Exact results for some extremal problems on expansions I

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Abstract

The expansion of a graph F, denoted by F^3, is the 3-graph obtained from F by adding a new vertex to each edge such that different edges receive different vertices. For large n, we establish tight upper bounds for:

- The maximum number of edges in an n-vertex 3-graph that does not contain T^3 for certain class $\mathcal{T}$ of trees, sharpening (partially) a result of Kostochka–Mubayi–Verstraëte.
- The minimum number of colors needed to color the complete n-vertex 3-graph to ensure the existence of a rainbow copy of F^3 when F is a graph obtained from some tree $T \in \mathcal{T}$ by adding a new edge, extending anti-Ramsey results on P_{2t}^3 by Gu–Li–Shi and C_{2t}^3 by Tang–Li–Yan.
- The maximum number of edges in an n-vertex 3-graph whose shadow does not contain the shadow of C_k^3 or T^3 for $T \in \mathcal{T}$, answering a question of Lv *et al.* on generalized Turán problems.

Keywords: hypergraph Turán problem, anti-Ramsey problem, generalized Turán problem, expansion of trees, linear cycles, triple systems, stability.

1 Introduction

Fix an integer $r \geq 2$, an r-graph $\mathcal{H}$ is a collection of r-subsets of some finite set V. We identify a hypergraph $\mathcal{H}$ with its edge set and use $V(\mathcal{H})$ to denote its vertex set. The size of $V(\mathcal{H})$ is denoted by $v(\mathcal{H})$. Given a family $\mathcal{F}$ of r-graphs, we say $\mathcal{H}$ is **$\mathcal{F}$-free** if it does not contain any member of $\mathcal{F}$ as a subgraph. The **Turán number** $\mathrm{ex}(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices. The study of $\mathrm{ex}(n, \mathcal{F})$ and its variant has been a central topic in extremal graph and hypergraph theory since the seminal work of Turán [47].

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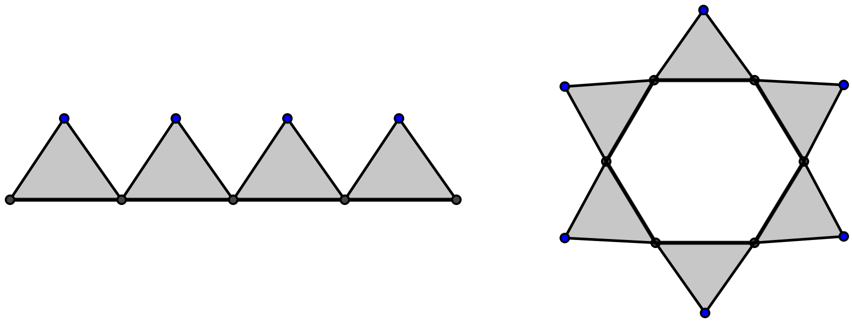

Figure 1: Expansions of P_4 and C_6 for $r = 3$.

Given a graph F, the **expansion** F^r of F is the r-graph obtained from F by adding a set of $r-2$ new vertices to each edge such that different edges receive disjoint sets (see Figure 1). Expansions are important objects in Extremal set theory and Hypergraph Turán problems. Its was introduced by Mubayi in [40] as a way to extend Turán's theorem to hypergraphs. There has been a significant amount of progress in the study of expansion over the last few decades, and we refer the reader to the survey [41] by Mubyai–Verstraëte for more related results.

In the present work, our primary focus will be extremal problems related to the expansion of trees and cycles when $r \in \{2, 3\}$. The following parameters will be crucial for us.

Following Frankl–Füredi [12], the **crosscut number** $\sigma(F)$ of graph F is

$$\sigma(F) := \min\left\{|I| + |F - I| \colon I \text{ is independent in } F\right\}.$$

Recall that the **covering number** $\tau(F)$ of a graph F is

$$\tau(F) := \min\left\{|S| \colon S \subseteq V(F) \text{ such that } |S \cap e| \geq 1 \text{ for all } e \in F\right\}.$$

Following the definition in [48], the **independent covering number** $\tau_{\mathrm{ind}}(F)$ of a bipartite graph F is

$$\tau_{\mathrm{ind}}(F) := \min\left\{|S| \colon S \subseteq V(F) \text{ such that } |S \cap e| = 1 \text{ for all } e \in F\right\}.$$

It follows trivially from the definitions that $\tau(F) \leq \sigma(F)$ holds for all graphs F, and $\tau(F) \leq \sigma(F) \leq \tau_{\mathrm{ind}}(F)$ holds for all bipartite graphs F.

Extending a definition in [17], we say a tree T is **strongly edge-critical** if

- $\tau(T) = \sigma(T) = \tau_{\mathrm{ind}}(T)$, and
- there exists an edge $e \in T$ such that $\sigma(T \setminus e) \leq \sigma(T) - 1$.

Path and cycle of length k will be denoted by P_k and C_k, respectively.

1.1 Turán problems for the expansion of trees

Determining $\mathrm{ex}(n, T)$ when T is a tree or hypertree is a central topic in Extremal combinatorics. Two well-known conjectures in this regard, the Erdős–Sós Conjecture on trees [7] and Kalai's conjecture on hypertrees (see [12]), are still open in general.

For $r \geq 4$ and large n, the exact value of $\mathrm{ex}(n, F^r)$ when F is a path, a cycle, or in a special class of trees, is now well-understood due to series works of Füredi [15], Füredi–Jiang [16, 17], Füredi–Jiang–Seiver [20]. Using methods significantly different from theirs, Kostochka–Mubyai–Verstraëte [29, 31] extended these results to $r = 3$, determining the exact value of $\mathrm{ex}(n, F^3)$ when F is a path or a cycle, and proving the following asymptotic result for trees.

Theorem 1.1 (Kostochka–Mubyai–Verstraëte [31])**.** *Suppose that T is a tree. Then*

$$\mathrm{ex}(n, T^3) = \left(\frac{\sigma(T) - 1}{2} + o(1)\right) n^2.$$

It is worth mentioning that the main term in the theorem above comes from the following construction. Let $n \geq t \geq 0$ be integers. Define the 3-graph

$$\mathcal{S}(n, t) := \left\{ e \in \binom{[n]}{3} : |e \cap [t]| \geq 1 \right\}.$$

It is easy to see that $\mathcal{S}(n, t)$ is T^3-free for all trees with $\tau(T) \geq t + 1$. Hence, $\mathrm{ex}(n, T^3) \geq |\mathcal{S}(n, t)|$ holds for all trees with $\tau(T) \geq t + 1$. Our main result in this subsection is an exact determination of $\mathrm{ex}(n, F^3)$ when F is a strongly edge-critical tree, thus sharpening (partially) Theorem 1.1.

Theorem 1.2. *Suppose that T is a tree satisfying $\sigma(T) = \tau_{\mathrm{ind}}(T)$ and containing a critical edge. Then*

$$\mathrm{ex}(n, T^3) \leq |\mathcal{S}(n, \sigma(T) - 1)| \quad \textit{for all sufficiently large } n.$$

In particular, T is a strongly edge-critical tree. Then

$$\mathrm{ex}(n, T^3) = |\mathcal{S}(n, \sigma(T) - 1)| \quad \textit{for all sufficiently large } n.$$

Remark.

- Note that one cannot hope that Theorem 1.2 holds for all trees, as paths of even length would be counterexamples (see [29]).
- For $r \geq 4$ Füredi–Jiang [17] determined $\mathrm{ex}(n, T^r)$ for all edge-critical trees T, where a tree T is edge-critical if $\tau(T) = \sigma(T)$ and there exists an edge $e \in T$ such that $\sigma(T \setminus e) \leq \sigma(T) - 1$. It would be interesting to see if the conclusion of Theorem 1.2 still holds under this weak constraint.

Theorem 1.2 is proved by using the stability method developed by Siminovits [44], and one of the key steps is to prove the following stability theorem for trees.

We say an n-vertex 3-graph $\mathcal{H}$ is **δ-close** to $\mathcal{S}(n, t)$ if there exists a set $L \subseteq V(\mathcal{H})$ of size t such that

$$|\mathcal{H} - L| \leq \delta n^2 \quad \text{and} \quad d_{\mathcal{H}}(v) \geq (1/2 - \delta)\, n^2 \text{ for all } v \in L.$$

Theorem 1.3 (Stability)**.** *Let T be a tree with $\sigma(T) = \tau_{\mathrm{ind}}(T)$. For every $\delta > 0$ there exist $\varepsilon > 0$ and n_0 such that the following holds for all $n \geq n_0$. Suppose that $\mathcal{H}$ is an n-vertex T^3-free 3-graph with*

$$|\mathcal{H}| \geq \left(\frac{\sigma(T) - 1}{2} - \varepsilon\right) n^2.$$

Then $\mathcal{H}$ is δ-close to $\mathcal{S}(n, \sigma(T) - 1)$.

The proofs for Theorems 1.2 and 1.3 will be presented in Sections 3.2 and 3.1, respectively.

1.2 Anti-Ramsey problems

Let F be an r-graph. The **anti-Ramsey number** $\mathrm{ar}(n, F)$ of F is the minimum number m such that any surjective map $\chi\colon K_n^r \to [m]$ contains a rainbow copy of F, i.e. a copy of F in which every edge receives a unique value under χ. It is easy to observe from the definition that for every r-graph F,

$$\mathrm{ex}\left(n, \{F \setminus e\colon e \in F\}\right) + 2 \le \mathrm{ar}(n, F) \le \mathrm{ex}(n, F) + 1.$$

The study of anti-Ramsey problems was initiated by Erdős–Simonovits–Sós [9] who proved that $\mathrm{ar}(n, K_{r+1}) = \mathrm{ex}(n, K_r) + 2$ for $r \ge 3$ and sufficiently large n. There has been lots of progress on this topic since their work and we refer the reader to a survey [14] by Fujita–Magnant–Ozeki for more details. In this subsection we are concerned with $\mathrm{ar}(n, F)$ when F is the expansion of a tree plus one edge.

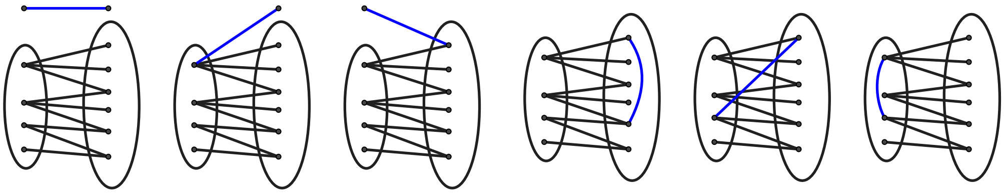

Figure 2: Possible positions for the new edge.

We say a graph F is an **augmentation** of a tree T if F can be obtained from T by adding a new edge (we do not require the new edge is contained in $V(T)$, see Figure 2). As another application of the stability theorem (Theorem 1.3) in the previous subsection, we prove the following anti-Ramsey result for augmentations of trees.

Theorem 1.4. *Suppose that T is a tree satisfying $\sigma(T) = \tau_{\mathrm{ind}}(T)$ and containing a critical edge, and F is an augmentation of T. Then*

$$\mathrm{ar}(n, F^3) \le \binom{n}{3} - \binom{n - \sigma(T) + 1}{3} + 2 \quad \textit{for sufficiently large } n.$$

Remark. In [22], Gu–Li–Shi determined $\mathrm{ar}(n, P_k^r)$ and $\mathrm{ar}(n, C_k^r)$ for certain combinations of k and r. Later, in [46], Tang–Li–Yan extended their result to all $r \ge 3$ and $k \ge 4$. Note that odd paths are strongly edge-critical, so Theorem 1.4 implies (and extends) their results on P_{2t}^3 (see [22, Theorem 2.1]) and C_{2t}^3 (see [46, Theorem 3]).

Theorem 1.4 will be proved in Section 4.

1.3 Generalized Turán problems

Let Q, F be two graphs. The **generalize Turán number** $\mathrm{ex}(n, Q, F)$ is the maximum number of copies of Q in an n-vertex F-free graph. When $Q = K_2$, this is the ordinary Turán number of F. The generalized Turán problem was first considered by Erdős in [6], in which he determined $\mathrm{ex}(n, K_s, K_t)$ for all $t > s \ge 3$. In [2], Alon–Shikhelman started a systematic study of $\mathrm{ex}(n, Q, F)$, and since then lots of effort has been devoted into this topic. Similar to the ordinary Turán problem, determining $\mathrm{ex}(n, Q, F)$ is much harder when $\chi(Q) \ge \chi(F)$, where $\chi(Q)$ and $\chi(F)$ denote the chromatic numbers of Q and F, respectively. In this subsection, we make some progress in the case $Q = K_3$ and F is the $\triangle$-blowup (defined below) of a cycle or a tree.

Given two graphs G_1 and G_2, the **join** $G_1 \Join G_2$ of G_1 and G_2 is obtained by taking vertex-disjoint copies of G_1 and G_2 and adding all pairs that have nonempty intersection with both $V(G_1)$ and $V(G_2)$, *i.e.*,

$$G_1 \Join G_2 := \{uv\colon u \in V(G_1),\ v \in V(G_2)\} \cup G_1 \cup G_2.$$

We use $T(n)$ to denote the balanced complete bipartite graph on n vertices, and use $T^+(n)$ to denote the n-vertex graph obtained from $T(n)$ by adding one edge into the smaller part. Given integers $n \ge t \ge 0$ let

$$S(n,t) := K_t \Join T(n-t) \quad \text{and} \quad S^+(n,t) := K_t \Join T^+(n-t).$$

Furthermore, we define two corresponding 3-graphs as follows:

$$\begin{aligned} \mathcal{S}_{\mathrm{bi}}(n,t) &:= \{A\colon S(n,t) \text{ induces a copy of } K_3 \text{ on } A\}, \quad \text{and} \\ \mathcal{S}^+_{\mathrm{bi}}(n,t) &:= \left\{A\colon S^+(n,t) \text{ induces a copy of } K_3 \text{ on } A\right\} \end{aligned}$$

Given a graph F the $\triangle$**-blowup** $F^{\triangle}$ of F is the graph obtained from F be replacing each edge with a copy of K_3 such that no two different copies of K_3 use the same new vertex (view Figure 1 as graphs).

Our first result in this subsection is an exact determination of $\mathrm{ex}(n, K_3, F^{\triangle})$ when F is a strongly edge-critical tree.

Theorem 1.5. *Suppose that T is a tree satisfying $\sigma(T) = \tau_{\mathrm{ind}}(T)$ and containing a critical edge. Then*

$$\mathrm{ex}(n, K_3, T^{\triangle}) \le |\mathcal{S}_{\mathrm{bi}}(n, \sigma(T)-1)| \quad \textit{for sufficiently large } n.$$

In particular, if T is a strongly edge-critical tree. Then

$$\mathrm{ex}(n, K_3, T^{\triangle}) = |\mathcal{S}_{\mathrm{bi}}(n, \sigma(T)-1)| \quad \textit{for sufficiently large } n.$$

Note the the lower bound for the 'In particular' part comes from the graph $S(n,t)$. In general, we have the following asymptotic bound.

Theorem 1.6. *Suppose that T is a tree. Then*

$$\mathrm{ex}(n, K_3, T^{\triangle}) = |\mathcal{S}_{\mathrm{bi}}(n, \sigma(T)-1)| + o(n^2).$$

Since odd paths are strongly edge-critical, we obtain the following corollary of Theorem 1.5.

Corollary 1.7. *Suppose that $t \ge 1$ is a fixed integer and n is sufficiently large. Then*

$$\mathrm{ex}(n, K_3, P^{\triangle}_{2t+1}) = |\mathcal{S}_{\mathrm{bi}}(n,t)|.$$

For paths of even length and cycles we prove the following results.

Theorem 1.8. *Suppose that $t \ge 1$ is a fixed integer and n is sufficiently large. Then*

$$\mathrm{ex}(n, K_3, P^{\triangle}_{2t+2}) = \left|\mathcal{S}^+_{\mathrm{bi}}(n,t)\right|.$$

Theorem 1.9. *Suppose that $t \ge 2$ is a fixed integer and n is sufficiently large. Then*

$$\mathrm{ex}(n, K_3, C^{\triangle}_{2t+1}) = |\mathcal{S}_{\mathrm{bi}}(n,t)| \quad \text{and} \quad \mathrm{ex}(n, K_3, C^{\triangle}_{2t+2}) = \left|\mathcal{S}^+_{\mathrm{bi}}(n,t)\right|.$$

Remark. In [39], Lv *et al.* determined the exact values of $\mathrm{ex}(n, K_3, P_3^{\triangle})$ and $\mathrm{ex}(n, K_3, C_3^{\triangle})$ using a very different method as we used in this paper. They also proposed a conjecture for the values of $\mathrm{ex}(n, K_3, P_k^{\triangle})$ and $\mathrm{ex}(n, K_3, C_k^{\triangle})$ for $k \geq 4$. Corollary 1.7, Theorem 1.8, and Theorem 1.9 confirm their conjecture (except for $C_4^{\triangle}$). Though the case $C_4^{\triangle}$ can be also handled using the same framework as in the proof of Theorem 1.9, it is a bit technical and needs some extra work. So we include its proof in a separate note.

Similar to the proof of Theorem 1.2, proofs for theorems above also use the stability method. A key step is to prove the following stability theorems.

We say an n-vertex graph G is δ**-close** to $S(n,t)$ if there exists a t-set $L \subseteq V(G)$ so that

- $d_G(v) \geq (1-\delta)n$ for all $v \in L$,
- $N(K_3, G-L) \leq \delta n^2$,
- $|G-L| \geq n^2/4 - \delta n^2$, and
- $G - L$ can be made bipartite by removing at most δn^2 edges.

Theorem 1.10 (Stability)**.** *Let T be a tree with $\sigma(T) = \tau_{\mathrm{ind}}(T)$. For every $\delta > 0$ there exist $\varepsilon > 0$ and n_0 such that the following holds for all $n \geq n_0$. Suppose that G is an n-vertex $T^{\triangle}$-free graph with*

$$N(K_3, G) \geq \left(\frac{\sigma(T)-1}{4} - \varepsilon\right) n^2.$$

Then G is δ-close to $S\left(n, \sigma(T)-1\right)$.

Recall from the definition that $\sigma(C_k) = \lfloor (k+1)/2 \rfloor = \lfloor (k-1)/2 \rfloor + 1$.

Theorem 1.11 (Stability)**.** *Let $k \geq 4$ be a fixed integer. For every $\delta > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose that G is an n-vertex $C_k^{\triangle}$-free graph with*

$$N(K_3, G) \geq \left(\frac{\sigma(C_k)-1}{4} - \varepsilon\right) n^2.$$

Then G is δ-close to $S\left(n, \sigma(C_k)-1\right)$.

Proofs for Theorems 1.5 and 1.10 are very similar to proofs of Theorems 1.2 and 1.3. So we include their proofs in the Appendix (see Sections I and H). The proof of Theorem 1.8 is similar to the proof of Theorem 1.9, so we include it in the Appendix as well (see Section J). The proof of Theorem 1.9 is presented in Section 5.2, and the proof of Theorem 1.11 is included in Section 5.1. In the next section, we introduce some definitions and prove some preliminary results.

2 Preliminaries

In this section, we present some definitions and preliminary results related to hypergraphs, trees, and expansions.

2.1 Hypergraphs

Given a 3-graph $\mathcal{H}$ the **shadow** $\partial\mathcal{H}$ of $\mathcal{H}$ is a 2-graph defined by

$$\partial\mathcal{H} := \left\{ uv \in \binom{V(\mathcal{H})}{2} : \exists w \in V(\mathcal{H}) \text{ such that } uvw \in \mathcal{H} \right\}.$$

The **link** $L_{\mathcal{H}}(v)$ of a vertex $v \in \mathcal{H}$ is

$$L_{\mathcal{H}}(v) := \{uw \in \partial\mathcal{H} \colon uvw \in \mathcal{H}\}.$$

The **degree** of v is $d_{\mathcal{H}}(v) := |L_{\mathcal{H}}(v)|$. We use $\delta(\mathcal{H})$ and $\Delta(\mathcal{H})$ to denote the minimum and maximum degree of $\mathcal{H}$, respectively. Given a pair of vertices $uv \subseteq V(\mathcal{H})$ the **neighborhood** of uv is

$$N_{\mathcal{H}}(uv) := \{w \in V(\mathcal{H}) \colon uvw \in \mathcal{H}\}.$$

The **codegree** of uv is $d_{\mathcal{H}}(uv) := |N_{\mathcal{H}}(uv)|$. We use $\delta_2(\mathcal{H})$ and $\Delta_2(\mathcal{H})$ to denote the minimum and maximum codegree of $\mathcal{H}$, respectively. For convenience, for every edge $uvw \in \mathcal{H}$ we let

$$\begin{aligned} \delta_2(uvw) &:= \min\{d_{\mathcal{H}}(uv),\ d_{\mathcal{H}}(vw),\ d_{\mathcal{H}}(vw)\}, \quad \text{and} \\ \Delta_2(uvw) &:= \max\{d_{\mathcal{H}}(uv),\ d_{\mathcal{H}}(vw),\ d_{\mathcal{H}}(vw)\} \end{aligned}$$

Let $k > d \geq 0$ be integers. We say a 3-graph $\mathcal{H}$ is **d-full** if $d_{\mathcal{H}}(uv) \geq d$ for all $uv \in \partial\mathcal{H}$. We say $\mathcal{H}$ is **(d,k)-superfull** if it is d-full and every edge in $\mathcal{H}$ contains at most one pair of vertices with codegree less than k.

The following simple lemma can be proved by greedily removing shadow edges with small codegree.

Lemma 2.1 ([29, Lemma 3.1])**.** *Let $d \geq 0$ be an integer. Every 3-graph $\mathcal{H}$ contains a $(d+1)$-full subgraph $\mathcal{H}'$ with*

$$|\mathcal{H}'| \geq |\mathcal{H}| - d|\partial\mathcal{H}|.$$

The following simple fact, which can be proved easily using a greedy argument, will be useful in our proofs.

Fact 2.2. *Let $\mathcal{H}$ be a 3-graph and F be a graph with $m \geq 1$ edges. Suppose that $\{e_1, \ldots, e_m\} \subseteq \partial\mathcal{H}$ is a copy of F and there exists $t \in [m]$ such that*

(i) there exist distinct vertices $w_1, \ldots, w_t \in V(\mathcal{H}) \setminus \left(\bigcup_{i\in m} e_i\right)$ with $e_i \cup \{w_i\} \in \mathcal{H}$ for $i \in [t]$, and

(ii) $d_{\mathcal{H}}(e_j) \geq 3m$ for all $j \in [t+1, m]$.

Then $F^3 \subseteq \mathcal{H}$.

A 3-graph $\mathcal{H}$ is **2-intersecting** if $|e_1 \cap e_2| = 2$ for all distinct edges $e_1, e_2 \in \mathcal{H}$. The following observation on the structure of 2-intersecting 3-graphs will be useful.

Fact 2.3. *Suppose that $\mathcal{H}$ is 2-intersecting. Then either $|\mathcal{H}| \leq 4$ or there exists a pair $\{u, v\} \subseteq V(\mathcal{H})$ such that all edges in $\mathcal{H}$ containing $\{u, v\}$.*

The following analogue for graphs is also useful.

Fact 2.4. *Suppose that the matching number of a graph G is at most one. Then either $|G| \le 3$ or there exists a vertex $v \in V(G)$ such that all edges in G containing v.*

Given a graph G, we associate a 3-graph with it by letting

$$\mathcal{K}_G := \left\{ e \in \binom{V(G)}{3} : G[e] \cong K_3 \right\}.$$

The following simple fact is clear from the definition.

Fact 2.5. *Let F be a graph. For every $F^{\triangle}$-free graph G, the 3-graph $\mathcal{K}_G$ is F^3-free. In particular,* $\mathrm{ex}(n, K_3, F^{\triangle}) \le \mathrm{ex}(n, F^3)$.

2.2 Trees

Let T be a tree. We say a pair (I, R) with $I \subseteq V(T)$ and $R \subseteq T$ is a **crosscut pair** if

$$I \text{ is an independent of } T, \quad R = T - I, \quad \text{and} \quad |I| + |R| = \sigma(T).$$

Recall that a vertex $v \in V(T)$ is a **leaf** if $d_T(v) = 1$. We say an edge $uv \in T$ is a **pendant** edge if $\min\{d_T(u), d_T(v)\} = 1$. An edge $e \in T$ is **crucial** if $\sigma(T \setminus e) \le \sigma(T) - 1$.

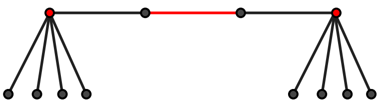

Figure 3: A tree whose crosscut-pair (I, R) is highlighted in red color.

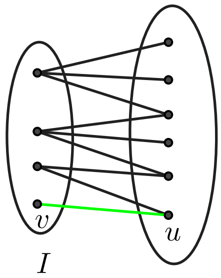

Figure 4: A strongly edge-critical tree in which I is an independent vertex-cover and uv is a critical edge.

In the proof of Theorems 1.3 and 1.10 we will use the following structural result on trees.

Proposition 2.6. *Suppose that T is a tree and (I, R) is a cross-cut pair of T. Then one of the following holds.*

(i) There exists a vertex $v \in I$ such that all but one vertex in $N_T(v)$ are leaves in T.

(ii) There exists an edge $e \in R$ which is a pendant edge in T.

In particular, if I is maximum, then (i) holds.

Proposition 2.6 can be proved easily by showing that the set system $\mathcal{T} := R \cup \{N_T(v) : v \in I\}$ define on $U := V(T) \setminus I$ is linear and acyclic. We include its short proof in the Appendix (see Section A).

The following structural result for trees with $\sigma(T) = \tau_{\mathrm{ind}}(T)$ will be crucial for proofs of Theorems 1.2 and 1.5.

Proposition 2.7. *Let T be a tree with $\sigma(T) = \tau_{\text{ind}}(T)$ and $I \subseteq V(T)$ be a minimum independent vertex cover. Suppose that T has a critical edge. Then there exists a pendant edge $e^* \in T$ whose leaf endpoint is contained in I such that $\sigma(T \setminus e^*) \leq \sigma(T) - 1$.*

Proof. Let $I \subseteq V(T)$ be a minimum independent vertex cover. It suffices to show that I contains a leaf as the edge containing this leaf will be a witness for e^*.

Suppose to the contrary that $d_T(v) \geq 2$ for all $v \in I$. Fix $e \in T$ such that $\sigma(T \setminus e) \leq \sigma(T) - 1$. Let $T' = T \setminus e$. For every set $S \subseteq I$, we have

$$\sum_{v \in S} d_{T'}(v) \geq \left(\sum_{v \in S} d_T(v) \right) - 1 \geq 2|S| - 1.$$

On the other hand, since the induced graph $T'[S \cup N_{T'}(S)]$ is a forest, we have

$$\sum_{v \in S} d_{T'}(v) = |T'[S \cup N_{T'}(S)]| \leq |S| + |N_{T'}(S)| - 1.$$

Combining these two inequalities, we obtain that $|N_{T'}(S)| \geq |S|$. Therefore, by Hall's theorem [25], T' contains a matching which saturates I, and it follows from the König's theorem (see e.g. [28, 4]) that $\tau(T') = |I| = \sigma(T)$. However, this implies that $\sigma(T') \geq \tau(T') = \sigma(T)$, contradicting the definition of e. ∎

Using Proposition 2.6, we obtain the following embedding result for the expansion of trees.

Proposition 2.8. *Let T be a tree with k vertices, and $t := \sigma(T) - 1$. Let $\mathcal{H}$ be a 3-graph on V, $S_1, S_2 \subseteq V$ are two distinct t-subsets, $V_1, V_2 \subset V \setminus (S_1 \cup S_2)$ are two sets with nonempty intersection. Suppose that there are two graphs G_1 and G_2 on V_1 and V_2 respectively such that for $i \in \{1, 2\}$*

(i) $d_{G_i}(v) \geq 3k$ *for all* $v \in V_i$, *and*

(ii) $G_i \subseteq L_{\mathcal{H}}(v)$ *for all* $v \in S_i$.

Then $T^3 \subseteq \mathcal{H}$.

Remark. We would like to remind the reader that for trees satisfying $\sigma(T) = \tau_{\text{ind}}(T)$, assumption (i) can be replaced by $d_{G_i}(v) \geq 1$. We state this general form as it might be useful for future research.

Proof of Proposition 2.8. Let (I, R) be a crosscut pair of T such that $|I|$ is maximized. By Proposition 2.6, there exists a vertex $v_* \in I$ such that all but one vertex $u_* \in N_T(v_*)$ are leaves. Let $N'(v_*) := N_T(v_*) \setminus \{u_*\}$, $T' := T - (\{v_*\} \cup N'(v_*))$, $I' := T \setminus \{v_*\}$, $J := V(T) \setminus I$, and $J' := J \setminus N_T(v_*)$. Observe that $|I'| + |R| = |I| + |R| - 1 = t$.

Fix $v'_* \in S_2 \setminus S_1$, $u'_* \in V_1 \cap V_2$, and a $(d_T(v_*) - 1)$-set $N' \subseteq V_2 \setminus \{u'_*\}$. Let $V'_1 := V_1 \setminus (\{u'_*\} \cup N')$. It follows from the assumptions that the induced bipartite graph of $\partial\mathcal{H}$ on $S_1 \cup V_1$ is complete. Combined with $\delta(G_1[V'_1]) \geq \delta(G_1) - d_T(v) - 1 \geq 3k - k \geq 2k$ and a simple greedy argument, it is easy to see that there exists an embedding $\phi\colon V(T') \to S_1 \cup V'_1$ such that $\phi(I') \subseteq S_1$, $\phi(T' - I') \subseteq G_1[V'_1]$, and moreover, $\phi(u_*) = u'_*$. We can extend ϕ by letting $\phi(v_*) = v'_*$ and $\phi(N'(v_*)) = N'$ to obtain an embedding of T into $\partial\mathcal{H}$ (see Figure 5). Observe that for every edge $e \in R$ we have $\phi(e) \subset G_1[V'_1]$, and $|S_1| - |\phi(I')| = t - |I'| = |R|$.

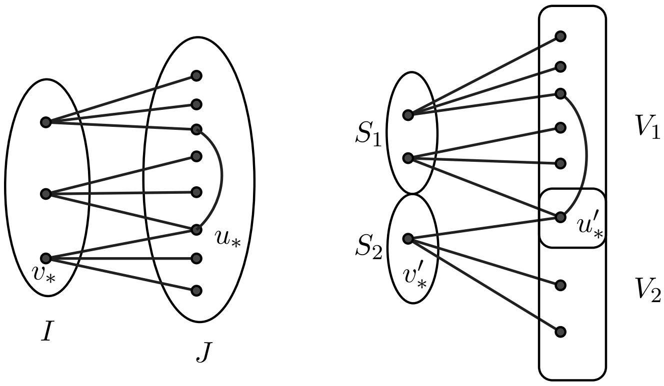

Figure 5: Embedding a tree.

Assigning each vertex in $S_1 \setminus \phi(I')$ to an edge $e \in R$ we obtain an embedding of R^3 into $\mathcal{H}$. In addition, notice that every pair in $S_1 \times V_1$ and $S_2 \times V_2$ has codegree at least $\min\{\delta(G_i),\ \delta(G_2)\} \geq 3k$, so it follows from Fact 2.2 that $T^3 \subseteq \mathcal{H}$. ∎

The following simple and crude upper bound for $\mathrm{ex}(n, K_3, F^{\triangle})$ will be crucial for the proofs of generalzied Turán theorems. Its proof is a standard application of the Kövari–Sós–Turán Theorem [32], the Triangle Removal Lemma (see e.g. [43, 1, 10]), and Simonovits' stability theorem [44], and we include it in the Appendix (see Section B).

Proposition 2.9. *Let F be a bipartite graph. For every $\delta > 0$ there exists $\varepsilon > 0$ such that for sufficiently large n, every n-vertex $F^{\triangle}$-free graph G with at least $n^2/4 - \varepsilon n^2$ edges can be made bipartite by removing at most δn^2 edges. In particular, every n-vertex $F^{\triangle}$-free graph G satisfies $|G| \leq n^2/4 + o(n^2)$.*

2.3 Expansion of trees

For convenience, we use $\mathcal{T}_k$ to denote the family of all trees with k edges. The following lemma can be proved using a simple greedy argument.

Lemma 2.10 ([29, Lemma 3.2])**.** *Let $k \geq 3$ be an integer. Every $3k$-full nonempty 3-graph $\mathcal{H}$ contains T_k^3 for all $T_k \in \mathcal{T}_k$.*

The following bound for the number of edges in an n-vertex T_k^3-free 3-graph with bounded codegree can be proved using the argument for [29, Proposition 3.8]. For completeness, we include its proof in the Appendix (see Section C).

Lemma 2.11. *Fix integers $k \geq 3$ and $C \geq 1$. Let $T_k \in \mathcal{T}_k$. If $\mathcal{H}$ is an n-vertex T_k^3-free 3-graph with $\Delta_2(\mathcal{H}) \leq C$, then $|\mathcal{H}| \leq 6Ckn = o(n^2)$.*

Though not specifically stated in [31], the following useful lemma can be proved using the argument for [31, Theorem 1.1] (see pages 467 and 468 and, in particular, Claim 4 in [31]).

Lemma 2.12 ([31])**.** *Let T be a tree with $\sigma(T) = \tau_{\mathrm{ind}}(T)$. For every constant $\varepsilon > 0$ there exists n_0 such that the following holds for all 3-graphs $\mathcal{H}$ with $n \geq n_0$ vertices. If there exists $E \subseteq \partial\mathcal{H}$ with $|E| \geq \varepsilon n^2$ and $d_{\mathcal{H}}(e) \geq \sigma(T)$ for all $e \in E$, then $T^3 \subseteq \mathcal{H}$.*

The proof for [31, Theorem 1.1] implies the following lemma (see [31, Section 6]).

Lemma 2.13 ([31]). *Let T be a tree. Every n-vertex $\sigma(T)$-full 3-graph $\mathcal{H}$ satisfies $|\mathcal{H}| = o(n^2)$.*

The following proposition follows easily from Lemmas 2.1 and 2.13.

Proposition 2.14. *Let T be a tree and $\varepsilon > 0$ be a constant. For sufficiently large n, every n-vertex T_k^3-free 3-graph $\mathcal{H}$ satisfies*

$$|\mathcal{H}| \le (\sigma(T) - 1)\, |\partial\mathcal{H}| + \varepsilon n^2.$$

In the proof of [29, Theorem 6.2], Kostochka–Mubayi–Verstraëte introduced the following process to obtain a large $(\lfloor (k-1)/2 \rfloor, 3k)$-superfull subgraph in a C_k^3-free (or P_k^3-free) 3-graph. We will use it in proofs of Theorems 1.3, 1.10 and 1.11.

Algorithm Cleaning Algorithm

Input: A triple $(\mathcal{H}, k, t)$, where $\mathcal{H}$ is a 3-graph and $k \ge t \ge 0$ are integers.
Output: An integer q, a sequence of edges $f_1, \ldots, f_q \in \partial\mathcal{H}$, and a sequence of subgraphs $\mathcal{H} \supseteq \mathcal{H}_0 \supseteq \cdots \supseteq \mathcal{H}_q$.
Initialization: Let

$$i = 0, \quad \mathcal{H}^* := \{e \in \mathcal{H} \colon \Delta_2(e) \le 3k\}, \quad \text{and} \quad \mathcal{H}_0 := \mathcal{H} \setminus \mathcal{H}^*$$

Operation: For every edge $e \in \partial\mathcal{H}_i$, we say e is of

- type-1 if $d_{\mathcal{H}_i}(e) \le t - 1$,
- type-2 if $d_{\mathcal{H}_i}(e) = t$ and there exists $f' \in \partial\mathcal{H}_i$ such that

$$|f' \cap e| = 1, \quad d_{\mathcal{H}_i}(f') = t, \quad \text{and } e \cup f' \in \mathcal{H}_i,$$

- type-3 if $t + 1 \le d_{\mathcal{H}_i}(e) \le 3k - 1$.

If $\partial\mathcal{H}_i$ contains no edges of types-1 $\sim$ 3, then we let $q = i$ and stop. Otherwise, we choose an edge $e \in \partial\mathcal{H}_i$ of minimum type. Let

$$f_{i+1} := e \quad \text{and} \quad \mathcal{H}_{i+1} := \{E \in \mathcal{H}_i \colon e \subseteq E\},$$

and then repeat this process to $\mathcal{H}_{i+1}$.

Repeating the proof of [29, Theorem 6.2] to a tree with $\sigma(T) = \tau_{\text{ind}}(T)$, it is easy to obtain the following lemma. For completeness, we include its proof in the Appendix (see Section D).

Lemma 2.15. *Let T be a fixed tree with $\sigma(T) = \tau_{\text{ind}}(T) =: t + 1$ and $k := |T|$. Fix constant $\varepsilon > 0$ and let n be sufficiently large. Let $\mathcal{H}$ be an n-vertex T^3-free 3-graph and $\mathcal{H}_q$ be the outputting 3-graph of Cleaning Algorithm with input $(\mathcal{H}, k, t)$ as defined here. If $|\mathcal{H}| \ge t|\partial\mathcal{H}| - \varepsilon n^2$, then $\mathcal{H}_q$ is $(t, 3k)$-superfull, and moreover,*

$$q \le 12k\varepsilon n^2, \quad |\mathcal{H}_q| \ge |\mathcal{H}| - 48k^2\varepsilon n^2, \quad \textit{and} \quad |\partial\mathcal{H}_q| \ge |\partial\mathcal{H}| - 50k^2\varepsilon n^2.$$

2.4 Expansion of cycles

In line with the previou subsection, we adopt the subsequent lemmas on C_k^3 from [29].

Lemma 2.16 ([29, Lemma 3.2]). *Let $k \geq 3$ be an integer. Every $3k$-full nonempty 3-graph $\mathcal{H}$ contains C_k^3.*

Lemma 2.17 ([29, Proposition 3.8]). *Let $k \geq 3$ and $C \geq 1$ be fix integers. Every n-vertex C_k^3-free 3-graph $\mathcal{H}$ with $\Delta_2(\mathcal{H}) \leq C$ satisfies $|\mathcal{H}| = o(n^2)$.*

Lemma 2.18 ([29, Theorem 6.1 (i)]). *Let $k \geq 4$ be a fixed integer. Every n-vertex $\sigma(C_k)$-full 3-graph $\mathcal{H}$ satisfies $|\mathcal{H}| = o(n^2)$.*

Similar to Proposition 2.14, the following proposition follows easily from Lemmas 2.1 and 2.18.

Proposition 2.19. *Let $k \geq 4$ be an integer and $\varepsilon > 0$ be a constant. For sufficiently large n, every n-vertex C_k^3-free 3-graph $\mathcal{H}$ satisfies*

$$|\mathcal{H}| \leq (\sigma(C_k) - 1)\, |\partial\mathcal{H}| + \varepsilon n^2.$$

Lemma 2.20 ([29, Lemma 5.1]). *Let $\varepsilon > 0$ and $k \geq 4$ be fixed and n be sufficiently large. Suppose that $\mathcal{H}$ is an n-vertex 3-graph such that there exists $E \subseteq \partial\mathcal{H}$ with $|E| \geq \varepsilon n^2$ so that*

(i) for even k, every $uv \in E$ satisfies $d_{\mathcal{H}}(uv) \geq \sigma(C_k)$,

(ii) for odd k, every $uv \in E$ satisfies $d_{\mathcal{H}}(uv) \geq \sigma(C_k)$, and there exists $w \in V(\mathcal{H})$ with $uvw \in \mathcal{H}$ such that

$$\min\{d_{\mathcal{H}}(uw),\ d_{\mathcal{H}}(vw)\} \geq 2 \quad \text{and} \quad \max\{d_{\mathcal{H}}(uw),\ d_{\mathcal{H}}(vw)\} \geq 3k.$$

Then $C_k^3 \subseteq \mathcal{H}$.

The proof of [29, Theorem 6.2] implies the following lemma. For completeness, we include its proof in the Appendix (see Section E).

Lemma 2.21. *Let $k \geq 4$ and $t = \lfloor (k-1)/2 \rfloor$ be integers. Fix a constant $\varepsilon > 0$ and let n be sufficiently large. Let $\mathcal{H}$ be an n-vertex C_k^3-free 3-graph and $\mathcal{H}_q$ be the outputting 3-graph of the Cleaning Algorithm with input $(\mathcal{H}, k, t)$ as defined here. If $|\mathcal{H}| \geq t|\partial\mathcal{H}| - \varepsilon n^2$, then $\mathcal{H}_q$ is $(t, 3k)$-superfull, and moreover,*

$$q \leq 12k\varepsilon n^2, \quad |\mathcal{H}_q| \geq |\mathcal{H}| - 48k^2\varepsilon n^2, \quad \text{and} \quad |\partial\mathcal{H}_q| \geq |\partial\mathcal{H}| - 50k^2\varepsilon n^2.$$

In the proof of Theorem 1.11, we need the following strengthen of [29, Lemma 3.5]. Since two proofs are essentially the same, we include the proof of the following lemma in the Appendix (see Section F).

Lemma 2.22. *Let $k \geq 4$, $t := \lfloor (k-1)/2 \rfloor$, and $\mathcal{H}$ be a t-superfull 3-graph containing two disjoint sets $W_1, W_2 \subseteq V(\mathcal{H})$ each of size at least $3k$ such that every pair of vertices $(w_1, w_2) \in W_1 \times W_2$ has codegree exactly t in $\mathcal{H}$. If $C_k^3 \not\subseteq \mathcal{H}$, then there exists a t-set $L \subseteq V(\mathcal{H}) \setminus (W_1 \cup W_2)$ such that $N_{\mathcal{H}}(w_1 w_2) = L$ for all pairs $(w_1, w_2) \in W_1 \times W_2$.*

3 Proofs for hypergraph Turán results

We prove Theorems 1.2 and 1.3 in this section. We will first prove the stability theorem (Theorem 1.3) and then use it to prove the exact result (Theorem 1.2).

3.1 Proof of Theorem 1.3

Building upon the foundation laid out in the proof of [29, Theorem 6.2], we begin with a near-extremal T^3-free 3-graph $\mathcal{H}$, and apply the Cleaning Algorithm to obtain a $(t, 3k)$-superfull subgraph $\mathcal{H}' \subseteq \mathcal{H}$ by removing a negligible number $o(n^2)$ of edges. After that, our proof diverges from the prior proof of [29, Theorem 6.2] due to the failure of a crucial lemma (Lemma 2.22) in the context of trees. To handle this technical difficulty, we first iteratively remove a small amount of edges from $\mathcal{H}'$ to shrink its vertex covering number to at most $\sqrt{n}$ (see Claim 4). Subsequently, we turn our attention to the common links of t-subsets within a covering set Z of the remaining 3-graph, which still contains most edges in $\mathcal{H}$. Using properties of trees (Proposition 2.8), we show that only one t-set in Z has large common link, and this particular t-set is the object we are looking for.

Proof of Theorem 1.3. Let T be a tree with $\sigma(T) = \tau_{\mathrm{ind}}(T)$. Fix $\delta < 0$. Let $0 \le \varepsilon \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \delta$ be sufficiently small constant, and let n be sufficiently large. Let $k := |T|$ and $t := \sigma(T) - 1$. Let $\mathcal{H}$ be a T^3-free 3-graph on n vertices with at least $(t/2 - \varepsilon)n^2$ edges. For convenience, let $V := V(\mathcal{H})$. Our aim is to show that $\mathcal{H}$ is δ-close to $\mathcal{S}(n, t)$.

First, observe from Proposition 2.14 that

$$|\partial \mathcal{H}| \ge \frac{|\mathcal{H}| - \varepsilon n^2}{t} \ge \frac{n^2}{2} - 2\varepsilon n^2. \tag{1}$$

Let $\mathcal{H}_q$ be the output 3-graph of the Cleaning Algorithm with input $(\mathcal{H}, k, t)$ as defined here. Since $|\mathcal{H}| \ge tn^2/2 - \varepsilon n^2 \ge t|\partial\mathcal{H}| - \varepsilon n^2$, it follows from Lemma 2.15 that $\mathcal{H}_q$ is $(t, 3k)$-superfull and

$$q \le 12k\varepsilon n^2, \quad |\mathcal{H}_q| \ge |\mathcal{H}| - 48k^2\varepsilon n^2, \quad \text{and} \quad |\partial\mathcal{H}_q| \ge |\partial\mathcal{H}| - 50k^2\varepsilon n^2. \tag{2}$$

Define graphs

$$G' := \left\{e \in \partial\mathcal{H}_q \colon d_{\mathcal{H}_q}(e) = t\right\}, \quad \text{and} \quad G'' := \left\{e \in \partial\mathcal{H}_q \colon d_{\mathcal{H}_q}(e) \ge 3k\right\}.$$

Since $\mathcal{H}_q$ is $(t, 3k)$-superfull, we have $G' \cup G'' = \partial\mathcal{H}_q$ and every edge in $\mathcal{H}_q$ contains at least two elements from G''.

Claim 3.1. *We have $|G''| \le kn$ and $|G'| \ge (1/2 - \varepsilon_1)\, n^2$.*

Proof. It follows from Fact 2.2 that G'' is T-free, Therefore, $|G''| \le kn$. In addition, by (1) and the third inequality in (2), we obtain

$$|G'| = |\partial\mathcal{H}_q| - |G''| \ge \frac{n^2}{2} - 2\varepsilon n^2 - 50k^2\varepsilon n^2 - \varepsilon n^2 \ge \frac{n^2}{2} - \varepsilon_1 n^2.$$

This proves Claim 3.1. ∎

Define a 3-graph $\mathcal{G} \subseteq \mathcal{H}_q$ as follows:

$$\mathcal{G} := \left\{e \in \mathcal{H}_q \colon \left|\binom{e}{2} \cap G'\right| = 1\right\}.$$

Since every pair in G' contributes exactly t triples to $\mathcal{G}$, it follows from Claim 3.1 that

$$|\mathcal{G}| \ge t|G'| \ge \left(\frac{1}{2} - \varepsilon_1\right) tn^2. \tag{3}$$

For every vertex $v \in V$ define the **light link** of v as

$$\widehat{L}(v) := \left\{ e \in G' \colon e \cup \{v\} \in \mathcal{H}_q \right\}.$$

It is easy to see from the definitions that

$$|\mathcal{G}| = \sum_{v \in V} |\widehat{L}(v)|, \quad \text{and} \quad \widehat{L}(v) \subseteq \binom{N_{G''}(v)}{2} \quad \text{for all} \quad v \in V(\mathcal{H}). \tag{4}$$

For every integer $i \geq 0$ define

$$Z_i := \left\{ v \in V \colon |\widehat{L}(v)| \geq \frac{n^{2-2^{-i}}}{(\log n)^2} \right\}, \quad U_i := V \setminus Z_i, \quad \text{and} \quad \mathcal{G}_i := \{ e \in \mathcal{G} \colon |e \cap Z_i| = 1 \}.$$

Observe that for all $i \geq 1$,

$$V \supseteq Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_i, \quad U_0 \subseteq U_1 \subseteq \cdots \subseteq U_i \subseteq V, \quad \text{and} \quad \mathcal{G} \supseteq \mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots \supseteq \mathcal{G}_i.$$

Claim 3.2. *We have $|Z_i| \leq 2kn^{2^{-i-1}} \log n$ and $|\mathcal{G}_i| \geq |\mathcal{G}| - \frac{2(i+1)n^2}{(\log n)^2}$ for all $i \geq 0$. In particular,*

$$\binom{|Z_k|}{t} \leq \sqrt{n} \quad \textit{and} \quad |\mathcal{G}_k| \geq \left(\frac{1}{2} - 2\varepsilon_1 \right) tn^2.$$

Proof. We prove it by induction on i. For the base case $i = 0$, observe from the second part of (4) that for every $v \in Z_0$, we have $d_{G''}(v) \geq \sqrt{|\widehat{L}(v)|} \geq n^{1/2}/\log n$ (as the worse case is that $\widehat{L}(v)$ is a complete graph on $d_{G''}(v)$ vertices). Therefore, by Claim 3.1,

$$2kn \geq 2|G''| \geq \sum_{v \in Z_0} d_{G''}(v) \geq \sum_{v \in Z_0} \sqrt{|\widehat{L}(v)|} \geq |Z_0| \frac{n^{1/2}}{\log n},$$

which implies that $|Z_0| \leq 2kn^{1/2} \log n$. Consequently, the number of edges in G' that have nonempty intersection with Z_0 is at most $|Z_0|n \leq 2kn^{3/2} \log n$. Observe that every triple in $\mathcal{G}$ with at least two vertices in Z_0 must contain a pair (in G') with (at least) one endpoint in Z_0. Therefore, the number of edges in $\mathcal{G}$ with at least two vertices in Z_0 is at most $2ktn^{3/2} \log n$ (recall that each edge in G' contributes t triples in $\mathcal{G}$). Therefore,

$$\begin{aligned} |\mathcal{G}_0| &\geq |\mathcal{G}| - \sum_{v \in V \setminus Z_0} |\widehat{L}(v)| - 2ktn^{3/2} \log n \geq |\mathcal{G}| - n \times \frac{n}{(\log n)^2} - 2ktn^{3/2} \log n \\ &\geq |\mathcal{G}| - \frac{2n^2}{(\log n)^2}. \end{aligned}$$

Now suppose that $|Z_i| \leq n^{2^{-i-1}} \log n$ holds for some $i \geq 0$. Repeating the argument above to Z_{i+1}, we obtain

$$2kn \geq 2|G''| \geq \sum_{v \in Z_{i+1}} d_{G''}(v) \geq \sum_{v \in Z_{i+1}} \sqrt{|\widehat{L}(v)|} \geq |Z_{i+1}| \frac{n^{1-2^{-i-2}}}{\log n},$$

which implies that $|Z_{i+1}| \leq 2kn^{2^{-i-2}} \log n$. So it follows from the inductive hypothesis that

$$\begin{aligned} |\mathcal{G}_{i+1}| &\geq |\mathcal{G}_i| - \sum_{v \in Z_i \setminus Z_{i+1}} |\widehat{L}(v)| - t|Z_{i+1}|n \geq |\mathcal{G}_i| - |Z_i| \times \frac{n^{2-2^{-i-1}}}{(\log n)^2} - 2ktn^{1+2^{-i-2}} \log n \\ &\geq |\mathcal{G}_i| - \frac{2n^2}{(\log n)^2} \geq |\mathcal{G}| - \frac{2(i+2)n^2}{(\log n)^2}. \end{aligned}$$

Here, $t|Z_{i+1}|n$ is an upper bound for the number of edges in $\mathcal{G}_i$ with at least two vertices in Z_{i+1}. ∎

Let $\mathcal{G}'_k$ denote the outputting 3-graph of the Cleaning Algorithm with input $(\mathcal{G}_k, k, t)$ as define here. Since $|\mathcal{G}_k| \geq tn^2/2 - 2\varepsilon_1 tn^2$, it follows from Lemma 2.15 that

$$|\mathcal{G}'_k| \geq \left(\frac{t}{2} - \varepsilon_2\right) n^2.$$

Recall from the definition of $\mathcal{G}_k$ that every edge in $\mathcal{G}'_k$ contains exactly one vertex in Z_k. Similar to the proof of Claim 3.1, the number of pairs in $\binom{U_k}{2}$ that have codegree exactly t in $\mathcal{G}'_k$ is at least $(1/2 - \varepsilon_3)n^2 - |Z_k|n \geq (1/2 - 2\varepsilon_3)n^2$. Therefore, the 3-graph

$$\mathcal{G}''_k := \left\{e \in \mathcal{G}'_k \colon \text{the pair } \binom{e}{2} \cap \binom{U_k}{2} \text{ has codegree exactly } t\right\}$$

satisfies

$$|\mathcal{G}''_k| \geq \left(\frac{1}{2} - 2\varepsilon_3\right) tn^2. \tag{5}$$

Define

$$\mathcal{Z} := \left\{T \in \binom{Z_k}{t} \colon |L_{\mathcal{G}''_k}(T)| \geq 4kn\right\},$$

where $L_{\mathcal{G}''_k}(T) := \bigcap_{v\in T} L_{\mathcal{G}''_k}(v)$. For every $T \in \mathcal{Z}$ let $L'(T)$ be a maximum induced subgraph of $L_{\mathcal{G}''_k}(T)$ with minimum degree at least $3k$. By greedily removing vertex with degree less than $3k$ we have

$$|L'(T)| \geq |L_{\mathcal{G}''_k}(T)| - 3kn > 0. \tag{6}$$

Let Supp_T denote the set of vertices in the graph $L'(T)$ with positive degree. The following claim follows easily from Proposition 2.8.

Claim 3.3. *We have* $\mathrm{Supp}_T \cap \mathrm{Supp}_{T'} = \emptyset$ *for all distinct sets* $T, T' \in \mathcal{Z}$.

Since $L'(T) \subseteq \binom{\mathrm{Supp}_T}{2}$ for all $T \in \mathcal{Z}$ and $\sum_{T\in\mathcal{Z}} |L_{\mathcal{G}''_k}(T)| = |\mathcal{G}''_k|/t$, it follows from (5) and (6) that

$$\begin{aligned}\sum_{T\in\mathcal{Z}} \binom{|\mathrm{Supp}_T|}{2} \geq \sum_{T\in\mathcal{Z}} |L'(T)| \geq \sum_{T\in\mathcal{Z}} \left(|L_{\mathcal{G}''_k}(T)| - kn\right) &= |\mathcal{G}''| - kn|Z_k| \\ &\geq \left(\frac{1}{2} - 2\varepsilon_3\right) n^2 - kn|Z_k| \\ &\geq \left(\frac{1}{2} - 3\varepsilon_3\right) n^2.\end{aligned}$$

It follows from Claim 3.3 that $\sum_{T\in\mathcal{Z}} |\mathrm{Supp}_T| \leq |U_k| \leq n$. Combined with the inequality above and using the fact $\sum_{i\geq 1} \binom{x_i}{2} \leq \binom{x_1}{2} + \binom{\sum_{i\geq 2} x_i}{2}$, we obtain

$$\max\{|\mathrm{Supp}_T| \colon T \in \mathcal{Z}\} \geq \left(1 - \sqrt{6\varepsilon_3}\right) n.$$

Let $T_* \in \mathcal{Z}$ be the (unique) t-set with $|\mathrm{Supp}_{T_*}| \geq \left(1 - \sqrt{6\varepsilon_3}\right) n$. Then, by Claim 3.3,

$$\begin{aligned}|L_{\mathcal{G}''_k}(T_*)| = |\mathcal{G}''_k| - \sum_{T\in\mathcal{Z}-T_*} |L_{\mathcal{G}''_k}(T)| &\geq \left(\frac{1}{2} - 2\varepsilon_3\right) n^2 - \binom{\sqrt{6\varepsilon_3}n}{2} \\ &\geq \left(\frac{1}{2} - \delta\right) n^2.\end{aligned}$$

In other words, the number of edges in $\mathcal{G}''_k \subseteq \mathcal{H}$ with exactly one vertex in T_* is at least $(1/2 - \delta)\, n^2$. This completes the proof of Theorem 1.3. ∎

3.2 Proof of Theorem 1.2

In this subsection, we use Theorem 1.3 to prove Theorem 1.2.

Proof of Theorem 1.2. Let T be a tree with $\sigma(T) = \tau_{\mathrm{ind}}(T)$. Let $I \cup J = V(T)$ be a partition such that I is a minimum independent vertex cover of T. Let $e_* := \{u_0, v_0\}$ be a pendant critical edge such that $v_0 \in I$ is a leaf (the existence of such an edge is guaranteed by Proposition 2.7). Choose $C > 0$ sufficiently large such that $\mathrm{ex}(N, T^3) \le CN^2/2$ holds for all integers $N \ge 0$. By the theorem of Kostochka–Mubayi–Verstraëte [31] (or Theorem 1.3), such a constant C exists. Let $0 < \delta \ll \delta_1 \ll C^{-1}$ be sufficiently small, $n \gg C$ be sufficiently large, $t := \sigma(T) - 1$, and $q := |\mathcal{S}(n,t)| = \binom{n}{3} - \binom{n-t}{3}$. Let $\mathcal{H}$ be an n-vertex T^3-free 3-graph with q edges. Our aim is to show that $\mathcal{H} \cong \mathcal{S}(n,t)$.

Let $V := V(\mathcal{H})$. Since $|\mathcal{H}| = q = (t - o(1))n^2/2$, by Theorem 1.3, there exists a t-set $L := \{x_1, \ldots, x_t\} \subseteq V$ such that

$$d_{\mathcal{H}}(x_i) \ge (1-\delta)\frac{n^2}{2} \quad \text{for all} \quad x_i \in L. \tag{7}$$

Let $V' := V \setminus L$,

$$\tau := \frac{n}{200}, \quad D := \left\{y \in V' \colon d_{\mathcal{H}}(yx_i) \ge \tau \text{ for all } x_i \in L\right\}, \quad \text{and} \quad \overline{D} := V' \setminus L.$$

Define

$$\mathcal{S} := \left\{e \in \binom{V}{3} \colon |e \cap L| \ge 1\right\}, \quad \mathcal{B} := \mathcal{H} - L = \mathcal{H}[V'], \quad \text{and} \quad \mathcal{M} := \mathcal{S} \setminus \mathcal{H}.$$

Let $m := |\mathcal{M}|$ and $b := |\mathcal{B}|$. Notice that we are done if $b = 0$. So we may assume that $b \ge 1$. Our aim is to show that $b < m$.

Claim 3.4. *We have $m \ge n|\overline{D}|/6$ and $|\overline{D}| \le 3\delta t n$.*

Proof. It follows from the definition of D that for every vertex $u \in \overline{D}$, there exists $x_i \in L$ such that the pair $\{u, x_i\}$ contributes at least $n - 2 - \tau$ members (called missing edges) in $\mathcal{M}$. Therefore, we have

$$m \ge \frac{1}{3}(n - 2 - \tau)|\overline{D}| \ge \frac{n|\overline{D}|}{6}. \tag{8}$$

On the other hand, it follows from (7) that

$$m \le |\mathcal{S}| - \sum_{x_i \in L} d_{\mathcal{H}}(x_i) \le t\left(\binom{n}{2} - (1-\delta)\frac{n^2}{2}\right) \le \frac{\delta t n^2}{2}.$$

Combined with (8), we obtain

$$|\overline{D}| \le \frac{\delta t n^2}{2 \times n/6} = 3\delta t n.$$

This proves Claim 3.4. ∎

Claim 3.5. *We have $\mathcal{B} \setminus \mathcal{B}\left[\overline{D}\right] = \emptyset$.*

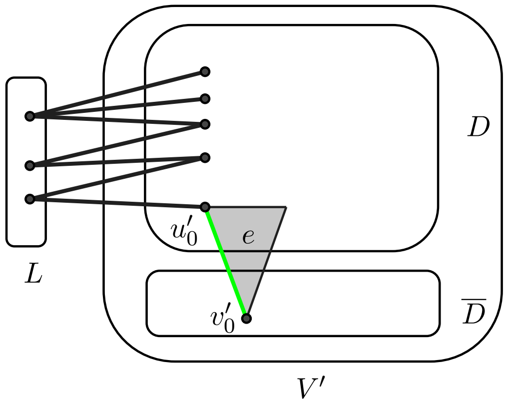

Figure 6: Embedding the tree T.

Proof. Suppose to the contrary that there exists an edge $e \in \mathcal{B} \setminus \mathcal{B}\left[\overline{D}\right]$. Then fix $u_0' \in e \cap D$ and $v_0' \in e \setminus \{u_0\}$. Let $T' := T - v_0$, $I' := I - v_0$, and $D' := D \setminus (e - u_0')$. Since the induced bipartite subgraph of $\partial\mathcal{H}$ on $L \cup D'$ is complete, there exists an embedding $\psi\colon T' \to \partial\mathcal{H}[L, D']$ such that $\psi(I') = L$, $\psi(J) \subseteq D$, and $\psi(u_0) = u_0'$. Since every pair in $\partial\mathcal{H}[L, D]$ has codegree at least $\tau \geq 3k$, by Fact 2.2, the map ψ can be greedily extended to be an embedding $\hat{\psi}\colon T^3 \to \mathcal{H}$ with $\hat{\psi}(e_*^3) = e$ (see Figure 6), a contradiction. Here, recall that e_*^3 is the expansion of e_*. ∎

Observe that the 3-graph $\mathcal{B}\left[\overline{D}\right]$ is T^3-free, so we have $|\mathcal{B}\left[\overline{D}\right]| \leq C|\overline{D}|^2$. It follows from Claim 3.5 that

$$b = |\mathcal{B}| = |\mathcal{B}\left[\overline{D}\right]| \leq C|\overline{D}|^2. \tag{9}$$

By Claim 3.4 and (9), we obtain

$$b \leq C|\overline{D}|^2 \leq C|\overline{D}| \times 3\delta t n < \frac{n|\overline{D}|}{6} = m.$$

Therefore,

$$q = |\mathcal{H}| = |\mathcal{S}| + |\mathcal{B}| - |\mathcal{M}| \leq q + b - m < q,$$

a contradiction. This proves Theorem 1.2. ∎

4 Proofs for anti-Ramsey results

We prove Theorem 1.4 in this section. Let us first present some useful lemmas.

Using Lemma 2.18 and the theorem of Kostochka–Mubayi–Verstraëte [31] on linear cycles (or Theorem 1.3), it is not hard to obtain the following crude bound for $\mathrm{ex}(n, F^3)$ when F is an augmentation of a tree, The detailed proof is included in the Appendix (see Section G).

Proposition 4.1. *Suppose that F is an augmentation of a tree T. Then $\mathrm{ex}(n, F^3) \leq 3\left(|T| + 1\right) n^2$ holds for sufficiently large n.*

The following lemma is a stability result for augmentations of trees.

Lemma 4.2. *Let T be a tree with $\sigma(T) = \tau_{\mathrm{ind}}(T)$ and F be an augmentation of T. For every $\delta > 0$ there exist $\varepsilon > 0$ and n_0 such that the following holds for all $n \geq n_0$. Suppose that $\chi\colon K_n^3 \to \mathbb{N}$ is an edge-coloring without rainbow copies of F^3 and $\mathcal{H} \subseteq K_n^3$ is a rainbow subgraph with*

$$|\mathcal{H}| \geq \left(\frac{\sigma(T)-1}{2} - \varepsilon\right) n^2.$$

Then there exists a $(\sigma(T)-1)$-set $L \subseteq [n]$ such that

$$d_{\mathcal{H}}(v) \geq \left(\frac{1}{2} - \delta\right) n^2 \quad \textit{for all} \quad v \in L.$$

Proof of Lemma 4.2. Let $\{e_*\} = F \setminus T$ and let x, y denote the endpoints of e_* in F. Let $\widehat{T}$ be the graph obtained from F by removing the edge e_* but keeping the endpoints of e_* (since e_* might be outside $V(T)$, two graphs T and $\widehat{T}$ can be different). Let $\mathcal{C}$ be a maximal collection of edge disjoint copies of $\widehat{T}^3$ in $\mathcal{H}$. Assume that $\mathcal{C} = \{\widehat{T}_1^3, \ldots, \widehat{T}_m^3\}$ for some integer $m \geq 0$. For $i \in [m]$ let x_i, y_i be two distinct vertices in $\widehat{T}_i^3$ such that there exists an isomorphism $\psi_i\colon \widehat{T}^3 \to \widehat{T}_i^3$ with $\psi_i(x) = x_i$ and $\psi_i(y) = y_i$. We call $\{x_i, y_i\}$ the **critical pair** of $\widehat{T}_i^3$ for $i \in [m]$. Let G denote the multi-graph on $[n]$ whose edge set is the multi-set $\{x_i y_i \colon i \in [m]\}$. Since the coloring χ does not contain rainbow copies of F^3, we obtain the following claim.

Claim 4.3. *For every $i \in [m]$ and every vertex $z \in [n] \setminus V(\widehat{T}_i^3)$ we have $\chi(z x_i y_i) \in \chi\left(\widehat{T}_i^3\right)$.*

Claim 4.4. *The graph G does not contain multi-edges.*

Proof. Suppose to the contrary that G contains multi-edges, and by symmetry, we may assume that $(x_1, y_1) = (x_2, y_2)$. Take any vertex z outside $V(\widehat{T}_1^3) \cup V(\widehat{T}_2^3)$. Then by Claim 4.3, we have $\chi(x_1 y_1 z) = \chi(x_2 y_2 z) \in \chi\left(\widehat{T}_1^3\right) \cap \chi\left(\widehat{T}_2^3\right) = \emptyset$ (note that $\mathcal{H}$ is rainbow, and $\widehat{T}_1^3, \widehat{T}_2^3 \subset \mathcal{H}$ are edge-disjoint), a contradiction. ∎

Claim 4.5. *We have $m = |G| \leq 2|V(F)|n$.*

Proof. Assume that the set of neighbors of x in F is $\{w_1, \ldots, w_\ell\} = N_F(x)$, where $\ell = d_F(x)$ and $w_1 = y$. Let $\{z_1, \ldots, z_\ell\}$ be a set of new vertices not contained in $V(F)$, and define a new graph

$$\widehat{F} = (F - x) \cup \{w_i z_i \colon i \in [\ell]\},$$

Observe that $\widehat{F}$ is acyclic and $|V(\widehat{F})| \leq 2|V(F)|$.

Suppose to the contrary that $|G| \geq 2|V(F)|n$. Then there exists an embedding $\phi\colon \widehat{F} \to G$. For every $e \in \phi(\widehat{F})$ we fix a member in $\mathcal{C}$ whose critical pair is e, and denote this member by $\widehat{T}_e$. Let $B_1 := \bigcup_{e \in \phi(\widehat{F})} V(\widehat{T}_e)$. Choose a vertex set $U_1 := \{w_e \colon e \in \phi(F - x)\} \subseteq [n] \setminus B_1$. Fix a vertex $v \in [n] \setminus (B_1 \cup U_1)$. Then, by Claim 4.3, the 3-graph

$$F' := \{e \cup w_e \colon e \in \phi(F - x)\} \cup \{v \phi(w_i) \phi(z_i) \colon i \in [\ell]\}$$

is rainbow under χ. Observe that F' is a copy of F^3 (with vertex v playing the role of x), contradicting the rainbow-F-freeness of χ. ∎

By Claim 4.5, the 3-graph $\mathcal{B} := \bigcup_{i\in[m]} \widehat{T}_i^3$ has size $m \times |F| \le 2|V(F)||F|n \le \varepsilon n^2$. Hence, $\mathcal{H}' := \mathcal{H} \setminus \mathcal{B}_1$ has size at least $\left(\frac{\sigma(T)-1}{2} - 2\varepsilon\right) n^2$. In addition, by the maximality of $\mathcal{C}$, the 3-graph $\mathcal{H}'$ is $\widehat{T}^3$-free, and hence, T^3-free. Therefore, it follows from Theorem 1.3 that there exists a $(\sigma(T)-1)$-set $L \subseteq [n]$ such that $d_{\mathcal{H}}(v) \ge (1/2 - \delta)\, n^2$ holds for all $v \in L$. ∎

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let T be a tree satisfying $\sigma(T) = \tau_{\mathrm{ind}}(T)$ and containing a critical edge. Let F be an augmentation of T. Let $\{f\} := F \setminus T$ and assume that $f = \{u_1, v_1\}$. Let $I \cup J = V(T)$ be a partition such that I is a minimum independent vertex cover of T and $f \not\subseteq I$ (such I exists due to the definition of augmentation). Let $e_* := \{u_0, v_0\}$ be a pendant critical edge such that $v_0 \in I$ is a leaf. The existence of such an edge is guaranteed by Proposition 2.7.

Choose $C > 0$ sufficiently large such that $\mathrm{ex}(N, F^3) \le CN^2$ holds for all integers $N \ge 0$. By Proposition 4.1, such a constant C exists and depends only on F. Let $t := \sigma(T) - 1$ and $q := |\mathcal{S}(n,t)| + 2 = \binom{n}{3} - \binom{n-t}{3} + 2$. Let $\chi\colon K_n^3 \to [q]$ be a surjective map. Our aim is to show that there exists a rainbow copy of $F^3 \subseteq K_n^3$ under the coloring χ.

Suppose to the contrary that there is no rainbow copy of F^3 under the coloring χ. Let $\mathcal{H} \subseteq K_n^3$ a rainbow subgraph with q edges. Since $|\mathcal{H}| = q = (t - o(1))\, n^2/2$, it follows from Lemma 4.2 that there exists a t-set $L := \{x_1, \dots, x_t\} \subseteq V := [n]$ such that

$$d_{\mathcal{H}}(v) \ge \left(\frac{1}{2} - \delta\right) n^2 \quad \text{for all} \quad x_i \in L. \tag{10}$$

Let $V' := V \setminus L$,

$$\tau := \frac{n}{200}, \quad D := \left\{y \in V' \colon d_{\mathcal{H}}(yx_i) \ge \tau \text{ for all } x_i \in L\right\}, \quad \text{and} \quad \overline{D} := V' \setminus L.$$

Define

$$\mathcal{S} := \left\{e \in \binom{V}{3} \colon |e \cap L| \ge 1\right\}, \quad \mathcal{B} := \mathcal{H} - L = \mathcal{H}[V'], \quad \text{and} \quad \mathcal{M} := \mathcal{S} \setminus \mathcal{H}.$$

Let $m := |\mathcal{M}|$ and $b := |\mathcal{B}|$.

Claim 4.6. *We have $m \ge n|\overline{D}|/6$ and $|\overline{D}| \le 6\delta t n$.*

Proof. Same as the proof of Claim 3.4. ∎

Claim 4.7. *We have $\left|\mathcal{B} \setminus \mathcal{B}[\overline{D}]\right| \le 1$.*

Proof. Suppose to the contrary that there exist two edges $e_1, e_2 \in \mathcal{B} \setminus \mathcal{B}[\overline{D}]$. We will consider several cases depending on the position of f in F. The most technical case would be Case 6.

Case 1: $f \cap V(T) = \emptyset$.

Take any triple $f' \subseteq D \setminus (e_1 \cup e_2)$. By symmetry, we may assume that $\chi(f') \ne \chi(e_1)$. Fix $u_0' \in e_1 \cap D$ and let $e_1' := e_1 \setminus \{u_0'\}$. Fix $v_0' \in e_1'$. Let $D' := D \setminus (f' \cup e_1')$. Let $\mathcal{H}'$ be the 3-graph be obtained from $\mathcal{H}$ by removing (at most one) edge with color $\chi(f')$. Observe that the induced bipartite subgraph of $\partial\mathcal{H}'$ on $L \cup D'$ is complete. Combined with

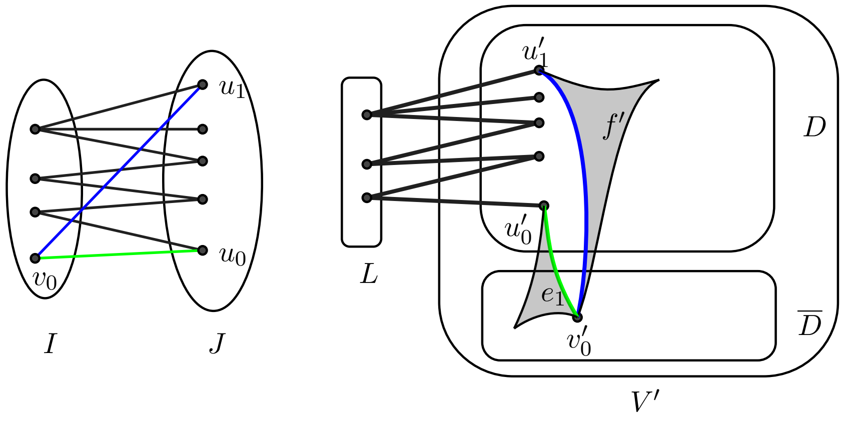

Figure 7: Embedding F.

$\sigma(T \setminus \{e_*\}) = |L|$, it is easy to see that there exists an embedding $\psi\colon V(T - v_0) \to L \cup D'$ such that $\psi(T \setminus \{e_*\}) \subseteq \partial\mathcal{H}'[L, D']$, $\psi(I \setminus \{v_0\}) = L$, and $\psi(u_0) = u_0'$. By definition, each pair in $\partial\mathcal{H}'[L, D']$ has codegree at least $\tau - 1 \geq 3k$. Therefore, by Fact 2.2, the map ψ can be extended greedily to be a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H}' \cup \{e_1, f'\}$ with $\hat{\psi}(v_0) = v_0'$, $\hat{\psi}(f^3) = f'$ and $\hat{\psi}(e_*^3) = e_1$.

Case 2: $|f \cap V(T)| = 1$.

Suppose that $f \cap e_* = \emptyset$ and $f \cap I \neq \emptyset$. Let us assume that $\{u_1\} = f \cap I$. Then take any triple $f' \subseteq L \cup D \setminus (e_1 \cup e_2)$ such that $|f' \cap L| = 1$. Let $\{u_1'\} := f' \cap L$. By symmetry, we may assume that $\chi(f') \neq \chi(e_1)$. Fix $u_0' \in e_1 \cap D$. Then similar to the proof in Case 1, it is easy to see that there exists a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H} \cup \{e_1, f'\}$ with $\hat{\psi}(u_0) = u_0'$, $\hat{\psi}(u_1) = u_1'$, $\hat{\psi}(f^3) = f'$, and $\hat{\psi}(e_*^3) = e_1$.

Suppose that $f \cap e_* = \emptyset$ and $f \cap J \neq \emptyset$. Let us assume that $\{u_1\} = f \cap J$. Then take any triple $f' \subseteq D \setminus (e_1 \cup e_2)$. By symmetry, we may assume that $\chi(f') \neq \chi(e_1)$. Fix $u_1' \in f' \cap D$ and $u_0' \in e_1 \cap D$. Then similar to the proof in Case 1, it is easy to see that there exists a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H} \cup \{e_1, f'\}$ with $\hat{\psi}(u_1) = u_1'$, $\hat{\psi}(u_0) = u_0'$, $\hat{\psi}(f^3) = f'$, and $\hat{\psi}(e_*^3) = e_1$.

Suppose that $f \cap e_* \neq \emptyset$ and $f \cap e_* \in J$. Let us assume that $\{u_1\} = \{u_0\} = f \cap e_*$. Take any triple $f' \subseteq D \setminus (e_1 \cup e_2)$ such that $|f' \cap e_1| = |f' \cap e_2| = 1$. By symmetry, we may assume that $\chi(f') \neq \chi(e_1)$. Let $\{u_1'\} := f' \cap e_1$. Then similar to the proof in Case 1, it is easy to see that there exists a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H} \cup \{e_1, f'\}$ with $\hat{\psi}(u_1) = \hat{\psi}(u_0) = u_1'$, $\hat{\psi}(f^3) = f'$, and $\hat{\psi}(e_*^3) = e_1$.

Suppose that $f \cap e_* \neq \emptyset$ and $f \cap e_* \in I$. Let us assume that $\{u_1\} = \{v_0\} = f \cap e_*$. Fix $u_0' \in e_1 \cap D$ and $u_0'' \in e_2 \cap D$. Let $e_1' := e_1 \setminus \{u_0'\}$ and $e_2' := e_2 \setminus \{u_0''\}$. Take any triple $f' \subseteq V'$ such that $|f' \cap e_1'| = |f' \cap e_2'| = 1$ and $f' \setminus (e_1' \cup e_2') \subseteq D$. By symmetry, we may assume that $\chi(f') \neq \chi(e_1)$. Let $v_0' \in e_1 \cap f'$ and fix $v_1' \in f' \setminus (e_1' \cup e_2')$. Then similar to the proof in Case 1, it is easy to see that there exists a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H} \cup \{e_1, f'\}$ with $\hat{\psi}(u_0) = u_0'$, $\hat{\psi}(u_1) = \hat{\psi}(v_0) = u_1'$, $\hat{\psi}(v_1) = v_1'$, $\hat{\psi}(f^3) = f'$, and $\hat{\psi}(e_*^3) = e_1$.

Case 3: $f \subseteq J$ and $e_* \cap f = \emptyset$.

Take any triple $f' \subseteq D$ such that $|f' \cap e_1| = |f' \cap e_2| = 0$. By symmetry, we may assume that $\chi(f') \neq \chi(e_1)$. Fix $\{u_1', v_1'\} \subseteq f'$ and $u_0' \subseteq e_1 \setminus \overline{D}$. Then similar to the proof in

Case 1, it is easy to see that there exists a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H} \cup \{e_1, f'\}$ with $\hat{\psi}(u_1) = u_1'$, $\hat{\psi}(v_1) = v_1'$, $\hat{\psi}(u_0) = u_0'$, $\hat{\psi}(f^3) = f'$, and $\hat{\psi}(e_*^3) = e_1$.

Case 4: $f \subseteq J$ and $e_* \cap f \neq \emptyset$.

Take any triple $f' \subseteq D$ such that $|f' \cap e_1| = |f' \cap e_2| = 1$. By symmetry, we may assume that $\chi(f') \neq \chi(e_1)$. Let $u_1' := f' \cap e_1$ and fix $v_1' \in f' \setminus e_1$. Then similar to the proof in Case 1, it is easy to see that there exists a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H} \cup \{e_1, f'\}$ with $\hat{\psi}(u_1) = \hat{\psi}(u_0) = u_1'$, $\hat{\psi}(v_1) = v_1'$, $\hat{\psi}(f^3) = f'$, and $\hat{\psi}(e_*^3) = e_1$.

Case 5: $f \cap I \neq \emptyset$, $f \cap J \neq \emptyset$, and $e_* \cap f = \emptyset$.

Assume that $\{u_1\} = f \cap I$ and $\{v_1\} = f \cap J$. Take any triple $f' \subseteq L \cup D$ such that $|f' \cap L| = 1$ and $|f' \cap e_1| = |f' \cap e_2| = 0$. By symmetry, we may assume that $\chi(f') \neq \chi(e_1)$. Let $u_1' := f' \cap L$. Fix $v_1' \in f' \setminus L$ and $u_0' \in e_1 \setminus \overline{D}$. Then similar to the proof in Case 1, it is easy to see that there exists a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H} \cup \{e_1, f'\}$ with $\hat{\psi}(u_1) = u_1'$, $\hat{\psi}(v_1) = v_1'$, $\hat{\psi}(u_0) = u_0'$, $\hat{\psi}(f^3) = f'$, and $\hat{\psi}(e_*^3) = e_1$.

Case 6: $f \cap I \neq \emptyset$, $f \cap J \neq \emptyset$, and $e_* \cap f \neq \emptyset$.

Assume that $\{u_1\} = f \cap I$ and $\{v_1\} = f \cap J$.

Suppose that $f \cap e_* \in J$. Then take any triple $f' \subseteq L \cup D$ such that $|f' \cap L| = 1$ and $|f' \cap e_1| = |f' \cap e_2| = 1$. By symmetry, we may assume that $\chi(f') \neq \chi(e_1)$. Let $u_1' := f' \cap L$ and $v_1' := f' \cap e_*$. Then similar to the proof in Case 1, it is easy to see that there exists a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H} \cup \{e_1, f'\}$ with $\hat{\psi}(u_1) = u_1'$, $\hat{\psi}(v_1) = v_1'$, $\hat{\psi}(f^3) = f'$, and $\hat{\psi}(e_*^3) = e_1$.

Suppose that $f \cap e_* \in I$. Then fix $u_0' \in e_1 \cap (D)$ and $u_0'' \in e_2 \cap (D)$ (it could be true that $u_0' = u_0''$). Let $e_1' := e_1 \setminus \{u_0'\}$ and $e_2' := e_2 \setminus \{u_0''\}$. Choose a triple $f' \subseteq V'$ such that $|f' \cap e_1'| = |f' \cap e_2'| = 1$ and $f' \setminus (e_1 \cup e_2) \in D$. By symmetry, we may assume that $\chi(f') \neq \chi(e_1)$. Let $\{v_0'\} := f' \cap e_1$ and $\{u_1'\} := f' \setminus (e_1 \cup e_2)$. Then similar to the proof in Case 1, it is easy to see that there exists a (rainbow) embedding $\hat{\psi}\colon F^3 \to \mathcal{H} \cup \{e_1, f'\}$ with $\hat{\psi}(u_1) = u_1'$, $\hat{\psi}(v_1) = \hat{\psi}(v_0) = v_0'$, $\hat{\psi}(u_0) = u_0'$, $\hat{\psi}(f^3) = f'$, and $\hat{\psi}(e_*^3) = e_1$ (see Figure 7).

Case 7: $f \subseteq I$ and $f \cap e_* = \emptyset$.

The proof is similar to Case 1, and the only difference is that in this case we choose the triple f' to satisfy $|f' \cap L| = 2$ and $|f' \cap e_1| = |f' \cap e_2| = 0$.

Case 8: $f \subseteq I$ and $f \cap e_* \neq \emptyset$.

First fix a vertex $u_0' \in e_1 \cap D$ and a vertex $u_0'' \in e_2 \cap D$. Let $e_1' := e_1 \setminus \{u_0'\}$ and $e_2' := e_2 \setminus \{u_0''\}$. Take a triple f' satisfying $|f' \cap L| = |f' \cap e_1'| = |f' \cap e_2'| = 1$. The rest part is similar to the proof of Case 6, and in this case we let u_0' play the role of u_0, let the vertex in $f' \cap e_1$ play the role of the vertex $f \cap e_*$, and let the vertex $f' \cap L$ play the role of $f \setminus e_*$. ∎

Claim 4.8. *We have $b \leq C|\overline{D}|^2 + 1$.*

Proof. For $i \in \{1, 2, 3\}$ let

$$\mathcal{B}_i := \left\{ e \in \mathcal{B} \colon |e \cap \overline{D}| = i \right\}.$$

Since $\mathcal{B}_3$ is F^3-free, it follows from the definition of C that $|\mathcal{B}_3| \le C|\overline{D}|^2$. By Claim 4.7, we have $|\mathcal{B}_1| + |\mathcal{B}_2| \le 1$. Therefore, $b = |\mathcal{B}_1| + |\mathcal{B}_2| + |\mathcal{B}_3| \le C|\overline{D}|^2 + 1$. ∎

Now it follows from Claims 4.6 and 4.8 that

$$|\mathcal{H}| = |\mathcal{S}| + |\mathcal{B}| - |\mathcal{M}| \le q - 2 + C|\overline{D}|^2 + 1 - m \le q - 1 - \left(\frac{n}{6} - C \times 6\delta t n\right)|\overline{D}| < q$$

a contradiction. ∎

5 Proofs for generalized Turán results

We prove Theorems 1.9 and 1.11 in this section. We will prove Theorem 1.11 first, and then use it to prove Theorem 1.9.

5.1 Proof of Theorem 1.11

Now we are ready to prove Theorem 1.11. The proof is an adaptation of the proof for [29, Theorem 6.2].

Proof of Theorem 1.11. Let $k \ge 4$ be a fixed integer and $t := \lfloor (k-1)/2 \rfloor$. Fix $\delta > 0$. Let $0 < \varepsilon \ll \varepsilon_1 \ll \varepsilon_2 \ll \delta$ be sufficiently small constants and n be a sufficiently large integer. Let G be an n-vertex $C_k^{\triangle}$-free graph with

$$N(K_3, G) \ge \left(\frac{t}{4} - \varepsilon\right) n^2. \tag{11}$$

Let $\mathcal{H} := \mathcal{K}_G$ and recall that

$$\mathcal{K}_G := \left\{ e \in \binom{V(G)}{3} : G[e] \cong K_3 \right\}.$$

Let $\mathcal{H}_q$ denote the outputting 3-graph of the Cleaning Algorithm with input $(\mathcal{H}, k, t)$ as defined here. Recall from Fact 2.5 that $\mathcal{H}$ is C_k^3-free. In addition, it follows from Proposition 2.9 and (11) that

$$|\mathcal{H}| = N(K_3, G) \ge t\frac{n^2}{4} - \varepsilon n^2 \ge t\left(|G| - \varepsilon n^2\right) - \varepsilon n^2 = t|\partial\mathcal{H}| - \varepsilon_1 n^2,$$

where $\varepsilon_1 := (t+1)\varepsilon$. Therefore, by Lemma 2.21, we have

$$q \le 12k\varepsilon_1 n^2, \quad |\mathcal{H}_q| \ge |\mathcal{H}| - 48k^2\varepsilon_1 n^2, \quad \text{and} \quad |\partial\mathcal{H}_q| \ge |\partial\mathcal{H}| - 50k^2\varepsilon_1 n^2. \tag{12}$$

Moreover, the 3-graph $\mathcal{H}_q$ is $(t, 3k)$-superfull.

Using the third inequality in (12) and Proposition 2.9, we obtain the following claim.

Claim 5.1. *We have* $|\partial\mathcal{H}_q| \ge (1/4 - \varepsilon_2)\, n^2$.

Define a graph

$$G' := \left\{ uv \in \partial\mathcal{H}_q \colon d_{\mathcal{H}_q}(uv) = t \right\}.$$

Claim 5.2. *We have* $|G'| \ge (1/4 - 2\varepsilon_2)\, n^2$.

Proof. Let $G^* := \partial\mathcal{H}_q \setminus G'$. Since $\mathcal{H}_q$ is $(t,3k)$-full, we have $d_{\mathcal{H}_q}(e) \geq 3k+1$ for all $e \in G^*$. Therefore, it follows from Lemma 2.20 and the $(t,3k)$-fullness of $\mathcal{H}_q$ that $|G^*| \leq \varepsilon_2 n^2$. Combined with Claim 5.1, we obatin $|G'| \geq |\partial\mathcal{H}_q| - |G^*| \geq (1/4 - 2\varepsilon_2)\, n^2$. ∎

Let $V_1 \cup V_2 = V(\mathcal{H})$ be a bipartition such that the number of edges (in G') crossing V_1 and V_2 is maximized. It follows from Claim 5.2 and Proposition 2.9 that $|G'[V_1, V_2]| \geq |G'| - \varepsilon_2 n^2 \geq (1/4 - 3\varepsilon_2)\, n^2$. Simple calculations show that

$$\frac{n}{2} - \sqrt{3\varepsilon_2} n \leq |V_i| \leq \frac{n}{2} - \sqrt{3\varepsilon_2} n \quad \text{for} \quad i \in \{1,2\}. \tag{13}$$

Define

$$\mathcal{W} := \left\{ (S,T) \in \binom{V_1}{3k} \times \binom{V_2}{3k} \colon G'[S,T] \cong K_{3k,3k} \right\}.$$

Since each nonedge in $G'[V_1, V_2]$ contributes at most $\binom{|V_1|-1}{3k-1}\binom{|V_2|-1}{3k-1}$ elements in $\binom{V_1}{3k} \times \binom{V_2}{3k} - \mathcal{W}$, it follows from (13) that

$$\begin{aligned} |\mathcal{W}| &\geq \binom{|V_1|}{3k}\binom{|V_2|}{3k} - 3\varepsilon_2 n^2 \binom{|V_1|-1}{3k-1}\binom{|V_2|-1}{3k-1} \\ &= \binom{|V_1|}{3k}\binom{|V_2|}{3k} - 3\varepsilon_2 n^2 \frac{3k}{|V_1|}\frac{3k}{|V_2|}\binom{|V_1|}{3k}\binom{|V_2|}{3k} \geq (1 - 144\varepsilon_2 k^2)\binom{|V_1|}{3k}\binom{|V_2|}{3k}, \end{aligned}$$

Here we used the identity $\binom{|V_i|-1}{3k-1} = \frac{3k}{|V_i|}\binom{|V_1|}{3k}$. By averaging, there exists an edge $e \in G'[V_1, V_2]$ that is contained in at least

$$\frac{(3k)^2|\mathcal{W}|}{|G'[V_1,V_2]|} \geq \frac{(3k)^2|\mathcal{W}|}{|V_1||V_2|} \geq (1 - 144\varepsilon_2 k^2)\binom{|V_1|-1}{3k-1}\binom{|V_2|-1}{3k-1}$$

members in $\mathcal{W}$. For every edge $f \in G'[V_1, V_2]$ that is disjoint from e, there are at most $\binom{|V_1|-2}{3k-2}\binom{|V_2|-2}{3k-2}$ copies of $K_{3k,3k}$ in $G'[V_1, V_2]$ containing both e and f. Hence, the set

$$G'' := \left\{ f \in G'[V_1, V_2] \colon \exists W \in \mathcal{W},\ \{e,f\} \subset W \right\}$$

has size at least

$$\begin{aligned} (3k-1)^2 \frac{(1 - 144\varepsilon_2 k^2)\binom{|V_1|-1}{3k-1}\binom{|V_2|-1}{3k-1}}{\binom{|V_1|-2}{3k-2}\binom{|V_2|-2}{3k-2}} &= (1 - 144\varepsilon_2 k^2)\,(|V_1|-1)\,(|V_2|-1) \\ &\geq \left(\frac{1}{4} - \delta\right) n^2, \end{aligned}$$

where the last inequality follows from (13) and the assumption that $\varepsilon_2 \ll \delta$. Let $L := N_{\mathcal{H}_q}(e)$. Then $|L| = t$ and it follows from Lemma 2.22 and the definitions of $\mathcal{W}, G''$ that

$$N_{\mathcal{H}_q}(f) = N_{\mathcal{H}_q}(e) = L \quad \text{for all} \quad f \in G''.$$

This completes the proof of Theorem 1.11. ∎

5.2 Proof of Theorem 1.9

In this subsection, we prove Theorem 1.9 by using Theorem 1.11. There are slight distinctions between the proofs for the even and odd cases, despite their shared framework. The proof for the odd case is less technical, so we include it in the Appendix (see Section K).

Proof of Theorem 1.9 for even k. Fix $k = 2t + 2 \geq 6$. Let $C > 0$ be a constant such that $\mathrm{ex}(N, C_k^3) \leq CN^2$ holds for all integers $N \geq 0$. The existence of such a constant C is guaranteed by the theorem of Kostochka–Mubayi–Verstraëte [29]. Let $0 < \delta \ll C^{-1}$ be sufficiently small and $n \gg C$ be sufficiently large. Let G be an n-vertex $C_k^{\triangle}$-free graph with

$$N(K_3, G) = |\mathcal{S}_{\mathrm{bi}}^{+}(n, t)|.$$

We may assume that every edge in G is contained in some triangle of G, since otherwise we can delete it from G and this does not change the value of $N(K_3, G)$. Our aim is to prove that $G \cong S_{\mathrm{bi}}^{+}(n, t)$.

Since $N(K_3, G) = tn^2/4 - o(n^2)$ and n is large, it follows from Theorem 1.11 that there exists a t-set $L := \{x_1, \ldots, x_t\} \subseteq V(G)$ such that

(i) $|G - L| \geq n^2/4 - \delta n^2$,

(ii) $N(K_3, G - L) \leq \delta n^2$,

(iii) $G - L$ can be made bipartite by removing at most δn^2 edges, and

(iv) $d_G(v) \geq (1 - \delta)n$ for all $v \in L$.

Let $V := V(G)$, $V' := V - L$, and $V_1 \cup V_2 = V'$ be a bipartition such that the number of edges (in G) crossing V_1 and V_2 is maximized. Define

$$\mathcal{S} := \left\{ e \in \binom{V}{3} \colon |e \cap L| \geq 1,\ |e \cap V_1| \leq 1,\ |e \cap V_2| \leq 1 \right\},$$

and let $\mathcal{H} := \mathcal{K}_G$, $\mathcal{B} := \mathcal{H} \setminus \mathcal{S}$, $\mathcal{M} := \mathcal{S} \setminus \mathcal{H}$. Let $m := |\mathcal{M}|$ and $b := |\mathcal{B}|$. It follows from Statements (i) and (iii) above that

$$|G[V_1, V_2]| \geq \frac{n^2}{4} - 2\delta n^2. \tag{14}$$

Combined with Statement (iv), for every $x_i \in L$ the intersection of links $L_{\mathcal{H}}(x_i)$ and $L_{\mathcal{S}'}(x_i)$ satisfies

$$|L_{\mathcal{H}}(x_i) \cap L_{\mathcal{S}'}(x_i)| = |L_{\mathcal{H}}(x_i) \cap G[V_1, V_2]| \geq |G[V_1, V_2]| - \delta n \times n \geq \frac{n^2}{4} - 3\delta n^2.$$

Therefore,

$$\begin{aligned} |\mathcal{M}| &= \sum_{x_i \in L} \left(|L_{\mathcal{S}'}(x_i)| - |L_{\mathcal{H}}(x_i) \cap L_{\mathcal{S}'}(x_i)| \right) \\ &\leq t \left(|V_1||V_2| - \left(\frac{n^2}{4} - 3\delta n^2 \right) \right) \leq 3\delta t n^2. \end{aligned} \tag{15}$$

Inequality (14) with some simple calculations also imply that

$$\left(\frac{1}{2} - \sqrt{2\delta} \right) n \leq |V_i| \leq \left(\frac{1}{2} + \sqrt{2\delta} \right) n \quad \text{for} \quad i \in \{1, 2\}. \tag{16}$$

Let

$$\tau := \frac{n}{200}, \quad D := \left\{ y \in V' \colon d_{\mathcal{H}}(yx_i) \geq \tau \text{ for all } x_i \in L \right\}, \quad \overline{D} := V' \setminus D.$$

Let $D_i := D \cap V_i$, $\overline{D}_i := V_i \setminus D_i$ for $i \in \{1, 2\}$. We also divide $\mathcal{M}$ further by letting

$$\mathcal{M}_1 := \left\{ e \in \mathcal{M} \colon e \cap \overline{D} \neq \emptyset \right\}, \quad \text{and} \quad \mathcal{M}_2 := \mathcal{M} \setminus \mathcal{M}_1.$$

Let $m_1 := |\mathcal{M}_1|$ and $m_2 := |\mathcal{M}_2|$.

Claim 5.3. *We have* $m_1 \geq 49n|\overline{D}|/100$ *and* $|\overline{D}| \leq 6\delta tn$.

Proof. By the definition of D, for every $i \in \{1,2\}$ and for every vertex $v \in \overline{D}_i$ there exists a vertex $x \in L$ (depending on v) such that

$$|N_{\mathcal{H}}(vx) \cap V_{2-i}| \leq \tau.$$

Hence, the pair $\{v, x\}$ contributes at least $|V_{2-i}| - \tau$ elements to $\mathcal{M}$. Therefore, it follows from (16) that

$$m_1 \geq (\min\{|V_1|, |V_2|\} - \tau)\left(|\overline{D}_1| + |\overline{D}_2|\right) \geq \left(\frac{n}{2} - \sqrt{2\delta}n - \tau\right)|\overline{D}| \geq \frac{49}{100}n|\overline{D}|.$$

Combined with (15), we obtain $|\overline{D}| \leq \frac{2\delta tn^2}{49n/100} \leq 6\delta tn$. ∎

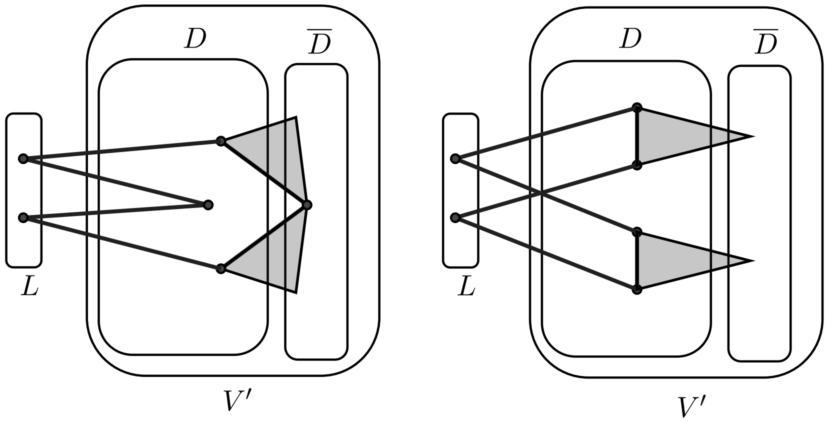

Figure 8: Embedding C_6 in two different cases.

Claim 5.4. *The* 3*-graph* $\mathcal{B}[V']$ *does not contain two edges* e_1 *and* e_2 *such that*

(i) $|e_1 \cap e_2| = 1$, $(e_1 \setminus e_2) \cap D \neq \emptyset$, *and* $(e_2 \setminus e_1) \cap D \neq \emptyset$, *or*

(ii) $\min\{|e_1 \cap D|,\ |e_2 \cap D|\} \geq 2$.

Proof. Suppose to the contrary that there exist two edges $e_1, e_2 \in \mathcal{B}[V']$ such that (i) holds. Let $\{v_0\} := e_1 \cap e_2$ and fix $v_i \in (e_i \cap D) \setminus \{v_0\}$ for $i \in \{1,2\}$. Let $D' := D \setminus (e_1 \cup e_2)$. Choose any $(t-1)$-set $\{u_1, \ldots, u_{t-1}\} \subseteq D'$. It follows from the definition of D that the induced bipartite graph of $\partial\mathcal{H}$ on $L \cup D$ is complete. Therefore, $F := v_1x_1u_1x_2u_2 \cdots x_{t-1}u_{t-1}x_tv_2$ is copy of P_{k-2} in the bipartite graph $\partial\mathcal{H}[L, D']$. This P_{k-2} together with $v_1v_0v_2$ (a copy of P_2) form a copy of C_k in $\partial\mathcal{H}$. Since all edges in F have codegree at least $\tau \geq 3k$ in $\mathcal{H}$, it follows from Fact 2.2 that $C_k^3 \subseteq \mathcal{H}$ (see Figure 8), a contradiction.

Suppose to the contrary that there exist two disjoint edges $e_1, e_2 \in \mathcal{B}[V']$ such that (ii) holds. Fix a 2-set $\{v_i, v_i'\} \subseteq e_i \cap D$ for $i \in \{1,2\}$. Choose a $(t-2)$-set $\{u_1, \ldots, u_{t-2}\} \subseteq D \setminus (e_1 \cup e_2)$. Similar to the proof above, the graph $F := v_1x_1u_1x_2 \cdots u_{t-2}x_{t-1}v_2v_2'x_tv_1'v_1$ is a copy of C_k in $\partial\mathcal{H}$. Note that all edges but v_1v_1', v_2v_2' in F have codegree at least $\tau \geq 3k$ in $\mathcal{H}$. So it follows from Fact 2.2 that $C_k^3 \subseteq \mathcal{H}$ (see Figure 8), a contradiction. ∎

Claim 5.5. *For every* $v \in \overline{D}$ *we have*

$$\min\{|N_G(v) \cap D_1|, |N_G(v) \cap D_2|\} \leq \sqrt{3\delta}n.$$

Proof. Suppose to the contrary that there exists a vertex $v \in \overline{D}$ such that the set $N_i := N_G(v) \cap D_i$ has size at least $\sqrt{3\delta}n$ for both $i \in \{1,2\}$. Observe from Claim 5.4 (ii) that the induced bipartite graph $G[N_1, N_2]$ does not contain two disjoint edges. Hence, $|G[N_1, N_2]| \le n$ and consequently,

$$|G[V_1, V_2]| \le |V_1||V_2| - (|N_1||N_2| - |G[N_1, N_2]|) \le \frac{n^2}{4} - (3\delta n^2 - n) < \frac{n^2}{4} - 2\delta n^2,$$

contradicting (14). ∎

Claim 5.6. *For every $i \in \{1,2\}$ and for every $v \in \overline{D}_i$ we have $|N_G(v) \cap D_i| \le 9\sqrt{\delta}tn$.*

Proof. Suppose to the contrary that there exist $i \in \{1,2\}$ and $v \in \overline{D}_i$ such that $|N_G(v) \cap D_i| > 9\sqrt{\delta}tn$. Then by the maximality of $G[V_1, V_2]$, we have $|N_G(v) \cap V_{2-i}| \ge |N_G(v) \cap V_i| \ge |N_G(v) \cap D_i| > 9\sqrt{\delta}tn$, since otherwise we can move v from V_i to V_{2-i} and this will result in a large bipartite subgraph of G. By Claim 5.3, this implies that $|N_G(v) \cap D_{2-i}| \ge |N_G(v) \cap V_{2-i}| - |\overline{D}| > 3\sqrt{\delta}n$, contradicting Claim 5.5. ∎

Claim 5.7. *We have $|\mathcal{B}[D]| \le n/2 + 11\sqrt{\delta}tn$.*

Proof. First, observe from Claim 5.4 that the 3-graph $\mathcal{B}[D]$ is 2-intersecting. Suppose to the contrary that $|\mathcal{B}[D]| > n/2 + 3\sqrt{\delta}n$. Then, by Fact 2.3, there exists a pair $\{v_1, v_2\} \subseteq D$ such that all edges in $\mathcal{B}[D]$ containing $\{v_1, v_2\}$. Then it follows from Claims 5.5, 5.6, and (16) that

$$|\mathcal{B}[D]| \le \max\{d_{G[D]}(v_1),\ d_{G[D]}(v_2)\} \le \frac{n}{2} + \sqrt{2\delta}n + 9\sqrt{\delta}tn \le \frac{n}{2} + 11\sqrt{\delta}tn.$$

This proves Claim 5.7. ∎

For $i \in \{0,1,2,3\}$ let

$$\mathcal{B}_i := \left\{E \in \mathcal{B} \colon |E \cap \overline{D}| = i\right\}.$$

Since $\mathcal{B}_3$ is C_k^3-free, it follows from the definition of C that

$$|\mathcal{B}_3| \le C|\overline{D}|^2. \tag{17}$$

Claim 5.8. *We have $|\mathcal{B}_2| \le n|\overline{D}|/3$.*

Proof. Let us partition $\mathcal{B}_2$ into two sets $\mathcal{B}_2'$ and $\mathcal{B}_2''$, where

$$\mathcal{B}_2' := \{e \in \mathcal{B}_2 \colon e \cap D \neq \emptyset\} \quad \text{and} \quad \mathcal{B}_2'' := \mathcal{B} \setminus \mathcal{B}_2.$$

Since every edge in $\mathcal{B}_2''$ contains exactly one vertex in L, we have

$$|\mathcal{B}_2''| \le |L|\binom{|\overline{D}|}{2} \le t|\overline{D}|^2. \tag{18}$$

Let

$$F := \binom{\overline{D}}{2} \cap \partial\mathcal{B}_2', \quad F_1 := \left\{e \in F \colon d_{\mathcal{B}_2'}(e) = 1\right\}, \quad \text{and} \quad F_2 := F \setminus F_1.$$

It is clear that F_1 contributes at most $\binom{|\overline{D}|}{2}$ edges to $\mathcal{B}_2'$. On the other hand, by Claim 5.4 (i), the graph F_2 is a matching. Combined with Claim 5.5, we obtain

$$|\mathcal{B}_2'| \leq \binom{|\overline{D}|}{2} + \left(\max\{|D_1|,\ |D_2|\} + \sqrt{3\delta n}\right)|F_2| \leq |\overline{D}|^2 + \left(\frac{n}{2} + 5\sqrt{\delta}n\right)\frac{|\overline{D}|}{2}.$$

Therefore, by Claim 5.3,

$$\begin{aligned}|\mathcal{B}_2| = |\mathcal{B}_2''| + |\mathcal{B}_2'| &\leq t|\overline{D}|^2 + |\overline{D}|^2 + \left(\frac{n}{2} + 5\sqrt{\delta}n\right)\frac{|\overline{D}|}{2} \\ &\leq \left(t \times 6\delta tn + 6\delta tn + \frac{n}{4} + \frac{5}{2}\sqrt{\delta}n\right)|\overline{D}| \leq \frac{n|\overline{D}|}{3}.\end{aligned}$$

This proves Claim 5.8. ∎

Claim 5.9. *We have*

$$|\mathcal{B}_1| \leq 10\sqrt{\delta}t^2 n|\overline{D}| + \begin{cases} 0, & \text{if } \mathcal{B}[D] \neq \emptyset, \\ \frac{n}{2} + 3\sqrt{\delta}n, & \text{if } \mathcal{B}[D] = \emptyset. \end{cases}$$

In particular,

$$|\mathcal{B}_1| + |\mathcal{B}[D]| \leq 10\sqrt{\delta}t^2 n|\overline{D}| + \frac{n}{2} + 11\sqrt{\delta}tn.$$

Proof. Let us partition $\mathcal{B}_1$ into two subsets $\mathcal{B}_1'$ and $\mathcal{B}_1''$, where

$$\mathcal{B}_1' := \{e \in \mathcal{B}_1 \colon e \cap L \neq \emptyset\} \quad \text{and} \quad \mathcal{B}_1'' := \mathcal{B}_1 \setminus \mathcal{B}_1'.$$

It follows from Claim 5.6 that

$$|\mathcal{B}_1'| \leq |\overline{D}| \times 9\sqrt{\delta}tn \times t = 9\sqrt{\delta}t^2 n|\overline{D}|.$$

Now we consider $\mathcal{B}_1''$. First, suppose that $\mathcal{B}[D] \neq \emptyset$. If there exists a vertex $v \in \overline{D}$ having degree at least four in $\mathcal{B}_1''$, then we can choose $e_2' \in \binom{D}{2}$ such that $|e_2' \cap e_1| \leq 1$. However, edges e_1 and $e_2 := e_2' \cup \{v\} \in \mathcal{B}_1''$ contradict Claim 5.4. Therefore, every vertex in $\overline{D}$ has degree at most three in $\mathcal{B}_1''$. Hence, $|\mathcal{B}_1''| \leq 3|\overline{D}|$, and

$$|\mathcal{B}| = |\mathcal{B}_1'| + |\mathcal{B}_1''| \leq 9\sqrt{\delta}t^2 n|\overline{D}| + 3|\overline{D}| \leq 10\sqrt{\delta}t^2 n|\overline{D}|.$$

Here we used the fact that $\sqrt{\delta}t^2 n \gg 1$.

Next, suppose that $\mathcal{B}[D] = \emptyset$. Let $v \in \overline{D}$. Observe that the link graph $L_{\mathcal{B}_1''}(v)$ is a subgraph of $G[D] \subseteq G[V']$. Claim 5.4 (ii) implies that the matching number of $L_{\mathcal{B}_1''}(v)$ is at most one. By Fact 2.4, either $|L_{\mathcal{B}_1''}(v)| \leq 3$ or there exists a vertex $v \in D$ such that all edges in $L_{\mathcal{B}_1''}(v)$ containing v. In the latter case, it follows from Claim 5.5 and (16) that

$$|L_{\mathcal{B}_1''}(v)| \leq d_{G[D]}(v_1) \leq \frac{n}{2} + \sqrt{2\delta}n + \sqrt{3\delta}n \leq \frac{n}{2} + 3\sqrt{\delta}n.$$

Similar to the proof above, it follows from Claims 5.4 that the number of vertices $v \in \overline{D}$ with $|L_{\mathcal{B}_1''}(v)| \geq 4$ is at most one. Therefore,

$$|\mathcal{B}_1''| \leq 3|\overline{D}| + \frac{n}{2} + 3\sqrt{\delta}n.$$

Therefore,

$$|\mathcal{B}_1| = |\mathcal{B}_1'| + |\mathcal{B}_1''| \leq 9\sqrt{\delta}t^2 n|\overline{D}| + 3|\overline{D}| + \frac{n}{2} + 3\sqrt{\delta}n \leq 10\sqrt{\delta}t^2 n|\overline{D}| + \left(\frac{n}{2} + 3\sqrt{\delta}n\right).$$

This proves Claim 5.9. ∎

To bound the size of $\mathcal{B}_0$ we divide each D_i into two further smaller subsets. More specifically, let

$$\tau' := \frac{n}{3}, \quad \text{and} \quad D_i' := \left\{v \in D_i \colon |N_G(v) \cap V_{2-i}| \geq \tau'\right\} \quad \text{for} \quad i \in \{1,2\}.$$

Let $\overline{D'} := D \setminus D'$ and $\overline{D'}_i := D_i \setminus D_i'$ for $i \in \{1,2\}$.

Claim 5.10. *We have $|\overline{D'}_i| \leq 14\delta n$ for $i \in \{1,2\}$, and hence, $|\overline{D'}| \leq 28\delta n$.*

Proof. Let $i \in \{1,2\}$. If $|\overline{D'}_i| > 14\delta n$, then, by (16), the number of edges in $G[V_1, V_2]$ would satisfy

$$|G[V_1,V_2]| \leq |V_1||V_2| - 14\delta n \times \left(\frac{n}{2} - \sqrt{2\delta}n - \frac{n}{3}\right) \leq \frac{n^2}{4} - 2\delta n^2,$$

which contradicts (14). ∎

Claim 5.11. *We have $|G[D_1'] \cup G[D_2']| \leq 1$.*

Proof. Suppose that $\{u,v\}$ is an edge in $G[D_i']$. Then it follows from the definition of D_i' that $\min\{|N_G(u) \cap V_{2-i}|, |N_G(v) \cap V_{2-i}|\} \geq n/3$. Due to (16) and the Inclusion-exclusion principle, we have

$$|V_{2-i} \cap N_G(u) \cap N_G(v)| \geq 2 \times \frac{n}{3} - \left(\frac{n}{2} + \sqrt{2\delta}n\right) \geq \frac{n}{100}.$$

This implies that the codegree of uv in $\mathcal{B}[V']$ is at least $n/100$, since every vertex in $V_{2-i} \cap N_G(u) \cap N_G(v)$ forms a copy of K_3 with $\{u,v\}$.

Suppose to the contrary that there exist two distinct edges $e_1', e_2' \in G[D_1'] \cup G[D_2']$. Then the argument above shows that there exist two edges $e_1, e_2 \in \mathcal{B}[V']$ such that $e_i' \subseteq e_i$ for $i \in \{1,2\}$ and $e_1 \cap e_2 = e_1' \cap e_2'$. If $e_1' \cap e_2' \neq \emptyset$, then e_1 and e_2 would contradict Claim 5.4 (i). If $e_1' \cap e_2' = \emptyset$, then e_1 and e_2 would contradict Claim 5.4 (ii). ∎

Recall that every triple in $\mathcal{M}_2$ has empty intersection with $\overline{D}$.

Claim 5.12. *We have $m_2 \geq tn|\overline{D'}|/7$.*

Proof. It follows from the definition of $\overline{D'}$ that for $i \in \{1,2\}$ every vertex $v \in \overline{D'}_i$ has at most τ' neighbors in D_{2-i}. Since every triple containing v, x_i, and a vertex in $D_{2-i} \setminus N_G(v)$ is a member in $\mathcal{M}_2$, the vertex v contributes at least $t\left(|V_{2-i}| - |\overline{D}| - \tau'\right)$ triples to $\mathcal{M}_2$. So, it follows from (16) and Claim 5.3 that

$$m_2 \geq \sum_{i \in \{1,2\}} t\left(|V_{2-i}| - |\overline{D}| - \tau'\right)|\overline{D'}_i| \geq t\left(\frac{n}{2} - \sqrt{2\delta}n - 6\delta tn - \frac{n}{3}\right)|\overline{D'}| \geq \frac{tn|\overline{D'}|}{7}.$$

This proves Claim 5.12. ∎

Claim 5.13. *We have $|\mathcal{B}_0 \setminus \mathcal{B}[D]| \leq 9\sqrt{\delta}t^2n|\overline{D'}| + t$.*

Proof. It follows from the definition that every triple in $\mathcal{B}_0 \setminus \mathcal{B}[D]$ contains one vertex from L and two vertices from D_i for some $i \in \{1,2\}$. By Claim 5.11, the number of triples in $\mathcal{B}_0 \setminus \mathcal{B}[D]$ that have empty intersection with $\overline{D'}$ is at most t. On the other hand, it follows from Claim 5.6 that the number of triples in $\mathcal{B}_0 \setminus \mathcal{B}[D]$ that have nonempty intersection with $\overline{D'}$ is at most $t \times 9\sqrt{\delta}tn\left(|\overline{D'}_1| + |\overline{D'}_2|\right) = 9\sqrt{\delta}t^2n|\overline{D'}|$. ∎

If $|\overline{D}|+|\overline{D'}|=0$, then it follows from Claim 5.11 and simple calculations that

$$|\mathcal{H}| \le \binom{t}{3}+\binom{t}{2}(n-t)+t|V_1||V_2|+\max\{|V_1|,\ |V_2|\}+t \le |\mathcal{S}^{+}_{\mathrm{bi}}(n,t)|,$$

and equality holds iff $G \cong S^{+}(n,t)$. So we may assume that $|\overline{D}|+|\overline{D'}| \ge 1$. Then it follows from (17), Claims 5.3, 5.8, 5.9, 5.12 5.13 that

$$\begin{aligned}
|\mathcal{H}| &= |\mathcal{S}|+|\mathcal{B}_0\setminus\mathcal{B}[D]|+|\mathcal{B}[D]|+|\mathcal{B}_1|+|\mathcal{B}_2|+|\mathcal{B}_3|-|\mathcal{M}_1|-|\mathcal{M}_2| \\
&\le |\mathcal{S}_{\mathrm{bi}}\left(n,t\right)|+9\sqrt{\delta}t^2n|\overline{D'}|+t+10\sqrt{\delta}t^2n|\overline{D}|+\frac{n}{2}+11\sqrt{\delta}tn+\frac{n|\overline{D}|}{3}+C|\overline{D}|^2 \\
&\quad -\frac{49}{100}n|\overline{D}|-\frac{tn|\overline{D'}|}{7} \\
&\le |\mathcal{S}_{\mathrm{bi}}\left(n,t\right)|+\left\lceil\frac{n-t}{2}\right\rceil+2t+11\sqrt{\delta}tn-\left(\frac{49}{100}n-10\sqrt{\delta}t^2n-\frac{n}{3}-C\times 10\delta tn\right)|\overline{D}| \\
&\quad -\left(\frac{tn}{7}-9\sqrt{\delta}t^2n\right)|\overline{D'}| \\
&\le |\mathcal{S}^{+}_{\mathrm{bi}}\left(n,t\right)|+t+11\sqrt{\delta}tn-\frac{n}{10}\left(|\overline{D}|+|\overline{D'}|\right) < |\mathcal{S}^{+}_{\mathrm{bi}}\left(n,t\right)|
\end{aligned}$$

a contradiction. This completes the proof of Theorem 1.9 for even k. ∎

6 Concluding remarks

• It is a natural open question to extend theorems concerning trees in the present paper to other classes of trees. In particular, for trees T satisfying $\tau(T)=\sigma(T)$ and containing critical edges. The main (and probably the only) barrier would be to extend stability theorems (Theorem 1.3 and 1.10) to these trees.

• Recall that one of the key ingredients in proofs of generalized Turán theorems is Kruskal–Katona type theorems (Propositions 2.14, 2.19) for F-free hypergraphs. It would be interesting to find applications of other Kruskal–Katona type theorems (see e.g. [33, 27, 41, 18, 19, 35, 36, 37, 21]) in generalized Turán problems.

• By utilizing a similar framework as presented in this paper, along with methods and results of Füredi [15], Füredi–Jiang [16, 17], and Füredi–Jiang–Seiver [20], theorems in Sections 1.2 and 1.3 can be extend to all values of $r \ge 4$. We plan to address this in future work [38].

• Given an r-graph and an integer $i \in [r-2]$, the i-th shadow $\partial_i\mathcal{H}$ of $\mathcal{H}$ is

$$\partial_i\mathcal{H} := \left\{A \in \binom{V(\mathcal{H})}{r-i} : \text{there exists } E \in \mathcal{H} \text{ such that } A \subseteq E\right\}.$$

In general, one could consider the following generalized Turán problem.

Fix integers $r \ge 3$, $i \in [r-2]$, and an r-graph F.

What is the maximum number of edges in an n-vertex r-graph $\mathcal{H}$ with $\partial_i F \not\subseteq \partial_i\mathcal{H}$?

Note that results in Section 1.3 answered this question when $(r,i)=(3,1)$ and F is the expansion of a cycle or certain trees. We hope that the framework used in these proofs could find more applications in this direction.

• By applying the following result, which is a slight modification of Lemma 5.3 in [31] (and the proof is the same), it becomes straightforward to extend the theorems concerning trees presented in this paper to encompass (some special class of) forests as well. One of the simplest examples would be a matching (see e.g. [8, 3, 26, 11, 42, 13, 34, 24, 45, 49]).

Proposition 6.1. *Let F be a k-vertex forest and $T_1, \dots, T_\ell$ are connected components of F. Suppose that (I_i, R_i) is a crosscut pair of T_i for $i \in [\ell]$. There exists a k-vertex tree T such that $F \subseteq T$, $\sigma(T) = \sigma(F)$, and $\left(\bigcup_{i\in[\ell]} I_i, \bigcup_{i\in[\ell]} R_i\right)$ is a crosscut pair of T. Moreover, if $e \in T_i$ is a critical edge in T_i for some $i \in [\ell]$, then e is also a critical edge in T. In particular,*

- *if T_i satisfies $\sigma(T_i) = \tau_{\text{ind}}(T_i)$ for all $i \in [\ell]$, and*
- *T_j contains a critical edge for some $j \in [\ell]$,*

then T satisfies $\sigma(T) = \tau_{\text{ind}}(T)$ and contains a critical edge.

Using arguments in [23, 5], it seems not hard to extend theorems in the present paper further to a special class of graphs whose connected components are trees and cycles. We leave the details to interested readers.

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A Proof of Proposition 2.6

Recall the following statement of Proposition 2.6.

Proposition A.1. *Suppose that T is a tree and (I, R) is a cross-cut pair of T. Then one of the following holds.*

(i) There exists a vertex $v \in I$ such that all but one vertex in $N_T(v)$ are leaves in T.

(ii) There exists an edge $e \in R$ which is a pendant edge in T.

In particular, if I is maximum, then (i) holds.

Proof. Define a set system $\mathcal{T}$ on $U := V \setminus I$ by letting

$$\mathcal{T} := R \cup \{N_T(v) \colon v \in I\}\,.$$

First we claim that $\mathcal{T}$ is linear, i.e. $|E_1 \cap E_2| \le 1$ for all distinct pairs $E_1, E_2 \in \mathcal{T}$. Indeed, suppose to the contrary that there exists a distinct pair $E_1, E_2 \in \mathcal{T}$ with $|E_1 \cap E_2| \ge 2$. Then clearly, E_1 and E_2 cannot be both contained in R. So, by symmetry, we may assume that $E_1 = N_T(v)$ for some $v \in I$. If $E_2 \in R$, then $R \subseteq E_1$, which implies that the induced subgraph of T on $\{v\} \cup E_2$ is a copy of K_3, a contradiction. If $E_2 = N_T(v')$ for some $v' \in I \setminus \{v\}$, then the induced subgraph of T on $\{v, v', u, u'\}$, where $\{u, u'\} \subseteq N_T(v) \cap N_T(v')$ are arbitrary two distinct vertices, is a copy of C_4, a contradiction. This proves that $\mathcal{T}$ is linear.

Next we claim that $\mathcal{T}$ is cyclic. Suppose to the contrary that there exist distinct vertices $u_1, \ldots, u_\ell \subseteq U$ and distinct edges $E_1, \ldots, E_\ell \in \mathcal{T}$ such that $\{v_i, v_{i+1}\} \subseteq E_i$ for $i \in [\ell]$ (the indices are taken module ℓ). Define

$$I_1 := \{i \in [\ell] \colon E_i \in \{N_T(v) \colon v \in I\}\}\,, \quad \text{and} \quad I_2 := [\ell] \setminus I_1.$$

For every $i \in I_1$ we assume that v_i is the vertex in I such that $E_i = N_T(v_i)$. Then it is easy to see that the induced subgraph of T on $\{v_i \colon i \in I_1\} \cup \bigcup_{j \in I_2} E_j$ contains a cycle of length $2|I_1| + |I_2|$, a contradiction. This proves that $\mathcal{T}$ is cyclic.

Let $\mathcal{P} \subseteq \mathcal{T}$ be a maximal path. Let us assume that $\mathcal{P} = E_1 \cdots E_m$ for some $m \geq 1$. Since $\mathcal{T}$ is linear and acyclic, the maximality of $\mathcal{P}$ implies that all but at most one vertex in E_m have degree one in $\mathcal{T}$, meaning that either (i) or (ii) hold. This proves Proposition 2.6. ∎

B Proof of Proposition 2.9

Recall the following statement of Proposition 2.9.

Proposition B.1. *Let F be a bipartite graph. For every $\delta > 0$ there exists $\varepsilon > 0$ such that for sufficiently large n, every n-vertex $F^{\triangle}$-free graph G with at least $n^2/4 - \varepsilon n^2$ edges can be made bipartite by removing at most δn^2 edges. In particular, every n-vertex $F^{\triangle}$-free graph G satisfies*

$$|G| \leq \frac{n^2}{4} + o(n^2).$$

Proof. Fix a constant $\delta > 0$. Let $\varepsilon > 0$ be sufficiently small and n be sufficiently large. We may assume that $\varepsilon \leq \delta/4$. Let G be an n-vertex $F^{\triangle}$-free graph with at least $n^2/4 - \varepsilon n^2$ edges. Since F is bipartite and the associated 3-graph $\mathcal{K}_G$ is F^3-free, it follows from the well-know Kövari–Sós–Turán Theorem [32] and [30, Proposition 1.1] that $|\mathcal{K}_G| \leq n \cdot \mathrm{ex}(n, F) + (|F| + v(F))\, n^2 \leq n^{3-\alpha}$, where $\alpha > 0$ is some constant depending only on F. This implies that $N(K_3, G) = |\mathcal{K}_G| \leq n^{3-\alpha}$. Therefore, by the Triangle Removal Lemma (see e.g. [43, 1, 10]), G contains a K_3-free subgraph G' with $|G'| \geq |G| - \varepsilon n^2 \geq n^2/4 - 2\varepsilon n^2$. Since ε is sufficiently small, it follows from the Stability theorem of Simonovits [44] that G' contains a bipartite subgraph G'' with $|G''| \geq |G'| - \delta n^2/4 \geq |G| - \delta n^2$. This completes the proof of Proposition 2.9. ∎

C Proof of Lemma 2.11

Proposition C.1. *Let $r \geq 2$ and $i \in [r]$ be integers. Every r-graph $\mathcal{H}$ contains a subgraph $\mathcal{H}'$ satisfying*

$$\Delta_i(\mathcal{H}') \leq 1 \quad \textit{and} \quad |\mathcal{H}'| \geq \frac{|\mathcal{H}|}{\binom{r}{i}\Delta_i(\mathcal{H})}. \tag{19}$$

Proof. Define an auxiliary graph G whose vertex set is $\mathcal{H}$ and a pair $\{e, e'\} \subseteq \mathcal{H}$ is an edge in G iff $|e \cap e'| = i$. Since every edge in $\mathcal{H}$ has exactly $\binom{r}{i}$ subsets of size i and each i-subset is contained in at most $\Delta_i(\mathcal{H})$ edges in $\mathcal{H}$, the maximum degree of G satisfies $\Delta(G) \leq \binom{r}{i}\Delta_i(\mathcal{H})$. A simply greedy argument shows that G contains an independent set of size at least $\frac{|V(G)|}{\Delta(G)} \geq \frac{|\mathcal{H}|}{\binom{r}{i}\Delta_i(\mathcal{H})}$. Since every independent set in G corresponds to a subgraph of $\mathcal{H}$ whose maximum i-degree is at most one, we can choose the largest independent set $\mathcal{H}'$, and it satisfies (19). ∎

Recall the following statement of Lemma 2.11.

Lemma C.2. *Fix integers $k \geq 3$ and $C \geq 1$. Let $T_k \in \mathcal{T}_k$. If $\mathcal{H}$ is an n-vertex T_k^3-free 3-graph with $\Delta_2(\mathcal{H}) \leq C$, then $|\mathcal{H}| \leq 6kCn = o(n^2)$.*

Proof. Suppose to the contrary that there exists an n-vertex T_k^3-free 3-graph $\mathcal{H}$ with $\Delta_2(\mathcal{H}) \le C$ and $|\mathcal{H}| \ge 6kCn$. By Proposition C.1, there exist a linear subgraph $\mathcal{H}' \subset \mathcal{H}$ of size $\frac{|\mathcal{H}|}{3C} \ge 2kn$. By removing vertices one by one, it is easy to see that $\mathcal{H}'$ contains an induced subgraph $\mathcal{H}''$ with minimum degree at least k. A simple greedy argument shows that $T_k^3 \subseteq \mathcal{H}''$, a contradiction. ∎

D Proof of Lemma 2.15

Recall the following statement of Lemma 2.15.

Lemma D.1. *Let T be a fixed tree with $\sigma(T) = \tau_{\mathrm{ind}}(T) =: t+1$ and $k := |T|$. Fix constant $\varepsilon > 0$ and let n be sufficiently large. Let $\mathcal{H}$ be an n-vertex T^3-free 3-graph and $\mathcal{H}_q$ be the outputting 3-graph of Cleaning Algorithm with input $(\mathcal{H}, k, t)$ as defined here. If $|\mathcal{H}| \ge t|\partial\mathcal{H}| - \varepsilon n^2$, then $\mathcal{H}_q$ is $(t, 3k)$-superfull, and moreover,*

$$q \le 12k\varepsilon n^2, \quad |\mathcal{H}_q| \ge |\mathcal{H}| - 48k^2\varepsilon n^2, \quad \text{and} \quad |\partial\mathcal{H}_q| \ge |\partial\mathcal{H}| - 50k^2\varepsilon n^2.$$

Proof. We keep using notations in the Cleaning Algorithm. First, it follows from Lemma 2.11 that $|\mathcal{H}^*| \le \varepsilon n^2$. Hence we have

$$|\mathcal{H}_0| \ge t|\partial\mathcal{H}| - 2\varepsilon n^2.$$

Note from the definition of the Cleaning Algorithm that $|\partial\mathcal{H}_q| \le |\partial\mathcal{H}| - q$. Combined with the inequality above, we obtain

$$|\mathcal{H}_0| = |\mathcal{H}| - |\mathcal{H}^*| \ge t\left(|\partial\mathcal{H}_q| + q\right) - 2\varepsilon n^2. \tag{20}$$

For $i \in \{1, 2, 3\}$ define

$$I_i := \{j \in [q] \colon f_j \text{ is of type-i in } \mathcal{H}_{j-1}\}.$$

It follows from the definition of type-3 and Lemma 2.12 that $|I_3| \le \varepsilon n^2$.

Observe from the definition of types-1 and 2 that if $j \in I_2$, then $j+1 \in I_1$. Therefore, $|I_2| \le |I_1|$, and combined with (20), we obtain

$$\begin{aligned}
|\mathcal{H}_q| \ge |\mathcal{H}_0| - (t-1)|I_1| - t|I_2| - 3k|I_3| &\ge |\mathcal{H}_0| - \left(t - \frac{1}{2}\right)\left(|I_1| + |I_2|\right) - 3k|I_3| \\
&\ge t\left(|\partial\mathcal{H}_q| + q\right) - 2\varepsilon n^2 - \left(t - \frac{1}{2}\right)q - 3k|I_3| \\
&= t|\partial\mathcal{H}_q| + \frac{q}{2} - 5k\varepsilon n^2.
\end{aligned}$$

Combined with Proposition 2.14, we obtain $q/2 - 5k\varepsilon n^2 \le \varepsilon' n^2$, which implies that $q \le 12k\varepsilon n^2$. Hence, it follows from the definition of the Cleaning Algorithm that

$$|\mathcal{H}_q| \ge |\mathcal{H}_0| - 3kq \ge |\mathcal{H}| - 48k^2\varepsilon n^2.$$

Combined with $t|\partial\mathcal{H}_q| \ge |\mathcal{H}_q| - \varepsilon n^2$ (by Proposition 2.14) and $|\mathcal{H}| \ge t|\partial\mathcal{H}| - \varepsilon n^2$ (by assumption), we obtain that $|\partial\mathcal{H}_q| \ge |\partial\mathcal{H}| - 50k^2\varepsilon n^2$. ∎

E Proof of Lemma 2.21

Recall the following statement of Lemma 2.21.

Lemma E.1. *Let $k \geq 4$ and $t = \lfloor (k-1)/2 \rfloor$ be integers. Fix a constant $\varepsilon > 0$ and let n be sufficiently large. Let $\mathcal{H}$ be an n-vertex C_k^3-free 3-graph and $\mathcal{H}_q$ be the outputting 3-graph of the Cleaning Algorithm with input $(\mathcal{H}, k, t)$. If $|\mathcal{H}| \geq t|\partial\mathcal{H}| - \varepsilon n^2$, then $\mathcal{H}_q$ is $(t, 3k)$-superfull, and moreover,*

$$q \leq 12k\varepsilon n^2, \quad |\mathcal{H}_q| \geq |\mathcal{H}| - 48k^2\varepsilon n^2, \quad \text{and} \quad |\partial\mathcal{H}_q| \geq |\partial\mathcal{H}| - 50k^2\varepsilon n^2.$$

Proof of Lemma E.1. We keep using notations in the Cleaning Algorithm. First, it follows from Lemma 2.17 that $|\mathcal{H}^*| \leq \varepsilon n^2$. Hence we have

$$|\mathcal{H}_0| \geq t|\partial\mathcal{H}| - 2\varepsilon n^2.$$

Note from the definition of the Cleaning Algorithm that $|\partial\mathcal{H}_q| \leq |\partial\mathcal{H}| - q$. Combined with the inequality above, we obtain

$$|\mathcal{H}_0| \geq t\left(|\partial\mathcal{H}_q| + q\right) - 2\varepsilon n^2. \tag{21}$$

For $i \in \{1, 2, 3\}$ define

$$I_i := \{j \in [q] \colon f_j \text{ is of type-i in } \mathcal{H}_{j-1}\}.$$

If k is even, then it follows from the definition of type-3 and Lemma 2.20 (i) that $|I_3| \leq \varepsilon n^2$. Suppose that k is odd. Note that for every $i \in I_3$, the pair e_i is contained in an edge $\widehat{e} \in \mathcal{H}_i$ with the property that every pair of vertices in $\widehat{e}$ has codegree at least $\ell \geq 2$. Since $d_{\mathcal{H}}(e_i) \geq d_{\mathcal{H}_i}(e_i) \geq \ell + 1$ and every edge in $\mathcal{H}$ contains a pair of vertices with codegree at least $3k+1$ in $\mathcal{H}$, it follows from Lemma 2.20 (ii) that $|I_3| \leq \varepsilon n^2$.

Observe from the definition of types-1 and 2 that if $j \in I_2$, then $j+1 \in I_1$. Therefore, $|I_2| \leq |I_1|$, and combined with (21), we obtain

$$\begin{aligned} |\mathcal{H}_q| \geq |\mathcal{H}_0| - (t-1)|I_1| - t|I_2| - 3k|I_3| &\geq |\mathcal{H}_0| - \left(t - \frac{1}{2}\right)\left(|I_1| + |I_2|\right) - 3k|I_3| \\ &\geq t\left(|\partial\mathcal{H}_q| + q\right) - 2\varepsilon n^2 - \left(t - \frac{1}{2}\right) q - 3k|I_3| \\ &= t|\partial\mathcal{H}_q| + \frac{q}{2} - 5k\varepsilon n^2. \end{aligned}$$

Combined with Proposition 2.19, we obtain $q/2 - 5k\varepsilon n^2 \leq \varepsilon' n^2$, which implies that $q \leq 12k\varepsilon n^2$. Hence, it follows from the definition of the Cleaning Algorithm that

$$|\mathcal{H}_q| \geq |\mathcal{H}_0| - 3kq \geq |\mathcal{H}| - 48k^2\varepsilon n^2.$$

Combined with $t|\partial\mathcal{H}_q| \geq |\mathcal{H}_q| - \varepsilon n^2$ (by Proposition 2.19) and $|\mathcal{H}| \geq t|\partial\mathcal{H}| - \varepsilon n^2$ (by assumption), we obtain that $|\partial\mathcal{H}_q| \geq |\partial\mathcal{H}| - 50k^2\varepsilon n^2$. ∎

F Proof of Lemma 2.22

Proposition F.1. *Let $\mathcal{H}$ be a 3-graph and $t \geq 1$ is an integer.*

(i) It there exists a cycle $w_1 \cdots w_t \subseteq \partial\mathcal{H}$ of length t and distinct vertices $v_1, \ldots, v_t \in V(\mathcal{H}) \setminus \{w_1, \ldots, w_t\}$ such that

$$\{w_i, v_{i+1}, w_{i+1}\} \in \mathcal{H}, \quad \text{and} \quad \min\{d_{\mathcal{H}}(w_i, v_{i+1}),\ d_{\mathcal{H}}(v_{i+1}, w_{i+1})\} \geq 6t + 6$$

for $i \in [t]$, where the indices are taken mod t. Then $C_\ell^3 \subseteq \mathcal{H}$ for all integers $\ell \in [t, 2t]$.

(ii) It there exists a path $w_1 \cdots w_{t+1} \subseteq \partial\mathcal{H}$ of length t and distinct vertices $v_1, \ldots, v_t \in V(\mathcal{H}) \setminus \{w_1, \ldots, w_t\}$ such that

$$\{w_i, v_{i+1}, w_{i+1}\} \in \mathcal{H}, \quad \text{and} \quad \min\{d_{\mathcal{H}}(w_i, v_{i+1}),\ d_{\mathcal{H}}(v_{i+1}, w_{i+1})\} \geq 6t + 6$$

for $i \in [t]$. Then $P_\ell^3 \subseteq \mathcal{H}$ for all integers $\ell \in [t, 2t]$.

Remark. In fact, $6t + 6$ in the assumptions above can be replaced by $4t$.

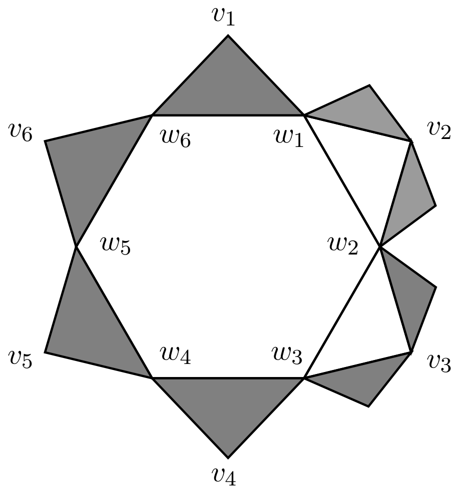

Figure 9: Varying the length of a cycle.

Proof of Proposition F.1. The proofs for both statements are essentially the same, so we will prove Statement (i) only.

Let $w_1, \ldots, w_t, v_1, \ldots, v_t$ be vertices in $V(\mathcal{H})$ satisfying the assumptions in Statement (i). Fix $\ell \in [t, 2t]$ and let $k := \ell - t$. Observe that $w_1 v_2 w_2 \cdots w_k v_{k+1} w_{k+1} w_{k+2} \cdots w_t w_1$ is a cycle of length ℓ in $\partial\mathcal{H}$ and it satisfies the assumptions in Fact 2.2. Therefore, $C_\ell^3 \subseteq \mathcal{H}$ (see Figure 9). ∎

Recall the following statement of Lemma 2.22.

Lemma F.2. *Let $k \geq 4$, $t = \lfloor (k-1)/2 \rfloor$, and $\mathcal{H}$ be a $(t, 3k)$-superfull 3-graph containing two disjoint sets $W_1, W_2 \subseteq V(\mathcal{H})$ of size at least $3k$ such that every pair of vertices $(w_1, w_2) \in W_1 \times W_2$ has codegree exactly t in $\mathcal{H}$. If $C_k^3 \not\subseteq \mathcal{H}$, then there exists a t-set $L \subseteq V(\mathcal{H}) \setminus (W_1 \cup W_2)$ such that $N_{\mathcal{H}}(w_1 w_2) = L$ for all $(w_1, w_2) \in W_1 \times W_2$.*

Proof. Let $V' := V(\mathcal{H}) \setminus (W_1 \cup W_2)$. Let $(w_1, w_2) \in W_1 \times W_2$. First, we prove that $N_{\mathcal{H}}(w_1 w_2) \subseteq V(\mathcal{H}) \setminus W$. Suppose to the contrary that there exists $w \in W_1 \cup W_2$ such that $w_1 w_2 w \in \mathcal{H}$. By symmetry, we may assume that $w \in W_2$. Since $\mathcal{H}$ is t-superfull and $d_{\mathcal{H}}(w_1 w_2) = t$, it follows from the definition that $d_{\mathcal{H}}(w_1 w) \geq 3k$, which contradicts the assumption that every pair of vertices $(w_1', w_2') \in W_1 \times W_2$ has codegree exactly t in $\mathcal{H}$. Therefore, $N_{\mathcal{H}}(w_1w_2) \subseteq V(\mathcal{H}) \setminus W$.

Next we prove that there exists a t-set $L \subseteq V(\mathcal{H}) \setminus (W_1 \cup W_2)$ such that $N_{\mathcal{H}}(w_1w_2) = L$ for all $(w_1, w_2) \in W_1 \times W_2$. Suppose to the contrary that there exist two distinct pairs $(w_1, w_2), (w_1', w_2') \in W_1 \times W_2$ such that $N_{\mathcal{H}}(w_1w_2) \neq N_{\mathcal{H}}(w_1'w_2')$. Since $\min\{|W_1|, |W_2|\} \geq 3k$, there exists $e_1 \in W_1 \times W_2$ sharing exactly one vertex with (w_1, w_2) and (w_1', w_2') respectively, and morover, $N_{\mathcal{H}}(e_1) \neq N_{\mathcal{H}}(w_1w_2)$ or $N_{\mathcal{H}}(e_1) \neq N_{\mathcal{H}}(w_1'w_2')$. By symmetry, we may assume that $N_{\mathcal{H}}(e_1) \neq N_{\mathcal{H}}(w_1w_2)$. Let $e_2 := \{w_1, w_2\}$.

Suppose that $t \geq 2$ and t is odd. Assume that $e_1 = \{w_1, w_{t+1}\}$. Choose distinct vertices $w_{2i+1} \in W_1 \setminus \{w_1\}$ for $1 \leq i \leq (t-1)/2$ and choose distinct vertices $w_{2i+2} \in W_2 \setminus \{w_2, w_{t+1}\}$ for $1 \leq i \leq (t-3)/2$. Let $e_i := \{e_{i-1}, e_i\}$ for $3 \leq i \leq t+1$. It is clear that $e_1e_2 \ldots e_{t+1}$ is a cycle of length $t+1$. Define an auxiliary bipartite graph $A[V_1, V_2]$, where $V_1 := \{e_1, \ldots, e_{t+1}\}$ and $V_2 := \bigcup_{i \in [t]} N_{\mathcal{H}}(e_i)$ and a pair $(e_i, v) \in V_1 \times V_2$ is an edge (in A) iff $e_i \cup \{v\} \in \mathcal{H}$. Since $|N_{\mathcal{H}}(e_i)| = t$ and $\left|\bigcup_{i \in [t]} N_{\mathcal{H}}(e_i)\right| \geq t+1$ (as $N_{\mathcal{H}}(e_1) \neq N_{\mathcal{H}}(e_2)$), Hall's condition is satisfied. Hence, A contains a matching of size $t+1$, which implies that there exist $t+1$ distinct vertices $v_1, \ldots, v_{t+1} \in V_2 \subseteq V'$ such that $e_i \cup \{v_i\} \in \mathcal{H}$ for all $i \in [t+1]$. Since $\mathcal{H}$ is $(t, 3k)$-superfull, $\min\{d_{\mathcal{H}}(w_{i-1}v_i),\ d_{\mathcal{H}}(w_iv_i)\} \geq 3k$ for all $i \in [t+1]$. Therefore, by Proposition F.1 (i), $C^3_{2t+1} \subseteq \mathcal{H}$ and $C^3_{2t+2} \subseteq \mathcal{H}$.

Suppose that $t \geq 2$ and t is even. Assume that $e_1 = \{w_1, w_{t+2}\}$. Choose distinct vertices $w_{2i+1} \in W_1 \setminus \{w_1\}$ for $1 \leq i \leq t/2$ and choose distinct vertices $w_{2i+2} \in W_2 \setminus \{w_2, w_{t+2}\}$ for $1 \leq i \leq (t-2)/2$. Let $e_i := \{e_{i-1}, e_i\}$ for $3 \leq i \leq t+2$. It is clear that $e_1e_2 \ldots e_{t+2}$ is a cycle of length $t+2$. Consider the first $t+1$ edges. Similar to the argument above, if follows from Hall's theorem that there exist distinct vertices $v_1, \ldots, v_{t+1} \subseteq V'$ such that $e_i \cup \{v_i\} \in \mathcal{H}$ for all $i \in [t+1]$. If there exists a vertex $v_{t+2} \in V' \setminus \{v_1, \ldots, v_{t+1}\}$ such that $e_{t+2} \cup \{v_{t+2}\} \in \mathcal{H}$, then it follows from Proposition F.1 (i) that $C^3_{2t+1} \subseteq \mathcal{H}$ and $C^3_{2t+2} \subseteq \mathcal{H}$. So we may assume that $N_{\mathcal{H}}(e_{t+2}) \subseteq \{v_1, \ldots, v_{t+1}\}$. Since $|N_{\mathcal{H}}(e_{t+2})| = t$, we have $N_{\mathcal{H}}(e_{t+2}) \cap \{v_1, v_{t+1}\} \neq \emptyset$. By symmetry, we may assume that $v_{t+1} \in N_{\mathcal{H}}(e_{t+2})$. Observe that $w_1v_2w_2 \cdots w_tv_{t+1}w_{t+1}v_{t+2}w_1$ is a copy of C_{2t+2} in $\partial\mathcal{H}$. Similarly, it follows from Proposition F.1 (ii) that $C^3_{2t+1} \subseteq \mathcal{H}$ and $C^3_{2t+2} \subseteq \mathcal{H}$.

Suppose that $t = 1$ and $k = 4$. Assume that $e_1 = \{w_1, w_4\}$. Choose a vertex $w_3 \in W_1 \setminus \{w_1\}$. Let $e_3 = \{w_3, w_4\}$ and $e_4 = \{w_4, w_1\}$. Since $N_{\mathcal{H}}(e_1) \neq N_{\mathcal{H}}(e_2)$, there exist distinct vertices $v_1, v_2 \in V'$ such that $e_1 \cup \{v_1\}, e_2 \cup \{v_2\} \in \mathcal{H}$. If there exist distinct vertices $v_3, v_4 \in V \setminus \{v_1, v_2\}$ such that $e_3 \cup \{v_3\}, e_4 \cup \{v_4\} \in \mathcal{H}$, then we are done. If $\{w_2, w_3, v_1\} \in \mathcal{H}$ (or $\{w_3, w_4, v_2\} \in \mathcal{H}$), then $v_1w_1v_2w_2v_1$ (or $v_1w_1v_2w_4v_1$) is a copy of C_4 and it follows from Fact 2.2 that this C_4 can be extended to be a copy of C^3_4 in $\mathcal{H}$. If $\{w_2, w_3, v_2\} \in \mathcal{H}$ and there exists a vertex $v_4 \in V' \setminus \{v_1, v_2\}$ such that $\{w_3, w_4, v_4\} \in \mathcal{H}$, then $w_1v_2w_3w_4w_1$ is a copy of C_4 and it follows from Fact 2.2 that this C_4 can be extended to be a copy of C^3_4 in $\mathcal{H}$. By symmetry, the case $\{w_3, w_4, v_1\} \in \mathcal{H}$ and there exists a vertex $v_3 \in V' \setminus \{v_1, v_2\}$ such that $\{w_2, w_3, v_3\} \in \mathcal{H}$ is also not possible. If $\{w_2, w_3, v_2\} \in \mathcal{H}$ and $\{w_3, w_4, v_1\} \in \mathcal{H}$, then $v_1w_1v_2w_3v_1$ is a copy of C_4 and it follows from Fact 2.2 that this C_4 can be extended to be a copy of C^3_4 in $\mathcal{H}$. This completes the proof of Lemma 2.22. ∎

G Proof of Proposition 4.1

Recall the following statement of Proposition 4.1.

Proposition G.1. *Suppose that T is a tree and F is obtained from T by adding one edge.*

Then $\mathrm{ex}(n, F^3) \leq 3\,(|T|+1)\,n^2$ *holds for sufficiently large* n.

Proof. Let T be a tree and F be a graph obtained from T by adding one edge. Let $k := |F| = |T|+1$. Let n be a sufficiently large integer and $\mathcal{H}$ be an n-vertex 3-graph with $3kn^2$ edges. Let $V := V(\mathcal{H})$ and $\mathcal{H}' \subseteq \mathcal{H}$ be a maximum $3k$-full subgraph. It follows from Lemma 2.18 that $|\mathcal{H}'| \geq |\mathcal{H}| - 3k\binom{n}{2} \geq 3kn^2/2$. We will show that $F^3 \subseteq \mathcal{H}'$.

If F is a forest, then it follows from the theorem of Kostochka–Mubayi–Verstraëte [31] (or Theorem 1.3) that $F^3 \subseteq \mathcal{H}$. So we may assume that F is not a forest. Then F contains a unique cycle of length ℓ for some integer $\ell \leq k$. Let $p := k - \ell$. By sequentially removing pendant edges we obtain a sequence of subgraphs

$$F =: F_0 \supseteq F_1 \cdots \supseteq F_p \cong C_\ell,$$

where $|F_i| = |F| - i$ for $i \in [0, p]$ and F_p is the unique cycle in F. In other words, for $i \in [0, p-1]$ we choose a pendant edge $e_i \in F_i$ and let $F_{i+1} := F_i - e_i$. We will prove by a backward induction on i such that $F_i^3 \subseteq \mathcal{H}'$.

The base case $i = p$ follows from the theorem of Kostochka–Mubayi–Verstraëte [29] on linear cycles. So we may assume that $i \in [p-1]$. Suppose that there exists an embedding $\psi\colon F_i^3 \to \mathcal{H}'$. We want to extend ψ to be an embedding of F_{i-1}^3 to $\mathcal{H}'$. To achieve this, let $e_{i-1} := \{u, v\}$ denote the edge in $F_{i-1} \setminus F_i$ and assume that v is a leaf in F_{i-1}. Observe from the definition that each F_j is connected, so there exists $w \in V(F_i)$ such that uw is an edge in F_i. This implies that $\psi(uw) \in \partial\mathcal{H}'$. Since $\mathcal{H}'$ is $3k$-full, there exists a vertex $v' \in V \setminus \psi(V(F_i^3))$ such that $\{\psi(u), \psi(w), v'\} \in \mathcal{H}'$. In particular, $\{\psi(u), v'\} \in \partial\mathcal{H}'$. Using the $3k$-fullness again, there exists a vertex $v'' \in V \setminus \psi(V(F_i^3))$ such that $\{\psi(u), v', v''\} \in \mathcal{H}'$. Observe that $\psi(F_{i-1}^3) \cup \{\{\psi(u), v', v''\}\}$ is a copy of F_i^3 in $\mathcal{H}'$. This completes the proof for the inductive step, and hence, completes the proof of Proposition 4.1. ∎

H Proof of Theorem 1.10

Recall the following stetement of Theorem 1.10.

Theorem H.1. *Let* T *be a tree with* $\sigma(T) = \tau_{\mathrm{ind}}(T)$. *For every* $\delta > 0$ *there exist* $\varepsilon > 0$ *and* n_0 *such that the following holds for all* $n \geq n_0$. *Suppose that* G *is an* n*-vertex* T^+*-free graph with*

$$N(K_3, G) \geq \left(\frac{\sigma(T) - 1}{4} - \varepsilon\right) n^2.$$

Then G *is* δ*-close to* $S(n, t)$.

Proof of Theorem H.1. Let T be a tree with $\sigma(T) = \tau_{\mathrm{ind}}(T)$. Fix $\delta > 0$. Let $\varepsilon > 0$ be sufficiently small and n be sufficiently large. Let $k := |T|$ and $t := \sigma(T) - 1$. Let G be an n-vertex $T^{\triangle}$-free graph with $N(K_3, G) \geq (t/4 - \varepsilon)\,n^2$. We may assume that every edge in G is contained in some triangle in G, since otherwise we can remove it from G and this does not affect the value of $N(K_3, G)$. Let $\mathcal{H} := \mathcal{K}_G$ and recall that

$$\mathcal{K}_G := \left\{ e \in \binom{V(G)}{3} : G[e] \cong K_3 \right\}.$$

Let $\mathcal{H}_q$ denote the outputting 3-graph of the Cleaning Algorithm with input $(\mathcal{H}, k, t)$ as defined here. Recall from Fact 2.5 that $\mathcal{H}$ is T^3-free. In addition, it follows from Proposition 2.9 that

$$|\mathcal{H}| = N(K_3, G) \geq t\frac{n^2}{4} - \varepsilon n^2 \geq t\left(|G| - \varepsilon n^2\right) - \varepsilon n^2 = t|\partial\mathcal{H}| - \varepsilon_1 n^2, \tag{22}$$

where $\varepsilon_1 := (t+1)\varepsilon$. Therefore, by Lemma 2.15,

$$q \leq 12k\delta_1 n^2, \quad |\mathcal{H}_q| \geq |\mathcal{H}| - 48k^2\delta_1 n^2, \quad \text{and} \quad |\partial\mathcal{H}_q| \geq |\partial\mathcal{H}| - 50k^2\delta_1 n^2. \tag{23}$$

First, observe from Proposition 2.14 that

$$|\partial\mathcal{H}| \geq \frac{|\mathcal{H}| - \varepsilon n^2}{t} \geq \frac{n^2}{4} - 2\varepsilon n^2. \tag{24}$$

Let $\mathcal{H}_q$ be the output 3-graph of the Cleaning Algorithm with input $(\mathcal{H}, k, t)$ as defined here. It follows from (22) and Lemma 2.15 that $\mathcal{H}_q$ is $(t, 3k)$-superfull and

$$q \leq 12k\varepsilon_1 n^2, \quad |\mathcal{H}_q| \geq |\mathcal{H}| - 48k^2\varepsilon_1 n^2, \quad \text{and} \quad |\partial\mathcal{H}_q| \geq |\partial\mathcal{H}| - 50k^2\varepsilon_1 n^2. \tag{25}$$

Define graphs

$$G' := \left\{e \in \partial\mathcal{H}_q \colon d_{\mathcal{H}_q}(e) = t\right\}, \quad \text{and} \quad G'' := \left\{e \in \partial\mathcal{H}_q \colon d_{\mathcal{H}_q}(e) \geq 3k\right\}.$$

Since $\mathcal{H}_q$ is $(t, 3k)$-superfull, we have $G' \cup G'' = \partial\mathcal{H}_q$ and every edge in $\mathcal{H}_q$ contains at least two elements from G''. Further, let $V_1 \cup V_2 = V(G)$ be a bipartition such that the number of edges in G' crossing V_1 and V_2 is maximized.

Claim H.2. *We have $|G''| \leq kn$ and $|G'[V_1, V_2]| \geq (1/4 - \varepsilon_2)\, n^2$.*

Proof. It follows from Fact 2.2 that G'' is T-free, Therefore, $|G''| \leq kn$. In addition, by (24) and the third inequality in (25), we obtain

$$|G'| = |\partial\mathcal{H}_q| - |G''| \geq \frac{n^2}{4} - 2\varepsilon n^2 - 50k^2\varepsilon_1 n^2 - \varepsilon n^2 \geq \frac{n^2}{4} - \frac{\varepsilon_2}{2} n^2.$$

By Proposition 2.9, G contains a bipartite graph G_{bi} such that $|G \setminus G_{\mathrm{bi}}| \leq \varepsilon n^2$. Therefore, G' contains a bipartite graph G'_{bi} with

$$|G'_{\mathrm{bi}}| = |G'| - |G' \setminus G'_{\mathrm{bi}}| \geq |G'| - |G \setminus G_{\mathrm{bi}}| \geq \frac{n^2}{4} - \frac{\varepsilon_2}{2} n^2 - \varepsilon n^2 \geq \frac{n^2}{4} - \varepsilon_2 n^2.$$

This proves Claim H.2. ∎

For convenience, let $G'_{\mathrm{bi}} := G'[V_1, V_2]$. Define a 3-graph $\mathcal{G} \subseteq \mathcal{H}_q$ as follows:

$$\mathcal{G} := \left\{e \in \mathcal{H}_q \colon \left|\binom{e}{2} \cap G'_{\mathrm{bi}}\right| = 1\right\}.$$

Since every pair in G'_{bi} contributes exactly t triples to $\mathcal{G}$, it follows from Claim H.2 that

$$|\mathcal{G}| \geq t|G'_{\mathrm{bi}}| \geq \left(\frac{1}{4} - \varepsilon_2\right) t n^2. \tag{26}$$

For every vertex $v \in V$ define the **light link** of v as

$$\widehat{L}(v) := \left\{e \in G'_{\mathrm{bi}} \colon e \cup \{v\} \in \mathcal{H}_q\right\}.$$

It is clear from the definitions that

$$|\mathcal{G}| = \sum_{v \in V} |\widehat{L}(v)|, \quad \text{and} \quad \widehat{L}(v) \subseteq \binom{N_{G''}(v)}{2} \quad \text{for all} \quad v \in V(G). \tag{27}$$

Let $V := V(\mathcal{H})$, and for every integer $i \geq 0$ define

$$Z_i := \left\{ v \in V : |\widehat{L}(v)| \geq \frac{n^{2-2^{-i}}}{(\log n)^2} \right\}, \quad U_i := V \setminus Z_i, \quad \text{and} \quad \mathcal{G}_i := \{ e \in \mathcal{G} : |e \cap Z_i| = 1 \}.$$

Observe that for all $i \geq 1$,

$$Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_i, \quad U_0 \subseteq U_1 \subseteq \cdots \subseteq U_i, \quad \text{and} \quad \mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots \supseteq \mathcal{G}_i.$$

Claim H.3. *We have $|Z_i| \leq 2kn^{2^{-i-1}} \log n$ and $|\mathcal{G}_i| \geq |\mathcal{G}| - \frac{2(i+1)n^2}{(\log n)^2}$ for all $i \geq 0$. In particular,*

$$\binom{|Z_k|}{t} \leq \sqrt{n} \quad \text{and} \quad |\mathcal{G}_k| \geq \left(\frac{1}{4} - 2\varepsilon_2\right) tn^2.$$

Proof. We prove it by induction on i. For the base case $i = 0$, observe from (27) that for every $v \in Z_0$ we have $d_{G''}(v) \geq \sqrt{|\widehat{L}(v)|} \geq n^{1/2}/\log n$ (note that the worse case is that $\widehat{L}(v)$ is a complete graph on $d_{G''}(v)$ vertices). Therefore, by Claim H.2,

$$2kn \geq 2|G''| \geq \sum_{v \in Z_0} d_{G''}(v) \geq \sum_{v \in Z_0} \sqrt{|\widehat{L}(v)|} \geq |Z_0| \frac{n^{1/2}}{\log n},$$

which implies that $|Z_0| \leq 2kn^{1/2} \log n$. Consequently, the number of edges in G' that have nonempty intersection with Z_0 is at most $|Z_0| n \leq 2kn^{3/2} \log n$. Observe that every edge in $\mathcal{G}$ with at least two vertices in Z_0 must contain an edge in G' with (at least) one endpoint in Z_0. Therefore, the number of edges in $\mathcal{G}$ with at least two vertices in Z_0 is at most $2ktn^{3/2} \log n$ (recall that each edge in G' contributes t triples in $\mathcal{G}$). Therefore,

$$\begin{aligned} |\mathcal{G}_0| \geq |\mathcal{G}| - \sum_{v \in V \setminus Z_0} |\widehat{L}(v)| - 2ktn^{3/2} \log n &\geq |\mathcal{G}| - n \times \frac{n}{(\log n)^2} - 2ktn^{3/2} \log n \\ &\geq |\mathcal{G}| - \frac{2n^2}{(\log n)^2}. \end{aligned}$$

Now suppose that $|Z_i| \leq n^{2^{-i-1}} \log n$ holds for some $i \geq 0$. Repeating the argument above, we obtain

$$2kn \geq 2|G''| \geq \sum_{v \in Z_{i+1}} d_{G''}(v) \geq \sum_{v \in Z_{i+1}} \sqrt{|\widehat{L}(v)|} \geq |Z_{i+1}| \frac{n^{1-2^{-i-2}}}{\log n},$$

which implies that $|Z_{i+1}| \leq 2kn^{2^{-i-2}} \log n$. So it follows from the inductive hypothesis that

$$\begin{aligned} |\mathcal{G}_{i+1}| \geq |\mathcal{G}_i| - \sum_{v \in Z_i \setminus Z_{i+1}} |\widehat{L}(v)| - t|Z_{i+1}|n &\geq |\mathcal{G}_i| - |Z_i| \times \frac{n^{2-2^{-i-1}}}{(\log n)^2} - 2ktn^{1+2^{-i-2}} \log n \\ &\geq |\mathcal{G}_i| - \frac{2n^2}{(\log n)^2} \geq |\mathcal{G}| - \frac{2(i+2)n^2}{(\log n)^2}. \end{aligned}$$

Here, $t|Z_{i+1}|n$ is an upper bound for the number of edges in $\mathcal{G}_i$ with at least two vertices in Z_{i+1}. ∎

Let $\mathcal{G}'_k$ denote the output 3-graph of the Cleaning Algorithm with input $(\mathcal{G}_k, k, t)$ as define here. Since $|\mathcal{G}_k| \ge tn^2/4 - 2\varepsilon_2 tn^2 \ge t|G| - 3\varepsilon_2 tn^2 \ge t|\partial\mathcal{G}_k| - 3\varepsilon_2 tn^2$, it follows from Lemma 2.15 that

$$|\mathcal{G}'_k| \ge \left(\frac{t}{4} - \varepsilon_3\right) n^2.$$

Recall from the definition that every edge in $\mathcal{G}'_k$ contains exactly one vertex in Z_k. Similar to the proof of Claim H.2, the number of pairs in $\binom{U_k}{2}$ that have codegree exactly t in $\mathcal{G}'_k$ is at least $(1/4 - \varepsilon_3)n^2 - |Z_k|n \ge (1/4 - 2\varepsilon_3)n^2$. Therefore, the 3-graph

$$\mathcal{G}''_k := \left\{ e \in \mathcal{G}'_k \colon e \cap \binom{U_k}{2} \text{ has codegree exactly } t \right\}$$

satisfies

$$|\mathcal{G}''_k| \ge \left(\frac{1}{4} - 2\varepsilon_3\right) tn^2. \tag{28}$$

Define

$$\mathcal{Z} := \left\{ T \in \binom{Z_k}{t} \colon |L_{\mathcal{G}''_k}(T)| \ge 4kn \right\},$$

where $L_{\mathcal{G}''_k}(T) := \bigcap_{v\in T} L_{\mathcal{G}''_k}(v)$. For every $T \in \mathcal{Z}$ let $L'(T)$ be a maximum induced subgraph of $L_{\mathcal{G}''_k}(T)$ with minimum degree at least $3k$. By greedily removing vertex with degree less than k we have $|L'(T)| \ge |L_{\mathcal{G}''_k}(T)| - 3kn > 0$.

Let Supp_T denote the set of vertices in the graph $L'(T)$ with positive degree. Using Proposition 2.8, we obtain the following claim.

Claim H.4. *We have* $\mathrm{Supp}_T \cap \mathrm{Supp}_{T'} = \emptyset$ *for all distinct sets* $T, T' \in \mathcal{Z}$.

Since $L'(T)$ is a bipartite graph on Supp_T for all $T \in \mathcal{Z}$ and $\sum_{T\in\mathcal{Z}} |L_{\mathcal{G}''_k}(T)| = |\mathcal{G}''_k|/t$, it follows from (28) that

$$\begin{aligned}
\sum_{T\in\mathcal{Z}} \frac{|\mathrm{Supp}_T|^2}{4} \ge \sum_{T\in\mathcal{Z}} |L'(T)| \ge \sum_{T\in\mathcal{Z}} \left(|L_{\mathcal{G}''_k}(T)| - kn\right) &\ge \left(\frac{1}{4} - 2\varepsilon_3\right) n^2 - kn|Z_k| \\
&\ge \left(\frac{1}{4} - 3\varepsilon_3\right) n^2.
\end{aligned}$$

It follows from Claim 3.3 that $\sum_{T\in\mathcal{Z}} |\mathrm{Supp}_T| \le |U_k| \le n$. Therefore, the inequality above and some simple calculations show that

$$\max\left\{|\mathrm{Supp}_T| \colon T \in \mathcal{Z}\right\} \ge \left(1 - \sqrt{6\varepsilon_3}\right) n.$$

Let $T_* \in \mathcal{Z}$ be the unique t-set with $|\mathrm{Supp}_{T_*}| \ge \left(1 - \sqrt{6\varepsilon_3}\right) n$. Then, by Claim 3.3,

$$\begin{aligned}
|L_{\mathcal{G}''_k}(T_*)| = |\mathcal{G}''_k| - \sum_{T\in\mathcal{Z}-T_*} |L_{\mathcal{G}''_k}(T)| &\ge \left(\frac{1}{4} - 2\varepsilon_3\right) n^2 - \binom{\sqrt{6\varepsilon_3}n}{2} \\
&\ge \left(\frac{1}{4} - \delta\right) n^2.
\end{aligned}$$

Here we used the inequality that $\sum_i x_i^2 \le \left(\sum_i x_i\right)^2$. In other words, the number of edges in $\mathcal{G}''_k \subseteq \mathcal{H}$ with exactly one vertex in T_* is at least $(1/4 - \delta)\, n^2$. This completes the proof of Theorem 1.10. ∎

I Proofs of Theorems 1.5 and 1.6

Recall the following statement of Theorem 1.6.

Theorem I.1. *For every tree T we have*

$$\mathrm{ex}(n, K_3, T^{\triangle}) \leq |\mathcal{S}_{\mathrm{bi}}(n, \sigma(T) - 1)| + o(n^2).$$

Proof of Theorem I.1. It follows immediately from Propositions 2.9 and 2.14. ∎

Recall the following statement of Theorem 1.5.

Theorem I.2. *Suppose that T is a strongly edge-critical tree. Then for sufficiently large n,*

$$\mathrm{ex}(n, K_3, T^{\triangle}) = |\mathcal{S}_{\mathrm{bi}}(n, \sigma(T) - 1)|.$$

Proof of Theorem I.2. Let T be a strongly edge-critical tree. Let $I \cup J = V(T)$ be a partition such that I is a minimum independent vertex cover of T. Let $e_* := \{u_0, v_0\}$ be a pendant critical edge such that $v_0 \in I$ is a leaf. Let $\delta > 0$ be sufficiently small and n be sufficiently large. Let $t := \sigma(T) - 1$, $q := |\mathcal{S}_{\mathrm{bi}}(n, t)|$, and G be an n-vertex $T^{\triangle}$-free graph with $N(K_3, G) = q$. Note that we may assume that every edge in G is contained in some triangle of G, since otherwise we may delete it from G and this does not change the value of $N(K_3, G)$. Recall that our aim is to prove that $G \cong S(n, t)$.

Since $N(K_3, G) = (t/4 - o(1))n^2$, it follows from Theorem 1.10 that there exists a t-set $L := \{x_1, \ldots, x_t\} \subseteq V(G)$ such that

(i) $|G - L| \geq (1/4 - \delta)\, n^2$,

(ii) $N(K_3, G - L) \leq \delta n^2$,

(iii) $G - L$ can be made bipartite by removing at most δn^2 edges, and

(iv) $d_G(v) \geq (1 - \delta)n$ for all $v \in L$.

Let $V := V(G)$, $V' := V - L$, and $V_1 \cup V_2 = V'$ be a bipartition of $V \setminus L$ such that the number of crossing edges between V_1 and V_2 is maximized. Define

$$\mathcal{S} := \left\{ E \in \binom{V}{3} : |E \cap L| \geq 1,\ |E \cap V_1| \leq 1,\ |E \cap V_2| \leq 1 \right\},$$
$$\mathcal{H} := \mathcal{K}_G = \left\{ E \in \binom{V}{3} : G[E] \cong K_3 \right\}, \quad \mathcal{B} := \mathcal{H} \setminus \mathcal{S}, \quad \text{and} \quad \mathcal{M} := \mathcal{S} \setminus \mathcal{H}.$$

It follows from Statements (i) and (iii) above that

$$|G[V_1, V_2]| \geq \frac{n^2}{4} - 2\delta n^2. \tag{29}$$

Combined with Statement (iv), for every $x_i \in L$ the intersection of links $L_{\mathcal{H}}(x_i)$ and $L_{\mathcal{S}'}(x_i)$ satisfies

$$|L_{\mathcal{H}}(x_i) \cap L_{\mathcal{S}'}(x_i)| = |L_{\mathcal{H}}(x_i) \cap G[V_1, V_2]| \geq |G[V_1, V_2]| - \delta n \times n \geq \frac{n^2}{4} - 3\delta n^2.$$

Therefore,

$$|\mathcal{M}| = \sum_{x_i \in L} \left(|L_{\mathcal{S}'}(x_i)| - |L_{\mathcal{H}}(x_i) \cap L_{\mathcal{S}'}(x_i)|\right) \le t\left(|V_1||V_2| - \left(\frac{n^2}{4} - 3\delta n^2\right)\right) \le 3\delta t n^2. \tag{30}$$

Inequality (29) with some simple calculations also imply that

$$\left(\frac{1}{2} - \sqrt{2\delta}\right) n \le |V_i| \le \left(\frac{1}{2} + \sqrt{2\delta}\right) n \quad \text{for} \quad i \in \{1,2\}. \tag{31}$$

Let

$$\tau := 3k, \quad D := \left\{y \in V' \colon d_{\mathcal{H}}(yx_i) \ge \tau \text{ for all } x_i \in L\right\}, \quad \overline{D} := V' \setminus D,$$
$$D_i := D \cap V_i \quad \text{and} \quad \overline{D}_i := V_i \setminus D_i \quad \text{for} \quad i \in \{1,2\}.$$

Claim I.3. *We have $m \ge 49n|\overline{D}|/100$ and $|\overline{D}| \le 6\delta t n$.*

Proof. By the definition of D, for every $i \in \{1,2\}$ and for every vertex $v \in \overline{D}_i$ there exists a vertex $x \in L$ (depending on v) such that

$$|N_{\mathcal{H}}(vx) \cap V_{2-i}| \le \tau.$$

Note that this pair $\{v,x\}$ contributes at least $|V_{2-i}| - \tau$ elements to $\mathcal{M}$. Therefore, it follows from (31) that

$$m \ge \left(\min\{|V_1|,|V_2|\} - \tau\right)\left(|\overline{D}_1| + |\overline{D}_2|\right) \ge \left(\frac{n}{2} - \sqrt{2\delta}n - \tau\right)|\overline{D}| \ge \frac{49}{100} n|\overline{D}|.$$

In addition, we obtain

$$|\overline{D}| \le \frac{2\delta t n^2}{49n/100} \le 6\delta t n.$$

This proves Claim I.3. ∎

Claim I.4. *We have $\mathcal{B}[V'] \setminus \mathcal{B}\left[\overline{D}\right] = \emptyset$.*

Proof. The proof is the same as the proof of Claim 3.5. ∎

Claim I.5. *For every $v \in \overline{D}$ we have*

$$\max\left\{|N_G(v) \cap D_1|, |N_G(v) \cap D_2|\right\} \le \sqrt{3\delta}n.$$

Proof. The proof is the same as the proof of Claim 5.5. ∎

Claim I.6. *For every $i \in \{1,2\}$ and for every $v \in \overline{D}_i$ we have $|N_G(v) \cap D_i| \le \sqrt{3\delta}n$.*

Proof. The proof is the same as the proof of Claim 5.6. ∎

For $i \in \{1,2,3\}$ let

$$\mathcal{B}_i := \left\{E \in \mathcal{B} \colon |E \cap \overline{D}| = i\right\}.$$

Since $\mathcal{B}_3$ is T^3-free, it follows from the definition of C that

$$|\mathcal{B}_3| \le C|\overline{D}|^2. \tag{32}$$

By Claim I.4, every edge in $\mathcal{B}_2$ must contain one vertex in L and two vertices in $\overline{D}$. Therefore,

$$|\mathcal{B}_2| \le t|\overline{D}|^2. \tag{33}$$

By Claim I.4 again, every edge in $\mathcal{B}_1$ must contain one vertex in L, one vertex in D_1, and one vertex in $\overline{D}_{2-i}$ for some $i \in \{1,2\}$. So it follows from Claim I.6 that

$$|\mathcal{B}_1| \le t\sqrt{3\delta}n\left(|\overline{D}_1| + |\overline{D}_2|\right) = \sqrt{3\delta}tn|\overline{D}|. \tag{34}$$

By Claim I.3, (32), (33), and (34), we obtain

$$\begin{aligned}
|\mathcal{L}| &= |\mathcal{S}| + |\mathcal{B}_1| + |\mathcal{B}_2| + |\mathcal{B}_3| + |\mathcal{M}| \\
&\le |\mathcal{S}_{\rm bi}(n,t)| + C|\overline{D}|^2 + t|\overline{D}|^2 + \sqrt{3\delta}tn|\overline{D}| - \frac{49}{100}n|\overline{D}| \\
&= |\mathcal{S}_{\rm bi}(n,t)| - \left(\frac{49}{100}n - C \times 6\delta n - t \times 6\delta n - \sqrt{3\delta}tn\right) \\
&\le |\mathcal{S}_{\rm bi}(n,t)|.
\end{aligned}$$

Nota that equality holds iff $\overline{D} = \emptyset$, which means that $G \cong S(n,t)$. This completes the proof of Theorem I.2. ∎

J Proof of Theorem 1.8

Recall the following statement of Theorem 1.8.

Theorem J.1. *Suppose that $t \ge 1$ is a fixed integer and n is sufficiently large. Then*

$$\mathrm{ex}(n, K_3, P_{2t+2}^{\triangle}) = \left|\mathcal{S}_{\rm bi}^{+}(n,t)\right|.$$

The proof for Theorem J.1 is almost identical to the proof of Theorem 1.9 for $k = 2t+2 \ge 6$. The only difference is the proof for the following claim.

Claim J.2. *The 3-graph $\mathcal{B}[V']$ does not contain two edges e_1 and e_2 such that*

(i) $|e_1 \cap e_2| = 1$, $(e_1 \setminus e_2) \cap D \neq \emptyset$, *and* $(e_2 \setminus e_1) \cap D \neq \emptyset$, *or*

(ii) $\min\{|e_1 \cap D|,\ |e_2 \cap D|\} \ge 2$.

Proof. Suppose to the contrary that there exist two edges $e_1, e_2 \in \mathcal{B}[V']$ such that (i) holds. Let $\{v_0\} := e_1 \cap e_2$ and fix $v_i \in (e_i \cap D) \setminus \{v_0\}$ for $i \in \{1,2\}$. Let $D' := D \setminus (e_1 \cup e_2)$. Choose any t-set $\{u_1, \ldots, u_t\} \subseteq D'$. It follows from the definition of D that the induced bipartite graph of $\partial\mathcal{H}$ on $L \cup D$ is complete. Therefore, $F := v_2x_1u_1x_2u_2\cdots x_tu_t$ is copy of P_{k-2} in the bipartite graph $\partial\mathcal{H}[L, D']$. This P_{k-2} together with $v_1v_0v_2$ (a copy of P_2)

form a copy of P_k in $\partial\mathcal{H}$. Since all edges in F have codegree at least $\tau \geq 3k$ in $\mathcal{H}$, it follows from Fact 2.2 that $P_k^3 \subseteq \mathcal{H}$, a contradiction.

Suppose to the contrary that there exist two disjoint edges $e_1, e_2 \in \mathcal{B}[V']$ such that (ii) holds. Fix a 2-set $\{v_i, v_i'\} \subseteq e_i \cap D$ for $i \in \{1,2\}$. Choose a $(t-1)$-set $\{u_1, \ldots, u_{t-1}\} \subseteq D \setminus (e_1 \cup e_2)$. Similar to the proof above, the graph $F := v_1' v_1 x_1 u_1 x_2 \cdots u_{t-1} x_t v_2 v_2'$ is a copy of P_k in $\partial\mathcal{H}$. Note that all edges but v_1v_1', v_2v_2' in F have codegree at least $\tau \geq 3k$ in $\mathcal{H}$. So it follows from Fact 2.2 that $P_k^3 \subseteq \mathcal{H}$, a contradiction. ∎

K Proof of Theorem 1.9

Recall the following statement of Theorem 1.9.

Theorem K.1. *Suppose that $t \geq 2$ is a fixed integer and n is sufficiently large. Then*

$$\mathrm{ex}(n, K_3, C_{2t+1}^{\triangle}) = \left|\mathcal{S}_{\mathrm{bi}}\left(n,t\right)\right|.$$

Proof of Theorem K.1. Fix $k = 2t+1 \geq 5$. Let $C > 0$ be a constant such that $\mathrm{ex}(N, C_k^3) \leq CN^2$ holds for all integers $N \geq 0$. The existence of such a constant C is guaranteed by the theorem of Kostochka–Mubayi–Verstraëte [29]. Let $0 < \delta \ll C^{-1}$ be sufficiently small and $n \gg C$ be sufficiently large. Let G be an n-vertex $C_k^{\triangle}$-free graph with

$$N(K_3, G) = |\mathcal{S}_{\mathrm{bi}}(n,t)|.$$

We may assume that every edge in G is contained in some triangle of G, since otherwise we can delete it from G and this does not change the value of $N(K_3, G)$. Our aim is to prove that $G \cong S_{\mathrm{bi}}(n,t)$.

Since $N(K_3, G) = tn^2/4 - o(n^2)$ and n is large, it follows from Theorem 1.11 that there exists a t-set $L := \{x_1, \ldots, x_t\} \subseteq V(G)$ such that

(i) $|G - L| \geq n^2/4 - \delta n^2$,

(ii) $N(K_3, G-L) \leq \delta n^2$,

(iii) $G - L$ can be made bipartite by removing at most δn^2 edges, and

(iv) $d_G(v) \geq (1-\delta)n$ for all $v \in L$.

Let $V := V(G)$, $V' := V - L$, and let $V_1 \cup V_2 = V'$ be a bipartition such that the number of edges (in G) crossing V_1 and V_2 is maximized. Define

$$\begin{aligned}
\mathcal{S} &:= \left\{ e \in \binom{V}{3} : |e \cap L| \geq 1,\ |e \cap V_1| \leq 1,\ |e \cap V_2| \leq 1 \right\}, \\
\mathcal{S}' &:= \left\{ e \in \binom{V}{3} : |e \cap L| = |e \cap V_1| = |e \cap V_2| = 1 \right\}, \\
\mathcal{H} &:= \mathcal{K}_G, \quad \mathcal{B} := \mathcal{H} \setminus \mathcal{S}, \quad \text{and} \quad \mathcal{M} := \mathcal{S}' \setminus \mathcal{H}.
\end{aligned}$$

Let $m := |\mathcal{M}|$ and $b := |\mathcal{B}|$. It follows from Statements (i) and (iii) above that

$$|G[V_1, V_2]| \geq \frac{n^2}{4} - 2\delta n^2. \tag{35}$$

Combined with Statement (iv), for every $x_i \in L$ the intersection of links $L_{\mathcal{H}}(x_i)$ and $L_{\mathcal{S}'}(x_i)$ satisfies

$$|L_{\mathcal{H}}(x_i) \cap L_{\mathcal{S}'}(x_i)| = |L_{\mathcal{H}}(x_i) \cap G[V_1, V_2]| \geq |G[V_1, V_2]| - \delta n \times n \geq \frac{n^2}{4} - 3\delta n^2.$$

Therefore,

$$|\mathcal{M}| = \sum_{x_i \in L} \left(|L_{\mathcal{S}'}(x_i)| - |L_{\mathcal{H}}(x_i) \cap L_{\mathcal{S}'}(x_i)|\right) \leq t\left(|V_1||V_2| - \left(\frac{n^2}{4} - 3\delta n^2\right)\right) \leq 3\delta t n^2. \tag{36}$$

Inequality (35) with some simple calculations also imply that

$$\left(\frac{1}{2} - \sqrt{2\delta}\right) n \leq |V_i| \leq \left(\frac{1}{2} + \sqrt{2\delta}\right) n \quad \text{for} \quad i \in \{1, 2\}. \tag{37}$$

Let

$$\tau := \frac{n}{200}, \quad D := \left\{y \in V' \colon d_{\mathcal{H}}(yx_i) \geq \tau \text{ for all } x_i \in L\right\}, \quad \overline{D} := V' \setminus D,$$
$$D_i := D \cap V_i \quad \text{and} \quad \overline{D}_i := V_i \setminus D_i \quad \text{for} \quad i \in \{1, 2\}.$$

We also divide $\mathcal{M}$ further by letting

$$\mathcal{M}_1 := \left\{e \in \mathcal{M} \colon e \cap \overline{D} \neq \emptyset\right\}, \quad \text{and} \quad \mathcal{M}_2 := \mathcal{M} \setminus \mathcal{M}_1.$$

Let $m_1 := |\mathcal{M}_1|$ and $m_2 := |\mathcal{M}_2|$.

Claim K.2. *We have $m_1 \geq 49n|\overline{D}|/100$ and $|\overline{D}| \leq 6\delta t n$.*

Proof. Same as the proof of Claim 5.3. ∎

Claim K.3. *The 3-graph $\mathcal{B}[V']$ does not contain an edge e with $|e \cap D| \geq 2$.*

Proof. Suppose to the contrary that there exists an edge $\{v_1, v_2, v_3\} \in \mathcal{B}[V']$ such that $\{v_1, v_2\} \subset D$. Let $D' := D \setminus \{v_1, v_2, v_3\}$. Choose $t-1$ vertices $u_1, \ldots, u_{t-1} \in D'$. It is easy to see that $v_1 x_1 u_1 x_2 \cdots u_{t-1} x_t v_2 v_1$ is a copy of C_k in $\partial\mathcal{H}$. Since all edges but $v_1 v_2$ in this C_k has codegree at least $\tau \geq 3k$ in $\mathcal{H}$, it follows from Fact 2.2 that $C_k^3 \subseteq \mathcal{H}$, a contradiction. ∎

Claim K.4. *The 3-graph $\mathcal{B}[V']$ does not contain two edges e_1 and e_2 such that*

$$|e_1 \cap e_2| = 1, \quad (e_1 \setminus e_2) \cap D \neq \emptyset, \quad \textit{and} \quad (e_2 \setminus e_1) \cap D \neq \emptyset. \tag{38}$$

Proof. Suppose to the contrary that there exist two edges $e_1, e_2 \in \mathcal{B}[V']$ such that (38) holds. Let $\{v_0\} := e_1 \cap e_2$ and fix $v_i \in (e_i \cap D) \setminus \{v_0\}$ for $i \in \{1, 2\}$. Let $D' := D \setminus (e_1 \cup e_2)$. Since $\tau = n/200 > |\overline{D}| + t + 5$, it follows from the definition of D that there exists a vertex $u_1 \in D'$ such that $\{v_1, u_1, x_1\} \in \mathcal{H}$. Choose any $(t-2)$-set $\{u_2, \ldots, u_{t-1}\} \subseteq D' \setminus \{u_1\}$. It follows from the definition of D again that $F := u_1 x_2 \cdots u_{t_1} x_t v_2$ is a copy of P_{2t-2} in $\partial\mathcal{H}$. This P_{2t-2} together with $u_1 v_1 v_0 v_2$ (a copy of P_3) forms a copy of C_k in $\partial\mathcal{H}$. Since every edge in F has codegree at least $t \geq 3k$ in $\mathcal{H}$, it follows from Fact 2.2 that $C_k^3 \subseteq \mathcal{H}$, a contradiction. ∎

Claim K.5. *For every $v \in \overline{D}$ we have*

$$\max\left\{|N_G(v) \cap D_1|, |N_G(v) \cap D_2|\right\} \leq \sqrt{3\delta} n.$$

Proof. Same as the proof of Claim 5.5. ∎

Claim K.6. *For every $i \in \{1,2\}$ and for every $v \in \overline{D}_i$ we have $|N_G(v) \cap D_i| \le 9\sqrt{\delta} tn$.*

Proof. Same as the proof of Claim 5.6. ∎

For $i \in \{1,2,3\}$ let

$$\mathcal{B}_i := \left\{E \in \mathcal{B} \colon |E \cap \overline{D}| = i\right\}.$$

Since $\mathcal{B}_3$ is C_k^3-free, it follows from the definition of C that

$$|\mathcal{B}_3| \le C|\overline{D}|^2. \tag{39}$$

Claim K.7. *We have $|\mathcal{B}_2| \le n|\overline{D}|/3$.*

Proof. It follows from the proof of Claim 5.8. ∎

Claim K.8. *We have $|\mathcal{B}_1| \le 9\sqrt{\delta} t^2 n|\overline{D}|$.*

Proof. It follows from Claim K.3 that every triple in $\mathcal{B}_1$ contains exactly one vertex from each set in $\{\overline{D}_i, D_{2-i}, L\}$ for some $i \in \{1,2\}$. Then, by Claim K.6, we have

$$|\mathcal{B}_1| \le t \times 9\sqrt{\delta} tn \times \left(|\overline{D}_1| + |\overline{D}_2|\right) = 9\sqrt{\delta} t^2 n|\overline{D}|,$$

which proves Claim K.8. ∎

Let

$$\tau' := \frac{n}{3} \quad \text{and} \quad D_i' := \left\{v \in D_i \colon |N_G(v) \cap V_{2-i}| \ge \tau'\right\} \quad \text{for} \quad i \in \{1,2\}.$$

Let $\overline{D'} := D \setminus D'$ and $\overline{D'}_i := D_i \setminus D_i'$ for $i \in \{1,2\}$.

Claim K.9. *We have $|\overline{D'}| \le 14\delta n$ for $i \in \{1,2\}$.*

Proof. Same as the proof of Claim 5.10. ∎

Claim K.10. *We have $|G[D_1'] \cup G[D_2']| = 0$.*

Proof. It follows from Claim K.3 and the proof of Claim 5.11. ∎

Claim K.11. *We have $|\mathcal{B}_0| \le \sqrt{3\delta} tn|\overline{D'}|$.*

Proof. It follows from Claim K.3 and the proof of Claim 5.13. ∎

Claim K.12. *We have $m_2 \ge tn|\overline{D'}|/7$.*

Proof. Same as the proof of Claim 5.12. ∎

By (39), Claims K.7, K.8, K.11, and K.12, we have

$$\begin{aligned}
|\mathcal{H}| &= |\mathcal{S}_{\mathrm{bi}}(n,t)| + \sum_{i=0}^{4} |\mathcal{B}_i| - |\mathcal{M}| - |\mathcal{M}'| \\
&\leq |\mathcal{S}_{\mathrm{bi}}(n,t)| + \sqrt{3\delta} tnz + 9\sqrt{\delta} t^2 n |\overline{D}| + \frac{n}{3}|\overline{D}| + C|\overline{D}|^2 - \frac{49}{100} n |\overline{D}| - \frac{t}{7} nz \\
&\leq |\mathcal{S}_{\mathrm{bi}}(n,t)| - \left(\frac{tn}{7} - \sqrt{3\delta} tn\right) z - \left(\frac{49}{100} n - 9\sqrt{\delta} t^2 n - \frac{n}{3} - C \times 6\delta tn\right) |\overline{D}| \\
&\leq |\mathcal{S}_{\mathrm{bi}}(n,t)|.
\end{aligned}$$

Note that the equality holds only if $|\overline{D}| = 0$ and $z = 0$. Further (simple) calculations show that equality holds iff $G \cong S(n,t)$, which proves Theorem K.1. ∎