

(*Throughout the Appendix, we assume $d \geq \frac{17}{50}$ & $d \leq \frac{7}{20}$ *)

$$\text{In}[*]:= (*\text{Lemma A.1. } a \geq \frac{17}{100} \& a \leq \frac{23}{100} *)$$

$$\text{Minimize}\left[\left\{a^3 + (2d - a^2) \sqrt{2d + a^2} - (4d - 1 + (1 - 2d)^{3/2})\right\}, \{a, d\}\right]$$

$$a \geq \frac{17}{100} \& a \leq \frac{23}{100} \& d \geq \frac{17}{50} \& d \leq \frac{7}{20}\}$$

$$\text{Out}[*]=$$

$$\left\{-\frac{347833}{1000000} - \frac{16\sqrt{2}}{125} + \frac{6271\sqrt{7329}}{1000000}, \left\{a \rightarrow \frac{23}{100}, d \rightarrow \frac{17}{50}\right\}\right\}$$

(*Lemma A.2. $a \geq \frac{17}{100}$ & $a \leq \frac{23}{100}$ *)

$$\text{Solve}\left[\frac{1}{2}y^2 + xy + a^2 - ax = d, y\right]$$

$$\text{Out}[*]=$$

$$\left\{\left\{y \rightarrow -x - \sqrt{-2a^2 + 2ax + 2d + x^2}\right\}, \left\{y \rightarrow -x + \sqrt{-2a^2 + 2ax + 2d + x^2}\right\}\right\}$$

$$\text{In}[*]:= \text{FullSimplify}\left[x y^2 + (a - x) a^2 + a (a + y)^2 + (y - a) (x + y)^2 / . y \rightarrow \left(-x + \sqrt{-2a^2 + 2ax + 2d + x^2}\right)\right]$$

$$\text{Out}[*]=$$

$$2a^3 - 3a^2x + x^3 + 2(ax + d)\sqrt{-2a^2 + 2(ax + d) + x^2} -$$

$$x^2\sqrt{-2a^2 + 2(ax + d) + x^2} + ax\left(x - 2\sqrt{-2a^2 + 2(ax + d) + x^2}\right)$$

$$\text{In}[*]:= D\left[2a^3 - 3a^2x + x^3 + ax^2 + (2d - x^2)\sqrt{-2a^2 + 2(ax + d) + x^2}, x\right]$$

$$\text{Out}[*]=$$

$$-3a^2 + 2ax + 3x^2 + \frac{(2a + 2x)(2d - x^2)}{2\sqrt{-2a^2 + x^2 + 2(d + ax)}} - 2x\sqrt{-2a^2 + x^2 + 2(d + ax)}$$

$$\text{In}[*]:= \text{NMinimize}\left[\left\{-3a^2 + 2ax + 3x^2 + \frac{(2a + 2x)(2d - x^2)}{2\sqrt{-2a^2 + x^2 + 2(d + ax)}} - 2x\sqrt{-2a^2 + x^2 + 2(d + ax)}, d \geq \frac{17}{50} \& d \leq \frac{7}{20} \& a \geq \frac{17}{100} \& a \leq \frac{23}{100} \& x \geq 0 \& x \leq a\right\}, \{x, a, d\}\right]$$

$$\text{Out}[*]=$$

$$\{0.0337748, \{x \rightarrow 0.142658, a \rightarrow 0.17, d \rightarrow 0.34\}\}$$

(*Lemma A.3. $a \geq \frac{1}{3}$ & $a \leq \frac{2}{5}$ *)

$$\text{In}[*]:= \text{FullSimplify}\left[\left(x + \frac{1}{5}\right)(y + a)^2 + \left(y - x - \frac{1}{5}\right)y^2 + a\left(x + \frac{1}{5}\right)^2 / . y \rightarrow \sqrt{2\left(d - ax - \frac{1}{5}a\right)}\right]$$

$$\text{Out}[*]=$$

$$\frac{1}{25}a(1 + 5x)(1 + 5a + 5x) + 2\sqrt{\frac{2}{5}}d\sqrt{-a + 5d - 5ax}$$

In[1]:= $D\left[\frac{1}{25} a (1 + 5 x) (1 + 5 a + 5 x) + 2 \sqrt{\frac{2}{5}} d \sqrt{-a + 5 d - 5 a x}, x\right]$

Out[1]= $\frac{1}{5} a (1 + 5 x) + \frac{1}{5} a (1 + 5 a + 5 x) - \frac{\sqrt{10} a d}{\sqrt{-a + 5 d - 5 a x}}$

In[2]:= $D\left[\frac{1}{5} a (1 + 5 x) + \frac{1}{5} a (1 + 5 a + 5 x) - \frac{\sqrt{10} a d}{\sqrt{-a + 5 d - 5 a x}}, x\right]$

Out[2]= $2 a - \frac{5 \sqrt{\frac{5}{2}} a^2 d}{(-a + 5 d - 5 a x)^{3/2}}$

In[3]:= $NMinimize\left[\left\{2 a - \frac{5 \sqrt{\frac{5}{2}} a^2 d}{(-a + 5 d - 5 a x)^{3/2}}, d \geq \frac{17}{50} \& d \leq \frac{7}{20} \& a \geq \frac{1}{3} \& a \leq \frac{2}{5} \& x \geq 0 \& x \leq \frac{3}{10}\right\}, \{x, a, d\}\right]$

Out[3]= $\{0.065669, \{x \rightarrow 0.3, a \rightarrow 0.4, d \rightarrow 0.34\}\}$

In[4]:= $NMinimize\left[\left\{\frac{1}{5} a (1 + 5 x) + \frac{1}{5} a (1 + 5 a + 5 x) - \frac{\sqrt{10} a d}{\sqrt{-a + 5 d - 5 a x}}, d \geq \frac{17}{50} \& d \leq \frac{7}{20} \& a \geq \frac{1}{3} \& a \leq \frac{2}{5} \& x \geq \frac{3}{10} \& x \leq \frac{4}{10}\right\}, \{x, a, d\}\right]$

Out[4]= $\{0.0459683, \{x \rightarrow 0.3, a \rightarrow 0.4, d \rightarrow 0.34\}\}$

In[5]:= $NMinimize\left[\left\{a^3 + (2 d - a^2) \sqrt{2 d + a^2} - \left(\frac{1}{5} a^2 + \frac{1}{25} a + 2 d \sqrt{2 \left(d - \frac{1}{5} a\right)}\right), d \geq \frac{17}{50} \& d \leq \frac{7}{20} \& a \geq \frac{1}{3} \& a \leq \frac{2}{5}\right\}, \{a, d\}\right]$

Out[5]= $\{0.0022329, \{a \rightarrow 0.4, d \rightarrow 0.34\}\}$

(*Lemma A.4. $a \geq \frac{1}{3} \& a \leq \frac{2}{5} */$

In[6]:= $Solve\left[\frac{1}{2} y^2 + x y + \frac{1}{5} a - \frac{1}{5} x = d, y\right]$

Out[6]= $\left\{\left\{y \rightarrow -x - \frac{\sqrt{-2 a + 10 d + 2 x + 5 x^2}}{\sqrt{5}}\right\}, \left\{y \rightarrow -x + \frac{\sqrt{-2 a + 10 d + 2 x + 5 x^2}}{\sqrt{5}}\right\}\right\}$

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In[1]:= FullSimplify[

$$\frac{1}{5} (y + a)^2 + \left(y - \frac{1}{5}\right) (x + y)^2 + x y^2 + (a - x) \left(\frac{1}{5}\right)^2 / . y \rightarrow \left(-x + \frac{\sqrt{-2 a + 10 d + 2 x + 5 x^2}}{\sqrt{5}}\right)$$
]

Out[1]=

$$\frac{1}{25} (a + 5 a^2 - 10 a x + 10 \sqrt{5} d \sqrt{-2 a + 10 d + x (2 + 5 x)} + x (-1 + 5 x (1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)})))$$


In[2]:= D[ $\frac{1}{25} (a + 5 a^2 - 10 a x + 10 \sqrt{5} d \sqrt{-2 a + 10 d + x (2 + 5 x)} + x (-1 + 5 x (1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)})))$ , x]

Out[2]=

$$\frac{1}{25} \left( -1 - 10 a + \frac{5 \sqrt{5} d (2 + 10 x)}{\sqrt{-2 a + 10 d + x (2 + 5 x)}} + 5 x (1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)}) + x \left( 5 x \left( 5 - \frac{\sqrt{5} (2 + 10 x)}{2 \sqrt{-2 a + 10 d + x (2 + 5 x)}} \right) + 5 (1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)}) \right) \right)$$


In[3]:= NMinimize[

$$\left\{ a^3 + (2 d - a^2) \sqrt{2 d + a^2} - \left( \frac{1}{25} (a + 5 a^2 - 10 a x + 10 \sqrt{5} d \sqrt{-2 a + 10 d + x (2 + 5 x)} + x (-1 + 5 x (1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)}))) \right), d \geq \frac{17}{50} \& d \leq \frac{7}{20} \& a \geq \frac{1}{3} \& a \leq \frac{2}{5} \& x \geq 0 \& x \leq \frac{1}{4} \right\}, \{x, a, d\}]$$


Out[3]=

$$\{0.0022329, \{x \rightarrow 0., a \rightarrow 0.4, d \rightarrow 0.34\}\}$$


In[4]:= NMinimize[

$$\left\{ \frac{1}{25} \left( -1 - 10 a + \frac{5 \sqrt{5} d (2 + 10 x)}{\sqrt{-2 a + 10 d + x (2 + 5 x)}} + 5 x (1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)}) + x \left( 5 x \left( 5 - \frac{\sqrt{5} (2 + 10 x)}{2 \sqrt{-2 a + 10 d + x (2 + 5 x)}} \right) + 5 (1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)}) \right) \right), d \geq \frac{17}{50} \& d \leq \frac{7}{20} \& a \geq \frac{1}{3} \& a \leq \frac{2}{5} \& x \geq \frac{1}{4} \& x \leq \frac{2}{5} \right\}, \{x, a, d\}]$$


Out[4]=

$$\{0.0106942, \{x \rightarrow 0.25, a \rightarrow 0.4, d \rightarrow 0.35\}\}$$


In[5]:= (*Lemma A.5. a ≥ 1/3 && a ≤ 2/5*)
FullSimplify[x + (y - x) y^2 + (1 - y) x^2 /. y → (x + √(x^2 - 2 x + 2 d))]

Out[5]=

$$x + 4 d x - 3 x^2 + x^3 + (2 d + (-2 + x) x)^{3/2}$$


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In[6]:= $D[x + 4 d x - 3 x^2 + x^3 + (2 d + (-2 + x) x)^{3/2}, x]$

Out[6]= $1 + 4 d - 6 x + 3 x^2 + \frac{3}{2} (-2 + 2 x) \sqrt{2 d + (-2 + x) x}$

In[7]:= $D\left[1 + 4 d - 6 x + 3 x^2 + \frac{3}{2} (-2 + 2 x) \sqrt{2 d + (-2 + x) x}, x\right]$

Out[7]= $-6 + 6 x + \frac{3 (-2 + 2 x)^2}{4 \sqrt{2 d + (-2 + x) x}} + 3 \sqrt{2 d + (-2 + x) x}$

In[8]:= $\text{NMinimize}\left[\left\{-6 + 6 x + \frac{3 (-2 + 2 x)^2}{4 \sqrt{2 d + (-2 + x) x}} + 3 \sqrt{2 d + (-2 + x) x}, d \geq \frac{17}{50} \& d \leq \frac{7}{20} \& x \geq \frac{1}{3} \& x \leq \frac{9}{20}\right\}, \{x, d\}\right]$

Out[8]= $\{0.648408, \{x \rightarrow 0.333333, d \rightarrow 0.35\}\}$

In[9]:= $x + 4 d x - 3 x^2 + x^3 + (2 d + (-2 + x) x)^{3/2} /. x \rightarrow \left(1 - \frac{1}{2} a - \frac{1 - 2 d}{2 a}\right)$

Out[9]= $1 - \frac{a}{2} - 3 \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right)^2 + \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right)^3 - \frac{1 - 2 d}{2 a} + 4 \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) d + \left(\left(-1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) + 2 d\right)^{3/2}$

In[10]:= $\text{NMinimize}\left[\left\{a^3 + (2 d - a^2) \sqrt{2 d + a^2} - \left(1 - \frac{a}{2} - 3 \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right)^2 + \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right)^3 - \frac{1 - 2 d}{2 a} + 4 \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) d + \left(\left(-1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) + 2 d\right)^{3/2}\right), d \geq \frac{17}{50} \& d \leq \frac{7}{20} \& a \geq \frac{1}{3} \& a \leq \frac{2}{5}\right\}, \{a, d\}\right]$

Out[10]= $\{0.0045254, \{a \rightarrow 0.4, d \rightarrow 0.35\}\}$

In[11]:= $\text{FullSimplify}\left[x + 4 d x - 3 x^2 + x^3 + (2 d + (-2 + x) x)^{3/2} /. x \rightarrow \left(2 d - \frac{1}{4}\right)\right]$

Out[11]= $\frac{1}{64} (-29 + ((3 - 8 d)^2)^{3/2} + 280 d - 448 d^2 + 512 d^3)$

In[12]:= $\text{NMinimize}\left[\left\{a^3 + (2 d - a^2) \sqrt{2 d + a^2} - \left(\frac{1}{64} (-29 + ((3 - 8 d)^2)^{3/2} + 280 d - 448 d^2 + 512 d^3)\right), d \geq \frac{17}{50} \& d \leq \frac{7}{20} \& a \geq \frac{1}{3} \& a \leq \frac{2}{5}\right\}, \{a, d\}\right]$

Out[12]= $\{0.000637872, \{a \rightarrow 0.4, d \rightarrow 0.34\}\}$