Generalized phase-shifting interferometry with arbitrary unknown phase steps for diffraction objects

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A general method of extracting the arbitrary unknown and unequal phase steps in phase-shift interferometry from interferograms recorded on the diffraction field of an object and then reconstructing the object wave front digitally with our derived formulas is proposed. The phase steps are first calculated based on the statistical nature of the diffraction field and are further improved by an iterative approach. This method is simple, highly accurate, and usable for any frame number N ($N \geq 3$) and for both smooth and diffusing objects, as is verified by a series of computer simulations. © 2004 Optical Society of America

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Recently, digital phase-shifting interferometry (PSI) has become a powerful tool for wave-front reconstruction and has found wide use in a variety of applications from phase measurement to microscopy.[1–5] Standard PSI requires a special and constant phase shift, $2\pi/N$, where the integer $N \geq 3$. This requirement, however, is often difficult to meet exactly in reality. Therefore special effort must be made to calibrate the phase shifter.[6,7] To remove this limitation, Greivenkamp introduced a general method with arbitrary phase steps that must be known precisely,[8] and Stoilov and Dragostinov suggested a five-frame approach in which the phase steps may be arbitrary but should still be equal.[9] In addition, phase-retrieval algorithms with unknown phase shifts have also been developed.[10–14] Usually these methods have substantial computation loads, and some need quite a few (as many as 15) interferograms[13] or additional optical devices.[12,14] Recently Guo *et al.*[15] and Cai *et al.*[16] introduced two algorithms to calculate the unknown phase shifts and then retrieve the object wave front. However, both require measurement of the object and the reference wave intensities, and the former also needs an approximate value of the phase step and can be used only for pure-phase objects.

In many applications the test object is located a certain distance from the recording (CCD) plane, and what we record in the CCD is actually the diffraction field from the object surface. In this case the statistical properties of the diffraction field may provide a helpful hint for data processing.[14,16] Based on this idea, in this Letter we propose a new, general method of PSI with arbitrary unknown phase shifts that can extract the phase steps and then reconstruct the object field with the use of only the measured constant reference wave intensity. This method can be used for any frame number $N \geq 3$. We first explain the principles of the method and then verify it by computer simulations.

Let us start with the simplest case, $N = 3$. Assume that the diffraction field of an object in recording plane P_H is $O(x, y) = A_o(x, y)\exp[i\varphi(x, y)]$ and the reference wave is a plane wave with constant amplitude A_r; the intensity distributions of three interferograms may then be written as

$$I_1(x, y) = A_r^2 + A_o^2(x, y) + 2A_rA_o(x, y)\cos[\varphi(x, y)], \quad (1)$$

$$I_2(x, y) = A_r^2 + A_o^2(x, y) + 2A_rA_o(x, y)\cos[\varphi(x, y) - \alpha_1], \quad (2)$$

$$I_3(x, y) = A_r^2 + A_o^2(x, y) + 2A_rA_o(x, y)\cos[\varphi(x, y) - \alpha_1 - \alpha_2], \quad (3)$$

where α_1 and α_2 ($0 < \alpha_j < \pi$, $j = 1, 2$) are two constant phase steps. We may introduce three new quantities formed by the three measured intensities above:

$$p = \langle |I_2 - I_1| \rangle = \langle |4A_rA_o \sin(\varphi - \alpha_1/2)| \rangle \sin(\alpha_1/2), \quad (4)$$

$$q = \langle |I_3 - I_2| \rangle = \langle |4A_rA_o \sin(\varphi - \alpha_1 - \alpha_2/2)| \rangle \sin(\alpha_2/2), \quad (5)$$

$$r = \langle |I_3 - I_1| \rangle = \langle |4A_rA_o \sin[\varphi - (\alpha_1 + \alpha_2)/2]| \rangle \sin[(\alpha_1 + \alpha_2)/2], \quad (6)$$

where $\langle\,\rangle$ is the average over the whole frame and the coordinate independence of A_o and φ is omitted here for conciseness. Supposing that $A_o(x, y)$ and $\varphi(x, y)$ are mutually independent and $\varphi(x, y)$ is a spatially random distribution that is due to Fresnel diffraction, we can reasonably expect all the terms $\langle | \ldots | \rangle$ on the right-hand side of Eqs. (4)–(6) to have approximately the same value c; then we have

$$p = c\sin(\alpha_1/2), \qquad q = c\sin(\alpha_2/2), \qquad r = c\sin[(\alpha_1 + \alpha_2)/2]. \quad (7)$$

It is not difficult to find the constant

$$c = 2pqr[2(p^2q^2 + p^2r^2 + q^2r^2) - (p^4 + q^4 + r^4)]^{-1/2} \quad (8)$$

from Eq. (7) by trigonometry, and α_1 and α_2 can then be calculated with

$$\alpha_1 = 2\arcsin(p/c), \qquad \alpha_2 = 2\arcsin(q/c). \quad (9)$$

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This result can be further improved by the iterative approach described below. First we substitute the values of α_1 and α_2 in Eq. (9) into Eq. (10) [Eqs. (10) and (14) below can be derived directly from the expressions of I_j ($j = 1, 2, \ldots, N$)[16]],

$$A_0 \exp(i\varphi) = \frac{1}{4A_r \sin(\alpha_2/2)} \left\{ \frac{\exp(i\alpha_1/2)}{\sin[(\alpha_1 + \alpha_2)/2]} (I_1 - I_3) - \frac{\exp[i(\alpha_1 + \alpha_2)/2]}{\sin(\alpha_1/2)} (I_1 - I_2) \right\}, \tag{10}$$

to get both the amplitude $A_o(x, y)$ and the phase $\varphi(x, y)$ by use of measured constant intensity $I_r = A_r^2$, and then we can construct an error function

$$\begin{aligned} \epsilon = \sum \{ & |I_1 - (A_r^2 + A_o^2 + 2A_r A_o \cos \varphi)| \\ & + |I_2 - [A_r^2 + A_o^2 + 2A_r A_o \cos(\varphi - \alpha_1)]| \\ & + |I_3 - [A_r^2 + A_o^2 + 2A_r A_o \cos(\varphi - \alpha_1 - \alpha_2)]| \}, \end{aligned} \tag{11}$$

where $\sum$ denotes summation over the frame. Theoretically ϵ is a smooth function of (α_1, α_2) and will reach its minimum value of zero when α_1 and α_2 are equal to their exact values that we used to record the interferograms. Therefore we can change α_1 and (or) α_2 gradually, step by step, to reduce ϵ until it reaches its minimum.

In the case of four-frame measurement, we can first treat this case as a three-frame measurement consisting of I_1, I_2, and I_3 and calculate the accurate values of α_1 and α_2 with the above method. Then using the expressions

$$\begin{aligned} I_4(x, y) = {} & A_r^2 + A_o^2(x, y) + 2A_r A_o(x, y) \\ & \times \cos[\varphi_o(x, y) - \alpha_1 - \alpha_2 - \alpha_3], \end{aligned} \tag{12}$$

$$\begin{aligned} s &= \langle |I_4 - I_3| \rangle \\ &= \langle |4A_o A_r \sin(\varphi - \alpha_1 - \alpha_2 - \alpha_3/2)| \rangle \sin(\alpha_3/2) \\ &= c \sin(\alpha_3/2), \end{aligned} \tag{13}$$

where α_3 is the third phase step and c is still the constant in Eq. (8), we can get an approximate α_3 from Eq. (13). With this α_3 and accurate values for α_1 and α_2, it is possible to retrieve the optical field by employing the expression for general four-frame PSI[16]:

$$\begin{aligned} A_o \exp(i\varphi) = {} & \frac{1}{4A_r \sin[(\alpha_1 + \alpha_3)/2]} \\ & \times \left\{ \frac{\exp[i(\alpha_1 + \alpha_2)/2]}{\sin[(\alpha_2 + \alpha_3)/2]} (I_2 - I_4) \right. \\ & \left. - \frac{\exp[i(\alpha_1 + \alpha_2/2 + \alpha_3/2)]}{\sin[(\alpha_1 + \alpha_2)/2]} (I_1 - I_3) \right\}. \end{aligned} \tag{14}$$

Once $A_o(x, y)$ and $\varphi(x, y)$ are calculated, we can use them to evaluate a new error function for α_3,

$$\begin{aligned} \epsilon' = \sum | I_4 - [& A_r^2 + A_o^2 \\ & + 2A_r A_o \cos(\varphi - \alpha_1 - \alpha_2 - \alpha_3)] |, \end{aligned} \tag{15}$$

and change α_3 step by step to minimize ϵ' as we did above. Therefore, we can see that, when α_1 and α_2 are precisely determined, a one-frame increase needs only a new error function consisting of only one variable. This is also true for the case with more frames.

A series of computer simulations has been carried out to verify quantitatively the effectiveness of our proposed method for phase-shift extraction. The test objects are spherical surfaces with different radii R, from 0.2 to 10 m, located a distance from plane P_H, and the computation process and parameters are the same as those we used before.[16] Two-dimensional Fresnel diffraction of the complex amplitude of original object plane P_o onto plane P_H serves as object wave $O(x, y)$. Different interferograms are computer generated with different nonstandard phase shifts. Both the three- and four-frame methods are tested.

For conciseness, here we give only a few calculation results of the four-frame method in Tables 1 and 2, the former is for an object surface of $R = 0.2$ m and the latter is for $R = 2$ m. In all these simulations we suppose that $\alpha_1 = 1.3963\,(80°)$, $\alpha_2 = 1.8326\,(105°)$, and $\alpha_3 = 1.5708\,(90°)$, and we consider both the smooth and the diffusing surfaces (with a Gaussian phase noise Δh of zero mean and $\lambda/12$ standard deviation added to h). From these tables we can see that the initial values of α_1, α_2, and α_3 directly calculated with Eqs. (9) and (13) have already reached quite high accuracy (the relative error is usually less than 0.4% and 1.7% in the worst case). In the iteration approach we have designed a simple program that can automatically adjust the step sizes of α_1 and α_2, calculate the error function, compare its values in the vicinity of (α_1, α_2), and then decide the direction in which α_1 and (or) α_2 should go to decrease ϵ. Specifically, the step sizes that we used here are 0.01 first, then 0.001, and finally 0.0001. The final accuracy, at least theoretically, is limited only by the step size.

From Tables 1 and 2 it is clear that the values of α_1 and α_2 can be retrieved exactly after 10–20 iterations, and the precise value of α_3 can be obtained even faster, within 10 steps. These tables also show that our algorithm works well for both the smooth and the diffusing surfaces, and usually it takes fewer iteration steps for a diffusing surface than for a smooth surface of the same radius R or for the same surface with smaller R. This fact can easily be explained, because the assumption of a random phase in the diffraction field is more likely satisfied by an object with stronger depth fluctuation.

Once the actual steps are obtained, wave front $A_o(x, y)\exp[i\varphi(x, y)]$ in plane P_H can be calculated from Eqs. (10) and (14), and the wave front in object plane P_o can be further determined with the use of digital inverse Fresnel diffraction.[14,16]

In conclusion, we have proposed a new generalized algorithm to extract the actual phase steps from the interferograms formed by the diffraction field of an object in general PSI. These phase steps are first

Table 1. Phase Step Retrieval Results for a Surface of Radius $R = 0.2$ m[a]

	Smooth Surface				Diffusing Surface		
	α_1	α_2	α_3		α_1	α_2	α_3
Assumed value	1.3963	1.8326	1.6581	Assumed value	1.3963	1.8326	1.6581
Value extracted by Eqs. (9) and (13)	1.3976	1.8354	1.6625	Value extracted by Eqs. (9) and (13)	1.3978	1.8315	1.6606
Iteration				Iteration			
4	1.3936	—	×	1	1.3968	—	×
7	1.3966	1.8324	×	2	—	1.8325	×
9	1.3964	—	×	7	1.3963	—	×
10	—	1.8325	×	8	—	1.8326	×
11	1.3963	—	×	11	—	—	1.6576
12	—	1.8326	×	16	—	—	1.6581
16	—	—	1.6585				
20	—	—	1.6581				

[a] In Tables 1 and 2 the values of α_1 and α_2 are shown only for the turning points in (α_1, α_2). ×, not used; —, unchanged. Iteration denotes the number of steps from the beginning.

Table 2. Phase Step Retrieval Results for a Surface of Radius $R = 2$ m

	Smooth Surface				Diffusing Surface		
	α_1	α_2	α_3		α_1	α_2	α_3
Assumed value	1.3963	1.8326	1.6581	Assumed value	1.3963	1.8326	1.6581
Value extracted by Eqs. (9) and (13)	1.4202	1.8334	1.6817	Value extracted by Eqs. (9) and (13)	1.4012	1.8354	1.6629
Iteration				Iteration			
2	1.4002	—	×	1	1.3912	—	×
7	1.3952	—	×	3	1.3932	—	×
8	1.3962	1.8324	×	6	1.3962	1.8324	×
10	1.3964	—	×	8	—	1.8326	×
11	1.3963	1.8325	×	9	1.3963	—	×
12	—	1.8326	×	14	—	—	1.6579
14	—	—	1.6617	16	—	—	1.6581
18	—	—	1.6577				
22	—	—	1.6581				

calculated approximately based on the statistical nature of the diffraction field and then improved by the iterative approach. This method has the following advantages over other similar methods. First, it can work well for arbitrary and unequal unknown phase steps $\{\alpha_j\}$ and for any frame number $N \geq 3$. Second, it is effective for both smooth and diffusing object surfaces. Third, measurement of the object wave intensity is not necessary as it is in some other methods.[15,16] Finally, the calculation procedure is simple, fast (usually within 20 iterations in $\sim$1 min), and able to yield results with a high degree of accuracy. These features have been verified by a series of computer simulations. We believe that this method can find practical application in PSI for wave-front measurement and reconstruction.

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