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(*Calculations for Proposition A.1 in ``Density Hajnal-
 Szemeredi theorem for cliques of size four''*)
(*Define the function f(b,x2,...,x10),
which corresponds to \eta(b,x2,...,x10)*)
f[y_, x2_, x3_, x4_, x5_, x6_, x7_, x8_, x9_, x10_] :=
  -3 x2^2/112 - 3 x3^2/56 - 3 x4^2/32 - x5^2/8 - 3/8 (x6 + x7 + x8 + x9 + x10)^2 -
   3/4 ((x2 + x3 + x4 + x5) x10 + (x3 + x4 + x5) x9 + (x4 + x5) x8 + x5 x7) + y/10 x2 +
   2 y/10 x3 + 3 y/10 x4 + 4 y/10 x5 + 5 y/10 x6 + 6 y/10 x7 + 7 y/10 x8 + 8 y/10 x9 + 9 y/10 x10;
(*Find the maxmimum of f(y,0,0,0,0,0,0,0,0,x10).*)
Expand[Simplify[MaxValue[
   {f[y, 0, 0, 0, 0, 0, 0, 0, 0, x10], y ≥ 0 && x10 ≥ 0 && x10 ≤ x && x ≥ 0}, {x10}]]]
(*Find the maxmimum of f(y,x2,0,0,0,0,0,0,x9,0).*)
Expand[Simplify[MaxValue[{f[y, x2, 0, 0, 0, 0, 0, 0, x9, 0],
    y ≥ 0 && x2 ≥ 0 && x9 ≥ 0 && x2 + x9 ≤ x && x ≥ 0}, {x2, x9}]]]
(*Find the maxmimum of f(y,x2,x3,0,0,0,0,x8,0,0).*)
Expand[Simplify[MaxValue[{f[y, x2, x3, 0, 0, 0, 0, x8, 0, 0],
    y ≥ 0 && x2 ≥ 0 && x3 ≥ 0 && x8 ≥ 0 && x2 + x3 + x8 ≤ x && x ≥ 0}, {x2, x3, x8}]]]
(*Find the maxmimum of f(y,x2,x3,x4,0,0,x7,0,0,0).*)
Expand[
 Simplify[MaxValue[{f[y, x2, x3, x4, 0, 0, x7, 0, 0, 0], y ≥ 0 && x2 ≥ 0 && x3 ≥ 0 &&
     x4 ≥ 0 && x7 ≥ 0 && x2 + x3 + x4 + x7 ≤ x && x ≥ 0}, {x2, x3, x4, x7}]]]
(*Find the maxmimum of f(y,x2,x3,x4,x5,x6,0,0,0,0).*)
Expand[Simplify[
  MaxValue[{f[y, x2, x3, x4, x5, x6, 0, 0, 0, 0], y ≥ 0 && x2 ≥ 0 && x3 ≥ 0 && x4 ≥ 0 &&
     x5 ≥ 0 && x6 ≥ 0 && x2 + x3 + x4 + x5 + x6 ≤ x && x ≥ 0}, {x2, x3, x4, x5, x6}]]]
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$$\begin{cases} \frac{27 y^2}{50} & 5 x \geq 6 y \,\&\&\, y \geq 0 \\ -\frac{3 x^2}{8} + \frac{9 x y}{10} & x \geq 0 \,\&\&\, 5 x \leq 6 y \\ -\infty & \text{True} \end{cases}$$

$$\begin{cases} \frac{13 y^2}{25} & 15 x \geq 44 y \,\&\&\, y \geq 0 \\ -\frac{3 x^2}{8} + \frac{4 x y}{5} & x \geq 0 \,\&\&\, 15 x \leq 14 y \\ -\frac{x^2}{40} + \frac{11 x y}{75} + \frac{343 y^2}{1125} & 15 x \geq 14 y \,\&\&\, 15 x \leq 44 y \\ -\infty & \text{True} \end{cases}$$

$$\begin{cases} \frac{91 y^2}{150} & 3 x \geq 14 y \,\&\&\, y \geq 0 \\ -\frac{3 x^2}{8} + \frac{7 x y}{10} & x \geq 0 \,\&\&\, 3 x \leq 2 y \\ -\frac{3 x^2}{64} + \frac{21 x y}{80} + \frac{7 y^2}{48} & 3 x \geq 2 y \,\&\&\, 15 x \leq 26 y \\ -\frac{3 x^2}{176} + \frac{7 x y}{44} + \frac{259 y^2}{1100} & 15 x \geq 26 y \,\&\&\, 3 x \leq 14 y \\ -\infty & \text{True} \end{cases}$$

$$\begin{cases} \frac{19 y^2}{25} & 15 x \geq 92 y \&\& y \geq 0 \\ -\frac{3 x^2}{8} + \frac{3 x y}{5} & x \geq 0 \&\& 5 x \leq 2 y \\ -\frac{3 x^2}{40} + \frac{9 x y}{25} + \frac{6 y^2}{125} & 5 x \geq 2 y \&\& 15 x \leq 16 y \\ -\frac{x^2}{32} + \frac{4 x y}{15} + \frac{22 y^2}{225} & 15 x \geq 16 y \&\& 3 x \leq 8 y \\ -\frac{3 x^2}{208} + \frac{23 x y}{130} + \frac{212 y^2}{975} & 3 x \geq 8 y \&\& 15 x \leq 92 y \\ -\infty & \text{True} \end{cases}$$

$$\begin{cases} \frac{151 y^2}{150} & 5 x \geq 38 y \&\& y \geq 0 \\ -\frac{3 x^2}{8} + \frac{x y}{2} & x \geq 0 \&\& 15 x \leq 2 y \\ -\frac{3 x^2}{32} + \frac{17 x y}{40} + \frac{y^2}{200} & 15 x \geq 2 y \&\& 3 x \leq 2 y \\ -\frac{3 x^2}{64} + \frac{29 x y}{80} + \frac{31 y^2}{1200} & 3 x \geq 2 y \&\& 15 x \leq 26 y \\ -\frac{x^2}{40} + \frac{43 x y}{150} + \frac{103 y^2}{1125} & 15 x \geq 26 y \&\& 15 x \leq 56 y \\ -\frac{3 x^2}{232} + \frac{57 x y}{290} + \frac{113 y^2}{435} & 15 x \geq 56 y \&\& 5 x \leq 38 y \\ -\infty & \text{True} \end{cases}$$

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(*We compare the five piecewise functions. We
  may assume y \neq 0. So Let x-axis be x/y and y-
 axis be max{f}/y^2. We define the following five piecewise functions.*)
```

$$\text{h1}[x\_] := \text{Piecewise}\left[\left\{\left\{-\frac{3x^2}{8}+\frac{9x}{10},\ x \geq 0\ \&\&\ x \leq \frac{6}{5}\right\},\ \left\{\frac{27}{50},\ x \geq \frac{6}{5}\right\}\right\}\right];$$

$$\text{h2}[x\_] := \text{Piecewise}\left[\left\{\left\{-\frac{3x^2}{8}+\frac{4x}{5},\ x \geq 0\ \&\&\ x \leq \frac{14}{15}\right\},\right.\right.$$
$$\left.\left.\left\{-\frac{x^2}{40}+\frac{11x}{75}+\frac{343}{1125},\ x \geq \frac{14}{15}\ \&\&\ x \leq \frac{44}{15}\right\},\ \left\{\frac{13}{25},\ x \geq \frac{44}{15}\right\}\right\}\right];$$

$$\text{h3}[x\_] := \text{Piecewise}\left[\left\{\left\{-\frac{3x^2}{8}+\frac{7x}{10},\ x \geq 0\ \&\&\ x \leq \frac{2}{3}\right\},\ \left\{-\frac{3x^2}{64}+\frac{21x}{80}+\frac{7}{48},\ x \geq \frac{2}{3}\ \&\&\ x \leq \frac{26}{15}\right\},\right.\right.$$
$$\left.\left.\left\{-\frac{3x^2}{176}+\frac{7x}{44}+\frac{259}{1100},\ x \geq \frac{26}{15}\ \&\&\ x \leq \frac{14}{3}\right\},\ \left\{\frac{91}{150},\ x \geq \frac{14}{3}\right\}\right\}\right];$$

$$\text{h4}[x\_] := \text{Piecewise}\left[\left\{\left\{-\frac{3x^2}{8}+\frac{3x}{5},\ x \geq 0\ \&\&\ x \leq \frac{2}{5}\right\},\right.\right.$$
$$\left\{-\frac{3x^2}{40}+\frac{9x}{25}+\frac{6}{125},\ x \geq \frac{2}{5}\ \&\&\ x \leq \frac{16}{15}\right\},\ \left\{-\frac{x^2}{32}+\frac{4x}{15}+\frac{22}{225},\ x \geq \frac{16}{15}\ \&\&\ x \leq \frac{8}{3}\right\},$$
$$\left.\left.\left\{-\frac{3x^2}{208}+\frac{23x}{130}+\frac{212}{975},\ x \geq \frac{8}{3}\ \&\&\ x \leq \frac{92}{15}\right\},\ \left\{\frac{19}{25},\ x \geq \frac{92}{15}\right\}\right\}\right];$$

$$\text{h5}[x\_] := \text{Piecewise}\left[\left\{\left\{-\frac{3x^2}{8}+\frac{x}{2},\ x \geq 0\ \&\&\ x \leq \frac{2}{15}\right\},\ \left\{-\frac{3x^2}{32}+\frac{17x}{40}+\frac{1}{200},\ x \geq \frac{2}{15}\ \&\&\ x \leq \frac{2}{3}\right\},\right.\right.$$
$$\left\{-\frac{3x^2}{64}+\frac{29x}{80}+\frac{31}{1200},\ x \geq \frac{2}{3}\ \&\&\ x \leq \frac{26}{15}\right\},\ \left\{-\frac{x^2}{40}+\frac{43x}{150}+\frac{103}{1125},\ x \geq \frac{26}{15}\ \&\&\ x \leq \frac{56}{15}\right\},$$
$$\left.\left.\left\{-\frac{3x^2}{232}+\frac{57x}{290}+\frac{113}{435},\ x \geq \frac{56}{15}\ \&\&\ x \leq \frac{38}{5}\right\},\ \left\{\frac{151}{150},\ x \geq \frac{38}{5}\right\}\right\}\right];$$

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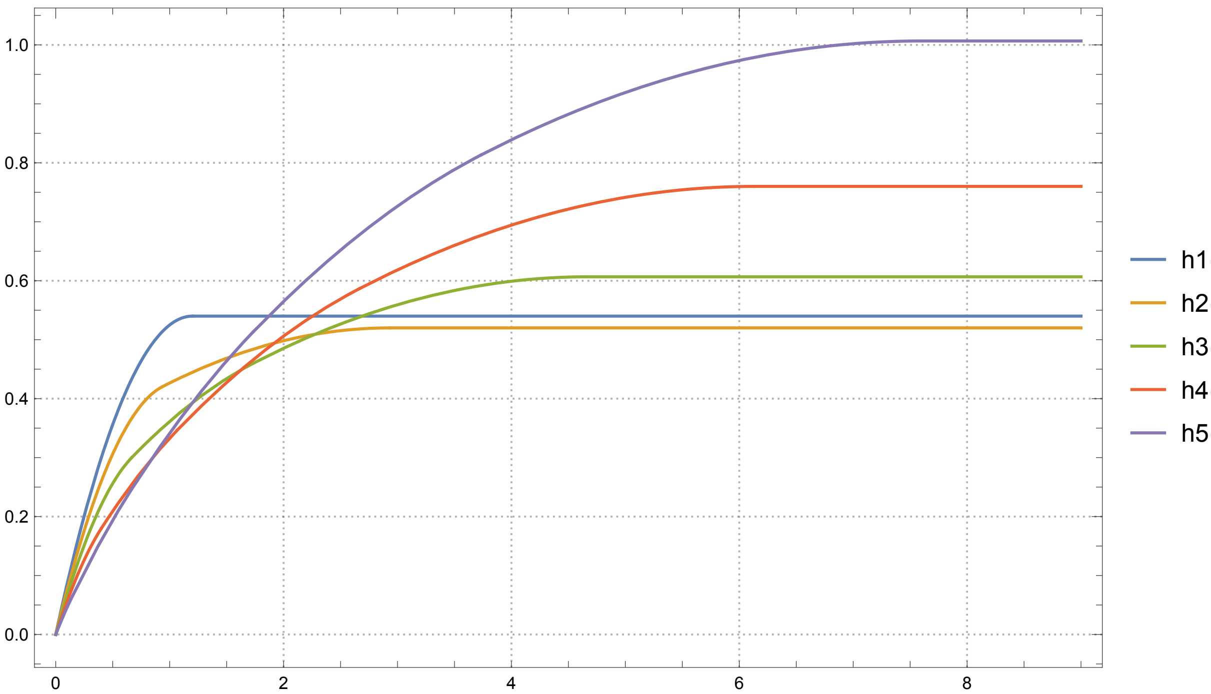
Show[Plot[{h1[x], h2[x], h3[x], h4[x], h5[x]},
  {x, 0, 9}, PlotTheme → "Detailed", ImageSize → Large]]
```



```
Simplify[PiecewiseExpand[Max[{h1[x], h2[x], h3[x], h4[x], h5[x]}], 0 ≤ x ≤ 8]]
```

$$\begin{cases} \frac{27}{50} & \frac{6}{5} < x \leq \frac{86}{15} - 4\sqrt{\frac{14}{15}} \\ \frac{151}{150} & x > \frac{38}{5} \\ -\frac{3}{40}\, x\, (-12 + 5\, x) & x \leq \frac{6}{5} \\ \frac{824 + 2580\, x - 225\, x^2}{9000} & \frac{86}{15} - 4\sqrt{\frac{14}{15}} < x \leq \frac{56}{15} \\ \frac{904 + 684\, x - 45\, x^2}{3480} & \text{True} \end{cases}$$

```
Plot[%17, {x, 0, 8}]
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