Bayesian hypothesis testing of mediation: Methods and the impact of prior odds

specifications

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Supplemental materials

Derivations of the formulas in Scenario 1: Independent paths

Proof of Eq. 7:

In the simple mediation model, we have $p(D|\alpha, \beta) = p(M|\alpha, X)p(Y|\beta, M, X)$. By dividing the numerator and denominator of Eq. 3 by $p(D|\alpha = 0, \beta = 0)$, the proposed BF^{med} becomes

$$
\begin{aligned}
BF^{med} &= \frac{p(D|\alpha \neq 0, \beta \neq 0)}{q_{00|0}p(D|\alpha = 0, \beta = 0) + q_{01|0}p(D|\alpha = 0, \beta \neq 0) + q_{10|0}p(D|\alpha \neq 0, \beta = 0)} \\
&= \frac{\frac{p(M|\alpha \neq 0, X)p(Y|\beta \neq 0, M, X)}{p(M|\alpha = 0, X)p(Y|\beta = 0, M, X)}}{q_{00|0} + q_{01|0}\frac{p(M|\alpha = 0, X)p(Y|\beta \neq 0, M, X)}{p(M|\alpha = 0, X)p(Y|\beta = 0, M, X)} + q_{10|0}\frac{p(M|\alpha \neq 0, X)p(Y|\beta = 0, M, X)}{p(M|\alpha = 0, X)p(Y|\beta = 0, M, X)}} \\
&= \frac{BF^{\alpha}BF^{\beta}}{q_{00|0} + q_{01|0}BF^{\beta} + q_{10|0}BF^{\alpha}}
\end{aligned}
$$

where $BF^{\alpha} = \frac{p(M|X, \alpha \neq 0)}{p(M|X, \alpha = 0)}$ and $BF^{\beta} = \frac{p(Y|M, X, \beta \neq 0)}{p(Y|M, X, \beta = 0)}$.

Proof of Eq. 8:

In the independent paths scenario where α and β are independent, we have

$$
\begin{aligned}
PriorOdds^{med} &= \frac{p(\alpha\beta \neq 0)}{p(\alpha\beta = 0)} \\
&= \frac{p(\alpha \neq 0, \beta \neq 0)}{p(\alpha = 0, \beta = 0) + p(\alpha = 0, \beta \neq 0) + p(\alpha \neq 0, \beta = 0)} \\
&= \frac{p(\alpha \neq 0)p(\beta \neq 0)}{p(\alpha = 0)p(\beta = 0) + p(\alpha = 0)p(\beta \neq 0) + p(\alpha \neq 0)p(\beta = 0)} \\
&= \frac{\frac{p(\alpha \neq 0)p(\beta \neq 0)}{p(\alpha = 0)p(\beta = 0)}}{1 + \frac{p(\alpha = 0)p(\beta \neq 0)}{p(\alpha = 0)p(\beta = 0)} + \frac{p(\alpha \neq 0)p(\beta = 0)}{p(\alpha = 0)p(\beta = 0)}} \\
&= \frac{PriorOdds^{\alpha}PriorOdds^{\beta}}{1 + PriorOdds^{\beta} + PriorOdds^{\alpha}}.
\end{aligned}
$$

Derivations for the two ways of prior odds specifications:

For the first way (i.e., specify the prior odds for the two paths separately), the conditional prior probabilities of the three scenarios under the null can be expressed as

$$q_{00|0} = \frac{p(\alpha=0)p(\beta=0)}{p(\alpha\beta=0)} = \frac{p(\alpha=0)p(\beta=0)}{p(\alpha=0)p(\beta=0)+p(\alpha=0)p(\beta\neq0)+p(\alpha\neq0)p(\beta=0)} = \frac{1}{1+PriorOdds^\beta+PriorOdds^\alpha},$$

$$q_{01|0} = \frac{p(\alpha=0)p(\beta\neq0)}{p(\alpha\beta=0)} = \frac{PriorOdds^\beta}{1+PriorOdds^\beta+PriorOdds^\alpha} \text{ and}$$

$$q_{10|0} = \frac{p(\alpha\neq0)p(\beta=0)}{p(\alpha\beta=0)} = \frac{PriorOdds^\alpha}{1+PriorOdds^\beta+PriorOdds^\alpha}.$$ Plugging these values into Eq. 7, the

mediation Bayes factor is

$$BF^{med} = \frac{BF^\alpha BF^\beta}{\frac{1}{1+PriorOdds^\beta+PriorOdds^\alpha} + \frac{PriorOdds^\beta}{1+PriorOdds^\beta+PriorOdds^\alpha}BF^\beta + \frac{PriorOdds^\alpha}{1+PriorOdds^\beta+PriorOdds^\alpha}BF^\alpha}$$

$$= \frac{BF^\alpha BF^\beta(1+PriorOdds^\beta+PriorOdds^\alpha)}{1+BF^\beta PriorOdds^\beta+BF^\alpha PriorOdds^\alpha};$$

and the mediation posterior odds is

$$PosteriorOdds^{med} = \frac{BF^\alpha BF^\beta \times \frac{PriorOdds^\alpha PriorOdds^\beta}{1+PriorOdds^\beta+PriorOdds^\alpha}}{\frac{1}{1+PriorOdds^\beta+PriorOdds^\alpha} + \frac{PriorOdds^\beta}{1+PriorOdds^\beta+PriorOdds^\alpha}BF^\beta + \frac{PriorOdds^\alpha}{1+PriorOdds^\beta+PriorOdds^\alpha}BF^\alpha}$$

$$= \frac{BF^\alpha BF^\beta \times PriorOdds^\alpha PriorOdds^\beta}{1+BF^\beta PriorOdds^\beta+BF^\alpha PriorOdds^\alpha}.$$

This completes the proofs of Eqs. 9-10.

For the second way (i.e., specify one path prior odds [e.g., $PriorOdds^\alpha$] and the mediation prior odds), we rewrite the equality constraint as $PriorOdds^{med}(1+PriorOdds^\beta+PriorOdds^\alpha) = PriorOdds^\alpha PriorOdds^\beta$. Then the prior odds of path β can be solved from this equation as $PriorOdds^\beta = \frac{PriorOdds^{med}(1+PriorOdds^\alpha)}{PriorOdds^\alpha-PriorOdds^{med}}$. The formulas in Eqs. 9-10 can then be used to obtain the Bayes factor and posterior odds for mediation, respectively.

Derivations of the formulas in Scenario 2: Dependent paths

For the first way of prior specification, when specifying the conditional prior probabilities $q_{00|0}, q_{01|0}$, and $q_{10|0}$ to all equal 1/3 and the mediation prior odds to equal 1, the two paths α and β are not independent. Specifically, under this specification, the joint prior probability of paths α and β being both present is $p_{11} = p(\alpha \neq 0, \beta \neq 0) = PriorOdds^{med}/(1+PriorOdds^{med}) = 1/2$, which is not equal to that under the independent paths assumption (4/9). This can be seen by noticing that the

marginal prior probability of the presence of path α is

$p(\alpha \neq 0) = p(\alpha \neq 0, \beta \neq 0) + p(\alpha \neq 0, \beta = 0) = p_{11} + (1 - p_{11})q_{10|0}$

$= 1/2 + (1 - 1/2)(1/3) = 2/3$, and similarly, the marginal prior probability of the presence

of path β is $p(\beta \neq 0) = p(\alpha \neq 0, \beta \neq 0) + p(\alpha = 0, \beta \neq 0) = p_{11} + (1 - p_{11})q_{01|0} = 2/3$. If

the two paths were independent, the joint prior probability of α and β being both present

should be $(2/3) * (2/3) = 4/9$. Since the joint prior probability is $1/2$, rather than $4/9$, the

two paths are not independent.

For the second way of prior odds specification, given the prior odds for the two paths

and the mediation effect, the conditional prior probabilities under the null can be

computed as $q_{10|0} = 1 - \frac{p(\alpha=0)}{p(\alpha\beta=0)} = \frac{PriorOdds^{\alpha} - PriorOdds^{med}}{1 + PriorOdds^{\alpha}}$,

$q_{01|0} = 1 - \frac{p(\beta=0)}{p(\alpha\beta=0)} = \frac{PriorOdds^{\beta} - PriorOdds^{med}}{1 + PriorOdds^{\beta}}$, and $q_{00|0} = 1 - q_{10|0} - q_{01|0}$.