Bayesian hypothesis testing of mediation: Methods and the impact of prior odds specifications

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Supplemental materials

Derivations of the formulas in Scenario 1: Independent paths

Proof of Eq. 7:

In the simple mediation model, we have $p(D|\alpha,\beta) = p(M|\alpha,X)p(Y|\beta,M,X)$. By dividing the numerator and denominator of Eq. 3 by $p(D|\alpha=0,\beta=0)$, the proposed BF^{med} becomes

$$\begin{aligned} BF^{med} &= \frac{p(D|\alpha\neq 0,\beta\neq 0)}{q_{00|0}p(D|\alpha=0,\beta=0)+q_{01|0}p(D|\alpha=0,\beta\neq 0)+q_{10|0}p(D|\alpha\neq 0,\beta=0)} \\ &= \frac{\frac{p(M|\alpha\neq 0,X)p(Y|\beta\neq 0,M,X)}{p(M|\alpha=0,X)p(Y|\beta=0,M,X)}}{q_{00|0}+q_{01|0}\frac{p(M|\alpha=0,X)p(Y|\beta\neq 0,M,X)}{p(M|\alpha=0,X)p(Y|\beta=0,M,X)}+q_{10|0}\frac{p(M|\alpha\neq 0,X)p(Y|\beta=0,M,X)}{p(M|\alpha=0,X)p(Y|\beta=0,M,X)}} \\ &= \frac{BF^{\alpha}BF^{\beta}}{q_{00|0}+q_{01|0}BF^{\beta}+q_{10|0}BF^{\alpha}} \end{aligned}$$

where $BF^{\alpha} = \frac{p(M|X,\alpha\neq 0)}{p(M|X,\alpha=0)}$ and $BF^{\beta} = \frac{p(Y|M,X,\beta\neq 0)}{p(Y|M,X,\beta=0)}$.

Proof of Eq. 8:

In the independent paths scenario where α and β are independent, we have

$$\begin{aligned} PriorOdds^{med} &= \frac{p(\alpha\beta\neq 0)}{p(\alpha\beta=0)} \\ &= \frac{p(\alpha\neq 0,\beta\neq 0)}{p(\alpha=0,\beta=0)+p(\alpha=0,\beta\neq 0)+p(\alpha\neq 0,\beta=0)} \\ &= \frac{p(\alpha\neq 0)p(\beta\neq 0)}{p(\alpha=0)p(\beta=0)+p(\alpha=0)p(\beta\neq 0)+p(\alpha\neq 0)p(\beta=0)} \\ &= \frac{\frac{p(\alpha\neq 0)p(\beta\neq 0)}{p(\alpha=0)p(\beta=0)}}{1+\frac{p(\alpha=0)p(\beta\neq 0)}{p(\alpha=0)p(\beta=0)}+\frac{p(\alpha\neq 0)p(\beta=0)}{p(\alpha=0)p(\beta=0)}} \\ &= \frac{PriorOdds^{\alpha}PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}. \end{aligned}$$

Derivations for the two ways of prior odds specifications:

For the first way (i.e., specify the prior odds for the two paths separately), the conditional prior probabilities of the three scenarios under the null can be expressed as

$q_{00|0} = \frac{p(\alpha=0)p(\beta=0)}{p(\alpha\beta=0)} = \frac{p(\alpha=0)p(\beta=0)}{p(\alpha=0)p(\beta=0)+p(\alpha=0)p(\beta\neq0)+p(\alpha\neq0)p(\beta=0)} = \frac{1}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}$,
$q_{01|0} = \frac{p(\alpha=0)p(\beta\neq0)}{p(\alpha\beta=0)} = \frac{PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}$ and
$q_{10|0} = \frac{p(\alpha\neq0)p(\beta=0)}{p(\alpha\beta=0)} = \frac{PriorOdds^{\alpha}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}$. Plugging these values into Eq. 7, the mediation Bayes factor is

$$\begin{aligned} BF^{med} &= \frac{BF^{\alpha}BF^{\beta}}{\frac{1}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}} + \frac{PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}BF^{\beta} + \frac{PriorOdds^{\alpha}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}BF^{\alpha}} \\ &= \frac{BF^{\alpha}BF^{\beta}(1+PriorOdds^{\beta}+PriorOdds^{\alpha})}{1+BF^{\beta}PriorOdds^{\beta}+BF^{\alpha}PriorOdds^{\alpha}}; \end{aligned}$$

and the mediation posterior odds is

$$\begin{aligned} PosteriorOdds^{med} &= \frac{BF^{\alpha}BF^{\beta} \times \frac{PriorOdds^{\alpha}PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}}{\frac{1}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}} + \frac{PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}BF^{\beta} + \frac{PriorOdds^{\alpha}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}BF^{\alpha}} \\ &= \frac{BF^{\alpha}BF^{\beta} \times PriorOdds^{\alpha}PriorOdds^{\beta}}{1+BF^{\beta}PriorOdds^{\beta}+BF^{\alpha}PriorOdds^{\alpha}}. \end{aligned}$$

This completes the proofs of Eqs. 9-10.

For the second way (i.e., specify one path prior odds [e.g., $PriorOdds^{\alpha}$] and the mediation prior odds), we rewrite the equality constraint as $PriorOdds^{med}(1+PriorOdds^{\beta}+PriorOdds^{\alpha}) = PriorOdds^{\alpha}PriorOdds^{\beta}$. Then the prior odds of path β can be solved from this equation as $PriorOdds^{\beta} = \frac{PriorOdds^{med}(1+PriorOdds^{\alpha})}{PriorOdds^{\alpha}-PriorOdds^{med}}$. The formulas in Eqs. 9-10 can then be used to obtain the Bayes factor and posterior odds for mediation, respectively.

Technical details on the theoretical connections between the proposed approach and the joint significance test of mediation.

The JS test can be connected to a special case of the proposed BHT approach where the conditional prior probability of both paths being absent when mediation is absent is set as $q_{00|0} = p(\alpha = 0, \beta = 0|\alpha\beta = 0)$. In this special case, results of the BHT for mediation (i.e., the mediation Bayes factor and posterior odds) are determined by the path that has a smaller prior probability to exist than the other path.

Particularly, in the independent paths scenario, the specification $q_{00|0} = 0$ holds if and only if $p(\alpha = 0) = 0$ or $p(\beta = 0) = 0$. Thus, when one's prior knowledge suggests that one path, for example, path β is less likely to exist than path α, $p(\alpha = 0) = 0$ or equivalently $PriorOdds^{\alpha} = \infty$. In Eqs. 9-10, when $PriorOdds^{\alpha} = \infty$, $BF^{med} = \frac{BF^{\alpha}BF^{\beta}(1+PriorOdds^{\beta}+PriorOdds^{\alpha})/PriorOdds^{\alpha}}{(1+BF^{\beta}PriorOdds^{\beta}+BF^{\alpha}PriorOdds^{\alpha})/PriorOdds^{\alpha}} = \frac{BF^{\alpha}BF^{\beta}}{BF^{\alpha}} = BF^{\beta}$, and $PosteriorOdds^{med} = \frac{BF^{\alpha}BF^{\beta}\times PriorOdds^{\alpha}PriorOdds^{\beta}/PriorOdds^{\alpha}}{(1+BF^{\beta}PriorOdds^{\beta}+BF^{\alpha}PriorOdds^{\alpha})PriorOdds^{\alpha}} = \frac{BF^{\alpha}BF^{\beta}\times PriorOdds^{\beta}}{BF^{\alpha}} = BF^{\beta} \times PriorOdds^{\beta} = PosteriorOdds^{\beta}$.

Derivations of the formulas in Scenario 2: Dependent paths

For the first way of prior specification, when specifying the conditional prior probabilities $q_{00|0}, q_{01|0}$, and $q_{10|0}$ to all equal 1/3 and the mediation prior odds to equal 1, the two paths α and β are not independent. Specifically, under this specification, the joint prior probability of paths α and β being both present is $p_{11} = p(\alpha \neq 0, \beta \neq 0) = PriorOdds^{med}/(1 + PriorOdds^{med}) = 1/2$, which is not equal to that under the independent paths assumption (4/9). This can be seen by noticing that the marginal prior probability of the presence of path α is $p(\alpha \neq 0) = p(\alpha \neq 0, \beta \neq 0) + p(\alpha \neq 0, \beta = 0) = p_{11} + (1 - p_{11})q_{10|0}$ $= 1/2 + (1 - 1/2)(1/3) = 2/3$, and similarly, the marginal prior probability of the presence of path β is $p(\beta \neq 0) = p(\alpha \neq 0, \beta \neq 0) + p(\alpha = 0, \beta \neq 0) = p_{11} + (1 - p_{11})q_{01|0} = 2/3$. If the two paths were independent, the joint prior probability of α and β being both present should be $(2/3) * (2/3) = 4/9$. Since the joint prior probability is 1/2, rather than 4/9, the two paths are not independent.

For the second way of prior odds specification, given the prior odds for the two paths and the mediation effect, the conditional prior probabilities under the null can be computed as $q_{10|0} = 1 - \frac{p(\alpha=0)}{p(\alpha\beta=0)} = \frac{PriorOdds^{\alpha}-PriorOdds^{med}}{1+PriorOdds^{\alpha}}$, $q_{01|0} = 1 - \frac{p(\beta=0)}{p(\alpha\beta=0)} = \frac{PriorOdds^{\beta}-PriorOdds^{med}}{1+PriorOdds^{\beta}}$, and $q_{00|0} = 1 - q_{10|0} - q_{01|0}$.