

(*Throughout the Appendix, we assume $d \geq \frac{17}{50}$ & $d \leq \frac{7}{20}$ *)

In[*]:= (*Lemma A.1. $a \geq \frac{17}{100}$ & $a \leq \frac{23}{100}$ *)

$$\text{Minimize} \left[\left\{ a^3 + (2d - a^2) \sqrt{2d + a^2} - (4d - 1 + (1 - 2d)^{3/2}), \right. \right. \\ \left. \left. a \geq \frac{17}{100} \text{ \& } a \leq \frac{23}{100} \text{ \& } d \geq \frac{17}{50} \text{ \& } d \leq \frac{7}{20} \right\}, \{a, d\} \right]$$

Out[*]=

$$\left\{ -\frac{347833}{1000000} - \frac{16\sqrt{2}}{125} + \frac{6271\sqrt{7329}}{1000000}, \left\{ a \rightarrow \frac{23}{100}, d \rightarrow \frac{17}{50} \right\} \right\}$$

(*Lemma A.2. $a \geq \frac{17}{100}$ & $a \leq \frac{23}{100}$ *)

$$\text{Solve} \left[\frac{1}{2} y^2 + x y + a^2 - a x == d, y \right]$$

Out[*]=

$$\left\{ \left\{ y \rightarrow -x - \sqrt{-2a^2 + 2ax + 2d + x^2} \right\}, \left\{ y \rightarrow -x + \sqrt{-2a^2 + 2ax + 2d + x^2} \right\} \right\}$$

In[*]:= FullSimplify[

$$x y^2 + (a - x) a^2 + a (a + y)^2 + (y - a) (x + y)^2 /. y \rightarrow \left(-x + \sqrt{-2a^2 + 2ax + 2d + x^2} \right)]$$

Out[*]=

$$2a^3 - 3a^2x + x^3 + 2(ax + d) \sqrt{-2a^2 + 2(ax + d) + x^2} - \\ x^2 \sqrt{-2a^2 + 2(ax + d) + x^2} + ax \left(x - 2 \sqrt{-2a^2 + 2(ax + d) + x^2} \right)$$

In[*]:= D[$2a^3 - 3a^2x + x^3 + ax^2 + (2d - x^2) \sqrt{-2a^2 + 2(ax + d) + x^2}$, x]

Out[*]=

$$-3a^2 + 2ax + 3x^2 + \frac{(2a + 2x)(2d - x^2)}{2 \sqrt{-2a^2 + x^2 + 2(d + ax)}} - 2x \sqrt{-2a^2 + x^2 + 2(d + ax)}$$

In[*]:= NMinimize[$\left\{ -3a^2 + 2ax + 3x^2 + \frac{(2a + 2x)(2d - x^2)}{2 \sqrt{-2a^2 + x^2 + 2(d + ax)}} - 2x \sqrt{-2a^2 + x^2 + 2(d + ax)}, \right.$

$$\left. d \geq \frac{17}{50} \text{ \& } d \leq \frac{7}{20} \text{ \& } a \geq \frac{17}{100} \text{ \& } a \leq \frac{23}{100} \text{ \& } x \geq 0 \text{ \& } x \leq a \right\}, \{x, a, d\}]$$

Out[*]=

$$\{0.0337748, \{x \rightarrow 0.142658, a \rightarrow 0.17, d \rightarrow 0.34\}\}$$

(*Lemma A.3. $a \geq \frac{1}{3}$ & $a \leq \frac{2}{5}$ *)

In[*]:= FullSimplify[$\left(x + \frac{1}{5} \right) (y + a)^2 + \left(y - x - \frac{1}{5} \right) y^2 + a \left(x + \frac{1}{5} \right)^2 /. y \rightarrow \sqrt{2 \left(d - ax - \frac{1}{5} a \right)}$]

Out[*]=

$$\frac{1}{25} a (1 + 5x) (1 + 5a + 5x) + 2 \sqrt{\frac{2}{5} d} \sqrt{-a + 5d - 5ax}$$

$$\text{In}[*]:= \text{D}\left[\frac{1}{25} a (1+5 x) (1+5 a+5 x) + 2 \sqrt{\frac{2}{5}} d \sqrt{-a+5 d-5 a x}, x\right]$$

Out[*]=

$$\frac{1}{5} a (1+5 x) + \frac{1}{5} a (1+5 a+5 x) - \frac{\sqrt{10} a d}{\sqrt{-a+5 d-5 a x}}$$

$$\text{In}[*]:= \text{D}\left[\frac{1}{5} a (1+5 x) + \frac{1}{5} a (1+5 a+5 x) - \frac{\sqrt{10} a d}{\sqrt{-a+5 d-5 a x}}, x\right]$$

Out[*]=

$$2 a - \frac{5 \sqrt{\frac{5}{2}} a^2 d}{(-a+5 d-5 a x)^{3/2}}$$

$$\text{In}[*]:= \text{NMinimize}\left[\left\{2 a - \frac{5 \sqrt{\frac{5}{2}} a^2 d}{(-a+5 d-5 a x)^{3/2}},\right.\right. \\ \left.\left.d \geq \frac{17}{50} \ \&\& \ d \leq \frac{7}{20} \ \&\& \ a \geq \frac{1}{3} \ \&\& \ a \leq \frac{2}{5} \ \&\& \ x \geq 0 \ \&\& \ x \leq \frac{3}{10}\right\}, \{x, a, d\}\right]$$

Out[*]=

$$\{0.065669, \{x \rightarrow 0.3, a \rightarrow 0.4, d \rightarrow 0.34\}\}$$

$$\text{In}[*]:= \text{NMinimize}\left[\left\{\frac{1}{5} a (1+5 x) + \frac{1}{5} a (1+5 a+5 x) - \frac{\sqrt{10} a d}{\sqrt{-a+5 d-5 a x}},\right.\right. \\ \left.\left.d \geq \frac{17}{50} \ \&\& \ d \leq \frac{7}{20} \ \&\& \ a \geq \frac{1}{3} \ \&\& \ a \leq \frac{2}{5} \ \&\& \ x \geq \frac{3}{10} \ \&\& \ x \leq \frac{4}{10}\right\}, \{x, a, d\}\right]$$

Out[*]=

$$\{0.0459683, \{x \rightarrow 0.3, a \rightarrow 0.4, d \rightarrow 0.34\}\}$$

$$\text{In}[*]:= \text{NMinimize}\left[\left\{a^3 + (2 d - a^2) \sqrt{2 d + a^2} - \left(\frac{1}{5} a^2 + \frac{1}{25} a + 2 d \sqrt{2 \left(d - \frac{1}{5} a\right)}\right),\right.\right. \\ \left.\left.d \geq \frac{17}{50} \ \&\& \ d \leq \frac{7}{20} \ \&\& \ a \geq \frac{1}{3} \ \&\& \ a \leq \frac{2}{5}\right\}, \{a, d\}\right]$$

Out[*]=

$$\{0.0022329, \{a \rightarrow 0.4, d \rightarrow 0.34\}\}$$

(*Lemma A.4. $a \geq \frac{1}{3} \ \&\& \ a \leq \frac{2}{5}$ *)

$$\text{In}[*]:= \text{Solve}\left[\frac{1}{2} y^2 + x y + \frac{1}{5} a - \frac{1}{5} x == d, y\right]$$

Out[*]=

$$\left\{\left\{y \rightarrow -x - \frac{\sqrt{-2 a + 10 d + 2 x + 5 x^2}}{\sqrt{5}}\right\}, \left\{y \rightarrow -x + \frac{\sqrt{-2 a + 10 d + 2 x + 5 x^2}}{\sqrt{5}}\right\}\right\}$$

`In[*]:= FullSimplify[`

$$\frac{1}{5} (y + a)^2 + \left(y - \frac{1}{5}\right) (x + y)^2 + x y^2 + (a - x) \left(\frac{1}{5}\right)^2 /. y \rightarrow \left(-x + \frac{\sqrt{-2 a + 10 d + 2 x + 5 x^2}}{\sqrt{5}}\right)]$$

`Out[*]=`

$$\frac{1}{25} \left(a + 5 a^2 - 10 a x + 10 \sqrt{5} d \sqrt{-2 a + 10 d + x (2 + 5 x)} + \right. \\ \left. x \left(-1 + 5 x \left(1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)} \right) \right) \right)$$

`In[*]:= D[`

$$\frac{1}{25} \left(a + 5 a^2 - 10 a x + 10 \sqrt{5} d \sqrt{-2 a + 10 d + x (2 + 5 x)} + \right. \\ \left. x \left(-1 + 5 x \left(1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)} \right) \right) \right), x]$$

`Out[*]=`

$$\frac{1}{25} \left(-1 - 10 a + \frac{5 \sqrt{5} d (2 + 10 x)}{\sqrt{-2 a + 10 d + x (2 + 5 x)}} + 5 x \left(1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)} \right) + \right. \\ \left. x \left(5 x \left(5 - \frac{\sqrt{5} (2 + 10 x)}{2 \sqrt{-2 a + 10 d + x (2 + 5 x)}} \right) + 5 \left(1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)} \right) \right) \right)$$

`In[*]:= NMinimize[`

$$\left\{ a^3 + (2 d - a^2) \sqrt{2 d + a^2} - \left(\frac{1}{25} \left(a + 5 a^2 - 10 a x + 10 \sqrt{5} d \sqrt{-2 a + 10 d + x (2 + 5 x)} + \right. \right. \right. \\ \left. \left. x \left(-1 + 5 x \left(1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)} \right) \right) \right) \right\}, \\ \left. d \geq \frac{17}{50} \ \&\& \ d \leq \frac{7}{20} \ \&\& \ a \geq \frac{1}{3} \ \&\& \ a \leq \frac{2}{5} \ \&\& \ x \geq 0 \ \&\& \ x \leq \frac{1}{4} \right\}, \{x, a, d\}]$$

`Out[*]=`

$$\{0.0022329, \{x \rightarrow 0., a \rightarrow 0.4, d \rightarrow 0.34\}\}$$

`In[*]:= NMinimize[`

$$\left\{ \frac{1}{25} \left(-1 - 10 a + \frac{5 \sqrt{5} d (2 + 10 x)}{\sqrt{-2 a + 10 d + x (2 + 5 x)}} + 5 x \left(1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)} \right) + x \right. \right. \\ \left. \left. \left(5 x \left(5 - \frac{\sqrt{5} (2 + 10 x)}{2 \sqrt{-2 a + 10 d + x (2 + 5 x)}} \right) + 5 \left(1 + 5 x - \sqrt{5} \sqrt{-2 a + 10 d + x (2 + 5 x)} \right) \right) \right) \right\}, \\ \left. d \geq \frac{17}{50} \ \&\& \ d \leq \frac{7}{20} \ \&\& \ a \geq \frac{1}{3} \ \&\& \ a \leq \frac{2}{5} \ \&\& \ x \geq \frac{1}{4} \ \&\& \ x \leq \frac{2}{5} \right\}, \{x, a, d\}]$$

`Out[*]=`

$$\{0.0106942, \{x \rightarrow 0.25, a \rightarrow 0.4, d \rightarrow 0.35\}\}$$

`In[*]:=`

$$(*\text{Lemma A.5. } a \geq \frac{1}{3} \ \&\& \ a \leq \frac{2}{5} *)$$

`FullSimplify`
$$\left[x + (y - x) y^2 + (1 - y) x^2 /. y \rightarrow \left(x + \sqrt{x^2 - 2 x + 2 d} \right) \right]$$

`Out[*]=`

$$x + 4 d x - 3 x^2 + x^3 + (2 d + (-2 + x) x)^{3/2}$$

In[*]:= $D[x + 4 d x - 3 x^2 + x^3 + (2 d + (-2 + x) x)^{3/2}, x]$

Out[*]=

$$1 + 4 d - 6 x + 3 x^2 + \frac{3}{2} (-2 + 2 x) \sqrt{2 d + (-2 + x) x}$$

In[*]:= $D[1 + 4 d - 6 x + 3 x^2 + \frac{3}{2} (-2 + 2 x) \sqrt{2 d + (-2 + x) x}, x]$

Out[*]=

$$-6 + 6 x + \frac{3 (-2 + 2 x)^2}{4 \sqrt{2 d + (-2 + x) x}} + 3 \sqrt{2 d + (-2 + x) x}$$

In[*]:= $\text{NMinimize}\left[\left\{-6 + 6 x + \frac{3 (-2 + 2 x)^2}{4 \sqrt{2 d + (-2 + x) x}} + 3 \sqrt{2 d + (-2 + x) x},\right.\right.$
 $\left.\left.d \geq \frac{17}{50} \ \&\& \ d \leq \frac{7}{20} \ \&\& \ x \geq \frac{1}{3} \ \&\& \ x \leq \frac{9}{20}\right\}, \{x, d\}\right]$

Out[*]=

$$\{0.648408, \{x \rightarrow 0.333333, d \rightarrow 0.35\}\}$$

In[*]:= $x + 4 d x - 3 x^2 + x^3 + (2 d + (-2 + x) x)^{3/2} /. x \rightarrow \left(1 - \frac{1}{2} a - \frac{1 - 2 d}{2 a}\right)$

Out[*]=

$$1 - \frac{a}{2} - 3 \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right)^2 + \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right)^3 - \frac{1 - 2 d}{2 a} +$$

$$4 \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) d + \left(\left(-1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) + 2 d\right)^{3/2}$$

In[*]:= $\text{NMinimize}\left[\right.$

$$\left\{a^3 + (2 d - a^2) \sqrt{2 d + a^2} - \left(1 - \frac{a}{2} - 3 \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right)^2 + \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right)^3 - \frac{1 - 2 d}{2 a} +\right.\right.$$

$$\left.\left.4 \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) d + \left(\left(-1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) \left(1 - \frac{a}{2} - \frac{1 - 2 d}{2 a}\right) + 2 d\right)^{3/2}\right\}, \{a, d\}\right]$$

$$d \geq \frac{17}{50} \ \&\& \ d \leq \frac{7}{20} \ \&\& \ a \geq \frac{1}{3} \ \&\& \ a \leq \frac{2}{5}\right\}, \{a, d\}]$$

Out[*]=

$$\{0.0045254, \{a \rightarrow 0.4, d \rightarrow 0.35\}\}$$

In[*]:= $\text{FullSimplify}\left[x + 4 d x - 3 x^2 + x^3 + (2 d + (-2 + x) x)^{3/2} /. x \rightarrow \left(2 d - \frac{1}{4}\right)\right]$

Out[*]=

$$\frac{1}{64} (-29 + ((3 - 8 d)^2)^{3/2} + 280 d - 448 d^2 + 512 d^3)$$

In[*]:= $\text{NMinimize}\left[\left\{a^3 + (2 d - a^2) \sqrt{2 d + a^2} - \left(\frac{1}{64} (-29 + ((3 - 8 d)^2)^{3/2} + 280 d - 448 d^2 + 512 d^3)\right),\right.\right.$
 $\left.\left.d \geq \frac{17}{50} \ \&\& \ d \leq \frac{7}{20} \ \&\& \ a \geq \frac{1}{3} \ \&\& \ a \leq \frac{2}{5}\right\}, \{a, d\}\right]$

Out[*]=

$$\{0.000637872, \{a \rightarrow 0.4, d \rightarrow 0.34\}\}$$