Efficient Portfolios*

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Given two random realized returns on an investment, which is to be preferred? This is a fundamental problem in finance that has no definitive solution except in the case one investment always returns more than the other. In 1952 Markowitz(Markowitz 1952) and Roy(Roy 1952) introduced the following criterion for risk vs. return in portfolio selection: if two portfolios have the same expected realized return then prefer the one with smaller variance. An efficient portfolio has the least variance among all portfolios having the same expected realized return.

In the one-period model every efficient portfolio belongs to a two-dimensional subspace of the set of all possible realized returns and is uniquely determined given its expected realized return. We show that if R is the (random) realized return of any efficient portfolio and R_0 and R_1 are the realized returns of any two linearly independent efficient portfolios then

$$R - R_0 = \beta (R_1 - R_0)$$

where $\beta = \text{Cov}(R - R_0, R_1 - R_0) / \text{Var}(R_1 - R_0)$. This generalizes the classical Capital Asset Pricing Model formula for the expected realized return of efficient portfolios. Taking expected values of both sides when $\text{Var}(R_0) = 0$ and R_1 is the "market" portfolio gives

$$E[R] - R_0 = \beta(E[R_1] - R_0)$$

where
$$\beta = \text{Cov}(R, R_1) / \text{Var}(R_1)$$
.

The primary contribution of this short note is observation that the CAPM formula holds for realized returns as random variables, not just their expectations. This follows directly from writing down a mathematical model for one period investments.

^{*}Peter Carr and David Shimko gave insightful feedback to make the exposition more accessible to finance professionals. Any remaining infelicities or omissions are my fault.

One-Period Model

The one-period model is parameterized directly by instrument prices. These have a clear financial interpretation and all other relevant financial quantities can be defined in terms of prices and portfolios.

Let I be the set of market instruments and Ω be the set of possible market outcomes over a single period. The one-period model specifies the initial instrument prices $x \in \mathbb{R}^I$ and the final instrument prices $X : \Omega \to \mathbb{R}^I$ depending on the outcome. We assume, as customary, that there are no cash flows associated with the instruments and transactions are perfectly liquid and divisible. The one period model also specifies a probability measure P on the space of outcomes.

It is common in the literature to write \mathbf{R}^n instead of \mathbf{R}^I where n is the cardinality of the set of instruments I. If $A^B = \{f : B \to A\}$ is the set of functions from B to A then $x \in \mathbf{R}^I$ is a function $x \colon I \to \mathbf{R}$ where $x(i) = x_i \in \mathbf{R}$ is the price of instrument $i \in I$. This avoids circumlocutions such as let $I = \{i_1, \ldots, i_n\}$ be the set of instruments and $x = (x_1, \ldots, x_n)$ be their corresponding prices $x_j = x(i_j), j = 1, \ldots, n$.

A portfolio $\xi \in \mathbf{R}^I$ is the number of shares initially purchased in each instrument. The value of a portfolio ξ given prices x is $\xi \cdot x = \sum_{i \in I} \xi_i x_i$. It is the cost of attaining the portfolio ξ . The realized return is $R(\xi) = \xi \cdot X/\xi \cdot x$ when $\xi \cdot x \neq 0$. Note $R(\xi) = R(t\xi)$ for any non-zero $t \in \mathbf{R}$ so there is no loss in assuming $\xi \cdot x = 1$ when considering returns. In this case $R(\xi) = \xi \cdot X$ is the realized return on the portfolio. It is common in the literature to use returns instead of realized returns where the return r is defined by $R = 1 + r\Delta t$ or $R = \exp(r\Delta t)$ where Δt is the time in years or a day count fraction of the period. Since we are considering a one period model there is no need to drag Δt into consideration.

Although portfolios and prices are both vectors they are not the same. A portfolio turns prices into a value. The function $\xi \mapsto \xi \cdot x$ is a linear functional from prices to values. Mathematically we say $\xi \in (\mathbf{R}^I)^*$, the dual space of \mathbf{R}^I . If V is any vector space its dual space is $V^* = \mathcal{L}(V, \mathbf{R})$ where $\mathcal{L}(V, W)$ is the space of linear transformations from the vector space V to the vector space W. If we write ξ' to denote the linear functional corresponding to ξ then $\xi' x = \xi \cdot x$ is the linear functional applied to x. We also write the dual pairing as $\langle x, \xi \rangle = \xi' x$.

Note that $x\xi'$ is a linear transformation from R^I to R^I defined by $(x\xi')y = x(\xi'y) = (\xi'y)x$ since $\xi'y \in \mathbf{R}$ is a scalar. Matrix multiplication is just composition of linear operators.

Model Arbitrage

There is model arbitrage if there exists a portfolio ξ with $\xi'x < 0$ and $\xi'X(\omega) \ge 0$ for all $\omega \in \Omega$: you make money on the initial investment and never lose money

when unwinding at the end of the period. This definition does not require a measure on Ω .

The one-period Fundamental Theorem of Asset Pricing states there is no model arbitrage if and only if there exists a positive measure Π on Ω with $x=\int_{\Omega}X(\omega)\,d\Pi(\omega)$. We assume X is bounded, as it is in the real world, and Π is a finitely additive measure. The dual space of bounded functions on Ω is the space of finitely additive measures on Ω with the dual pairing $\langle X,\Pi\rangle=\int_{\Omega}X\,d\Pi$ (Dunford and Schwartz 1963) Chapter III.

If $x = \int_{\Omega} X d\Pi$ for a positive measure Π then all portfolios have the same expected realized return $R = 1/\|\Pi\|$ where $\|\Pi\| = \int_{\Omega} 1 d\Pi$ is the mass of Π and the expected value is with respect to the *risk-neutral* probability measure $Q = \Pi/\|\Pi\|$. This follows from $E[\xi'X] = \xi'x/\|\Pi\| = R\xi'x$ for any portfolio ξ .

Note Q is not the probability of anything, it is simply a positive measure with mass 1. The above statements are geometrical, not probabilistic.

Efficient Portfolio

A portfolio ξ is efficient if its variance $\text{Var}(R(\xi)) \leq \text{Var}(R(\eta))$ for every portfolio η having the same expected realized return as ξ .

If $\xi'x=1$ then $\mathrm{Var}(R(\xi))=E[(\xi'X)^2]-(E[\xi'X])^2=E[\xi'XX'\xi]-E[\xi'X]E[X'\xi]=\xi'V\xi$, where $V=\mathrm{Var}(X)=E[XX']-E[X]E[X']$. We can find efficient portfolios using Lagrange multipliers. For a given realized return ρ we can solve

$$\min \frac{1}{2}\xi' V \xi - \lambda(\xi' x - 1) - \mu(\xi' E[X] - \rho)$$

for ξ , λ , and μ . The first order conditions for an extremum are $V\xi - \lambda x - \mu E[X] = 0$, $\xi' x = 1$, and $\xi' E[X] = \rho$.

Riskless Portfolio

A portfolio ζ is *riskless* if its realized return is constant. In this case $0 = \operatorname{Var}(R(\zeta)) = \zeta' V \zeta$ assuming, as we may, $\zeta' x = 1$. If another riskless portfolio exists with different realized return then arbitrage exists. By removing redundant assets we can assume there is exactly one riskless portfolio ζ with $\zeta' x = 1$.

Let $P_{\parallel}=\zeta\zeta'/\zeta'\zeta$. Note $P_{\parallel}\zeta=\zeta$ and $P_{\parallel}\xi=0$ if $\zeta'\xi=0$ so it is the orthogonal projection onto the space spanned by ζ . Let $P_{\perp}=I-P_{\parallel}$ be the projection onto its orthogonal complement, $\{\zeta\}^{\perp}=\{y\in \mathbf{R}^I:\zeta'y=0\}$, so $V=VP_{\perp}+VP_{\parallel}$. Below we analyze the first order conditions for an extremum on each subspace. Note P_{\parallel} commutes with V so these subspaces are invariant under V. Let $y_{\parallel}=P_{\parallel}y$ be the component of y parallel to ζ and $y_{\perp}=P_{\perp}y$ be the component of y orthogonal to ζ for $y\in\mathbf{R}^I$.

The first order condition $V\xi = \lambda x + \mu E[X]$ implies $V\xi_{\parallel} = \lambda x_{\parallel} + \mu E[X]_{\parallel}$. Since ξ_{\parallel} is a scalar multiple of ζ we have $0 = \lambda + \mu R$ so $\lambda = -\mu R$. On the orthogonal complement $V\xi_{\perp} = -\mu Rx_{\perp} + \mu E[X]_{\perp}$ so $\xi_{\perp} = V^{\dashv}(E[X] - Rx)$ where V^{\dashv} is the generalized (Moore-Penrose) inverse of V. Letting $\alpha = \xi_{\perp} = V^{\dashv}(E[X] - Rx)$, every efficient portfolio can be written $\xi = \mu \alpha + \nu \zeta$. We can and do assume $\alpha' x = 1$ so $1 = \mu + \nu$ and $\xi = \mu \alpha + (1 - \mu)\zeta$. Multiplying both sides by X we have $\xi' X = \mu \alpha' X + (1 - \mu)R$ hence

$$R(\xi) - R = \mu(R(\alpha) - R).$$

This implies the classical CAPM formula by taking expected values where α is the "market portfolio". It also shows the Lagrange multiplier $\mu = \text{Cov}(R(\xi), R(\alpha)) / \text{Var}(R(\alpha))$ is the classical beta.

Non-singular Variance

If V is invertible the Appendix shows solution is $\lambda = (C - \rho B)/D$, $\mu = (-B + \rho A)/D$, and

$$\xi = \frac{C - \rho B}{D} V^{-1} x + \frac{-B + \rho A}{D} V^{-1} E[X]$$

where $A = xV^{-1}x$, $B = x'V^{-1}E[X] = E[X']V^{-1}x$, $C = E[X]V^{-1}E[X]$, and $D = AC - B^2$. The variance of the efficient portfolio is

$$Var(R(\xi)) = (C - 2B\rho + A\rho^2)/D.$$

If ξ_0 and ξ_1 are any two independent efficient portfolios then they belong to the subspace spanned by $V^{-1}x$ and $V^{-1}E[X]$. Every efficient portfolio can be written $\xi = \beta_0 \xi_0 + \beta_1 \xi_1$ for some scalars β_0 and β_1 . Assuming $\xi'_j x = 1$ for j = 0, 1 then $R(\xi_j) = \xi'_j X$. Assuming $\xi' x = 1$ so $R(\xi) = \xi' X$ then $\beta_0 + \beta_1 = 1$ and $\xi = (1 - \beta)\xi_0 + \beta\xi_1$ where $\beta = \beta_1$. Multiplying both sides by X we have $\xi' X = (1 - \beta)\xi'_0 X + \beta\xi'_1 X$ hence

$$R(\xi) - R(\xi_0) = \beta(R(\xi_1) - R(\xi_0))$$

as functions on Ω where $\beta = \text{Cov}(R(\xi) - R(\xi_0), R(\xi_1) - R(\xi_0)) / \text{Var}(R(\xi_1) - R(\xi_0))$. The classical CAPM formula follows from taking expected values of both sides when ξ_1 is the "market portfolio" and ξ_0 is a *riskless portfolio*.

Note that A, B, C, and D depend only on x, E[X], and E[XX']. Classical literature focuses mainly on the latter three which may explain why prior authors overlooked our elementary but stronger result.

Appendix

Lagrange Multiplier Solution

Let's find the minimum value of $\operatorname{Var}(R(\xi))$ given $E[R(\xi)] = \rho$. If $\xi' x = 1$ then $R(\xi) = \xi' E[X]$ and $\operatorname{Var}(R(\xi)) = \xi' V \xi$ where V = E[XX'] - E[X]E[X'].

We use Lagrange multipliers and solve

$$\min \frac{1}{2}\xi' V \xi - \lambda(\xi' x - 1) - \mu(\xi' E[X] - \rho)$$

for ξ , λ , and μ .

The first order conditions for an extremum are

$$0 = V\xi - \lambda x - \mu E[X]$$

$$0 = \mathcal{E}'x - 1$$

$$0 = \xi' E[X] - \rho$$

Assuming V is invertible $\xi = V^{-1}(\lambda x + \mu E[X])$. Note every extremum lies in the (at most) two dimensional subspace spanned by $V^{-1}x$ and $V^{-1}E[X]$.

The constraints $1 = x'\xi$ and $\rho = E[X']\xi$ can be written

$$\begin{bmatrix} 1 \\ \rho \end{bmatrix} = \begin{bmatrix} \lambda x' V^{-1} x + \mu x' V^{-1} E[X] \\ \lambda E[X'] V^{-1} x + \mu E[X'] V^{-1} E[X] \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}$$

with $A=xV^{-1}x$, $B=x'V^{-1}E[X]=E[X']V^{-1}x$, and $C=E[X]V^{-1}E[X]$. Inverting gives

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \frac{1}{D} \begin{bmatrix} C & -B \\ -B & A \end{bmatrix} \begin{bmatrix} 1 \\ \rho \end{bmatrix} = \begin{bmatrix} (C - \rho B)/D \\ (-B + \rho A)/D \end{bmatrix}$$

where $D = AC - B^2$. The solution is $\lambda = (C - \rho B)/D$, $\mu = (-B + \rho A)/D$, and

$$\xi = \frac{C - \rho B}{D} V^{-1} x + \frac{-B + \rho A}{D} V^{-1} E[X].$$

A straightforward calculation shows the variance is

$$Var(R(\xi)) = \xi' V \xi = (C - 2B\rho + A\rho^2)/D.$$

Fundamental Theorem of Asset Pricing

The one-period Fundamental Theorem of Asset Pricing states there is no model arbitrage if and only if there exists a positive measure Π on Ω with $x = \int_{\Omega} X(\omega) \, d\Pi(\omega)$. We assume X is bounded, as it is in the real world, and Π is finitely additive.

If such a measure exists and $\xi \cdot X \ge 0$ then $\xi \cdot x = \int_{\Omega} \xi \cdot X \, d\Pi \ge 0$ so arbitrage cannot occur. The other direction is less trivial.

Lemma. If $x \in \mathbb{R}^n$ and C is a closed cone in \mathbb{R}^n with $x \notin C$ then there exists $\xi \in \mathbb{R}^n$ with $\xi \cdot x < 0$ and $\xi \cdot y \ge 0$ for $y \in C$.

Recall that a *cone* is a subset of a vector space closed under addition and multiplication by a positive scalar, that is, $C + C \subseteq C$ and $tC \subseteq C$ for t > 0. The set of arbitrage portfolios is a cone.

Proof. Since C is closed and convex there exists $x^* \in C$ with $0 < ||x^* - x|| \le ||y - x||$ for all $y \in C$. Let $\xi = x^* - x$. For any $y \in C$ and t > 0 we have $ty + x^* \in C$ so $||\xi|| \le ||ty + \xi||$. Simplifying gives $t^2 ||y||^2 + 2t\xi \cdot y \ge 0$. Dividing by t > 0 and letting t decrease to 0 shows $\xi \cdot y \ge 0$. Take $y = x^*$ then $tx^* + x^* \in C$ for $t \ge -1$. By similar reasoning, letting t increase to 0 shows $\xi \cdot x^* \le 0$ so $\xi \cdot x^* = 0$. Now $0 < ||\xi||^2 = \xi \cdot (x^* - x) = -\xi \cdot x$ hence $\xi \cdot x < 0$. ■

Since the set of non-negative finitely additive measures is a closed cone and $X \mapsto \int_{\Omega} X \, d\Pi$ is positive, linear, and continuous $C = \{ \int_{\Omega} X \, d\Pi : \Pi \ge 0 \}$ is also a closed cone. The contrapositive follows from the lemma.

The proof also shows how to find an arbitrage when one exists.

References

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