Numerical Kawahara Equation Solver and Stability Analysis

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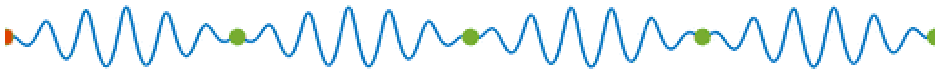
I: Intro to KdV and Kawahara Equations

A General 5th – order ***KdV****:*

$$u_t = Vu_x + \alpha u_{xxx} + \beta u_{5x} + \sigma(u^{p+1})_x$$

- The **KdV** (Korteweg–De Vries) **equation** is a nonlinear, dispersive PDE with 2 variables: time ***t*** and space ***x***.
  - dispersive: waves of different wavelength propagate at different phase velocities.



Application: Modeling **shallow water** waves with **weakly non-linear** restoring forces.

I: Intro to KdV and Kawahara Equations

- The **Kawahara equation** is a special case of KdV where p = 1.

A General 5th – order ***KdV****:*

$$u_t = Vu_x + \alpha u_{xxx} + \beta u_{5x} + \sigma(u^{p+1})_x$$

The ***Kawahara*** *Equation:*

$$u_t = Vu_x + \alpha u_{xxx} + \beta u_{5x} + \sigma(u^2)_x$$

Application: Modeling **shallow water** waves under the effect of **gravity** and **surface tension**. (ie. water surface covered by ice sheet)

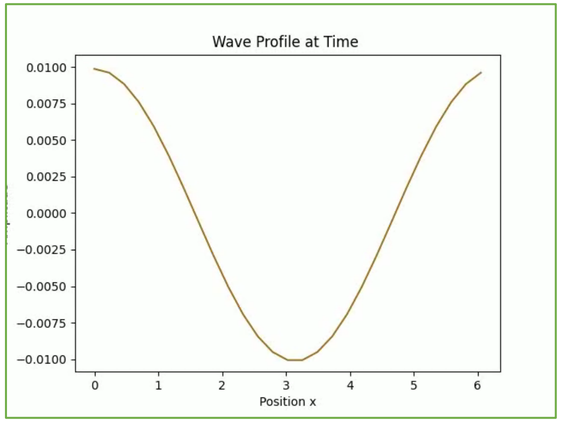
I: Intro to KdV and Kawahara Equations

*The **Kawahara** Equation:*

$$u_t = V u_x + \alpha u_{xxx} + \beta u_{5x} + \sigma (u^2)_x$$

Initial Condition (a small amplitude travelling wave):

$$u(x,t) = \boldsymbol{u}^{(0)}(\boldsymbol{x}) + \delta e^{\lambda t} \boldsymbol{u}^{(1)}(\boldsymbol{x}) + \cancel{O(\delta^2)}$$



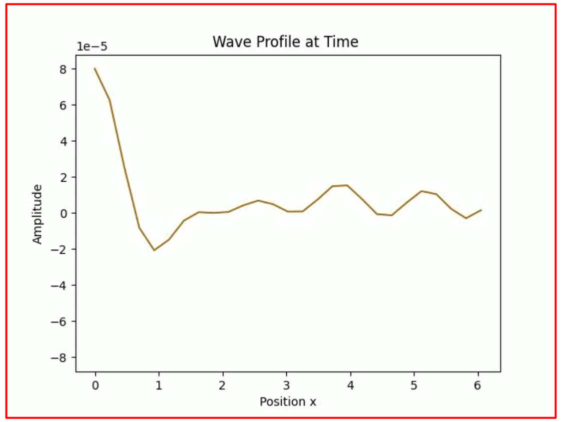
Stable Part $\boldsymbol{u}^{(0)}$

$$\boldsymbol{u}^{(0)}(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} \hat{\boldsymbol{u}}_k^{(0)} \boldsymbol{e}^{ikx}$$

Unstable Part $\boldsymbol{u}^{(1)}$

(if Re$\{\delta\}$ > 0)

$$\boldsymbol{u}^{(1)}(\boldsymbol{x}) = \boldsymbol{e}^{i\mu x} \sum_{m=-\infty}^{\infty} \hat{\boldsymbol{u}}_k^{(1)} \boldsymbol{e}^{imx} + \boldsymbol{c.c.}$$

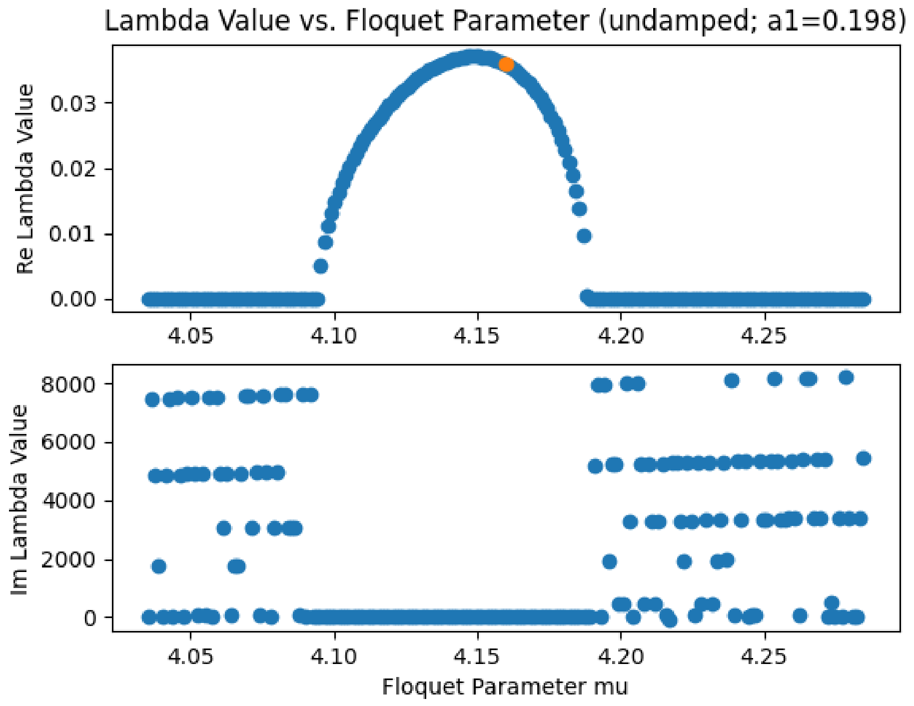


I: Intro to KdV and Kawahara Equations

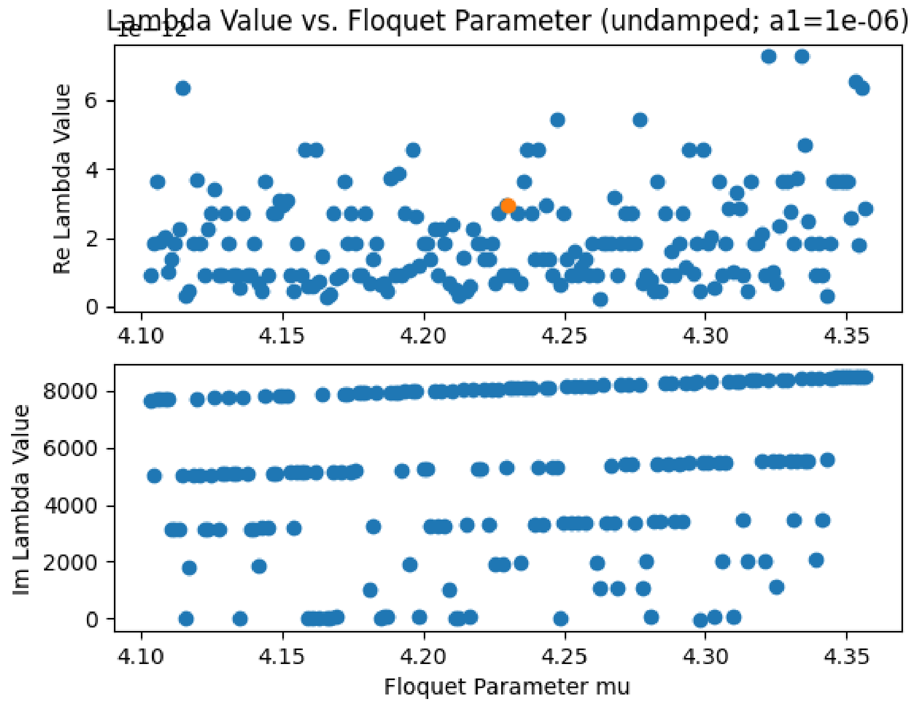
- Floquet Parameter

Previous theories predict the stability of an initial condition given to the Kawahara equation:

1) Floquet Parameter μ 2) Eigenvalue λ



Expected Unstable



Expected Stable

arXiv:1806.08445v1 [nlin.PS] 21 Jun 2018

I: Intro to KdV and Kawahara Equations

- To develop a ***time-dependent*** Kawahara equation solver that takes ***any*** form of initial condition and determines the stability of ***general cases***.

II: Numerical Kawahara Solver

Class `waveEquation`

"Kawahara and Numerical Integration Related Functions"

`kawahara_model`
`u0_leading_coeff`
`fourierCoeffMatrix`
`u1_leading_terms`
`kawahara_stationary_solution`
`kawahara_combined_solution`
`kdv_soliton_solution`
`solve_kawahara`

Class `visual`

"Plot and Video Related Functions"

`plot_all`
`plot_profile`
`plotInitial`
`plot_video`
`amplitude`
`simple_plot`

Class `numAnalysis`

"Solution Data Acquisition Related Functions"

`collect_u0`
`fourierCoeffMatrix`
`find_stable_lamb`
`collect_u1`
`dampingAnalysis`
`optimize_u0Coeff`
`lambdaFloquet`
`optimize_Floquet_a1`
`numError`
`spectrogram`

Class `analytical`

"Floquet, Error, and Stability Analysis Related Functions"

II: Numerical Kawahara Solver

- Computation Workflow and Time

CPU Spec: AMD Ryzen 7 3700X 8 Core 16 Processes Overclocked 4.05 GHz

RAM Spec: Corsair DDR-4 32 Gb; 3600 MHz

Case Specific: 1 – 60 s
Damping, Optimization Calculation

130 – 140 s
Wave Equation Calculation

Initial Condition u
0.3 – 0.4 s

Methods:
Differentiation: *Numpy* Spectral Derivative
Integration: *Scipy* RK-45

1 – 3 s
Solution Analysis Calculation

Methods:
Time-Frequency: *Matplotlib* DTFT w/ Hanning Window

All frames in RAM

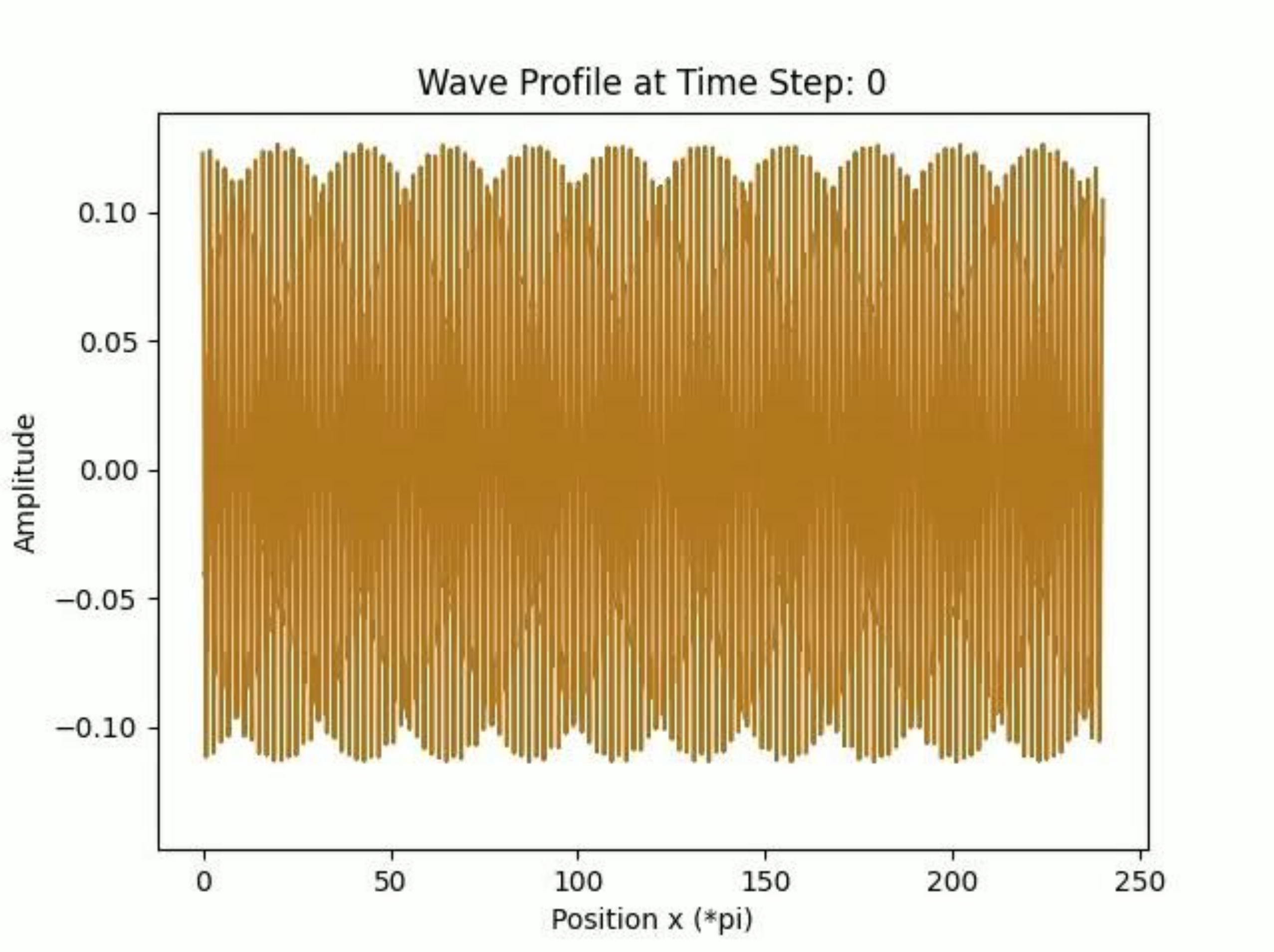
Case Specific: 2 – 10 min
Final Wave Profile Video Construction

× 14 Parallel Processes (depending on # of CPU cores)

Most Cases: under 10 min

III: Stability

- Special Cases Demonstration



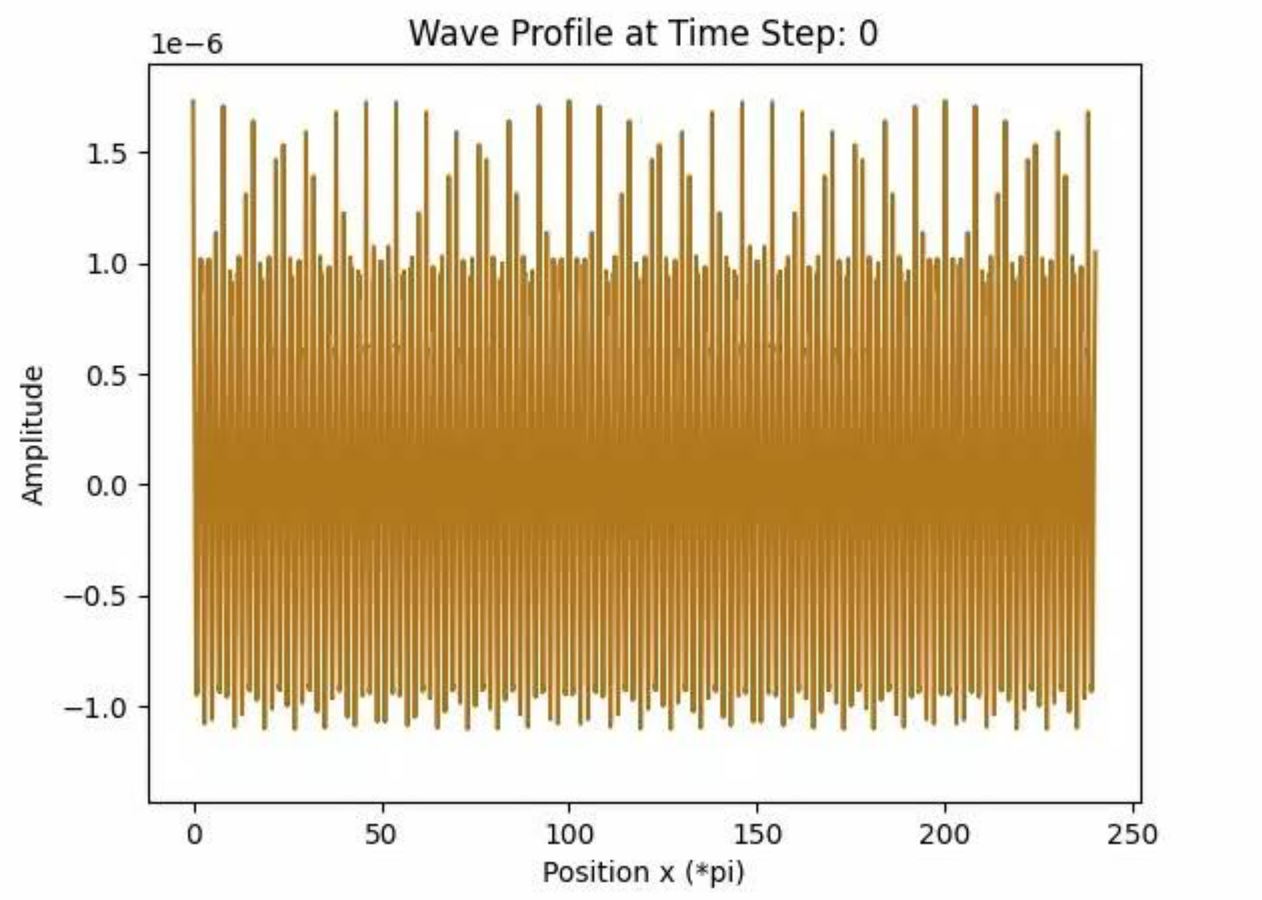
Stable

Recorded Feature: Wave Profile

Parameter	Value
Floquet Parameter μ	5.09
λ	i62.82
a1	10^{-1}
L	240π
β	$\frac{3}{160}$
δ	10^{-2}
t	1
t-steps	400

III: Stability

- Special Cases Demonstration

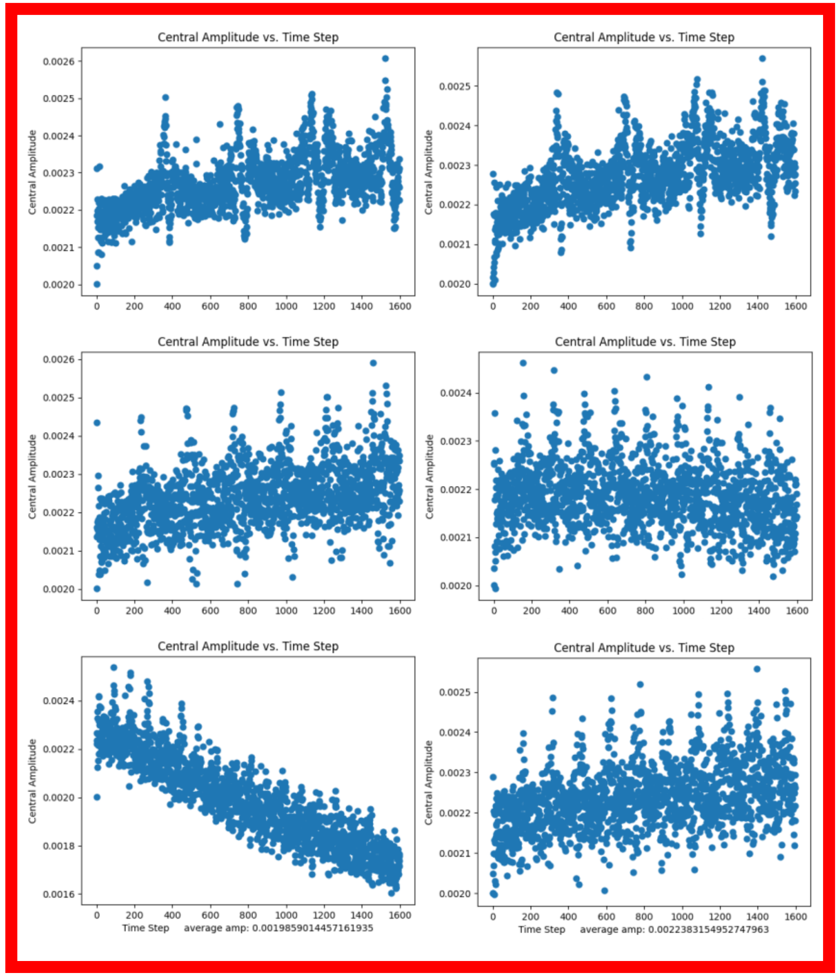


Unstable

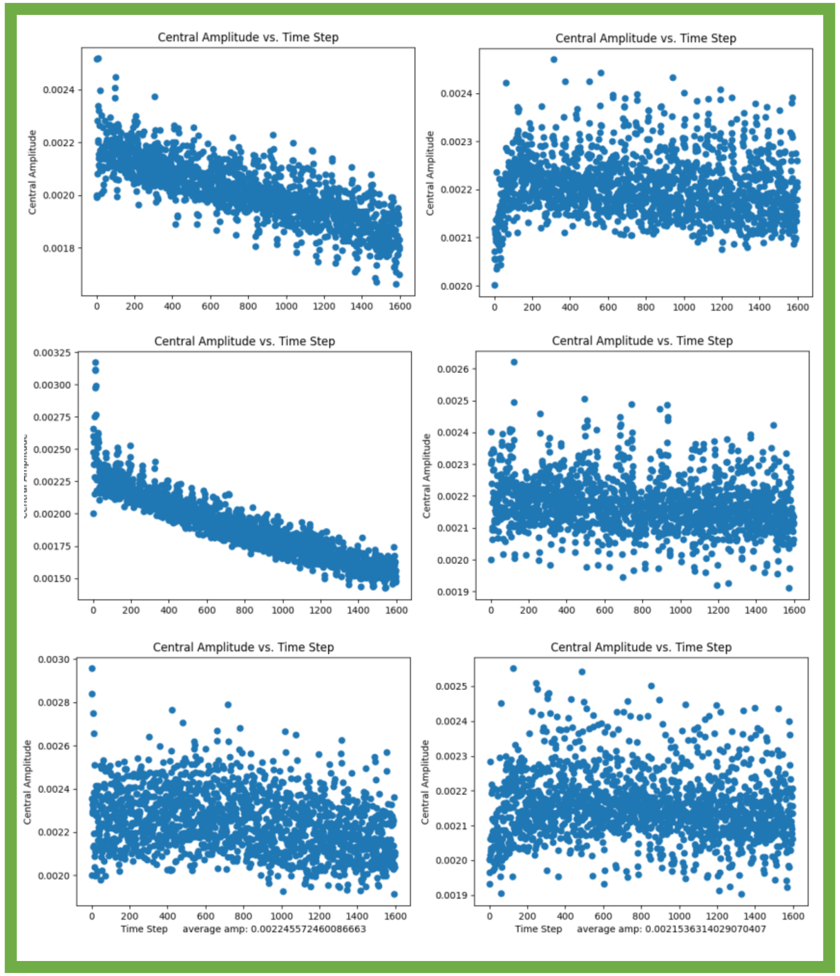
Recorded Feature: Wave Profile

Parameter	Value
Floquet Parameter μ	0.74
λ	-i64.87
a1	10^{-6}
L	240π
β	$\frac{3}{160}$
δ	10^{-2}
t	2
t-steps	2000

III: Stability



Expected Unstable



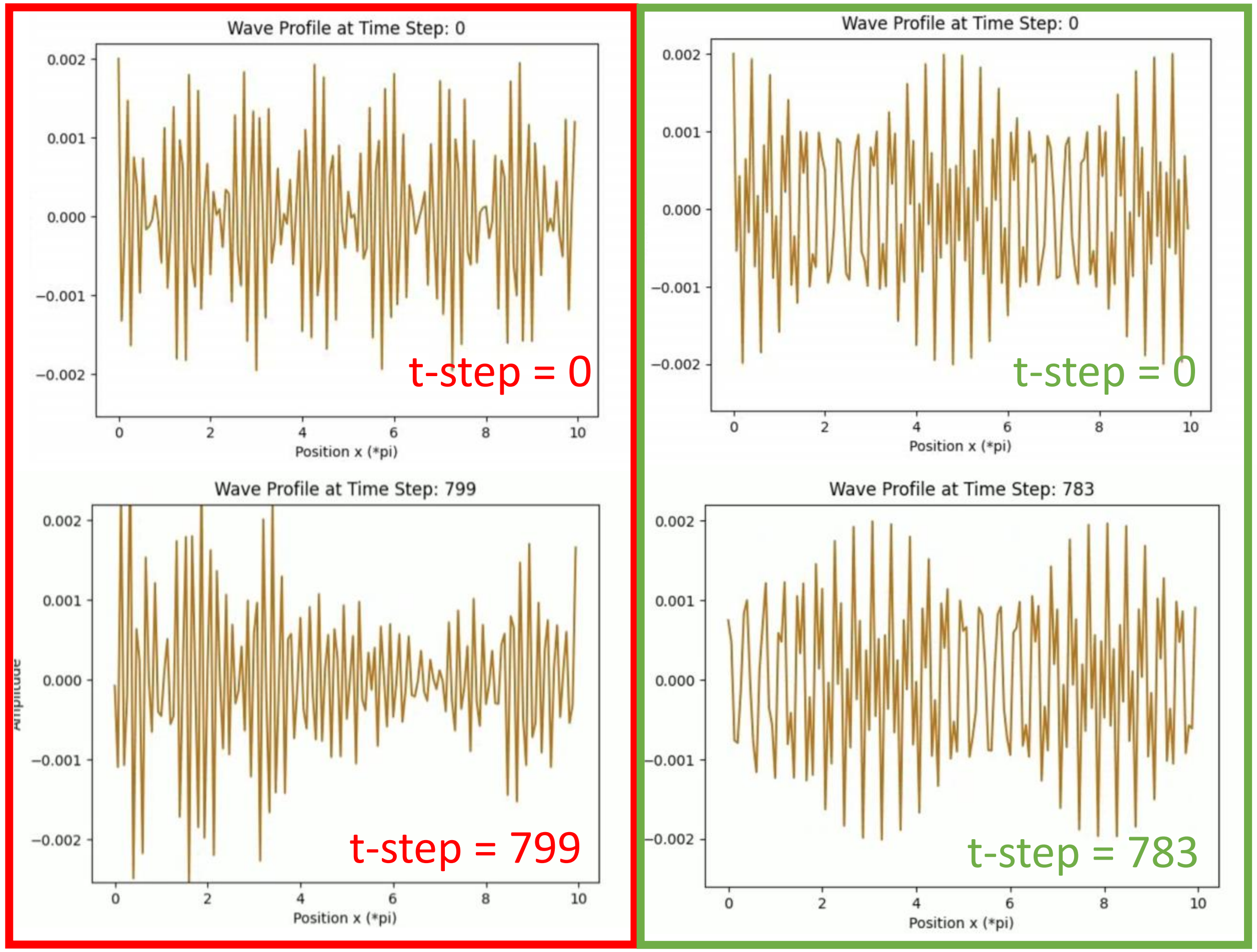
Expected Stable

Recorded Feature:
Max Amplitude

Parameter	Value
Floquet Parameter μ	-
λ	-
a1	10^{-6}
L	240π
β	$\frac{3}{160}$
δ	10^{-3}
t	5
t-steps	1600

III: Stability

- Wave Profile Deformation

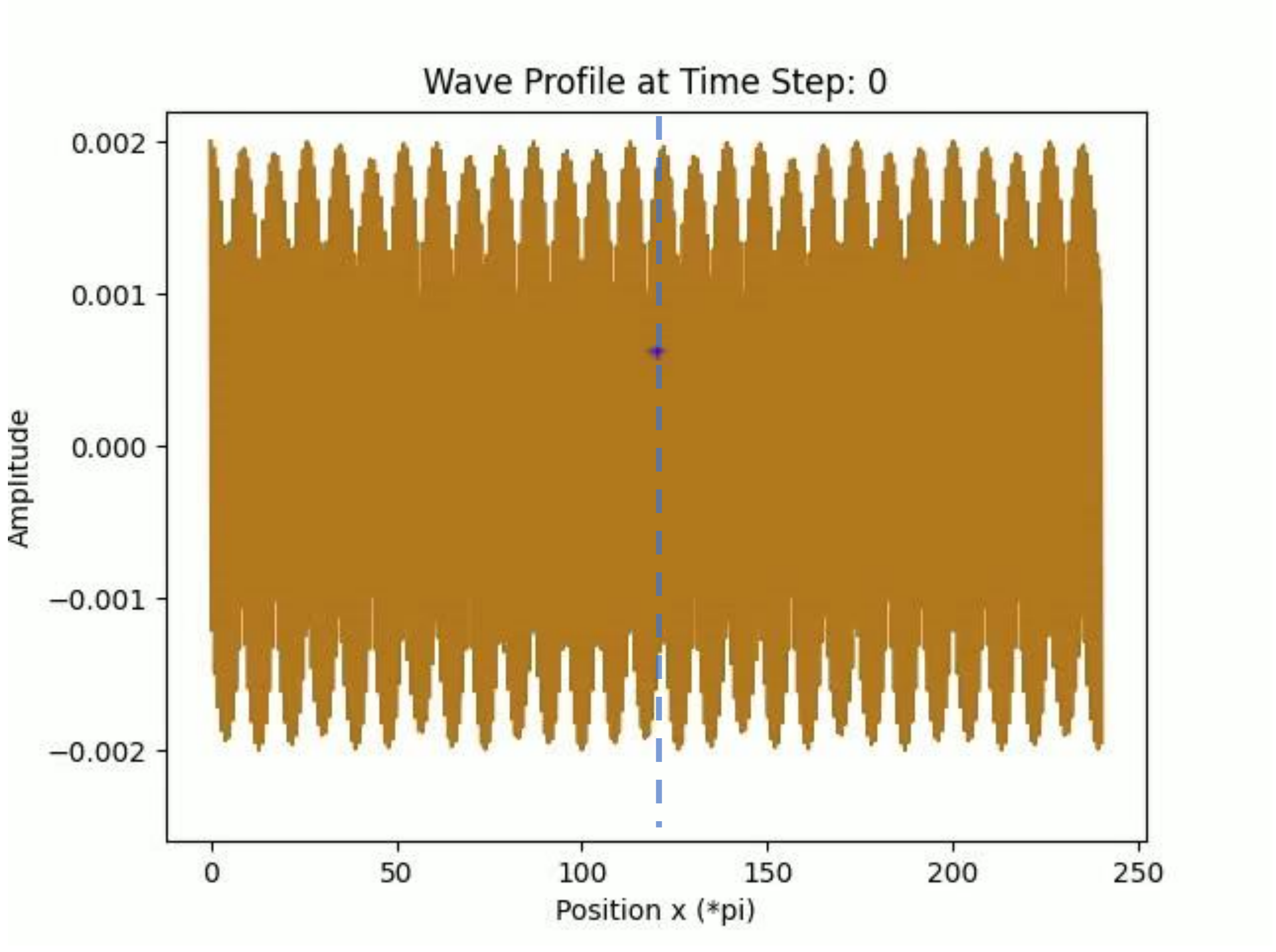


Expected Unstable

Expected Stable

III: Stability

- Amplitude Data Acquisition Demonstration

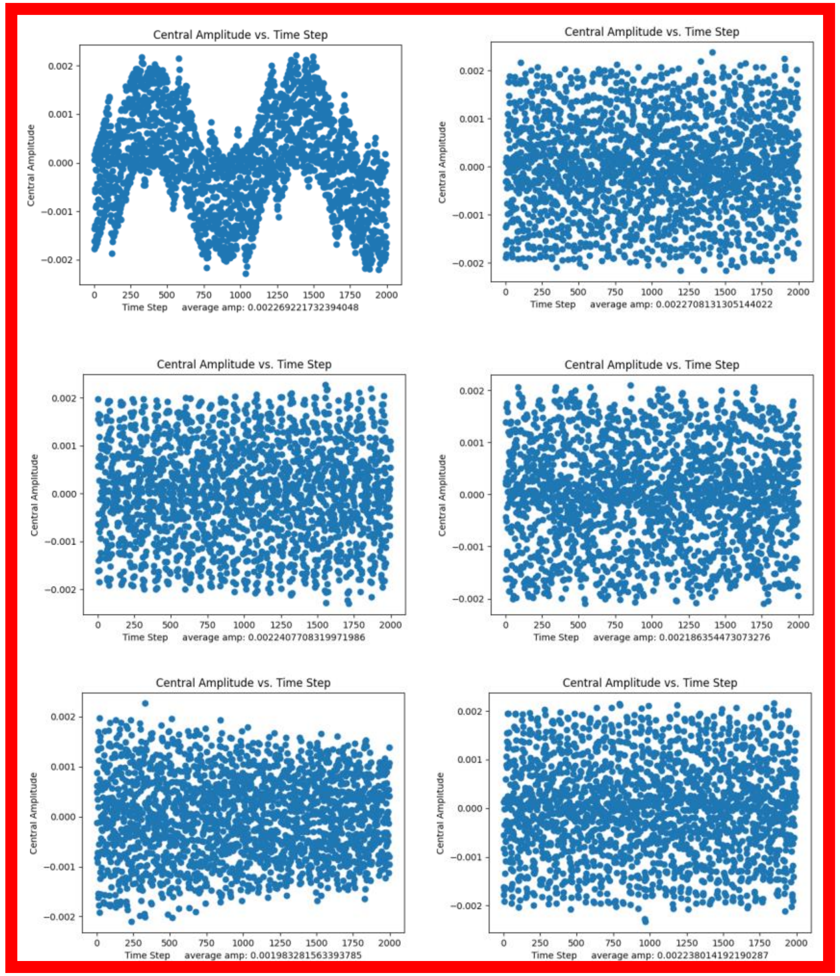


Recorded Feature:

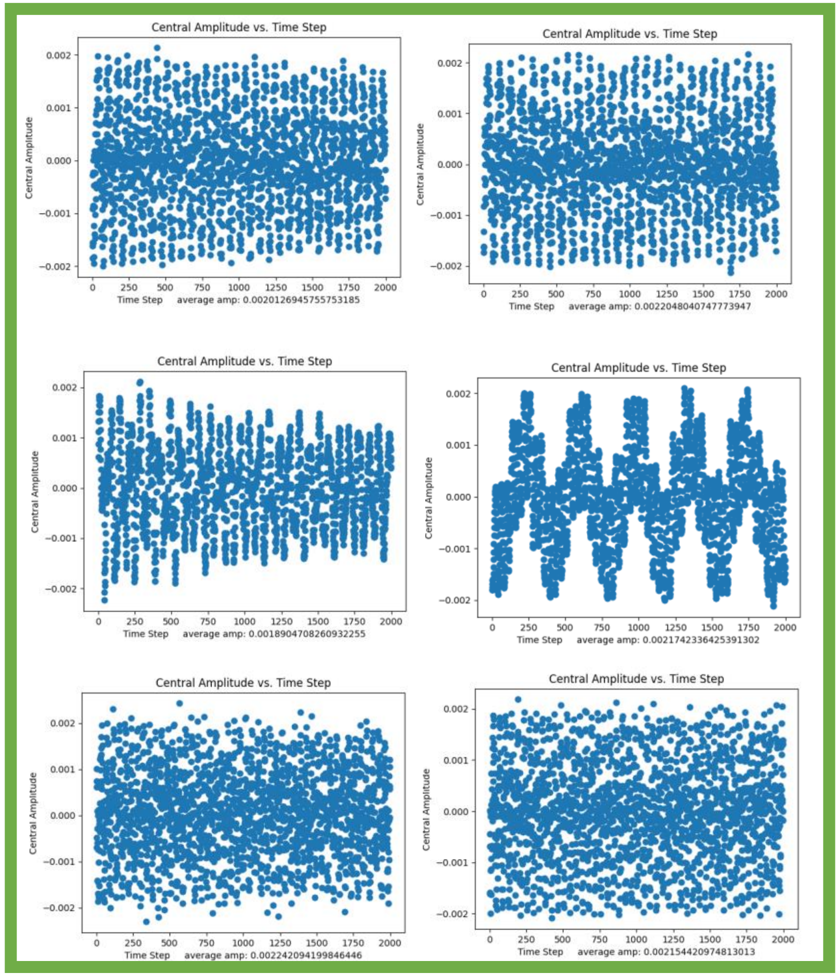
Central Amplitude Time Series

Parameter	Value
Floquet Parameter μ	-3.23
λ	-i0.305
a1	10^{-6}
L	240π
β	$\frac{3}{160}$
δ	10^{-2}
t	5
t-steps	2000

III: Stability



Expected Unstable

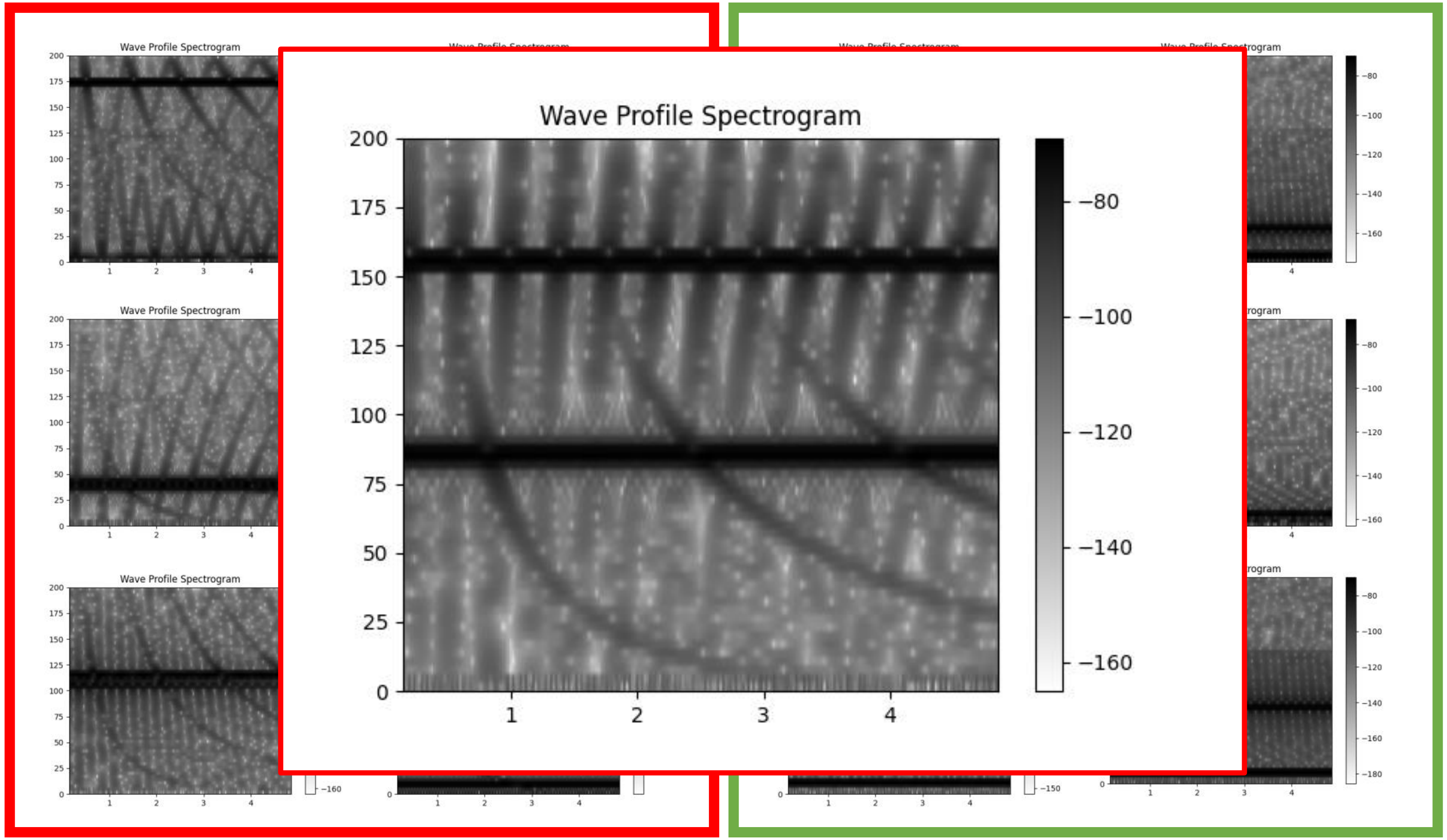


Expected Stable

Recorded Feature:
Central Amplitude Time Series

Parameter	Value
Floquet Parameter μ	-
λ	-
a1	10^{-6}
L	240π
β	$\frac{3}{160}$
δ	10^{-3}
t	5
t-steps	2000

III: Stability



Unstable

Stable

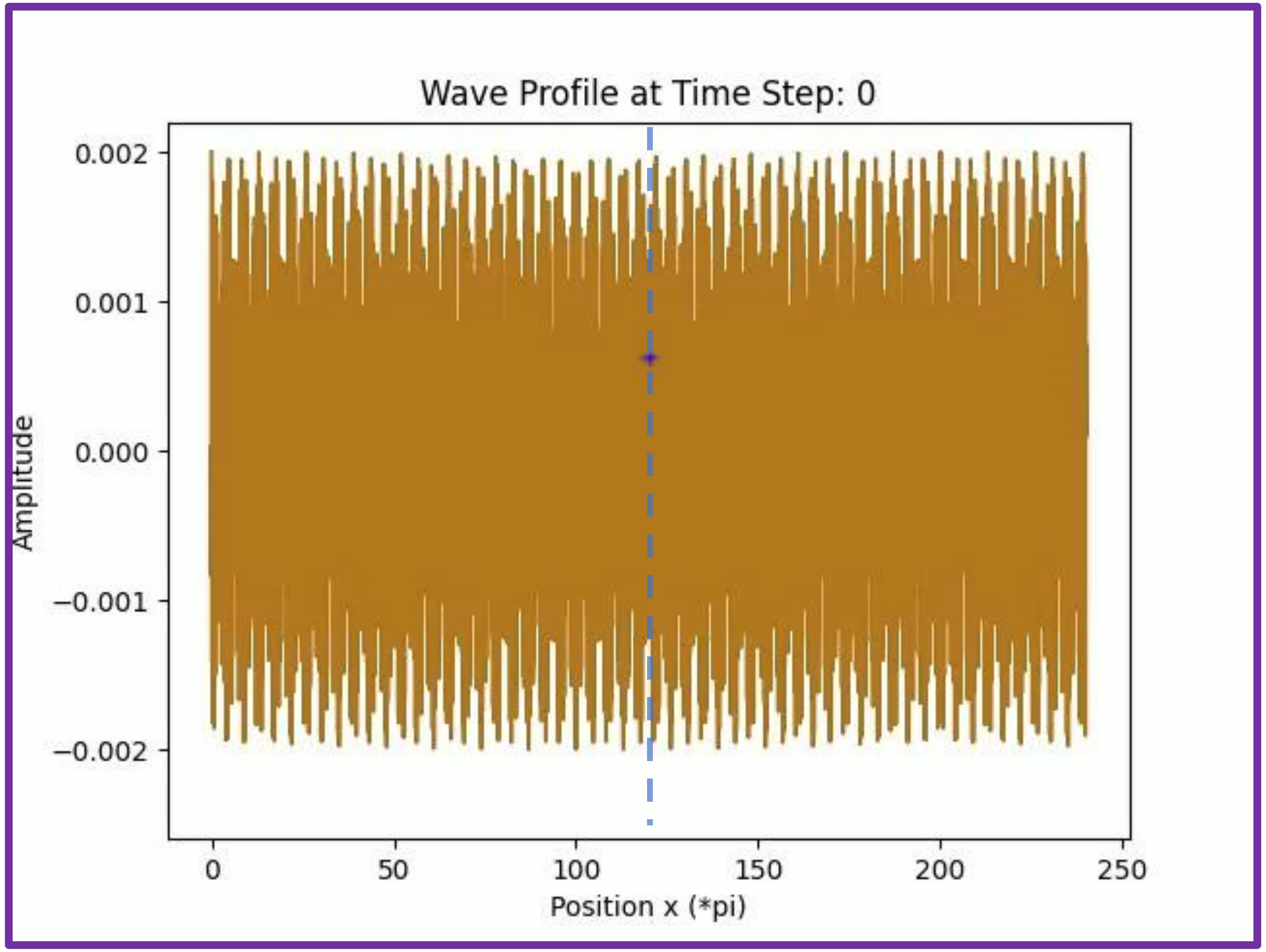
Recorded Feature:

Central Amplitude Spectrogram

Parameter	Value
Floquet Parameter μ	-
λ	-
a1	10^{-6}
L	240π
β	$\frac{3}{160}$
δ	10^{-3}
t	5
t-steps	2000

IV: Damping

- A Way to Stabilize Any Case



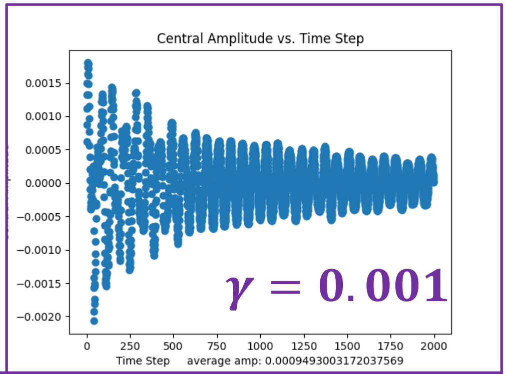
Damped

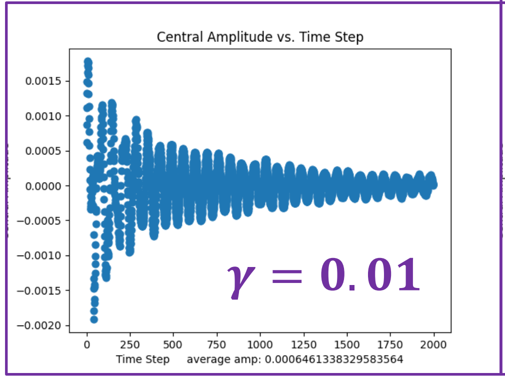
The Damped ***Kawahara*** *Equation:*

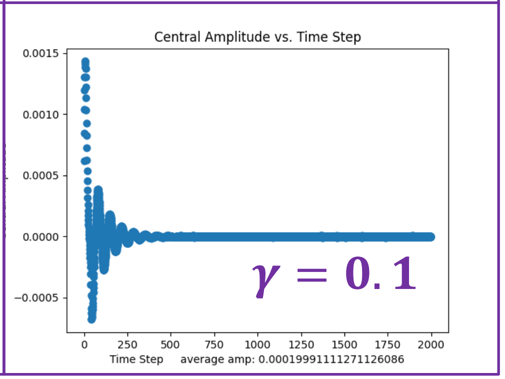
$$u_t = Vu_x + \alpha u_{xxx} + \beta u_{5x} + \sigma(u^2)_x + \boldsymbol{\gamma u_{xx}}$$

Recorded Feature:

Central Amplitude Time Series

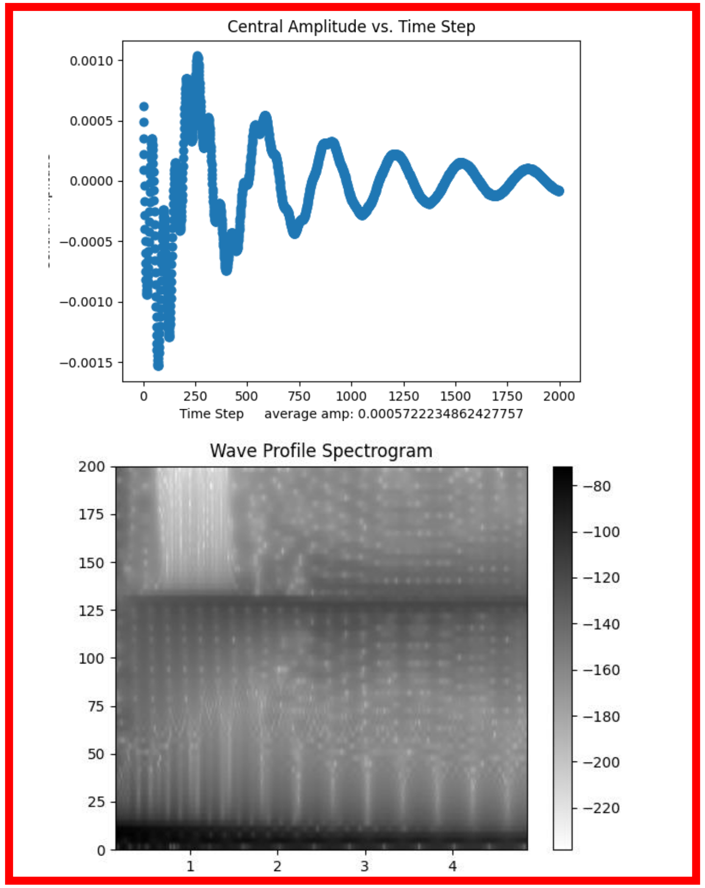




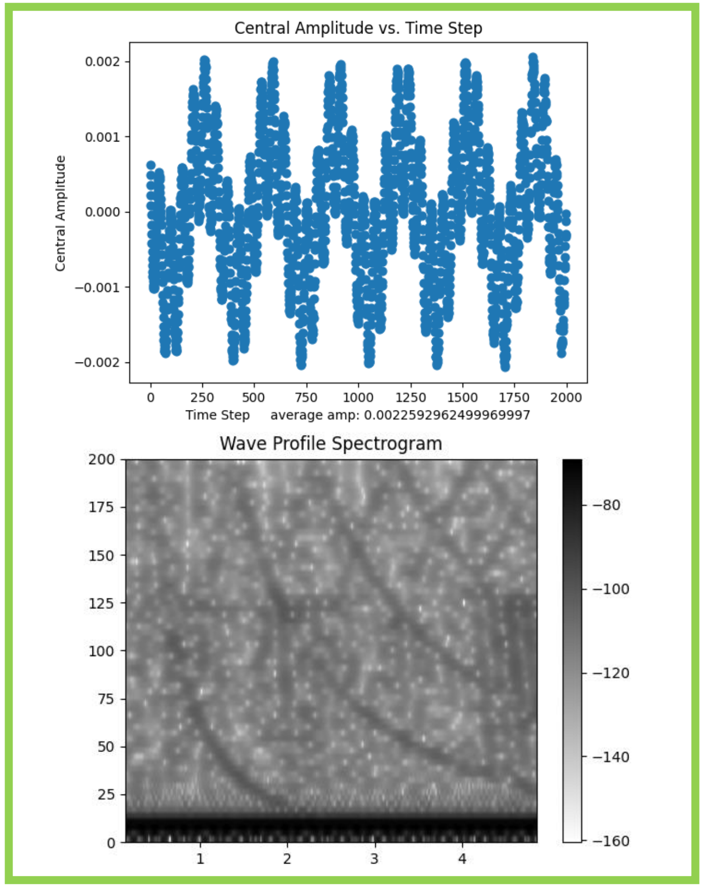


IV: Damping

- A Way to Stabilize Any Case



Hard Damping



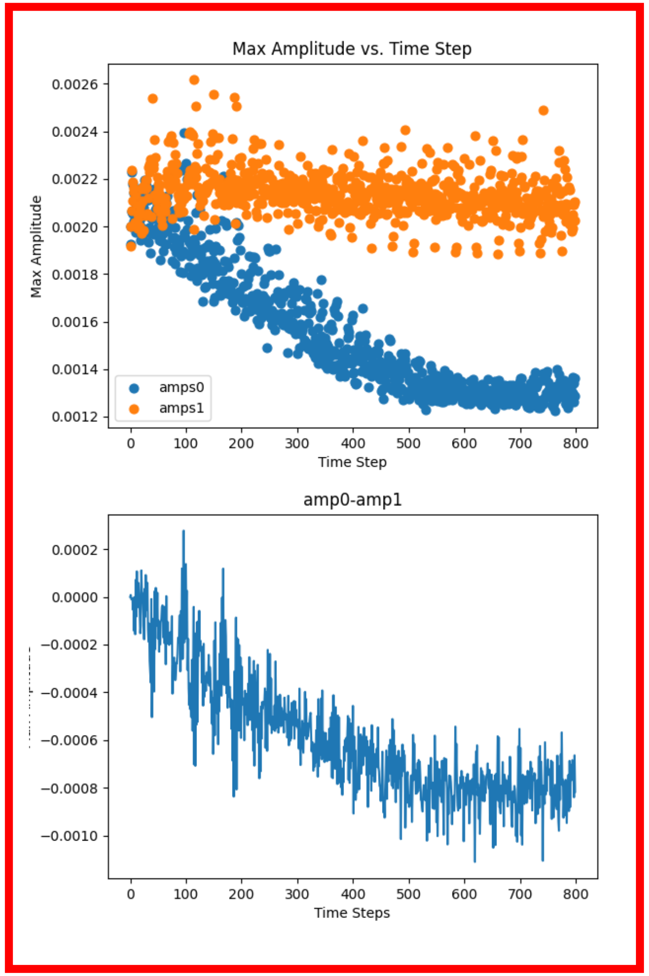
No Damping

Recorded Feature:

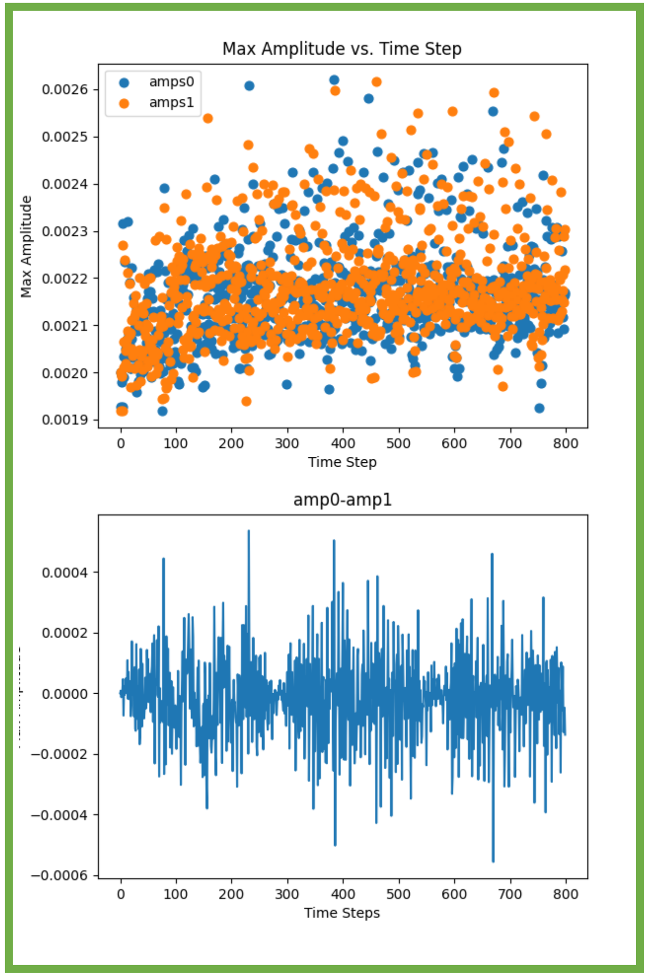
Central Amplitude and Spectrogram

Parameter	Value
Floquet Parameter μ	-3.23
λ	-i0.305
a1	10^{-6}
Damping	10^{-1}
L	240π
β	$\frac{3}{160}$
δ	10^{-3}
t	5
t-steps	2000

V: Numerical Error



Major Error



Minor Error

Recorded Feature:
Max Amplitude

Parameter	Value
Floquet Parameter μ	5.09
λ	i62.82
a1	10^{-6}
L	240π
β	$\frac{3}{160}$
δ	10^{-3}
t	8 | 2
t-steps	800 | 2400
t-steps Ratio	1:3

VI: Conclusions

- A numerical Kawahara equation solver was developed.
- A general method for determining the stability of a numerical solution to the Kawahara was developed and tested.
- Damping was proven to be an effective method for imposing stability to any cases. (*although the exact damping parameter that removes instability but does not suppress the overall amplitude of a wave can be difficult to find*)