ECON 35101 International Macroeconomics and Trade: Comprehension Check 1

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1 Discrete Choice with Gumble

Consumers maximize utility

$$U_i = \ln \frac{y}{p_i} + \nu_i$$

where ν_i satisfies Gumble with PDF $f(\nu_i) = e^{-\nu_i} e^{-e^{-\nu_i}}$ and CDF $F(\nu_i) = e^{-e^{-\nu_i}}$. Given ν_i, the probability of choosing i over other alternatives is given by

$$\begin{aligned}
P_i(\nu_i) &= Pr\left(\ln \frac{y}{p_i} + \nu_i \geq \max_j \{\ln \frac{y}{p_j} + \nu_j\}\right) \\
&= Pr\left(\nu_j < \ln \frac{p_j}{p_i} + \nu_i, \ \forall j\right) \\
&= \prod_{j \neq i} e^{-e^{-\left(\ln \frac{p_j}{p_i} + \nu_i\right)}}
\end{aligned}$$

Integrating over the distribution of ν_i, we have

$$\begin{aligned}
P_i &= \int \prod_{j \neq i} e^{-e^{-\left(\ln \frac{p_j}{p_i} + \nu_i\right)}} e^{-\nu_i} e^{-e^{-\nu_i}} d\nu_i \\
&= \int \prod_{j} e^{-e^{-\left(\ln \frac{p_j}{p_i} + \nu_i\right)}} e^{-\nu_i} d\nu_i \\
&= \int \prod_{j} e^{-\frac{p_i}{p_j} e^{-\nu_i}} e^{-\nu_i} d\nu_i \\
&= \int \exp\left(-\sum_j \frac{p_i}{p_j} e^{-\nu_i}\right) e^{-\nu_i} d\nu_i \\
&= \int \exp\left(-e^{-\nu_i} \sum_j \frac{p_i}{p_j}\right) e^{-\nu_i} d\nu_i
\end{aligned}$$

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Define $s_i = \exp(-\nu_i)$, change of variable implies that

$$\begin{aligned}
P_i &= \int_0^\infty \exp\left(-s_i \sum_j \frac{p_i}{p_j}\right) ds_i \\
&= \frac{\exp\left(-s_i \sum_j \frac{p_i}{p_j}\right)}{-\sum_j \frac{p_i}{p_j}}\Bigg|_0^\infty \\
&= \frac{1}{\sum_j \frac{p_i}{p_j}} \\
&= \frac{p_i^{-1}}{\sum_j p_j^{-1}}
\end{aligned}$$

The demand for good i is thus

$$D_i = P_i \frac{y}{p_i} = \frac{p_i^{-2}}{\sum_j p_j^{-1}} y \tag{1}$$

2 Discrete choice with Frechet

Consumers maximize utility

$$U_i = \varepsilon_i \frac{y}{p_i}$$

where ε_i satisfies Frechet with PDF $f(\varepsilon_i) = \theta T \varepsilon_i^{-\theta-1} e^{-T\varepsilon_i^{-\theta}}$ and CDF $F(\varepsilon_i) = e^{-T\varepsilon_i^{-\theta}}$. Given ε_i, the probability of choosing i over other alternatives is given by

$$\begin{aligned}
P_i(\varepsilon_i) &= Pr\left(\varepsilon_i \frac{y}{p_i} \geq \max_j \{\varepsilon_j \frac{y}{p_j}\}\right) \\
&= Pr\left(\varepsilon_j < \varepsilon_i \frac{p_j}{p_i},\ \forall j\right) \\
&= \prod_{j \neq i} e^{-T\left(\varepsilon_i \frac{p_j}{p_i}\right)^{-\theta}}
\end{aligned}$$

Integrating over the distribution of ε_i, we have

$$\begin{aligned}
P_i &= \int \prod_{j \neq i} e^{-T\left(\varepsilon_i \frac{p_j}{p_i}\right)^{-\theta}} \theta T \varepsilon_i^{-\theta-1} e^{-T\varepsilon_i^{-\theta}} d\varepsilon_i \\
&= \int \prod_j e^{-T\left(\varepsilon_i \frac{p_j}{p_i}\right)^{-\theta}} \theta T \varepsilon_i^{-\theta-1} d\varepsilon_i \\
&= \int \exp\left(-\sum_j T \left(\varepsilon_i \frac{p_j}{p_i}\right)^{-\theta}\right) \theta T \varepsilon_i^{-\theta-1} d\varepsilon_i \\
&= \int \exp\left(-T\varepsilon_i^{-\theta} \sum_j \left(\frac{p_j}{p_i}\right)^{-\theta}\right) \theta T \varepsilon_i^{-\theta-1} d\varepsilon_i
\end{aligned}$$

Define $t_i = T\varepsilon_i^{-\theta}$, change of variable implies that

$$\begin{aligned}
P_i &= \int_0^{\infty} \exp\left(-t_i \sum_j \left(\frac{p_j}{p_i}\right)^{-\theta}\right) dt_i \\
&= \frac{\exp\left(-t_i \sum_j \left(\frac{p_j}{p_i}\right)^{-\theta}\right)}{-\sum_j \left(\frac{p_j}{p_i}\right)^{-\theta}}\Bigg|_0^{\infty} \\
&= \frac{1}{\sum_j \left(\frac{p_j}{p_i}\right)^{-\theta}} \\
&= \frac{p_i^{-\theta}}{\sum_j p_j^{-\theta}}
\end{aligned}$$

The demand for good i is thus

$$D_i = P_i \frac{y}{p_i} = \frac{p_i^{-\theta-1}}{\sum_j p_j^{-\theta}} y \tag{2}$$

3 Constant elasticity of substitution preferences

Consumers maximize utility

$$U = \left(\sum_i q_i^{\rho}\right)^{\frac{1}{\rho}}$$

s.t. $\sum_i p_i q_i \leq y$. The FOC is given by

$$\begin{aligned}
&\left(\sum_i q_i^{\rho}\right)^{\frac{1}{\rho}-1} q_i^{\rho-1} = \lambda p_i \\
\implies &\frac{p_i}{p_j} = \frac{q_i^{\rho-1}}{q_j^{\rho-1}} \\
\implies &\frac{q_i}{q_j} = \left(\frac{p_i}{p_j}\right)^{\frac{1}{\rho-1}} \\
\implies &\frac{p_i q_i}{p_j q_j} = \left(\frac{p_i}{p_j}\right)^{\frac{\rho}{\rho-1}} \\
\implies &\frac{y}{p_j q_j} = (p_j)^{\frac{\rho}{1-\rho}} \sum_i (p_i)^{\frac{\rho}{\rho-1}}
\end{aligned}$$

We obtain the demand function under CES utility as

$$q_j = \frac{(p_j)^{\frac{1}{\rho-1}}}{\sum_i (p_i)^{\frac{\rho}{\rho-1}}} y \tag{3}$$

Note that $\rho < 1$ and q_j is decreasing in p_j. Moreover, we can define the price index to be $P = \left(\sum_i (p_i)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$.

4 Comparisons

We see that (2) and (3) are equivalent when $\theta = \frac{\rho}{1-\rho}$. (1) is a special case of (2) or (3) when $\theta = 1$ or $\rho = \frac{1}{2}$. Note that if we modify the utility function in the case of Gumble distribution to be

$$U_i = \beta \ln \frac{y}{p_i} + \nu_i$$

Then, the demand function is

$$D_i = P_i \frac{y}{p_i} = \frac{p_i^{-\beta-1}}{\sum_j p_j^{-\beta}} y$$

and the three demand systems are equivalent.