal in the presence of a constant phase-shift error. Some interesting features of these algorithms can be visualized through a Fourier representation[10,11] of the sampling amplitudes of these algorithms.

We define the sampling functions of the numerator $[f_1(\alpha)]$ and the denominator $[f_2(\alpha)]$ of the general algorithm expressed in Eq. (2) in terms of the sampling amplitudes as[10]

$$f_1(\alpha) = \sum_{i=1}^{m} b_i\delta(\alpha - \alpha_i), \quad (34)$$

$$f_2(\alpha) = \sum_{i=1}^{m} a_i\delta(\alpha - \alpha_i), \quad (35)$$

where $\delta(\alpha)$ is the Dirac delta function and the other parameters are as defined above.

Using the sampling functions and Parseval's identity, we can rewrite Eq. (2) as

$$\phi = \arctan\left[\frac{\int f_1(\alpha)I(\alpha)\mathrm{d}\alpha}{\int f_2(\alpha)I(\alpha)\mathrm{d}\alpha}\right] = \arctan\left[\frac{\int F_1(\nu)J(\nu)\mathrm{d}\nu}{\int F_2(\nu)J(\nu)\mathrm{d}\nu}\right], \quad (36)$$

where $F_1(\nu)$, $F_2(\nu)$, and $J(\nu)$ are the Fourier transforms of $f_1(\alpha)$, $f_2(\alpha)$, and the signal irradiance $I(\alpha)$, respectively, and the integrals are taken over all real values of α and ν, respectively. From Eq. (36) we see that the dependence of the algorithm on the harmonic components of the signal is determined by the products of the Fourier transform of each sampling function and the Fourier transform of the signal.

Larkin and Oreb showed[10] that, when a phase-shifting algorithm is insensitive to the harmonic components up to the pth order as well as to a constant phase-shift error, the Fourier transforms of its sampling functions have the following two properties:

(1) Both functions have matched gradients at the fundamental frequency $\nu = \nu_0$; i.e., the magnitudes of $\mathrm{d}F_1/\mathrm{d}\nu$ and $\mathrm{d}F_2/\mathrm{d}\nu$ are equal, and their phases are in quadrature at $\nu = \nu_0$.

(2) Both functions take zero magnitudes and zero gradients at harmonic frequencies $\nu = 2\nu_0, \ldots, p\nu_0$, where $\nu_0 = 2\pi/T$ and T is the period of the phase-shift parameter α. In our present discussion ν_0 is equal to unity, since T is now equal to 2π rad.

For the seven-sample algorithm defined by Eqs. (26) and (27) we can show that the Fourier transforms $F_1(\nu)$ and $F_2(\nu)$ are given by

$$F_1(\nu) = (\sqrt{-1}/4)[3\sin(\pi\nu/2) - \sin(3\pi\nu/2)], \quad (37)$$

$$F_2(\nu) = {}^1\!/\!_2[(1 - \cos(\pi\nu)]. \quad (38)$$

Figure 1 shows the Fourier transforms $F_1(\nu)$ and $F_2(\nu)$ for the seven-sample algorithm. We can observe that the functions have matched gradients at the fundamental frequency $\nu = \nu_0$ and zero magnitudes and gradients at even-order harmonics $\nu = 2\nu_0, 4\nu_0, \ldots$. Since the seven-sample algorithm is insensitive to the second-harmonic component and a constant phase-shift error, these results are consistent with properties (1) and (2) above.

Similarly, we can calculate the two Fourier transforms of the 11-sample phase-shifting algorithm, which has been defined in Eq. (31). Figure 2 shows the Fourier transforms $F_1(\nu)$ and $F_2(\nu)$ of the algorithm. From the figure we can observe that again the functions have matched gradients at the fundamental frequency $\nu = \nu_0$, and zero magnitudes and gradients at the harmonic frequencies $\nu = 2\nu_0$, $3\nu_0$, and $4\nu_0$. These results are consistent with the fact that the 11-sample algorithm is insensitive to the effects of harmonic components up to the fourth order in the presence of a constant phase-shift error.

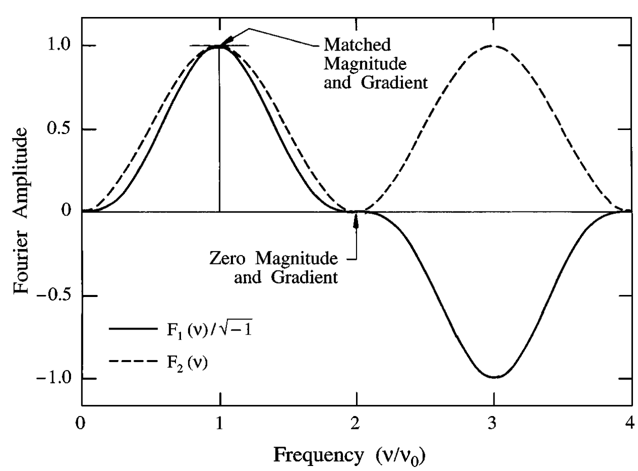

Fig. 1. Fourier transform of the sampling amplitudes b_i $[F_1(\nu)]$ and a_i $[F_2(\nu)]$ for the seven-sample algorithm.

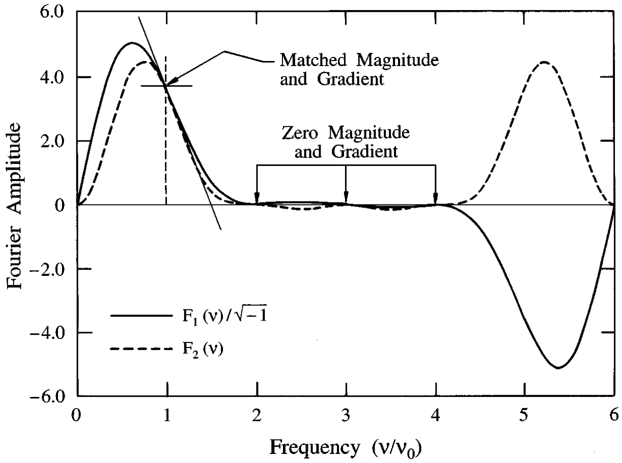

Fig. 2. Fourier transforms of the sampling amplitudes b_i $[F_1(\nu)]$ and a_i $[F_2(\nu)]$ for the 11-sample algorithm.

5. CONCLUSION

Phase errors in phase-shifting systems that are due to nonsinusoidal signals and a phase-shift error were discussed. It has been shown that, to a first-order approximation, these errors can be eliminated by taking enough samples. It has also been shown that at least $2j + 3$ samples are necessary to eliminate the effects of harmonic components up to the jth order in the presence of a constant phase-shift error and that the phase-shift interval must be set to $\leq 2\pi/(j + 2)$ rad. A general procedure for constructing this new class of phase-shifting algorithm with extended symmetry was derived in terms of simultaneous equations of the sampling amplitudes of the algorithms. As examples, several specific forms of the algorithms were derived. The procedure can be easily applied to construct a phase-shifting algorithm with larger numbers of samples for measurements involving signals with many harmonics.

APPENDIX A

The minimum number of samples necessary to eliminate the effects of harmonic components in the presence of a constant phase-shift error is calculated here. In Section 2 it was shown that, if we are to suppress the effects of harmonic components up to the jth order, the sampling amplitudes of the algorithm should satisfy the nine equations (7)–(10) and (20)–(24), which are repeated here for convenience:

$$\sum_{i=1}^{m} a_1 \sin(k\alpha_i) = 0, \qquad k = 1, \ldots, j, \tag{A1}$$

$$\sum_{i=1}^{m} a_1 \cos(k\alpha_i) = \delta(k, 1), \qquad k = 0, 1, \ldots, j, \tag{A2}$$

$$\sum_{i=1}^{m} b_i \sin(k\alpha_i) = \delta(k, 1), \qquad k = 1, \ldots, j, \tag{A3}$$

$$\sum_{i=1}^{m} b_i \cos(k\alpha_i) = 0, \qquad k = 0, 1, \ldots, j, \tag{A4}$$

$$\sum_{i=1}^{m} a_i(i - l)\sin(k\alpha_i) = 0, \qquad k = 2, \ldots, j, \tag{A5}$$

$$\sum_{i=1}^{m} a_i(i - l)\cos(k\alpha_i) = 0, \qquad k = 1, \ldots, j, \tag{A6}$$

$$\sum_{i=1}^{m} b_i(i - l)\sin(k\alpha_i) = 0, \qquad k = 1, \ldots, j, \tag{A7}$$

$$\sum_{i=1}^{m} b_i(i - l)\cos(k\alpha_i) = 0, \qquad k = 2, \ldots, j, \tag{A8}$$

$$\sum_{i=1}^{m} a_i(i - l)\sin \alpha_i = -\sum_{i=1}^{m} b_i(i - l)\cos \alpha_i\,. \tag{A9}$$

The minimum number of samples necessary is determined by the number of independent equations among Eqs. (A1)–(A9). In what follows, we show that the minimum number of samples is $2j + 3$.

For the simplest case of $j = 1$ we can see by substitution that all of Eqs. (A1)–(A9), except for Eqs. (A5) and (A8), are independent for any phase-shift interval $2\pi/n$ satisfying $n \geq 3$. However, we note that Eqs. (A5) and (A8) are not necessary for the case $j = 1$. Equations (A1)–(A7) give four independent equations for each of the amplitudes a_i and b_i, and Eq. (A9) gives one for both amplitudes. The minimum number of components for each amplitude necessary to satisfy all of Eqs. (A1)–(A9) is 5. Therefore at least five samples are necessary for the case $j = 1$.

For any larger integer j we confine our discussion to the case in which the algorithm has a phase-shift interval of $2\pi/(j + 2)$ rad, that is, $n = j + 2$. This case is most interesting, since the number of independent equations becomes larger for any larger value of n.

We note the two identities

$$\sum_{i=1}^{m} a_i \cos[(j - p)\alpha_i] = \sum_{i=1}^{m} a_i \cos[(p + 2)\alpha_i], \tag{A10}$$

$$\sum_{i=1}^{m} a_i \sin[(j - p)\alpha_i] = -\sum_{i=1}^{m} a_i \sin[(p + 2)\alpha_i], \tag{A11}$$

where p is an arbitrary integer and $\alpha_i = 2\pi(i - l)/(j + 2)$.

First, consider the case in which integer j is even and greater than unity. Using $n = j + 2$ and Eq. (A10), one can show that Eq. (A2) for $k = j - p$ and $k = p + 2$ gives identical equations, where p is an arbitrary integer satisfying $0 \leq p \leq j/2 - 1$. This means that Eq. (A2) for $k = 0, 1, \ldots, j$ results in only $j/2 + 2$ independent equations.

Similarly, we can show that Eq. (A6) for $k = 1, \ldots, j$ results in only $j/2 + 1$ independent equations. Using Eq. (A11), we can also show that Eq. (A1) for $k = 1, \ldots, j$ gives $j/2$ independent equations, and Eq. (A5) for $k = 2, \ldots, j$ gives $j/2 - 1$ independent equations, where we note that Eqs. (A1) and (A5) for $k = j/2 + 1$ become trivial equations.

Therefore, when j is even, Eqs. (A1), (A2), (A5), and (A6) give $2j + 2$ independent equations for amplitudes a_i. The same discussion can be used for amplitudes b_i; that is, Eqs. (A3), (A4), (A7), and (A8) give $2j + 2$ independent equations. Therefore there are $2j + 2$ independent equations for each of the two amplitudes and one independent equation [Eq. (A9)] for both amplitudes. The minimum number of samples or, in other words, the number of amplitudes necessary for satisfying all the equations, is thus $2j + 3$ for even j.

A similar discussion can be applied for the case in which integer j is odd. In this case we can show that, toward the total number of independent equations for amplitudes a_i, Eq. (A1) gives $(j+1)/2$, Eq. (A2) gives $(j+3)/2$, Eq. (A5) gives $(j-1)/2$, and Eq. (A6) gives $(j+1)/2$.

Similarly, we can show that, toward the total number of independent equations for amplitudes b_i, Eq. (A3) gives $(j+1)/2$, Eq. (A4) gives $(j+3)/2$, Eq. (A7) gives $(j+1)/2$, and Eq. (A8) gives $(j-1)/2$.

Therefore, when j is odd, Eqs. (A1), (A2), (A5), and (A6) give $2j+2$ independent equations for amplitudes a_i, and Eqs. (A3), (A4), (A7), and (A8) give $2j+2$ independent equations for amplitudes b_i.

Again, there are $2j+2$ independent equations for each of the two amplitudes and one independent equation for both amplitudes. The minimum number of samples necessary to satisfy all the equations is thus $2j+3$.

From these results it can be concluded that, for a phase-shifting algorithm with phase-shift interval of $2\pi/(j+2)$ rad, the minimum number of samples necessary to eliminate the effects of harmonic components up to the jth order, in the presence of a constant phase-shift error, is $2j+3$. Note that for the case $j=1$ the minimum number of samples is 5 for any phase-shift interval.

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