Bayesian hypothesis testing of mediation: Methods and the impact of prior odds specifications

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Supplemental materials

Derivations of the formulas in Scenario 1 (independent paths)

In the independent paths scenario, $PriorOdds^{med} = \frac{p(\alpha\neq0)p(\beta\neq0)}{p(\alpha=0)p(\beta=0)+p(\alpha=0)p(\beta\neq0)+p(\alpha\neq0)p(\beta=0)} = \frac{PriorOdds^{\alpha}PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}$. In the simple mediation model, we have$p(D|\alpha,\beta) = p(M|\alpha,X)p(Y|\beta,M,X)$. By dividing the numerator and denominator of Eq. 4 by $p(D|\alpha=0,\beta=0)$, the proposed BF^{med} becomes

$$\begin{aligned} BF^{med} &= \frac{\frac{p(M|\alpha\neq0,X)p(Y|\beta\neq0,M,X)}{p(M|\alpha=0,X)p(Y|\beta=0,M,X)}}{q_{00|0} + q_{01|0}\frac{p(M|\alpha=0,X)p(Y|\beta\neq0,M,X)}{p(M|\alpha=0,X)p(Y|\beta=0,M,X)} + q_{10|0}\frac{p(M|\alpha\neq0,X)p(Y|\beta=0,M,X)}{p(M|\alpha=0,X)p(Y|\beta=0,M,X)}} \\ &= \frac{BF^{\alpha}BF^{\beta}}{q_{00|0} + q_{01|0}BF^{\beta} + q_{10|0}BF^{\alpha}} \end{aligned}$$

where $BF^{\alpha} = \frac{p(M|X,\alpha\neq0)}{p(M|X,\alpha=0)}$ and $BF^{\beta} = \frac{p(Y|M,X,\beta\neq0)}{p(Y|M,X,\beta=0)}$.

Next we provide derivations for the two ways of prior odds specifications. For the first way (i.e., specify the prior odds for the two paths separately), the conditional prior probabilities of the three scenarios under the null can be expressed as
$q_{00|0} = \frac{p(\alpha=0)p(\beta=0)}{p(\alpha\beta=0)} = \frac{p(\alpha=0)p(\beta=0)}{p(\alpha=0)p(\beta=0)+p(\alpha=0)p(\beta\neq0)+p(\alpha\neq0)p(\beta=0)} = \frac{1}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}$,
$q_{01|0} = \frac{p(\alpha=0)p(\beta\neq0)}{p(\alpha\beta=0)} = \frac{PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}$ and
$q_{10|0} = \frac{p(\alpha\neq0)p(\beta=0)}{p(\alpha\beta=0)} = \frac{PriorOdds^{\alpha}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}$. Plugging these values into Eqs. 8 and 9, the mediation Bayes factor is

$$\begin{aligned} BF^{med} &= \frac{BF^{\alpha}BF^{\beta}}{\frac{1}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}} + \frac{PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}BF^{\beta} + \frac{PriorOdds^{\alpha}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}BF^{\alpha}} \\ &= \frac{BF^{\alpha}BF^{\beta}(1+PriorOdds^{\beta}+PriorOdds^{\alpha})}{1+BF^{\beta}PriorOdds^{\beta}+BF^{\alpha}PriorOdds^{\alpha}}; \end{aligned}$$

and the mediation posterior odds is $PosteriorOdds^{med} = \frac{BF^{\alpha}BF^{\beta}\times\frac{PriorOdds^{\alpha}PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}}{\frac{1}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}+\frac{PriorOdds^{\beta}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}BF^{\beta}+\frac{PriorOdds^{\alpha}}{1+PriorOdds^{\beta}+PriorOdds^{\alpha}}BF^{\alpha}} = \frac{BF^{\alpha}BF^{\beta}\times PriorOdds^{\alpha}PriorOdds^{\beta}}{1+BF^{\beta}PriorOdds^{\beta}+BF^{\alpha}PriorOdds^{\alpha}}$.

For the second way (i.e., specify one path prior odds [e.g., $PriorOdds^{\alpha}$] and the mediation prior odds), we rewrite the equality constraint as $PriorOdds^{med}(1 + PriorOdds^{\beta} + PriorOdds^{\alpha}) = PriorOdds^{\alpha}PriorOdds^{\beta}$. Then the prior odds of path β can be solved from this equation as $PriorOdds^{\beta} = \frac{PriorOdds^{med}(1+PriorOdds^{\alpha})}{PriorOdds^{\alpha}-PriorOdds^{med}}$. The formulas in Eqs. 11 and 12 can then be used to obtain the Bayes factor and posterior odds for mediation, respectively.

Derivations for Footnotes 8 and 9 are presented below. Specifically, given $PriorOdds^{med}$ and $q_{00|0}$, the other two conditional prior probabilities under the null need to satisfy $q_{01|0} + q_{10|0} = 1 - q_{00|0}$ and $q_{01|0}q_{10|0} = PriorOdds^{med}q_{00|0}$. Solving this equation set, we have $(q_{01|0}, q_{10|0}) = (\frac{c\pm\sqrt{c^2-4d}}{2}, \frac{c\mp\sqrt{c^2-4d}}{2})$, where $c = 1 - q_{00|0}, d = PriorOdds^{med}q_{00|0}$ (Footnote 9). In order for $c^2 - 4d \geq 0$, $PriorOdds^{med}$ and $q_{00|0}$ need to satisfy $PriorOdds^{med} \leq (1 - q_{00|0})^2/4q_{00|0}$ (Footnote 8).