Improved iterative least-squares algorithm for phase-shifting interferometry

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Iterative Least-Squares (ILS) algorithms are used when having a Phase-Shifting Interferogram (PSI) sequence where we do not know the inter-frame phase-shifts and these are not necessary constant. Basically, ILS algorithms uses the PSI least-squares model in two main steps: 1) Estimate the modulating phase with a set of guessed phase-shifts, and 2) Estimate the phase-shifts using the previously estimated phase. These two steps are performed iteratively. In this class of algorithms, a source of error is when is estimating the phase-shifts while the background illimination varies spatially. This is because the least-squares model assumes constat background illumination. To improve this, in this paper we divede in subregions the interferogram frames and apply the least-squares model on each subregion. Thus, each subregion may have its own background illimination, being less restrictive than applying the least-squares model to the global frame. The numerical experiments presented here, show that this approach enhance the obtained results of the well known published ILS algorithms. For demonstration purposes, the source code of this algorithm is provided in the references for download.

OCIS codes: 120.0120, 120.3180, 120.3940, 120.5050, 120.2650, 050.5080

1. Introduction

Nowadays, Phase Shifting Interferometry (PSI) techniques are some of the most used techniques in optical metrology [?]. In PSI, one obtains a small sequence of at least 3 interferograms with a phase-shift among them [?]. To recover the modulating phase, there are standard demodulation PSI methods, the well known 3-, 4-, and 5- step phase-shifting algorithms. Knowing the inter-frame phase-shifts (or temporal carrier), the standard methods recover the 2π modulus phase map with the minimum possible error [?, ?, ?]. If we do not know the phase-shifts exactly, we obtain a phase map with an unavoidable detuning error whose magnitude depends on the number of interferograms employed and on how far we are from the actual phase-shifts [?, ?, ?, ?]. This unfortunate case can occur when the optical interferometer setup is uncalibrated or when perturbations from the environment affect the interferometer's optical path. For example, for most phase shifters, such as a piezoelectric (PZT), there is a repeatability problem arising from hysteresis, non linearity and temperature linear drift [?, ?]; curiously, the first phase-shifting algorithms were self-tuning nonlinear algorithms [?, ?]. To reduce detuning errors, other approaches propose error compensating algorithms that basically use redundant data, such as the Schwider- Hariharan 5- step algorithm [?, ?, ?], and more recently, the 9-step algorithm shown in Ref. [?] has been used to do it by constructing a wide-band frequency response of the phase-shifting algorithm. Further methods use the Fourier transform in order to estimate the inter-frame phase-shifts, and others are based on the least-squares scheme, iteratively estimating the inter-frame phase-shifts and phase [?, ?]. Ref. [?] presentes an approach that estimates the local temporal carrier (the phase-shift) as the average of the phase difference between two consecutive phase maps obtained from two realizations of the tunable 3-step algorithm. What we are going to show in this work is a regularized self-tuning demodulation technique that obtains the analytical image (complex interferogram) and inter-frame phase-shifts from an interferogram sequence. Thus, we can recover the modulating phase 2π modulus and the inter-frame phase shifts in the same process. Here, it is not necessary to know the inter-frame phase-shifts. These inter-frame phase-shifts can vary arbitrarily.

The main difference between the demodulation method presented here and the ones reported in [?], [?] and [?] is that our demodulation method is based on a regularization technique that is robust to non-constant modulation variations and non-constant background illumination, which is an issue that introduces errors in the methods in works [?] and [?]. Besides, we do not require to estimate the fringe orientation, as the method in work [?].

2. Method

In general, an interferogram sequence with arbitrary inter-frame phase-shifts can be modeled as:

$$I_{x,y}^k = a_{x,y} + b_{x,y} cos(\phi_{x,y} + \alpha_k),\ k = 0, 1, 2, ..., L-1, \tag{1}$$

where $I_{x,y}^k$ is the intensity at the site x, y of the k-interferogram in a sequence of $L-1$ interferograms, being $a_{x,y}$ its background illumination, $b_{x,y}$ its contrast, $\phi_{x,y}$ the modulating phase under test and α_k the phase-shift of the k-interferogram.

2.A. Least-squares method

As we can see from Eq. (2), conventional phase-shifting algorithms assume that the background illumination and the modulation amplitude do not have frame-to-frame variation; i.e., they are functions of pixels only. Defining a new set of variables as $\varphi_{x,y} = b_{x,y}cos(\phi_{x,y})$, $\psi_{x,y} = b_{x,y}sin(\phi_{x,y})$, $C_k = cos(\alpha_k)$, $S_k = sin(\alpha_k)$, we can express Eq. (2) as

$$I^k_{x,y} = a_{x,y} + \varphi_{x,y}C_k - \psi_{x,y}S_k,\ k = 0, 1, 2, ..., L - 1. \tag{2}$$

If we know α_k, there are $3MN$ unknowns (where M and N are the interferogram dimension). These unknowns can be solved using the least-squares method. An energy cost function that described above can be written as

$$U(a_{x,y}, \varphi_{x,y}, \psi_{x,y}) = \sum_{k=0}^{L-1}[a_{x,y} + \varphi_{x,y}C_k - \psi_{x,y}S_k - I'_{x,y,k}]^2, \tag{3}$$

where $I'_{x,y,k}$ is the k-th experimentally measured intensity of the interferogram sequence. For the known α_k, the least-squares criteria require to make zero the gradient of Eq. (2.A) as

$$\nabla U(a_{x,y}, \varphi_{x,y}, \psi_{x,y}) = 0. \tag{4}$$

Eq. (2.A) yields

$$X = A^{-1}B, \tag{5}$$

where

$$A = \begin{bmatrix} M \times N & \sum C_k & \sum S_k \\ \sum C_k & \sum C_k^2 & \sum C_k S_k \\ \sum S_k & \sum S_k C_k & \sum S_k^2 \end{bmatrix}, \tag{6}$$

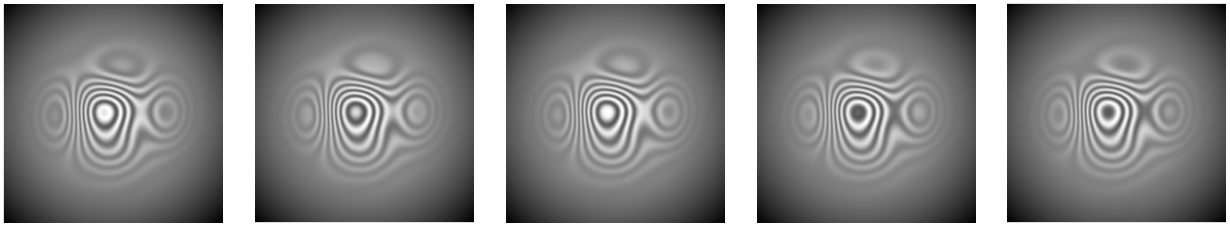
$$B = \begin{bmatrix} \sum I'_{x,y,k} & \sum I'_{x,y,k}C_k & \sum I'_{x,y,k}S_k \end{bmatrix}^T, \tag{7}$$

$$X = \begin{bmatrix} a_{x,y} & \varphi_{x,y} & \psi_{x,y} \end{bmatrix}^T, \tag{8}$$

where the sum $\sum$ runs over $k = 0, 1, 2, ..., L - 1$. To ensure that A is nonsingular, Eq. (2.A) requires at least three different phase-steps. From Eqs. (2.A)-(2.A) phase ϕ at point x, y can be determined from

$$\phi_{x,y} = \arctan(-\psi_{x,y}/\varphi_{x,y}). \tag{9}$$

Note that the inverse of A is performed only once because its components depend only on the α_k steps.



I_0 I_1 I_2 I_3 I_4

Fig. 1. Interferogram sequence used to test the AIA and the proposed algorithm (RST).

2.B. Robust self-tuning (RST) method

To determine the steps, we propose the following regularized cost functional

$$\begin{aligned} U(a_{x,y}, C^k_{x,y}, S^k_{x,y}) &= \sum_{x,y}\sum_{k=0}^{L-1}\sum_{m=-1}^{1}\sum_{n=-1}^{1}\left[a_{x,y} + \varphi_{x+m,y+n}C^k_{x,y} - \psi_{x+m,y+n}S^k_{x,y} - I'_{x+m,y+n,k}\right]^2 \\ &\quad + \sum_{x,y}\sum_{k=0}^{L-1}\left[\lambda\frac{\nabla[a_{x,y}]}{L} + \mu\nabla[C^k_{x,y}] + \mu\nabla[S^k_{x,y}]\right]^2, \end{aligned} \tag{10}$$

where λ and μ are the regularization parameters that control the smoothness of $a_{x,y}$, $C^k_{x,y}$ and $S^k_{x,y}$. Operator $\nabla[*]$ takes the first order differences along the x and y directions as follows:

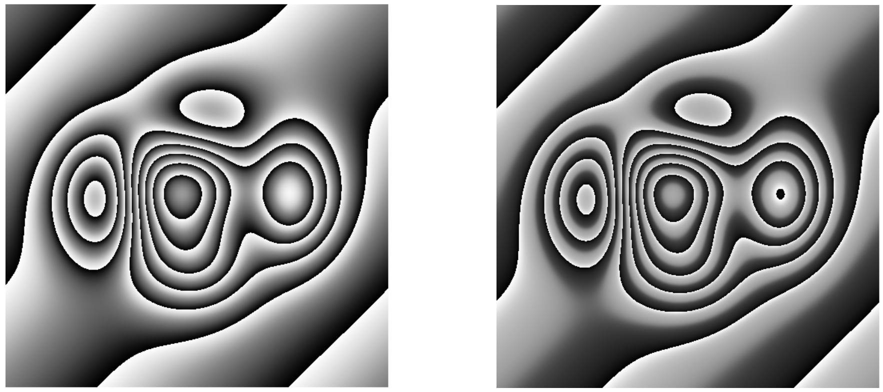
$$\nabla[f_{x,y}] = [a_{x,y} - a_{x-1,y}, a_{x,y} - a_{x,y-1}]^T. \tag{11}$$

To find our unkown steps we need to minimize the cost functional in Eq 10. Equating the partial gradient with respect to $a_{x,y}$, $C^k_{x,y}$ and $S^k_{x,y}$ to zero, and solving, we obtain a closed formula for iteratively computing the background illumination and the phase-shift steps, as follows:

$$a_{x,y} = \frac{\sum_{k=0}^{L-1}\sum_{m=-1}^{1}\sum_{n=-1}^{1}\left[-\varphi_{x+m,y+n}C^k_{x,y} + \psi_{x+m,y+n}S^k_{x,y} + I'_{x+m,y+n,k}\right] + \lambda R[a_{x,y}]}{9L + N\lambda}, \tag{12}$$

$$C^k_{x,y} = \frac{\sum_{m=-1}^{1}\sum_{n=-1}^{1}\left[-a_{x,y}\varphi_{x+m,y+n} + \psi_{x+m,y+n}\varphi_{x+m,y+n}S^k_{x,y} + I'_{x+m,y+n,k}\varphi_{x+m,y+n}\right] + \mu R[C^k_{x,y}]}{\sum_{m=-1}^{1}\sum_{n=-1}^{1}\varphi^2_{x+m,y+n} + N\mu}, \tag{13}$$

$$S^k_{x,y} = \frac{\sum_{m=-1}^{1}\sum_{n=-1}^{1}\left[a_{x,y}\psi_{x+m,y+n} + \psi_{x+m,y+n}\varphi_{x+m,y+n}C^k_{x,y} - I'_{x+m,y+n,k}\psi_{x+m,y+n}\right] + \mu R[C^k_{x,y}]}{\sum_{m=-1}^{1}\sum_{n=-1}^{1}\psi^2_{x+m,y+n} + N\mu}, \tag{14}$$



(a) RST Phase error = $7.0570\text{x}10^{-06}$. (b) AIA Phase error = 0.0608

Fig. 2. Phase Comparison.(a) shows the recovered phase and error using the regularized self-tuning (RST) method proposed here. (b) shows the recovered phase and error using the AIA method. The error shown (in radians) is the variance with respec to the true phase map. The interferogram frames has a size of 512 x 512.

were $R[a_{x,y}] = a_{x-1,y} + a_{x+1,y} + a_{x,y-1} + a_{x,y+1}$, $R[C^k_{x,y}] = C^k_{x-1,y} + C^k_{x+1,y} + C_{x,y-1} + C^k_{x,y+1}$ and $R[S^k_{x,y}] = S^k_{x-1,y} + S^k_{x+1,y} + S_{x,y-1} + S^k_{x,y+1}$. As we see from Eqs 2.B and 2.B, we can compute the phase-shift in every site of each interferogram. Then, to obtain the phase-shift steps, we use

$$\alpha_k = \arctan(-S^k_m / C^k_m), \tag{15}$$

where S^k_m and C^k_m are the mode, the most frequent value, calculated for each interferogram. Now, to determine the modulated phase, we use the least squares method and the proposed method alternately until the values of the phase-shifts converge to a desired value.

3. Numerical Experiments and Results

To test the robust self-tuning algorithm developed here, we simulated 50 interferogram sequences with different temporal frequencies, and 5 interferograms each sequence. An example of these interferogram sequences is shown in Fig.1. These simulated interferograms have poor fringe visibility, since we have added non-constant background illumination and contrast. To reproduce the interferogram sequences that we used in these tests, the reader can use Eq. 2 with the following parameters:

$$a_{x,y} = -\frac{(x-256)^2 + (y-256)^2}{1.5259x10^{-5}}, \tag{16}$$

$$b_{x,y} = e^{-\frac{(x-256)^2+(y-256)^2}{100^2}}, \quad (17)$$

$$\phi_{x,y} = 30(-1 + \frac{x}{2} - x^5 - y^3)e^{(-x^2-y^2)} + \frac{4\pi x}{512} + \frac{4\pi y}{512}, \quad (18)$$

and

$$\alpha_k = k\frac{\pi}{3} + \eta_k,\ k = 0, 1, .., 5, \quad (19)$$

where η_k is a random number with mean zero and a variance of 0.5 that changes at each step. In our numerical experiments, we always start the temporal carrier $\alpha_k = 0, 1, .., 5$. Regularization parameters λ and μ were set at 100 and 500 respectively (see Eq. 10). Using the Gauss-Seidel method to find the phase-steps (Eqs. 13, 14 and 15) 50 iterations were required. Finally, to estimate the phase in the AIA and our proposed method, 20 iterations were required.

In Table. 1 , we show the error values of the estimated phase-shifts using our regularized method and the estimated using the AIA method. These errors are calculated as $|\alpha_k - \hat{\alpha}_k|$, where α_k and $\hat{\alpha}_k$ are the values of the actual and estimated phase-shifts for the $k-$frame, respectively. On the other hand, in Fig. 2 , we show the recovered phase. Fig. 2(a) shows the recovered phase using the regularized self-tuning demodulation method presented here, while Fig. 2(b) shows the recovered phase using the AIA method. In this figure, we can see in this figure that our proposed regularized self-tuning demodulation method recovers the phase with less error than the AIA method. The errors shown in Fig. 2(a) and Fig. 2(b) are calculated as the variance between the recovered phase map and the true phase map used to generate the interferograms.

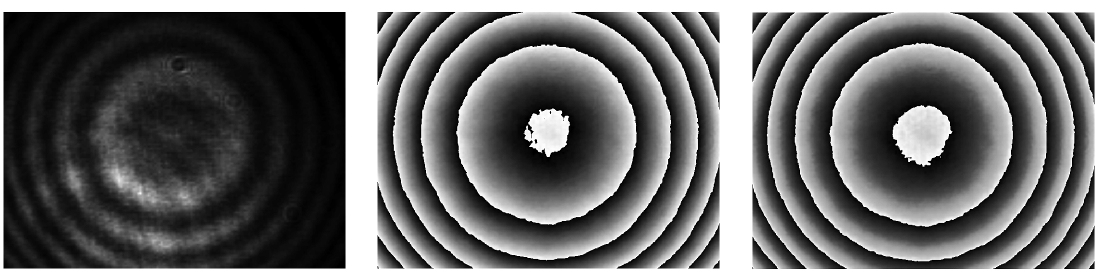
Steps	Actual	AIA	RST
α_0	0	0	0
α_1	1.6953	2.2511	1.6994
α_2	0.6961	0.6284	0.6769
α_3	3.3038	3.2427	3.3031
α_4	4.0793	4.2983	4.0678
e^{ave}		0.1807	0.0071

Table 1. Numerical results. Comparison between the phase-shift estimation and the actual phase-shift for the AIA and the proposed method, and the phase-shift average error.

On the other hand, Table 2 shows the average error of 50 random tests, with the same parameters listed above. As seen from these results, the proposed method recovers the phase with three orders of magnitude less than the AIA method. Finally, in Fig 3(b) shows the recovered phase with the RST method using a 4-frame experimental sequence, while Fig 3(c) shows the recovered phase using the AIA method. As we see in Fig 3(c), the AIA method recovers a phase map with detuning error. The interferogram shown in Fig 3(a) is the first frame of the sequence used. For demonstration purposes, we modify the contrast and background illumination of the original experimental sequence as follows: $I_n = a + bI$, where I_n is the new interferogram sequence, a is a parabola centered at the origin with an amplitude of one, b is a Gaussian with a variance of 7 and I is the original experimental sequence.

	AIA	RST
Step-error	0.1902	0.0153
Phase-error	4.57x10^{-2}	5.11x10^{-5}

Table 2. This table compares the average error obtained in 50 random tests for the proposed method and the AIA method. The first row shows the average error of estimated steps, while the second shows the variance between the estimated phase map and the one used to generate the sequence of interferograms.



(a) Interferogram. (b) RST method. (c) AIA method.

Fig. 3. Experimental test.(a) shows an experimental interferogram with background illumination and contrast modifed. (b) shows the recovered phase map using the RST method and (C) using the AIA method.

4. Conclusions

We have presented a regularized self-tuning phase-shifting demodulation method for interferogram sequences that have arbitrary variations of the inter-frame phase-shifts. This method is robust to non-constant background illumination and contrast in their fringes. As shown in the results, our demodulation method is able to recover the modulating phase and the inter-frame phase-shifts with a minimum error. The demodulation method presented here provides stable convergence and accurate phase demodulation with as few as three interferograms, even when the phase-shifts are completely random.