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Structural Controllability of Multiplex Networks with the Minimum
Number of Driver Nodes

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This Supplementary Material is organized as follows. In Section I, the proposed “Minimum-cost Flow based Driver-node Identification” (MFDI) method is compared with the existing methods in terms of the minimum number of driver nodes required to ensure the structural controllability of two-layer multiplex networks. In Section II, we demonstrate that using the method designed for single-layer networks on two-layer networks can result in the allocation of driver nodes that violate the constraint. In Section III, we extend the proposed MFDI algorithm to handle the different number of nodes per layer. In Section IV, we include the actual controllability simulation results.

1 Comparative Analysis of MFDI with Existing Techniques

To the best of our knowledge, our work is the first to study the problem of finding the minimum number of driver nodes to ensure the structural controllability of multiplex networks. Our work shares a common interest in the structural controllability of multiplex networks with several other significant studies but features some distinct characteristics [14, 18, 19]. In [14], the authors examined finding the minimum number of controllers on multiplex multi-time-scale networks within discrete dynamics, providing a solution using the max flow method. However, this work may not be directly comparable to our work and existing works on continuous dynamics [18, 19] due to the significant difference in the dynamics framework. In the following, we shall conduct the comparisons between the MFDI algorithm and the algorithms proposed in [18, 19] from problem formulation and numerical results perspectives.

In [18, 19], the authors delved into the problem of finding the minimum number of controllers required to ensure target controllability on two-layer multiplex networks. These two studies respectively employed a maximum flow algorithm and a minimum-cost maximum flow algorithm, respectively. When their methods are applied to the problem of structural controllability, the target set is considered as the whole network’s node sets. Specifi-

cally, their formulation of the problem is as follows. In a two-layer multiplex network with n nodes in each layer, of which the wiring diagram is $\bar{A} = \begin{bmatrix} \bar{A}_{11} & \mathbf{0} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \in \{0, \star\}^{2n \times 2n}$, the problem can be formulated as

$$\begin{aligned} & \min_{\bar{B}_1 \in \{0, \star\}^{n \times m}} m, \\ & \text{s. t. } \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \mathbf{0} \end{bmatrix} \in \{0, \star\}^{2n \times m}, \\ & \dot{\mathbf{x}}(t) = \bar{A}\mathbf{x}(t) + \bar{B}\mathbf{u}(t), \\ & (\bar{A}, \bar{B}) \text{ is structurally controllable.} \end{aligned} \tag{S1}$$

where m represents the number of external controllers.

In our paper, we study the problem of finding the minimum number of driver nodes necessary to ensure the structural controllability of multiplex networks. On a two-layer multiplex network, of which the wiring diagram is denoted as $\bar{A} = \begin{bmatrix} \bar{A}_{11} & \mathbf{0} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \in \{0, \star\}^{2n \times 2n}$, we tackle the optimization problem

$$\begin{aligned} & \min_{\bar{B}_1 \in \{0, \star\}^{n \times n}} \|\bar{B}\|_0, \\ & \text{s. t. } \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \mathbf{0} \end{bmatrix} \in \{0, \star\}^{2n \times n}, \\ & \dot{\mathbf{x}}(t) = \bar{A}\mathbf{x}(t) + \bar{B}\mathbf{u}(t), \\ & (\bar{A}, \bar{B}) \text{ is structurally controllable.} \end{aligned} \tag{S2}$$

The main distinction between formulations (S1) and (S2) is that while (S1) focuses on finding the minimum number of controllers, (S2) is concerned with identifying the minimum number of driver nodes required to ensure the structural controllability of multiplex

networks. Both equations (S1) and (S2) impose a constraint that controllers can only be connected to nodes in the first layer.

The proposed “Minimum-cost Flow based Driver-node Identification” (MFDI) method is compared with the methods described in [18, 19] in terms of the driver nodes required to ensure the structural controllability of two-layer multiplex networks. The results of comparisons on synthetic multiplex networks are presented in Table S1, where each layer consists of n nodes and μ_i denotes the average nodal degree in the i^{th} layer. In Table S1, each value is represented as the mean \pm standard deviation calculated from 100 independent simulations. Moreover, Table S2 showcases the comparisons made on real-life multiplex networks where each layer consists of n nodes and $|E_i|$ denotes the number of edges in the i^{th} layer. The real-life network datasets are accessible on the website of CoMuNe Laboratory¹, the Research Unit of Bruno Kessler Foundations Center for Information and Communication Technologies. For those networks with more than two layers, we only use the data of the first two layers in the simulations. In Table S2, each real-life multiplex network is represented where n marks the number of nodes in each layer, and $|E_i|$ denotes the number of edges in the i^{th} layer.

¹<https://manliodedomenico.com/data.php>

Table S1: Comparisons between the number of driver nodes obtained by different methods on two-layer synthetic multiplex networks ($n = 1000$)

	maximum flow algorithm [18]	minimum-cost maximum flow algorithm [19]	proposed MFDI algorithm
ER-ER, $\mu_1 = 2, \mu_2 = 2$	330.99 ± 10.48	330.99 ± 10.48	330.62 ± 10.11
ER-ER, $\mu_1 = 3, \mu_2 = 3$	131.44 ± 9.10	131.44 ± 9.10	127.95 ± 8.87
ER-ER, $\mu_1 = 4, \mu_2 = 4$	44.04 ± 5.52	44.04 ± 5.52	41.88 ± 5.40
ER-ER, $\mu_1 = 5, \mu_2 = 5$	18.10 ± 4.33	18.10 ± 4.33	14.28 ± 3.55
ER-ER, $\mu_1 = 8, \mu_2 = 8$	6.64 ± 2.17	6.64 ± 2.17	1.36 ± 0.66
SF-SF, $\mu_1 = 2, \mu_2 = 2$	776.58 ± 7.89	776.58 ± 7.89	775.16 ± 7.37
SF-SF, $\mu_1 = 3, \mu_2 = 3$	693.82 ± 8.92	693.82 ± 8.92	692.28 ± 8.21
SF-SF, $\mu_1 = 10, \mu_2 = 10$	301.46 ± 9.85	301.46 ± 9.85	300.66 ± 9.88
SF-SF, $\mu_1 = 15, \mu_2 = 15$	129.81 ± 8.07	129.81 ± 8.07	128.64 ± 8.16
SF-SF, $\mu_1 = 20, \mu_2 = 20$	26.41 ± 7.07	26.41 ± 7.07	23.22 ± 6.93
SF-SF, $\mu_1 = 30, \mu_2 = 30$	6.28 ± 1.71	6.28 ± 1.71	1.00 ± 0.00

Table S2: Comparisons between the Number of Driver Nodes Obtained by Different Methods on Two-layer Real-life Multiplex Networks

	maximum flow algorithm [18]	minimum-cost maximum flow algorithm [19]	proposed MFDI algorithm
CKM physicians' innovation [43] $n = 246, E _1 = 480, E _2 = 565$	133	133	127
Lazega law firm [44] $n = 71, E _1 = 892, E _2 = 575$	9	9	6
C. Elegans connectome [47] $n = 279, E _1 = 1031, E _2 = 1639$	135	135	60
C. Elegans GPI [48] $n = 3879, E _1 = 5557, E _2 = 313$	3817	3817	3810
Arabidopsis GPI [48] $n = 6980, E _1 = 13857, E _2 = 4411$	6597	6597	6564

From Tables S1 and S2, it can be observed that the proposed MFDI consistently finds the smallest number of driver nodes on either synthetic or real-life networks. This is the comparison between the number of driver nodes when minimizing the number of controllers versus the actual minimum number of driver nodes. However, it's important to note that we should not claim our algorithm "outperforms" the algorithms in [18,19] because their optimization targets are different from ours. Table S2 also outlines the number of driver nodes needed when minimizing the number of controllers (using the maximum flow algorithm [18] and the minimum-cost maximum flow algorithm [19]) versus the actual minimum number of driver nodes (using the MFDI algorithm) in four real-life networks. The comparisons demonstrate the different results under different control objectives. It may be worth mentioning that the differences tend to be smaller in highly sparse networks, as can be observed in the last two rows of Table R1. Further future studies are needed to

verify whether such is always the case in networks with different clustering and community structures.

2 Applying Single-Layer Methods to Two-Layer Networks

The fundamental distinction between identifying the minimum number of driver nodes required to ensure structural controllability in single-layer and two-layer multiplex networks is that, in the latter, the driver nodes must be situated in the first layer. Casting a two-layer network into a single-layer network and using the method in [22] on two-layer networks may result in the allocation of driver nodes that violates this constraint. We further elucidate this with mathematical formulations and supportive empirical data.

Specifically, in [22], the authors studied the problem of finding the minimum number of driver nodes on single-layer networks, which could be formulated as

$$\begin{aligned} \min_{\bar{B} \in \{0, \star\}^{n \times n}} & \|\bar{B}\|_0, \\ \text{s. t. } & \dot{\mathbf{x}}(t) = \bar{A}\mathbf{x}(t) + \bar{B}\mathbf{u}(t), \\ & (\bar{A}, \bar{B}) \text{ is structurally controllable,} \end{aligned} \tag{S3}$$

where $\bar{A} \in \{0, \star\}^{n \times n}$ denotes the wiring diagram of the network, and $\|\bar{B}\|_0$ denotes the number of non-zero entries in B , i.e., the number of driver nodes. This problem was solved by a minimum-weighted maximum matching method.

In our paper, we study the problem of finding the minimum number of driver nodes needed to ensure the structural controllability of multiplex networks, where the driver nodes can only be situated on the first layer of the network. On a two-layer multiplex network, of which the wiring diagram is denoted as $\bar{A} = \begin{bmatrix} \bar{A}_{11} & \mathbf{0} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \in \{0, \star\}^{2n \times 2n}$, the

problem is formulated as

$$\begin{aligned}
& \min_{\bar{B}_1 \in \{0, \star\}^{n \times n}} \|\bar{B}\|_0, \\
& \text{s. t. } \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \mathbf{0} \end{bmatrix} \in \{0, \star\}^{2n \times n}, \\
& \dot{\mathbf{x}}(t) = \bar{A}\mathbf{x}(t) + \bar{B}\mathbf{u}(t), \\
& (\bar{A}, \bar{B}) \text{ is structurally controllable.}
\end{aligned} \tag{S4}$$

We shall then illustrate that the minimum-weighted maximum matching method in [22] may violate the constraints in our problem (S4). In order to keep the dimensions consistent, we rewrite the formulation (S3) as

$$\begin{aligned}
& \min_{\bar{B} \in \{0, \star\}^{2n \times n}} \|\bar{B}\|_0, \\
& \text{s. t. } \dot{\mathbf{x}}(t) = \bar{A}\mathbf{x}(t) + \bar{B}\mathbf{u}(t), \\
& (\bar{A}, \bar{B}) \text{ is structurally controllable,}
\end{aligned} \tag{S5}$$

where $\bar{A} \in \{0, \star\}^{2n \times 2n}$. The distinction between (S4) and (S5) is that (S5) does not confine the locations of driver nodes. As such, the lower half of $\bar{B} \in \{0, \star\}^{2n \times n}$ obtained by using minimum-weighted maximum matching method [22] may contain non-zero entries, violating the restriction set in (S4). Table S3 records the proportion of instances where the method proposed in [22] fails to meet the constraints of our problem (S4) on synthetic networks. This implies the scenarios where the lower half of the obtained $\bar{B} \in \{0, \star\}^{2n \times n}$ has non-zero entries. In Table S3, μ_i represents the mean degree of the i^{th} layer. For each network configuration, the result is generated from simulations run on 100 independent networks. Table S4 illustrates cases from real-life multiplex networks, assessing whether the calculated $\bar{B} \in \{0, \star\}^{2n \times n}$ obtained using the minimum-weighted maximum matching method [22], violates the constraints outlined in (S4). The real-life network datasets are accessible on the website of CoMuNe Laboratory², the Research Unit of Bruno Kessler

²<https://manliodedomenico.com/data.php>

Foundations Center for Information and Communication Technologies. For those networks with more than two layers, we only use the data of the first two layers in the simulations. In Table S4, for each real-life multiplex network, n denotes the number of nodes in each layer and $|E_i|$ denotes the number of edges in the i^{th} layer. The symbol \checkmark denotes that \bar{B} , calculated using the minimum-weighted maximum matching method, complies with the constraints specified in (S4). Conversely, \times indicates that the calculated \bar{B} violates the constraints.

Table S3: The proportion of instances where the calculated B using the minimum-weighted maximum matching method [22] on synthetic networks fails to comply with the constraints specified in problem (S4)

two-layer ER-ER multiplex networks, $n = 1000$	$\mu_1 = 2$	$\mu_1 = 3$	$\mu_1 = 4$	$\mu_1 = 5$
$\mu_2 = 2$	100%	100%	100%	100%
$\mu_2 = 3$	100%	100%	100%	100%
$\mu_2 = 4$	84%	83%	94%	100%
$\mu_2 = 5$	46%	42%	43%	67%
two-layer SF-SF multiplex networks, $n = 1000$	$\mu_1 = 2$	$\mu_1 = 3$	$\mu_1 = 4$	$\mu_1 = 5$
$\mu_2 = 2$	100%	100%	100%	100%
$\mu_2 = 3$	100%	100%	100%	100%
$\mu_2 = 4$	92%	100%	100%	100%
$\mu_2 = 5$	86%	95%	100%	100%

Table S4: Assessment of constraint violations in (S4) by the minimum-weighted maximum matching method [22] on real-life multiplex networks

CKM physicians' innovation [43], $n = 246$, $ E _1 = 480$, $ E _2 = 565$	✗
Lazega law firm [44], $n = 71$, $ E _1 = 892$, $ E _2 = 575$	✓
C. Elegans connectome [47], $n = 279$, $ E _1 = 1031$, $ E _2 = 1639$	✗
C. Elegans GPI [48], $n = 3879$, $ E _1 = 5557$, $ E _2 = 313$	✗
Arabidopsis GPI [48], $n = 6980$, $ E _1 = 13857$, $ E _2 = 4411$	✗

From Tables S3 and S4, it can be seen that on either synthetic or real-life networks, results generated by the minimum-weighted maximum matching method [22] may not fulfill the constraints in (S4).

3 Extension of MFDI to Handle Different Numbers of Nodes Per Layer

While Theorems 1 - 4 are initially formulated for multiplex networks with the same number of nodes in each layer and one-to-one directed interlayer connections, their main principles largely remain applicable when being extended to multilayer networks with varying numbers of nodes across layers, connected by directed edges from layer 1 to layer 2. However, for mathematical rigor, certain modifications in notation are required. For instance, if layer 1 contains n_1 nodes and layer 2 contains n_2 nodes, then the node set of the multilayer network becomes $V_A = V_1 \cup V_2$ with $V_1 = \{v_1, \dots, v_{n_1}\}$ and $V_2 = \{v_{n_1+1}, \dots, v_{n_1+n_2}\}$; the edge set is $E_A = E_1 \cup E_2 \cup E_I$ where $E_1 \subseteq V_1 \times V_1$, $E_2 \subseteq V_2 \times V_2$ and $E_I \subseteq V_1 \times V_2$. When a feasible solution exists for the extended problem, by modifying these symbols in Theorems 1 - 4 and slightly expanding the MFDI to an extended MFDI algorithm (as shown in Algorithm S1), one can identify the minimum number of driver nodes located only in the first layer to ensure the system's structural controllability. As our later discussions

may refer to some of the conclusions in Theorems 1-4, for your easy reference, we attached Theorems 1 - 4 here.

Theorem 1: In $D(V_A, E_A)$, any set of vertices from which all network vertices are reachable contains at least one vertex of each source SCC.

Theorem 2: In $D(V', E', b, d)$, for any node v_i^{in} with an incoming edge $(q_s, v_i^{in}) \in E_s$ satisfying $f(q_s, v_i^{in}) = 1$, its corresponding vertex $v_i \in V_A$ is an initial node of a path in \mathcal{P} in $D(V_A, E_A)$ where V_A can be covered by $\mathcal{P} \cup \mathcal{C}$.

Theorem 3: The cardinality of the minimum subtraction node set, $|(V_c \setminus V_r)^*|$, equals the total cost of the minimum-cost flow in $D(V'', E'', c, b, d)$.

Theorem 4: In a network $D(V_A, E_A)$ where $V_A = V_1 \cup V_2$ and $E_A = E_1 \cup E_2 \cup \{(v_i, v_{n+i})\}_{i=1}^n$, the cardinality of the minimum set of driver nodes for ensuring its structural controllability $|V_d^*|$ satisfies

$$|V_d^*| = |V_r| + |(V_c \setminus V_r)^*|,$$

where $|V_r| = k_0$ is the number of source SCCs in $D(V_1, E_1)$, and

$$|(V_c \setminus V_r)^*| = \sum_{(v_i, v_j) \in E''} f^*(v_i, v_j) \cdot c(v_i, v_j)$$

equals the total cost of the minimum-cost flow in $D(V'', E'', b, c, d)$.

The remaining gap to address is identifying the cases where no feasible solution exists, for which we introduce Lemma S1.

Lemma S1: In a multilayer network where each layer has a different number of nodes, a set of driver nodes solely on layer 1 that ensures the structural controllability of the system exists unless:

- a) The multilayer network contains a source Strongly Connected Component (SCC) that does not include any nodes from layer 1; or
- b) There is no feasible flow in the network $D(V'', E'', b, c, d)$ constructed in Step 1 of the MFDI algorithm.

Proof. (\implies) The sufficiency is relatively straightforward. When neither of the aforementioned conditions is satisfied, the extended MFDI algorithm can be applied to find a feasible solution that ensures the structural controllability of the system, as per Theorems 1 - 4.

(\impliedby) The necessity proof has two parts. Condition a) follows from Lemma 3 and Theorem 1, which require that driver nodes for structural controllability include at least one node from each source SCC. The presence of a driver node in layer 2 contradicts the requirement for all driver nodes to be on layer 1. For condition b), in the absence of a feasible flow in $D(V'', E'', b, c, d)$, it is impossible to find a set of vertex-disjoint paths and cycles, i.e., $\mathcal{P} \cup \mathcal{C}$, that covers all nodes in the multilayer network with each path's start node in layer 1, thus contradicting the necessity condition b) of Lemma 3 and rendering the problem infeasible.

■

In the extended MFDI algorithm, we have incorporated lines 4 - 5 and 17 - 18, to assess feasibility in line with the conditions a) and b) outlined in Lemma S1. Specifically, lines 4 - 5 are designed to check for the existence of any source SCC in layers other than the first layer. Lines 17-18 assess the existence of a feasible flow in $D(V'', E'', b, c, d)$. The remaining code segments are consistent with those for the original MFDI.

Algorithm S1: Extended Minimum-cost Flow based Driver-node Identification Algorithm
(Part 1)

- 1: **Input:** Network $D(V_A, E_A)$ where $V_A = V_1 \cup V_2$, $E_A = E_1 \cup E_2 \cup E_I$.
 - 2: **Output:** Minimum driver node set V_d^* .
 - 3: **Step 0: Feasibility Check:**
 - 4: **if** There is a source SCC found in layers other than the first **then**
 - 5: **Return** False
 - 6: **else**
 - 7: **Step 1: Associate graph construction:**
 - 8: Find all k_0 source SCCs in $D(V_1, E_1)$ as S_1, S_2, \dots, S_{k_0} , respectively.
 - 9: Create a fictitious vertex set $V_{\mathcal{R}} \leftarrow \{r_i\}_{i=1}^{k_0}$ consisting of k_0 vertices.
 - 10: Define a vertex set $V'' \leftarrow V^{in} \cup V^{out} \cup \{q_s, q_t\} \cup V_{\mathcal{R}}$, where $V^{in} \leftarrow \{v_i^{in} : v_i \in V_A\}$,
and $V^{out} \leftarrow \{v_i^{out} : v_i \in V_A\}$.
 - 11: Define an edge set $E'' \leftarrow E'_A \cup E_s \cup E_t \cup \{(q_s, q_t)\} \cup \{(q_t, r_i)\}_{i=1}^{k_0} \cup \{(r_i, v_j^{in}) : v_j \in S_i\}_{i=1}^{k_0}$,
where $E'_A \leftarrow \{(v_i^{out}, v_j^{in}) : (v_i, v_j) \in E_A\}$, $E_s \leftarrow \{(q_s, v^{in}) : v \in V_1\}$, $E_t \leftarrow \{(v^{out}, q_t) : v \in V_A\}$.
 - 12: Set $b(e) \leftarrow 1$, for every $e \in E'' \setminus \{(q_t, q_s)\}$, and $b(q_t, q_s) \leftarrow +\infty$.
 - 13: Set $c(e) \leftarrow 1$, for every $e \in E_s$; Set $c(q_t, r_i) \leftarrow -1$, for every $(q_t, r_i) \in \{(q_t, r_i)\}_{i=1}^{k_0}$,
and $c(r_i, v_j^{in}) \leftarrow 1$, for every $(r_i, v_j^{in}) \in \{(r_i, v_j^{in}) : v_j \in S_i\}_{i=1}^{k_0}$; and set $c(e) \leftarrow 0$, for
every $e \in E'_A \cup E_t \cup \{(q_s, q_t)\}$.
 - 14: Set $d(v_i^{in}) \leftarrow 1$, for every $v_i^{in} \in V^{in}$, and $d(v_j^{out}) \leftarrow -1$, for every $v_j^{out} \in V^{out}$; and
set $d(v) \leftarrow 0$, for every $v \in V'' \setminus (V^{in} \cup V^{out})$.
 - 15: Obtain the associate graph $D(V'', E'', b, c, d)$.
 - Step 2: Driver nodes identification:**
 - 16: Obtain the minimum-cost flow f^* in $D(V'', E'', b, c, d)$.
 - 17: **if** There is no feasible f^* **then**
 - 18: **Return** False
 - 19: **else**
-

Algorithm S1: Extended Minimum-cost Flow based Driver-node Identification Algorithm
(Part 2)

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20:   Obtain edge set  $E_{p1} \leftarrow \{(q_s, v_i^{in}) : f^*(q_s, v_i^{in}) > 0, v_i^{in} \in V^{in}\}$ , and  $E_{p2} \leftarrow$ 
       $\{(r_i, v_j^{in}) : f^*(r_i, v_j^{in}) > 0, v_j^{in} \in V^{in}, r_i \in V_{\mathcal{R}}\}$ .
21:   Initialize the driver node set  $V_d^* \leftarrow \emptyset$ .
22:   for  $(q_s, v_i^{in}) \in E_{p1}$  do
23:      $V_d^* \leftarrow V_d^* \cup \{v_i\}$ 
24:   end for
25:   for  $(r_i, v_j^{in}) \in E_{p2}$  do
26:      $V_d^* \leftarrow V_d^* \cup \{v_j\}$ 
27:      $V_{\mathcal{R}} \leftarrow V_{\mathcal{R}} \setminus \{r_i\}$ 
28:   end for
29:   if  $V_{\mathcal{R}} \neq \emptyset$  then
30:     for  $r_i \in V_{\mathcal{R}}$  do
31:        $V_d^* \leftarrow V_d^* \cup \{\text{a node from } S_i\}$ 
32:     end for
33:   end if
34:   Return  $V_d^*$ 
35: end if
36: end if

```

4 Actual Controllability Simulation Results

In this section, we include the actual controllability simulation results. This provides interested readers with comprehensive access to in-depth results. We may discuss the actual controllability by making use of the example shown in Fig. 1(c). To delve into actual controllability, we begin by replacing the structural patterns of the system matrix

$\bar{A} \in \{0, \star\}^{n \times n}$ and input matrix $\bar{B} \in \{0, \star\}^{n \times m}$ with the corresponding weights. For simplicity and without loss of generality, we replace the parameter \star with 1. In this case, we have

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In control theory, a networked system is controllable if it can be driven to any desired state with appropriate inputs. Without loss of generality, we set the system's desired state as $\mathbf{0}$, while the initial state is denoted as $\mathbf{x}_0 \in \mathbb{R}^{n \times 1}$. We denote the required control time as t_f . Given a controllable system (A, B) , when the input signal

$$\mathbf{u}(t) = -B^T e^{A^T(t_f - t)} [W_B]^{-1} e^{At_f} \mathbf{x}_0 \quad (\text{S3})$$

is applied, where $W_B = \int_0^{t_f} e^{A(t_f-t)} B B^T e^{A^T(t_f-t)} dt$, the states could reach the origin at time $t = t_f$, i.e., $\mathbf{x}(t_f) = e^{At_f} \mathbf{x}_0 - W_B (W_B)^{-1} e^{At_f} \mathbf{x}_0 = \mathbf{0}$ [18–21, 31]. We generate the initial state \mathbf{x}_0 from the normal distribution with a mean of 0 and a variance of 1 and we set $t_f = 2$. Upon applying the input signal as defined by (S3), the states of all network nodes are simulated and the results are presented in Fig. S1. All node states converge to zero at $t = t_f$.

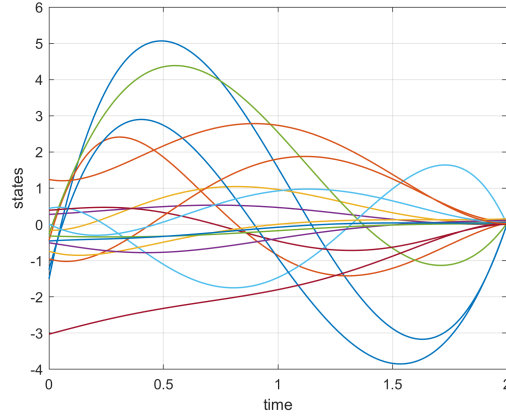


Fig. S1: Actual controllability of the networked system (A, B).

References (in sequence as they appear in the Supplementary Material)

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