Bayesian hypothesis testing of mediation: Methods and the impact of prior odds

specifications

Xiao Liu, Lijuan Wang and Zhiyong Zhang

University of Notre Dame

Bayesian hypothesis testing of mediation: Methods and the impact of prior odds specifications

Supplemental materials

Derivations of the formulas in Scenario 1 (independent paths)

In the independent paths scenario,

$PriorOdds^{med} = \frac{p(\alpha \neq 0)p(\beta \neq 0)}{p(\alpha=0)p(\beta=0)+p(\alpha=0)p(\beta\neq0)+p(\alpha\neq0)p(\beta=0)} = \frac{PriorOdds^\alpha PriorOdds^\beta}{1+PriorOdds^\beta+PriorOdds^\alpha}$. In the

simple mediation model, we have $p(D|\alpha, \beta) = p(M|\alpha, X)p(Y|\beta, M, X)$. By dividing the

numerator and denominator of Eq. **??** by $p(D|\alpha = 0, \beta = 0)$, the proposed BF^{med} becomes

$$BF^{med} = \frac{\frac{p(M|\alpha\neq0,X)p(Y|\beta\neq0,M,X)}{p(M|\alpha=0,X)p(Y|\beta=0,M,X)}}{q_{00|0} + q_{01|0}\frac{p(M|\alpha=0,X)p(Y|\beta\neq0,M,X)}{p(M|\alpha=0,X)p(Y|\beta=0,M,X)} + q_{10|0}\frac{p(M|\alpha\neq0,X)p(Y|\beta=0,M,X)}{p(M|\alpha=0,X)p(Y|\beta=0,M,X)}}$$
$$= \frac{BF^\alpha BF^\beta}{q_{00|0} + q_{01|0}BF^\beta + q_{10|0}BF^\alpha}$$

where $BF^\alpha = \frac{p(M|X,\alpha\neq0)}{p(M|X,\alpha=0)}$ and $BF^\beta = \frac{p(Y|M,X,\beta\neq0)}{p(Y|M,X,\beta=0)}$.

Next we provide derivations for the two ways of prior odds specifications. For the

first way (i.e., specify the prior odds for the two paths separately), the conditional prior

probabilities of the three scenarios under the null can be expressed as

$q_{00|0} = \frac{p(\alpha=0)p(\beta=0)}{p(\alpha\beta=0)} = \frac{p(\alpha=0)p(\beta=0)}{p(\alpha=0)p(\beta=0)+p(\alpha=0)p(\beta\neq0)+p(\alpha\neq0)p(\beta=0)} = \frac{1}{1+PriorOdds^\beta+PriorOdds^\alpha}$,

$q_{01|0} = \frac{p(\alpha=0)p(\beta\neq0)}{p(\alpha\beta=0)} = \frac{PriorOdds^\beta}{1+PriorOdds^\beta+PriorOdds^\alpha}$ and

$q_{10|0} = \frac{p(\alpha\neq0)p(\beta=0)}{p(\alpha\beta=0)} = \frac{PriorOdds^\alpha}{1+PriorOdds^\beta+PriorOdds^\alpha}$. Plugging these values into Eqs. 8 and 9, the

mediation Bayes factor is

$$BF^{med} = \frac{BF^\alpha BF^\beta}{\frac{1}{1+PriorOdds^\beta+PriorOdds^\alpha} + \frac{PriorOdds^\beta}{1+PriorOdds^\beta+PriorOdds^\alpha}BF^\beta + \frac{PriorOdds^\alpha}{1+PriorOdds^\beta+PriorOdds^\alpha}BF^\alpha}$$
$$= \frac{BF^\alpha BF^\beta(1 + PriorOdds^\beta + PriorOdds^\alpha)}{1 + BF^\beta PriorOdds^\beta + BF^\alpha PriorOdds^\alpha};$$

and the mediation posterior odds is $PosteriorOdds^{med} =$

$\frac{BF^\alpha BF^\beta \times \frac{PriorOdds^\alpha PriorOdds^\beta}{1+PriorOdds^\beta+PriorOdds^\alpha}}{\frac{1}{1+PriorOdds^\beta+PriorOdds^\alpha}+\frac{PriorOdds^\beta}{1+PriorOdds^\beta+PriorOdds^\alpha}BF^\beta+\frac{PriorOdds^\alpha}{1+PriorOdds^\beta+PriorOdds^\alpha}BF^\alpha} =$

$\frac{BF^\alpha BF^\beta \times PriorOdds^\alpha PriorOdds^\beta}{1+BF^\beta PriorOdds^\beta+BF^\alpha PriorOdds^\alpha}$.

For the second way (i.e., specify one path prior odds [e.g., $PriorOdds^\alpha$] and the mediation prior odds), we rewrite the equality constraint as $PriorOdds^{med}(1 + PriorOdds^\beta + PriorOdds^\alpha) = PriorOdds^\alpha PriorOdds^\beta$. Then the prior odds of path β can be solved from this equation as $PriorOdds^\beta = \frac{PriorOdds^{med}(1+PriorOdds^\alpha)}{PriorOdds^\alpha - PriorOdds^{med}}$. The formulas in Eqs. 11 and 12 can then be used to obtain the Bayes factor and posterior odds for mediation, respectively.

Derivations for Footnotes 8 and 9 are presented below. Specifically, given $PriorOdds^{med}$ and $q_{00|0}$, the other two conditional prior probabilities under the null need to satisfy $q_{01|0} + q_{10|0} = 1 - q_{00|0}$ and $q_{01|0}q_{10|0} = PriorOdds^{med}q_{00|0}$. Solving this equation set, we have $(q_{01|0}, q_{10|0}) = (\frac{c \pm \sqrt{c^2 - 4d}}{2}, \frac{c \mp \sqrt{c^2 - 4d}}{2})$, where $c = 1 - q_{00|0}, d = PriorOdds^{med}q_{00|0}$ (Footnote 9). In order for $c^2 - 4d \geq 0$, $PriorOdds^{med}$ and $q_{00|0}$ need to satisfy $PriorOdds^{med} \leq (1 - q_{00|0})^2/4q_{00|0}$ (Footnote 8).