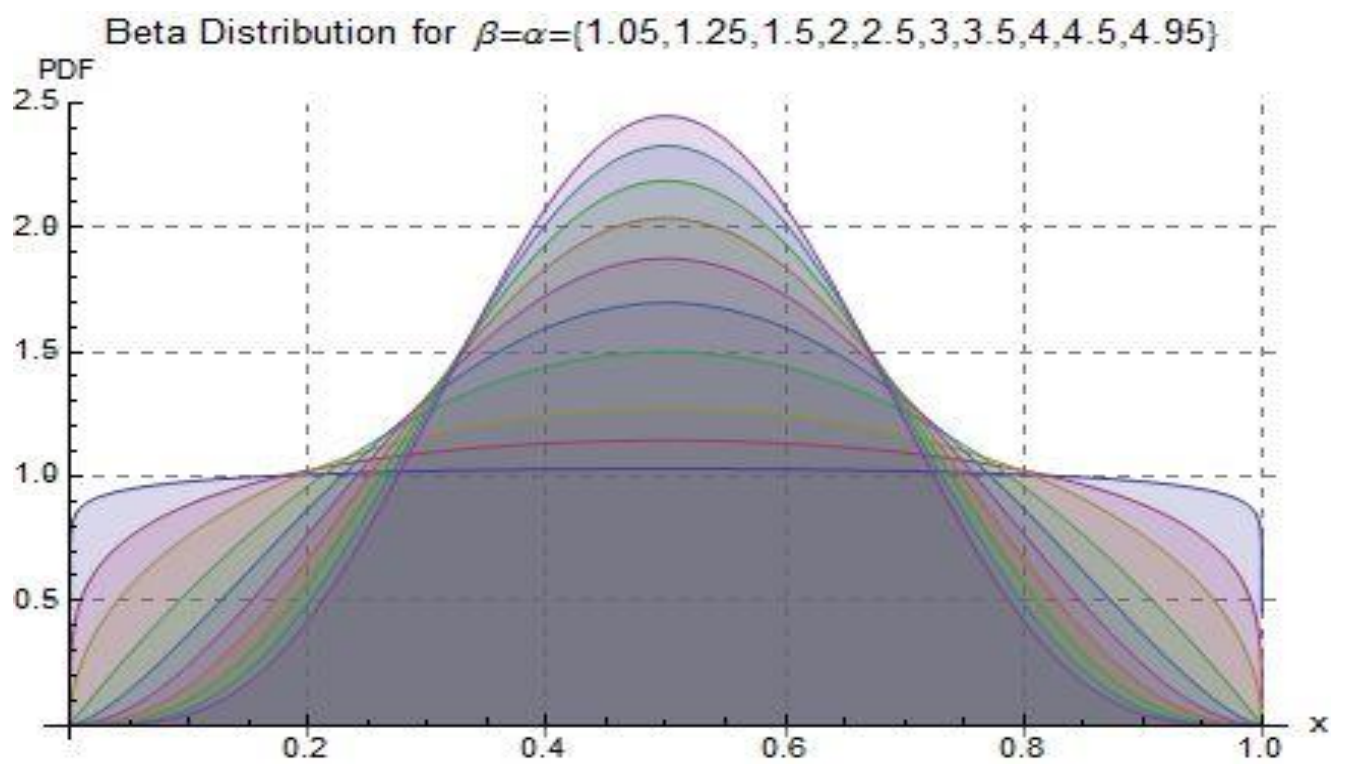
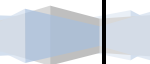


BETA DISTRIBUTION



Arun Mahanta

**Associate Professor
Dept. of Mathematics,
Kaliabor College.**



Introduction:

In probability theory and statistics, the **beta distribution** is a family of continuous probability distributions defined on the interval $[0, 1]$ with two positive shape parameters, denoted by α and β , that appear as exponents of the random variable and control the shape of the distribution.

The beta distribution has been applied to model the behavior of random variables limited to intervals of finite length in a wide variety of disciplines. For example, it has been used as a statistical description of allele frequencies in population genetics; time allocation in project management / control systems; sunshine data; variability of soil properties; proportions of the minerals in rocks in stratigraphy; and heterogeneity in the probability of HIV transmission.

In Bayesian inference, the beta distribution is the conjugate prior probability distribution for the Bernoulli, binomial and geometric distributions. For example, the beta distribution can be used in Bayesian analysis to describe initial knowledge concerning probability of success such as the probability that a space vehicle will successfully complete a specified mission. The beta distribution is a suitable model for the random behavior of percentages and proportions.

The usual formulation of the beta distribution is also known as the beta distribution of the first kind, whereas *beta distribution of the second kind* is an alternative name for the beta prime distribution.

GAMMA FUNCTION

Definition.

For x positive we define the **Gamma function** by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

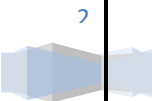
This integral cannot be easily evaluated in general, therefore we first look at the Gamma function at two important points. We start with $x = 1$:

$$\begin{aligned} \Gamma(1) &= \int_0^{\infty} t^0 e^{-t} dt = \int_0^{\infty} e^{-t} dt = \left[-e^{-t} \right]_0^{\infty} \\ &= \lim_{t \rightarrow \infty} \left(\frac{-1}{e^t} \right) - (-1) = 0 + 1 = 1. \end{aligned}$$

Now we look at the value at $x = 1/2$:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{-1/2} e^{-t} dt = \left| \begin{array}{l} y = \sqrt{t} \\ 2dy = t^{-1/2} dt \\ t = y^2 \end{array} \right| = 2 \int_0^{\infty} e^{-y^2} dy = \sqrt{\pi}.$$

To find the value of the Gamma function at other points we deduce an interesting identity using integration by parts:



$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt = \left| \begin{array}{l} f = t^x \quad g' = e^{-t} \\ f' = x t^{x-1} \quad g = -e^{-t} \end{array} \right| \\
&= \left[-t^x e^{-t} \right]_0^\infty - \int_0^\infty -x t^{x-1} e^{-t} dt = -\lim_{t \rightarrow \infty} \left(\frac{t^x}{e^t} \right) + x \int_0^\infty t^{x-1} e^{-t} dt \\
&= 0 + x\Gamma(x) = x\Gamma(x).
\end{aligned}$$

The limit is evaluated using l'Hospital's rule several times. We see that for x positive we have

$$\Gamma(x+1) = x\Gamma(x).$$

If we apply this to a positive integer n , we get

$$\begin{aligned}
\Gamma(n) &= (n-1)\Gamma(n-1) = (n-1) \cdot (n-2)\Gamma(n-2) = \dots \\
&= (n-1) \cdot (n-2) \cdots 2 \cdot 1 \cdot \Gamma(1) = (n-1)!.
\end{aligned}$$

So, we see that the Gamma function is a generalization of the factorial function.

BETA FUNCTION

Definition.

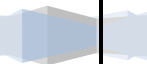
For x, y positive we define the **Beta function** by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Using the substitution $u = 1 - t$ it is easy to see that

$$B(x, y) = B(y, x).$$

To evaluate the Beta function we usually use the Gamma function. To find their relationship, one has to do a rather complicated calculation involving change of variables (from rectangular into tricky polar) in a double integral. This is beyond the scope of this section, but I include the calculation for



the sake of completeness

$$\begin{aligned}
 \Gamma(x)\Gamma(y) &= \int_0^\infty u^{x-1} e^{-u} du \int_0^\infty v^{y-1} e^{-v} dv = \int_0^\infty \int_0^\infty u^{x-1} v^{y-1} e^{-u-v} du dv \\
 &= \left| \begin{array}{l} u = \varrho \cos^2(\varphi) \quad v = \varrho \sin^2(\varphi) \\ J = \det \begin{pmatrix} \cos^2(\varphi) & -2\varrho \cos(\varphi) \sin(\varphi) \\ \sin^2(\varphi) & 2\varrho \sin(\varphi) \cos(\varphi) \end{pmatrix} = 2\varrho \sin(\varphi) \cos(\varphi) \end{array} \right| \\
 &= 2 \int_0^\infty \int_0^{\pi/2} e^{-\varrho} \varrho^{x+y-1} \cos^{2x-1}(\varphi) \sin^{2y-1}(\varphi) d\varrho d\varphi \\
 &= 2 \int_0^\infty e^{-\varrho} \varrho^{x+y-1} d\varrho \cdot \int_0^{\pi/2} \cos^{2x-1} \sin^{2y-1}(\varphi) d\varphi \\
 &= \left| \begin{array}{l} \cos^2(\varphi) = t \\ dt = 2 \sin(\varphi) \cos(\varphi) d\varphi \end{array} \right| \\
 &= \Gamma(x+y) \cdot \int_0^1 t^{x-1} (1-t)^{y-1} dt = \Gamma(x+y) B(x, y).
 \end{aligned}$$

Thus

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

BETA DISTRIBUTION:

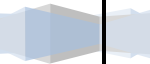
Definition: The family of beta distributions is composed of all distribution with probability density function of the form:

$$f(Y; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \frac{(y-a)^{\alpha-1} (b-y)^{\beta-1}}{(b-a)^{\alpha+\beta-2}}; a \leq y \leq b, \text{ with } \alpha > 0, \beta > 0 \text{ --- (1)}$$

If we make the transformation:

$$X = \frac{Y - a}{b - a};$$

we obtain the probability density function of Beta distribution as:



$$f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}; 0 \leq x \leq 1 \text{ --- (2)}$$

= 0 ; otherwise.

In this case we write $X \sim \text{Beta}(\alpha, \beta)$.

This is the standard form of the beta distribution with shape parameters $\alpha > 0, \beta > 0$. The graph of f for various values of α and β are shown below:

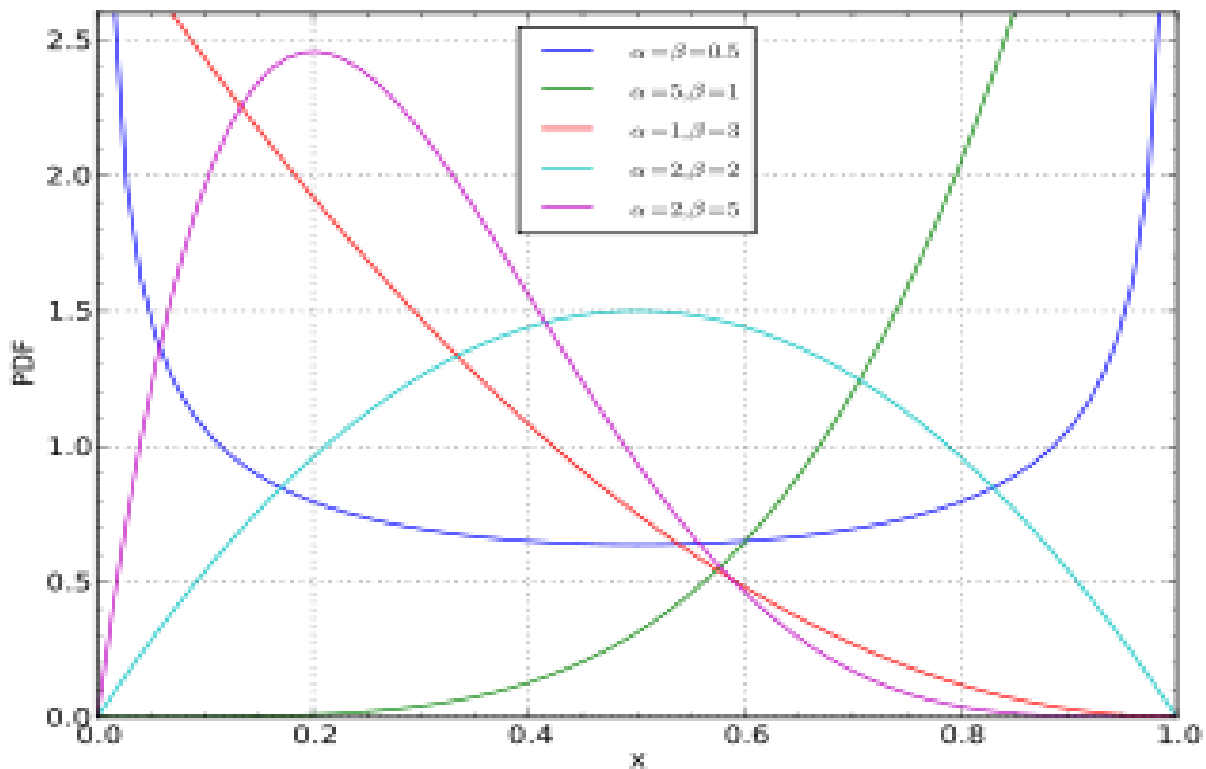


Fig. 1

Cumulative Distribution Function: The cumulative distribution function is

$$F(X; \alpha, \beta) = I_x(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt \text{ --- (3)}$$

Here $I_x(\alpha, \beta)$ is called incomplete beta function ratio. The word 'ratio' which distinguishes (3) from the incomplete beta function:

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt \quad \text{-----(4)}$$

is often omitted.

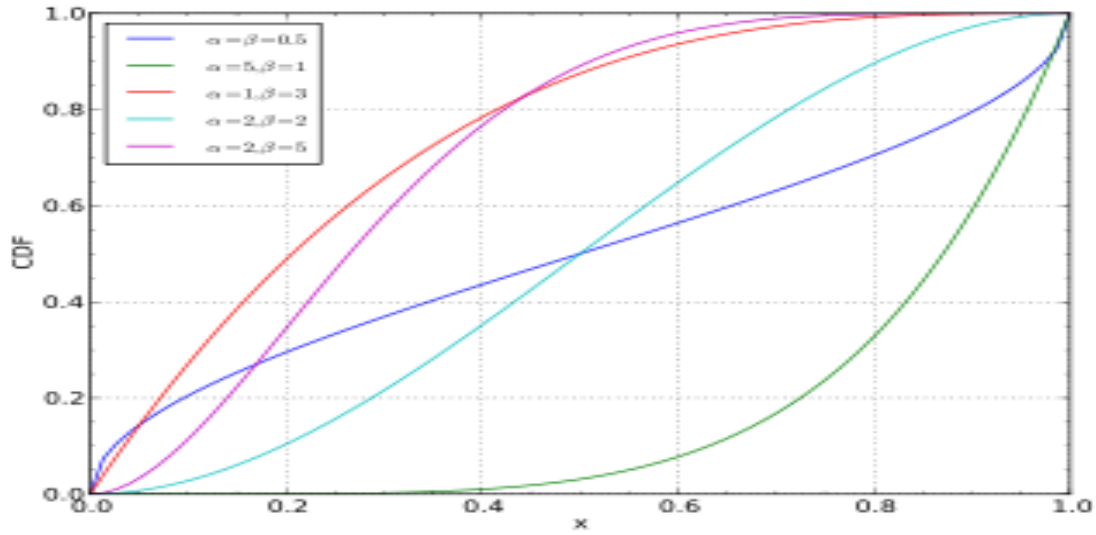


Fig.2: Graph of cumulative distribution function for some values of alpha and beta:

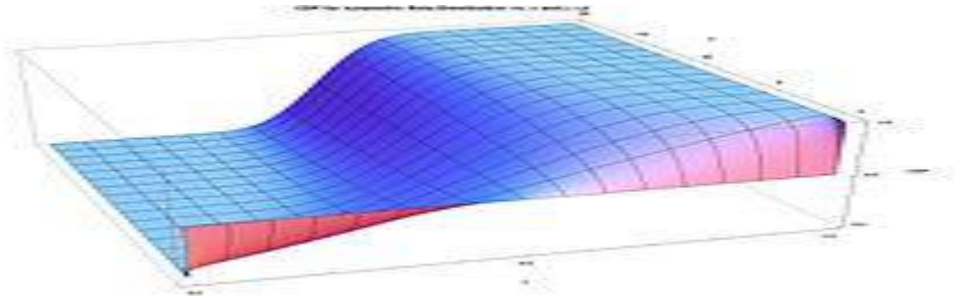


Fig.3 : CDF for symmetric beta distribution vs. x and alpha=beta:

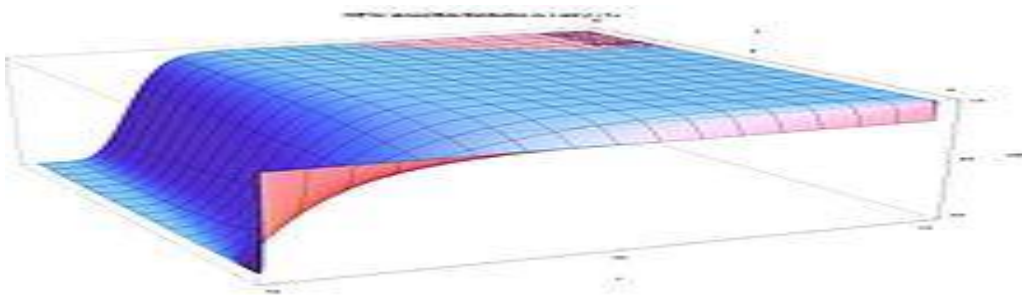


Fig.4: CDF for skewed beta distribution vs. x and beta= 5 alpha

GENESIS:

In 'normal theory', the beta distribution arises naturally as the distribution of

$V^2 = \frac{X_1^2}{(X_1^2 + X_2^2)}$ where X_1^2 , X_2^2 are independent random variables, and X_j^2 is distributed as χ^2 with ν_j degrees of freedom ($j=1,2$). the distribution of V^2 is then a standard beta distribution, as in (2), with $\alpha = \frac{1}{2}\nu_1, \beta = \frac{1}{2}\nu_2$.

An extension of this result is that if $X_1^2, X_2^2, \dots, X_k^2$ are mutually independent random variables with X_j^2 distributed as χ^2 with ν_j degrees of freedom ($j=1,2,\dots,k$) then,

$$V_1^2 = X_1^2 / (X_1^2 + X_2^2)$$

$$V_2^2 = (X_1^2 + X_2^2) / (X_1^2 + X_2^2 + X_3^2)$$

.....

$$V_{k-1}^2 = (X_1^2 + X_2^2 + \dots + X_{k-1}^2) / (X_1^2 + \dots + X_k^2)$$

are mutually independent random variables, each with a beta distribution, the value of α, β for V_j^2 being $\frac{1}{2} \sum_{i=1}^j \nu_i, \frac{1}{2} \nu_{j+1}$ respectively. Under these conditions, the product of any successive set of V_j^2 's also has a beta distribution. This property also holds when the ν 's are any positive numbers.

Another way in which the beta distribution arises is as the distribution of an ordered variable from a rectangular distribution. If Y_1, Y_2, \dots, Y_n are independent random variables each having the standard rectangular distribution, so that

$$P_{Y_j}(y) = 1; (0 \leq y \leq 1)$$

and the corresponding order statistics are $Y_1' \leq Y_2' \leq \dots \leq Y_n'$, the s^{th} order statistic Y_s' has the beta distribution

$$f(y) = P_{Y_s'}(y) = \frac{1}{B(r, n-r+1)} y^{r-1} (1-y)^{n-r}; (0 \leq y \leq 1) \dots (5)$$

This result may be used to generate beta distributed random variables from standard rectangularly distributed variables. Using this method, only integer values can be obtained for n and $(n-s)$. A method

applicable for fractional values of n and $(n-s)$, has been constructed by Jöhnk. He was shown that if X and Y are independent standard rectangular variables the conditional distribution of $X^{1/n}$ given that $X^{1/n} + Y^{1/r} \leq 1$, is a standard beta distribution with parameters $n+1$ and r .

PROPERTIES:

1. MEAN OF BETA DISTRIBUTION:

Mean $= \mu = E(x)$

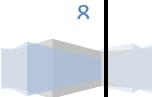
$$\begin{aligned}
 &= \int_0^1 \frac{xx^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)} dx \\
 &= \frac{1}{B(\alpha, \beta)} \int_0^1 x^{(\alpha+1)-1} (1-x)^{\beta-1} dx \\
 &= \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} \\
 &= \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\
 &= \frac{\alpha\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)} \\
 &= \frac{\alpha}{\alpha+\beta} = \frac{1}{1+\frac{\beta}{\alpha}}
 \end{aligned}$$

Note1:

For $\alpha=\beta$, we have, $\mu = 1/2$ i.e. mean is at the center of the distribution. Thus for $\alpha=\beta$, the distribution is symmetric.

Note2:

We have for $\beta/\alpha \rightarrow 0$, or for $\alpha/\beta \rightarrow \infty$, the mean is located at the right end, $x = 1$. For these limit ratios, the beta distribution becomes a one-point degenerate distribution with a Dirac delta function spike at the right end, $x = 1$, with probability 1, and zero probability everywhere else. There is 100% probability (absolute certainty) concentrated at the right end, $x = 1$.



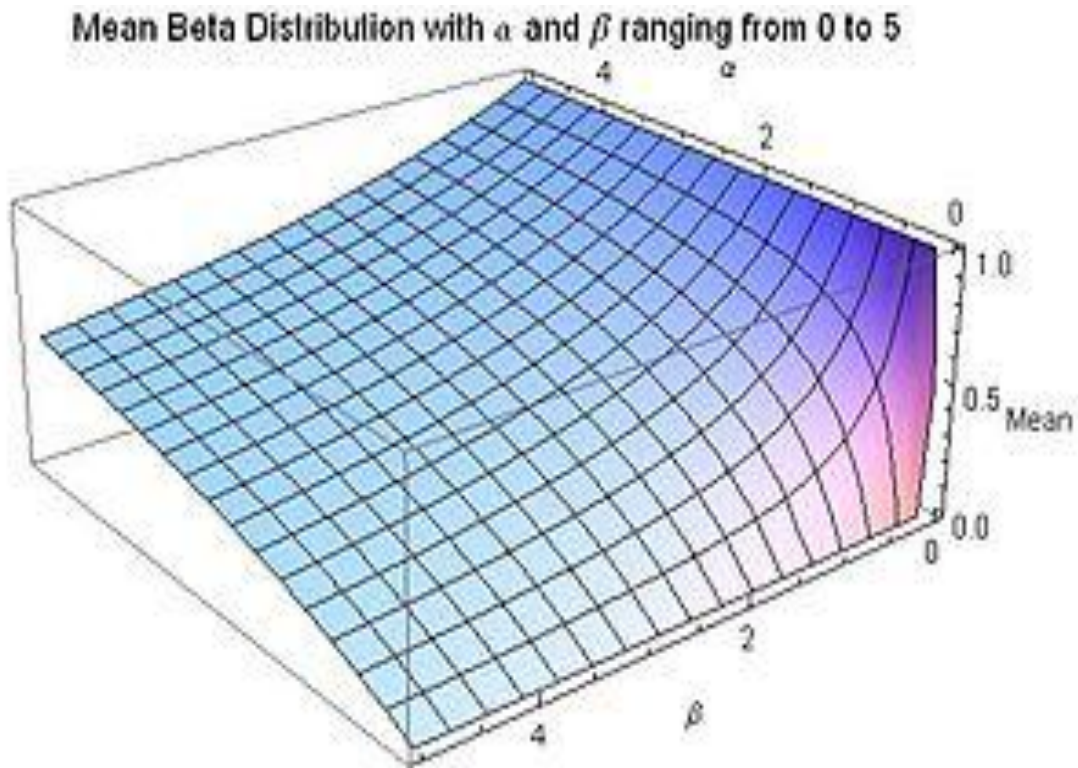


Figure 5:

2. MEDIAN OF BETA DISTRIBUTION: There is no general closed formula for the median of the beta distribution for arbitrary values of the parameter α and β . The median function denoted by $m(\alpha, \beta)$, is the function that satisfies,

$$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{m(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{1}{2}$$

Median of beta distribution for some particular values of α and β are given below:

- ❖ For symmetric cases $\alpha = \beta$, $m(\alpha, \beta) = 1/2$
- ❖ For $\alpha = 1$ and $\beta > 0$, $m(\alpha, \beta) = 1 - 2^{-1/\beta}$
- ❖ For $\alpha > 0$ and $\beta = 1$, $m(\alpha, \beta) = 2^{-1/\alpha}$
- ❖ For $\alpha = 3$ and $\beta = 2$, $m(\alpha, \beta) = 0.6142724318676105\dots$, the real solution to the [quartic equation](#) $1 - 8x^3 + 6x^4 = 0$, which lies in $[0, 1]$.
- ❖ For $\alpha = 2$ and $\beta = 3$, $m(\alpha, \beta) = 0.38572756813238945\dots$

3. MODE OF BETA DISTRIBUTION:

The mode occurs where the distribution reaches a maximum i.e., where the derivative is zero.

$$\begin{aligned}
 \therefore \frac{df(x)}{dx} &= 0 \\
 \Rightarrow \frac{1}{B(\alpha, \beta)} \{(\alpha - 1)x^{\alpha-2}(1-x)^{\beta-1} - (\beta - 1)x^{\alpha-1}(1-x)^{\beta-2}\} &= 0 \\
 \Rightarrow x^{\alpha-2}(1-x)^{\beta-2} \{(\alpha - 1)(1-x) - (\beta - 1)x\} &= 0 \\
 \Rightarrow (\alpha - 1) - x(\alpha + \beta - 2) &= 0 \\
 \Rightarrow x &= \frac{\alpha - 1}{\alpha + \beta - 2}
 \end{aligned}$$

Therefore, mode = $(\alpha-1)/(\alpha+\beta-2)$.

Mode Beta Distribution with α and β ranging from 1 to 5

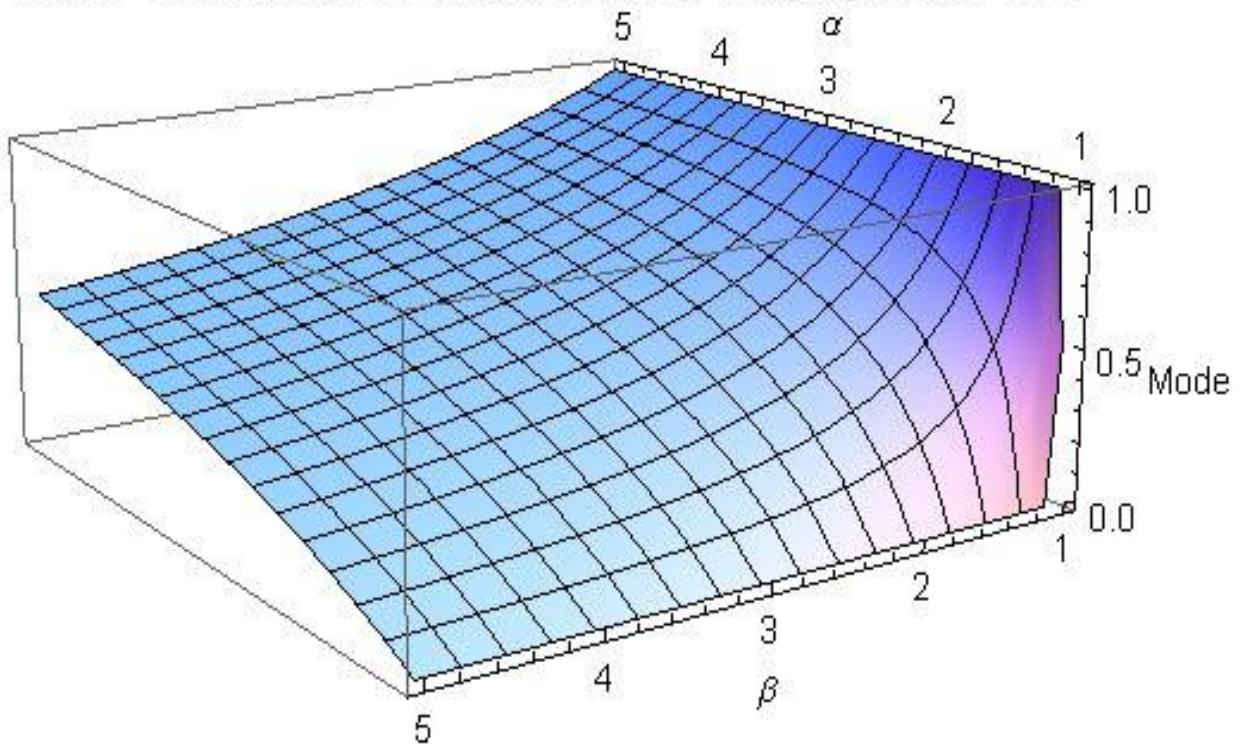


Fig.5: Mode Beta Distribution with α and β ranging from 1 to 5

4. Recurrence Relation for Expectation:

$$\begin{aligned}
 E(x^k) &= \frac{1}{B(\alpha, \beta)} \int_0^1 x^k x^{\alpha-1} (1-x)^{\beta-1} dx \\
 &= \frac{\Gamma(\alpha+k)\Gamma(\beta)}{\Gamma(\alpha+\beta+k)} \frac{1}{B(\alpha, \beta)} \\
 &= \frac{(\alpha+k-1)\Gamma(\alpha+k-1)\Gamma(\beta)}{(\alpha+\beta+k-1)\Gamma(\alpha+\beta+k-1)} \frac{1}{B(\alpha, \beta)} \\
 &= \frac{(\alpha+k-1)}{(\alpha+\beta+k-1)} \frac{B(\alpha+k-1, \beta)}{B(\alpha, \beta)} \\
 &= \frac{(\alpha+k-1)}{(\alpha+\beta+k-1)} E(X^{k-1})
 \end{aligned}$$

Thus, $E(X^k) = \frac{(\alpha+k-1)}{(\alpha+\beta+k-1)} E(X^{k-1}) \text{-----(6)}$

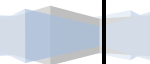
5. Variance of Beta Distribution:

The variance (the second moment about mean) of a random variable X which follows beta distribution with parameters α and β is:

$$\begin{aligned}
 \text{Var}(X) &= E[(X - \mu)^2] \\
 &= E(X^2) - [E(X)]^2 \\
 &= \frac{(\alpha+1)}{(\alpha+\beta+1)} E(X) - [E(X)]^2 \text{ (using (6))} \\
 &= \frac{(\alpha+1)}{(\alpha+\beta+1)} \frac{\alpha}{(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2} \\
 &= \frac{(\alpha^2 + \alpha)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
 &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}
 \end{aligned}$$

Note1: Letting $\alpha=\beta$ in the above expression we obtains:

$$\text{Var}(X) = \frac{1}{4(2\beta+1)}$$



which shows that for $\alpha=\beta$ the variance decreases monotonically as α ($=\beta$) increases.

Note2: Setting $\alpha=\beta=0$ in this expression, one can find the maximum variance as $\text{var}(X) = 1/4$ (Which only occurs by approaching the limit, at $\alpha=\beta=0$).

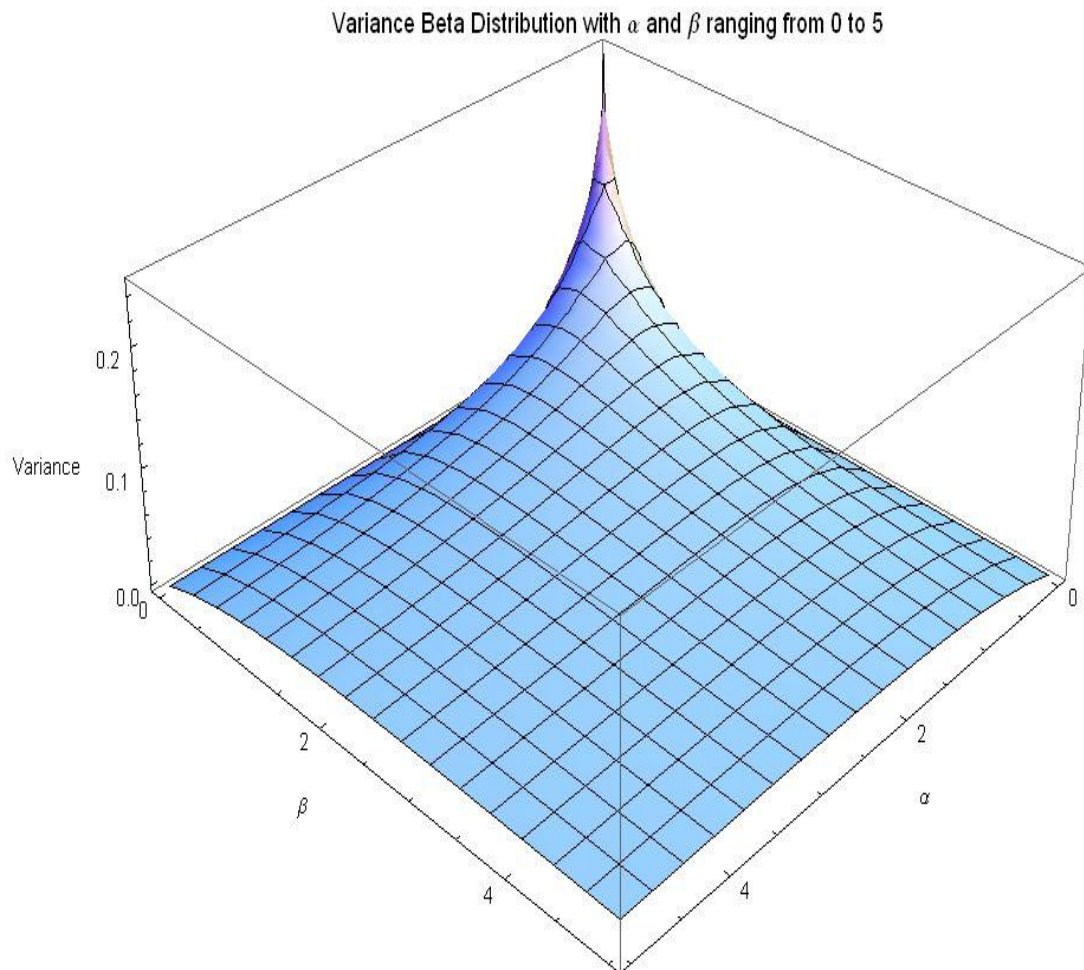
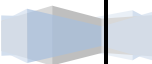


Fig.6



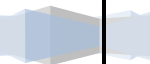
6.Higher Moments:

The k^{th} moment of a Beta random variable X is:

$$\begin{aligned}
 \mu_X(k) &= E(X^k) \\
 &= \frac{(\alpha + k - 1)}{(\alpha + \beta + k - 1)} E(X^{k-1}) \\
 &= \frac{(\alpha + k - 1)(\alpha + k - 2)}{(\alpha + \beta + k - 1)(\alpha + \beta + k - 2)} E(X^{k-2}) \\
 &= \frac{(\alpha + k - 1)(\alpha + k - 2) \cdots (\alpha + 1)}{(\alpha + \beta + k - 1)(\alpha + \beta + k - 2) \cdots (\alpha + \beta + 1)} E(X) \\
 &= \frac{(\alpha + k - 1)(\alpha + k - 2) \cdots (\alpha + 1)}{(\alpha + \beta + k - 1)(\alpha + \beta + k - 2) \cdots (\alpha + \beta + 1)} \frac{\alpha}{(\alpha + \beta)} \\
 &= \prod_{n=0}^{k-1} \frac{(\alpha + n)}{(\alpha + \beta + n)}
 \end{aligned}$$

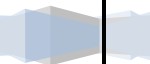
7.Moment Generating Function: The moment generating function is:

$$M_X(\alpha; \beta, t)$$



$$\begin{aligned}
&= E(e^{tX}) \\
&= \int_{-\infty}^{\infty} e^{tx} f(x; \alpha, \beta) dx \\
&= \int_0^1 e^{tx} \frac{x^{\alpha-1} (1-x)^{\beta-1}}{B(\alpha, \beta)} dx \\
&= \frac{1}{B(\alpha, \beta)} \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(tx)^k}{k!} \right) x^{\alpha-1} (1-x)^{\beta-1} dx \\
&= \frac{1}{B(\alpha, \beta)} \sum_{k=0}^{\infty} \int_0^1 \frac{(tx)^k}{k!} x^{\alpha-1} (1-x)^{\beta-1} dx \\
&= \frac{1}{B(\alpha, \beta)} \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 x^{(\alpha+k)-1} (1-x)^{\beta-1} dx \\
&= \frac{1}{B(\alpha, \beta)} \sum_{k=0}^{\infty} \frac{t^k}{k!} B(\alpha+k, \beta) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{B(\alpha+k, \beta)}{B(\alpha, \beta)} \\
&= 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \frac{B(\alpha+k, \beta)}{B(\alpha, \beta)} \\
&= 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} E(X^k) \\
&= 1 + E(X) \frac{t}{1!} + E(X^2) \frac{t^2}{2!} + E(X^3) \frac{t^3}{3!} + \dots - \infty \\
&= 1 + \frac{\alpha}{(\alpha+\beta)} \frac{t}{1!} + \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \frac{t^2}{2!} + \dots - \infty \\
&= {}_1F_1(\alpha; \alpha+\beta; t)
\end{aligned}$$

where, ${}_1F_1(\alpha; \alpha+\beta; t) = \sum_{k=0}^{\infty} \frac{\alpha^{(k)} t^k}{(\alpha+\beta)^{(k)} k!}$ is the Kummer's confluent hypergeometric function (of the first kind) and



$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$$

is the **rising factorial**, also called the "Pochhammer symbol". Thus, the moment generating function of Beta distribution is the Kummer's confluent hypergeometric function ${}_1F_1(\alpha; \alpha + \beta; t)$. The value of the moment generating function for $t = 0$, is one; i.e.

$$M_X(\alpha; \beta, 0) = 1$$

8. Characteristics Function:

The characteristic function is the Fourier transform of the probability density function. The characteristic function of the beta distribution is Kummer's confluent hypergeometric function (of the first kind). In the moment generating function of Beta distribution, replacing t by it , where i is the complex root of unity, we get characteristics function as:

$$\begin{aligned}\varphi_X(\alpha; \beta; t) &= E[e^{itX}] \\ &= \int_0^1 e^{itx} f(x; \alpha, \beta) dx \\ &= {}_1F_1(\alpha; \alpha + \beta; it) \\ &= \sum_{n=0}^{\infty} \frac{\alpha^{(n)} (it)^n}{(\alpha + \beta)^{(n)} n!} \\ &= 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha + r}{\alpha + \beta + r} \right) \frac{(it)^k}{k!}\end{aligned}$$

where

$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$$

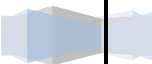
is the **rising factorial**, also called the "Pochhammer symbol". The value of the characteristic function for $t = 0$, is one:

$$\varphi_X(\alpha; \beta; 0) = {}_1F_1(\alpha; \alpha + \beta; 0) = 1.$$

Also, the real and imaginary parts of the characteristic function enjoy the following symmetries with respect to the origin of variable t :

$$\operatorname{Re} [{}_1F_1(\alpha; \alpha + \beta; it)] = \operatorname{Re} [{}_1F_1(\alpha; \alpha + \beta; -it)]$$

$$\operatorname{Im} [{}_1F_1(\alpha; \alpha + \beta; it)] = -\operatorname{Im} [{}_1F_1(\alpha; \alpha + \beta; -it)]$$



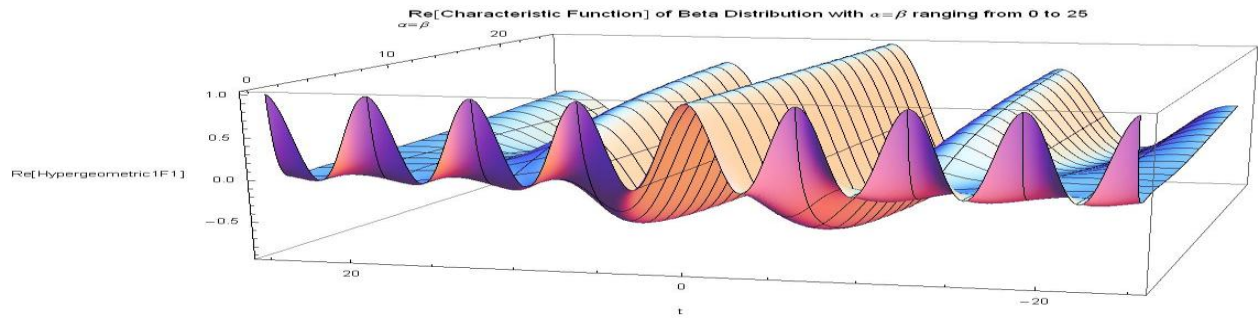


Fig.7

9.Relation to Binomial Distribution

(i) Relation to the uniform distribution:

Proposition 1.1: A Beta distribution with parameters α and β is a uniform distribution on the interval .

Proof: When $\alpha=1$ and $\beta=1$, we have,

$$\begin{aligned}
 \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \\
 &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} x^0 (1-x)^0 \\
 &= \frac{1!}{0!0!} && \text{(because } \Gamma(n) = (n-1)!) \\
 &= 1
 \end{aligned}$$

Therefore, the probability density function of a Beta distribution with parameters α and β can be written as

$$\begin{aligned}
 f_X(x; \alpha = 1, \beta = 1) &= 1, \text{ if } x \in [0,1] \\
 &= 0, \text{ if } x \notin [0,1]
 \end{aligned}$$

Which is the probability density function of a uniform distribution of X on the interval $[0,1]$.

(ii) Relation to Binomial Distribution:

Proposition 1.2 : Suppose $X \sim B(\alpha, \beta)$. Let Y be another random variable such that its distribution conditional on X is a binomial distribution with parameters n and x . Then, the

conditional distribution of X given $Y=y$ is a Beta distribution with parameters $\alpha+y$ and $\beta+n-y$.

Proof: We are dealing with one continuous random variable X and one discrete random variable Y . With a slight abuse of notation, we will proceed as if Y is also continuous, treating its probability mass function as if it were a probability density function. Rest assured that this can be made fully rigorous (by defining a probability density function with respect to a counting measure on the support of Y). By assumption Y has a binomial distribution conditional on X , so that its conditional probability mass function is:

$$f_{Y|X=x}(y) = {}^nC_y x^y (1-x)^{n-y}$$

Also, by assumption X has a beta distribution, so its probability density function is

$$f_X(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

Therefore, the joint probability density function of X and Y is:

$$\begin{aligned} f_{XY}(x,y) &= f_{Y|X=x}(y) f_X(x) \\ &= {}^nC_y x^y (1-x)^{n-y} \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \\ &= \frac{1}{B(\alpha, \beta)} {}^nC_y x^{(\alpha+y)-1} (1-x)^{(\beta+n-y)-1} \\ &= \frac{B(\alpha+y, \beta+n-y)}{B(\alpha, \beta)} {}^nC_y \frac{x^{(\alpha+y)-1} (1-x)^{(\beta+n-y)-1}}{B(\alpha+y, \beta+n-y)} \end{aligned}$$

Thus, we have factorized the joint probability density function as

$$f_{XY}(x,y) = g(x,y) h(y), \text{ where,}$$

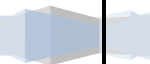
$$g(x,y) = \frac{1}{B(\alpha+y, \beta+n-y)} x^{(\alpha+y)-1} (1-x)^{(\beta+n-y)-1}$$

is the probability density function of Beta distribution with parameters $\alpha+y$ and $\beta+n-y$ and the function

$$h(y) = \frac{B(\alpha+y, \beta+n-y)}{B(\alpha, \beta)} {}^nC_y$$

does not depend on x

So, by a result of 'Factorization of joint probability density functions', this implies that the probability density function of X given $Y=y$ is a Beta distribution with parameters $\alpha+y$ and $\beta+n-y$.



By combining proposition 1.1 and 1.2, we obtain the following result:

Proposition 1.3: Suppose X is a random variable having a uniform distribution. Let Y be another random variable such that its distribution conditional on X is a binomial distribution with parameters n and X . Then, the conditional distribution of X given $Y=y$ is a Beta distribution with parameters $1+y$ and $1+n-y$.

10. APPLICATIONS:

(i) BAYESIAN INFERENCE:

The use of Beta distributions in Bayesian inference is due to the fact that they provide a family of conjugate prior probability distributions for binomial(including Bernoulli) and geometric distributions. The domain of the beta distribution can be viewed as a probability, and in fact the beta distribution is often used to describe the distribution of a probability value.

(ii) SUBJECTIVE LOGIC:

In standard logic, propositions are considered to be either true or false. In contradistinction, subjective logic assumes that humans cannot determine with absolute certainty whether a proposition about the real world is absolutely true or false. In subjective logic the posteriori probability estimates of binary events can be represented by beta distributions.

(iii) WAVELET ANALYSIS:

A wavelet is a wave-like oscillation with an amplitude that starts out at zero, increases, and then decreases back to zero. It can typically be visualized as a "brief oscillation" that promptly decays. Wavelets can be used to extract information from many different kinds of data, including – but certainly not limited to – audio signals and images. Thus, wavelets are purposefully crafted to have specific properties that make them useful for signal processing. Wavelets are localized in both time and frequency whereas the standard Fourier transform is only localized in frequency. Therefore, standard Fourier Transforms are only applicable to stationary processes, while wavelets are applicable to non-stationary processes. Continuous wavelets can be constructed based on the beta distribution. Beta wavelets^[71] can be viewed as a soft variety of Haar wavelets whose shape is fine-tuned by two shape parameters α and β .

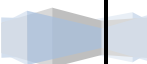
11. RELATED DISTRIBUTION:

(i) If X has Beta distribution (pdf given by (2)), then by the transformation:

$$T = X/(1 - X)$$

we obtain a distribution with probability density function

$$P_T(y)$$



$$\begin{aligned}
&= \frac{1}{B(\alpha, \beta)} \left(\frac{t}{1+t}\right)^{\alpha-1} \left(\frac{1}{1+t}\right)^{\beta-1} \frac{1}{(1+t)^2} \\
&= \frac{1}{B(\alpha, \beta)} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}, (t > 0)
\end{aligned}$$

This is a standard form of **Pearson Type VI** distribution, sometimes called a **beta-prime** distribution.

(ii) Let $X \sim \text{Beta}(\alpha, \beta)$. Suppose, α and β are positive integers and that for $\alpha + \beta = s (\geq 2)$ fixed, α is equally likely to take values $1, 2, \dots, (s-1)$ then the probability density function of X , given $\alpha + \beta = s$ is:

$$P_X(x|s) = \frac{(s-1) \sum_{p=1}^{s-2} s^{-2} C_{p-1} x^{p-1} (1-x)^{s-2-(p-1)}}{(s-1)} = 1; (0 < x < 1)$$

that is, the distribution is rectangular. It follows that whatever the distribution of $(\alpha + \beta)$, the compound distribution is rectangular if the conditional distribution of α , given $(\alpha + \beta)$ is discrete rectangular as described above.

(iii) As described in the 'genesis' part Beta distribution can be generated as the distributions of ratios $X_1/(X_1 + X_2)$ where X_1, X_2 are independent random variables having chi-square distribution.

If one or both of X_1 and X_2 have non-central chi-square distributions the distribution of the ratio is called a non-central Beta distribution.

BIBLIOGRAPHY:

- Johnson Norman L. & Kotz Samuel. Continuous Univariate Distribution-2, John Wiley and Sons, New York, 37-53.
- Jambunathan, M.V. (1954), Some properties of beta and gamma distributions, Annals of the Mathematical Statistics, 25, 401-405.
- Johnson, N.L. (1949). Systems of frequency curves generated by methods of translation, Biometrika, 47, 93-102.
- Feller, William (1971). An Introduction to Probability Theory and its Applications, Vol-2. Wiley. ISBN 978-0471257097.
- Gupta (Editor), Arjun K. (2004). Handbook of Beta Distribution and its Applications. CRC Press. ISBN 978-0824753962.
- Panik, Michael J. (2005). Advanced Statistics from an Elementary point of view. Academic Press. ISBN 978-0120884940.
- Rose, Colin; Smith, Murry D. (2002). Mathematical Statistics with MATHEMATICA. Springer. ISBN 978-0387952345.

○ _____ X _____

