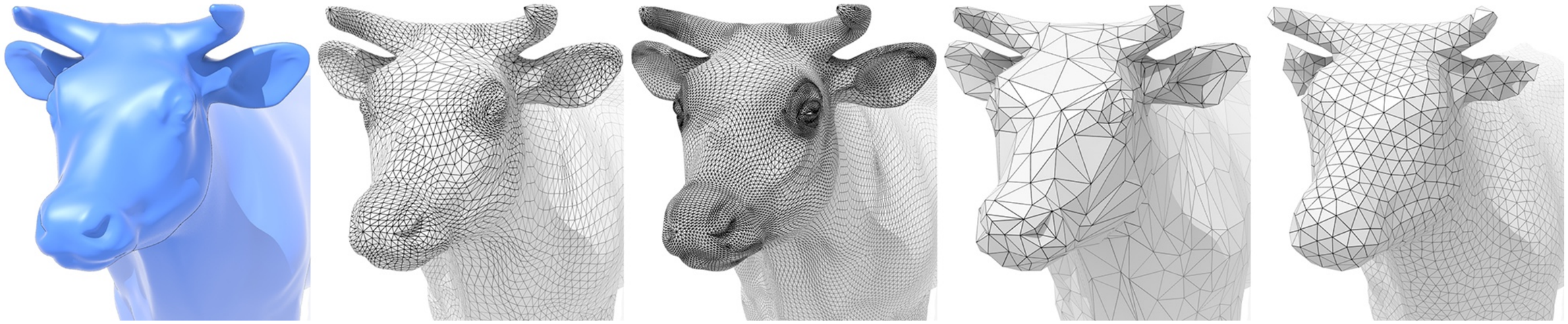
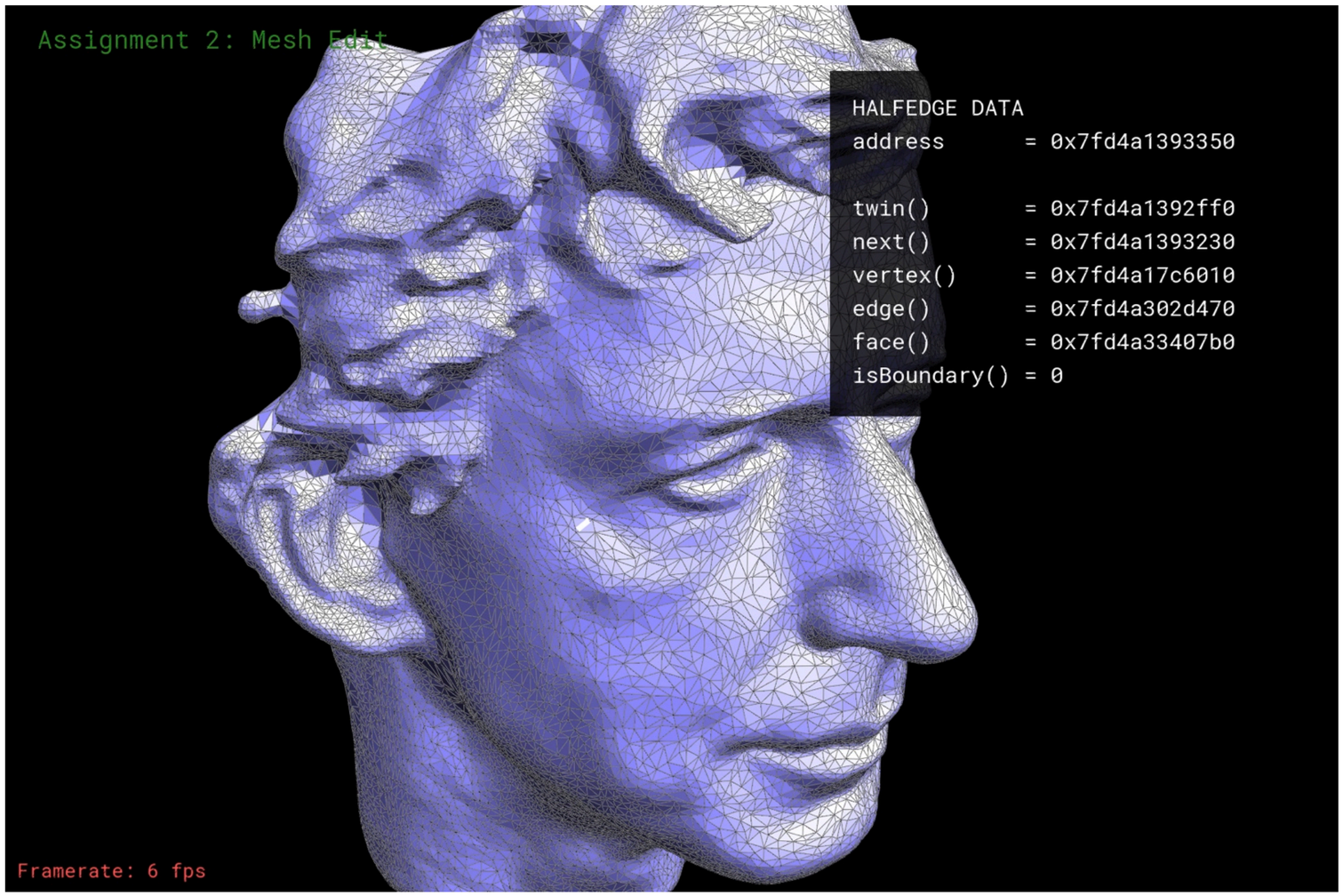
Lecture 7:

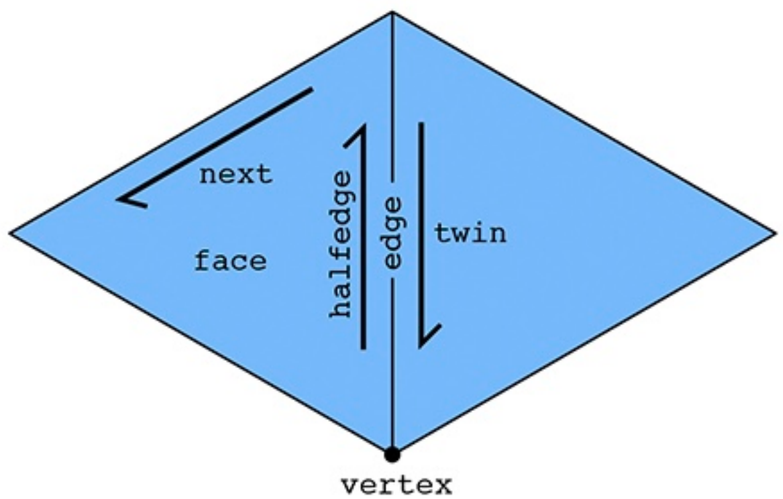
Curves, Surfaces & Meshes

Computer Graphics
CMU 15-462/15-662, Fall 2015

Assignment 2 is out!







Last time: overview of geometry

- Many types of geometry in nature
- Demand sophisticated representations
- Two major categories:
  - IMPLICIT - "tests" if a point is in shape
  - EXPLICIT - directly "lists" points
- Lots of representations for both
- Today:
  - what is a surface, anyway?
  - nuts & bolts of polygon meshes

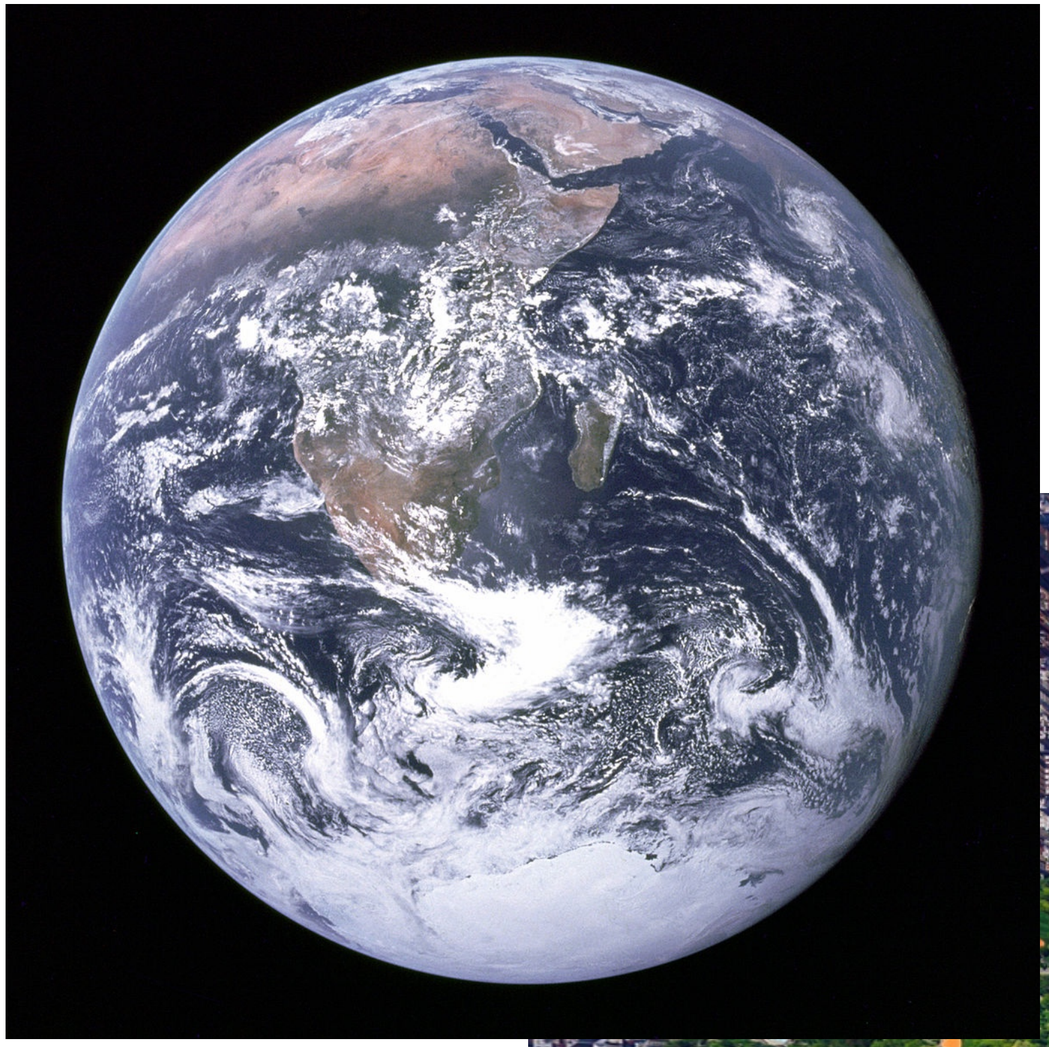
Geometry

Q: What is a “surface?”

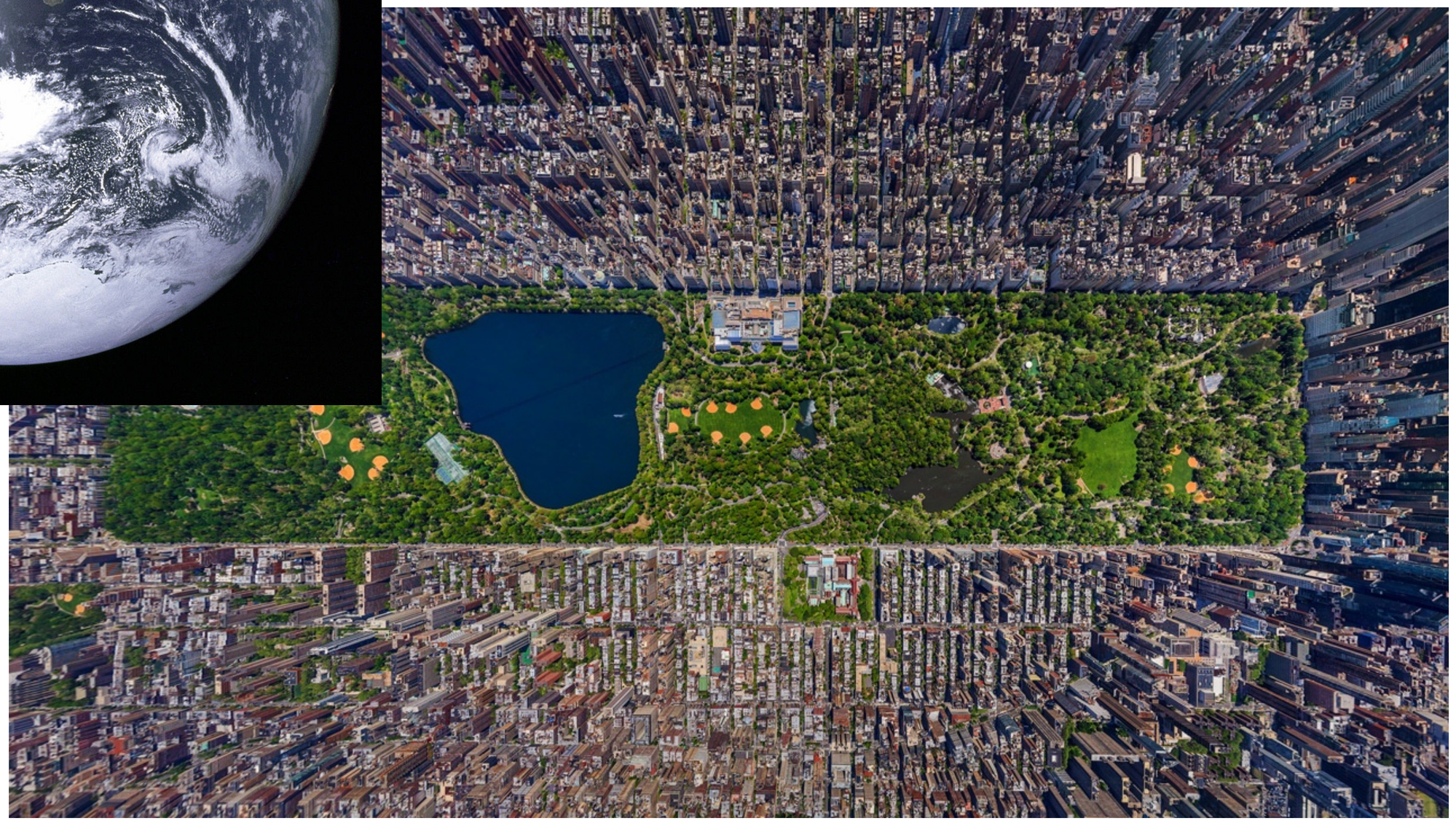
A: Oh, it's a 2-dimensional manifold.

Q: Ok… but what the heck is a *manifold?*

The Earth looks flat, if you get close enough



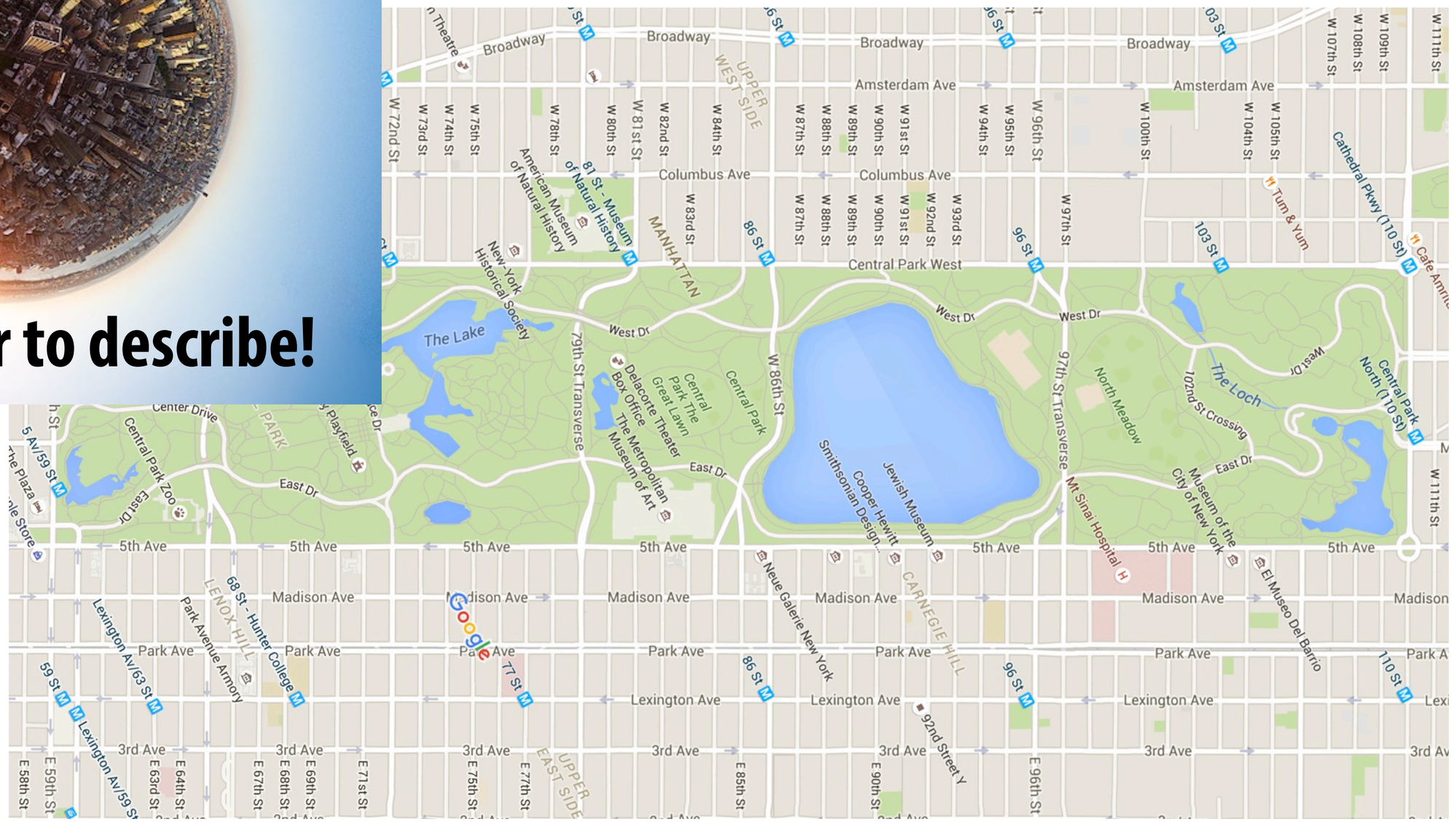
Can pretend we're on a grid:



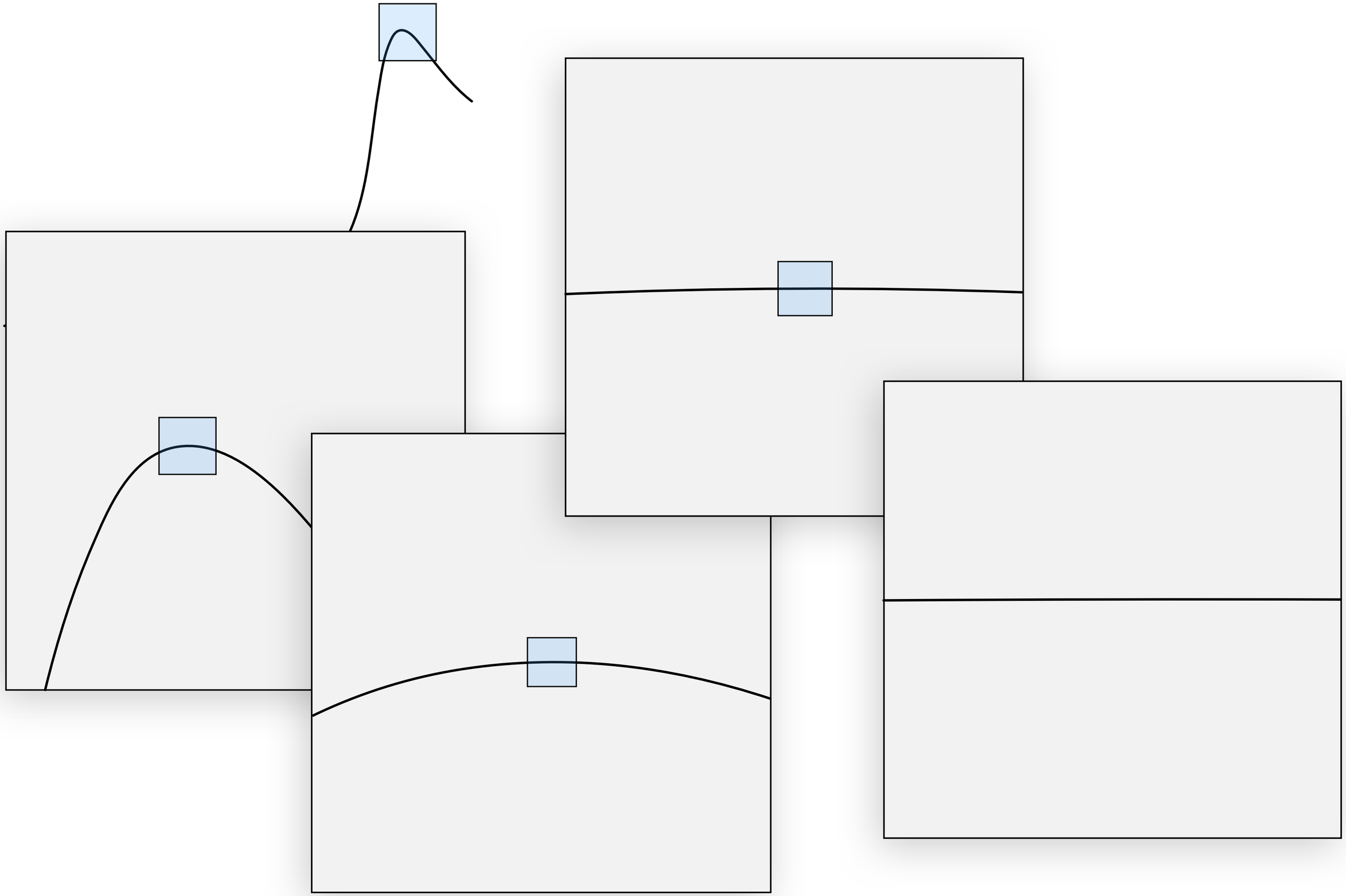
The Earth looks flat, if you get close enough

Much harder to describe!

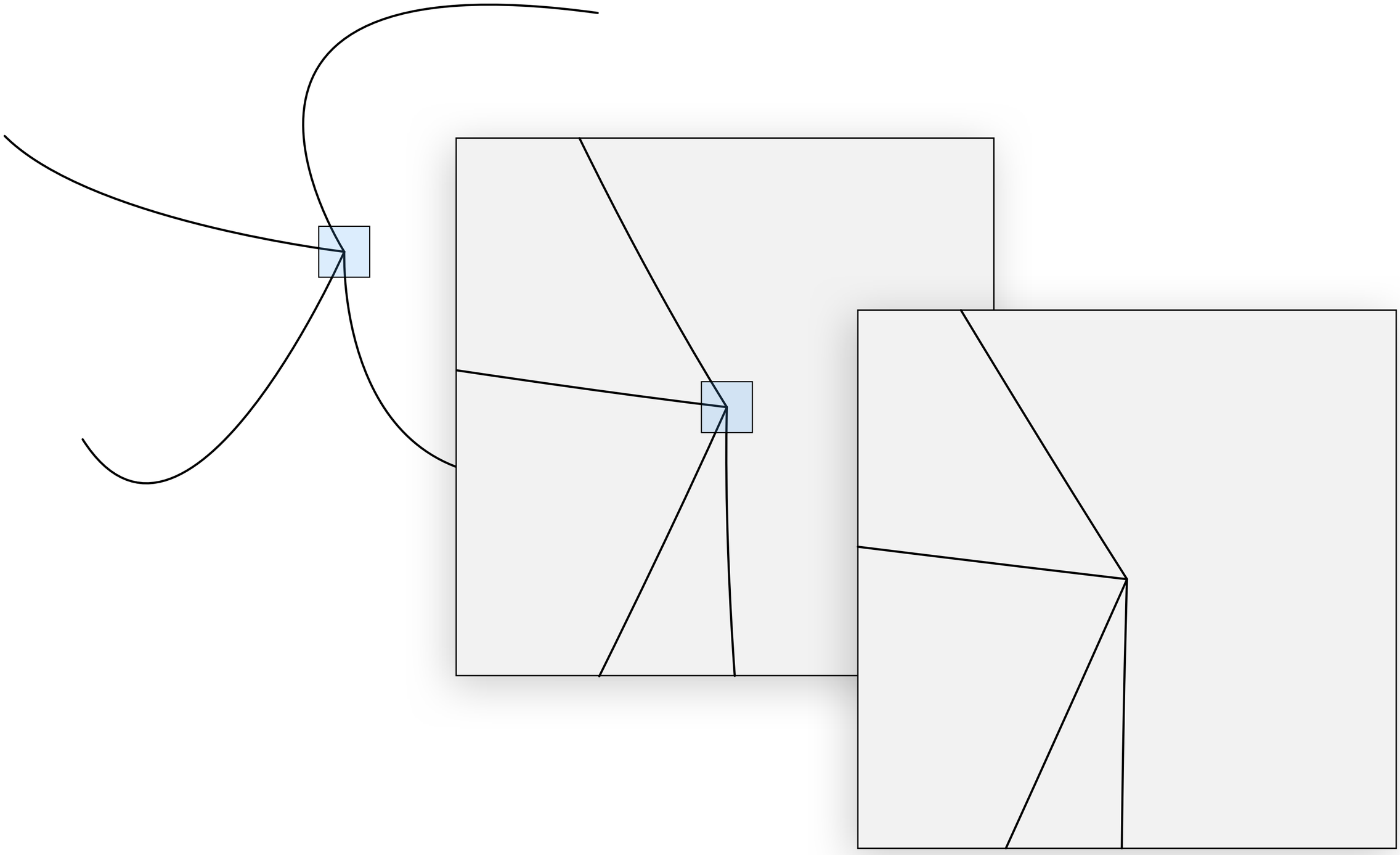
Can pretend we're on a grid:



A *smooth manifold* also looks flat close up

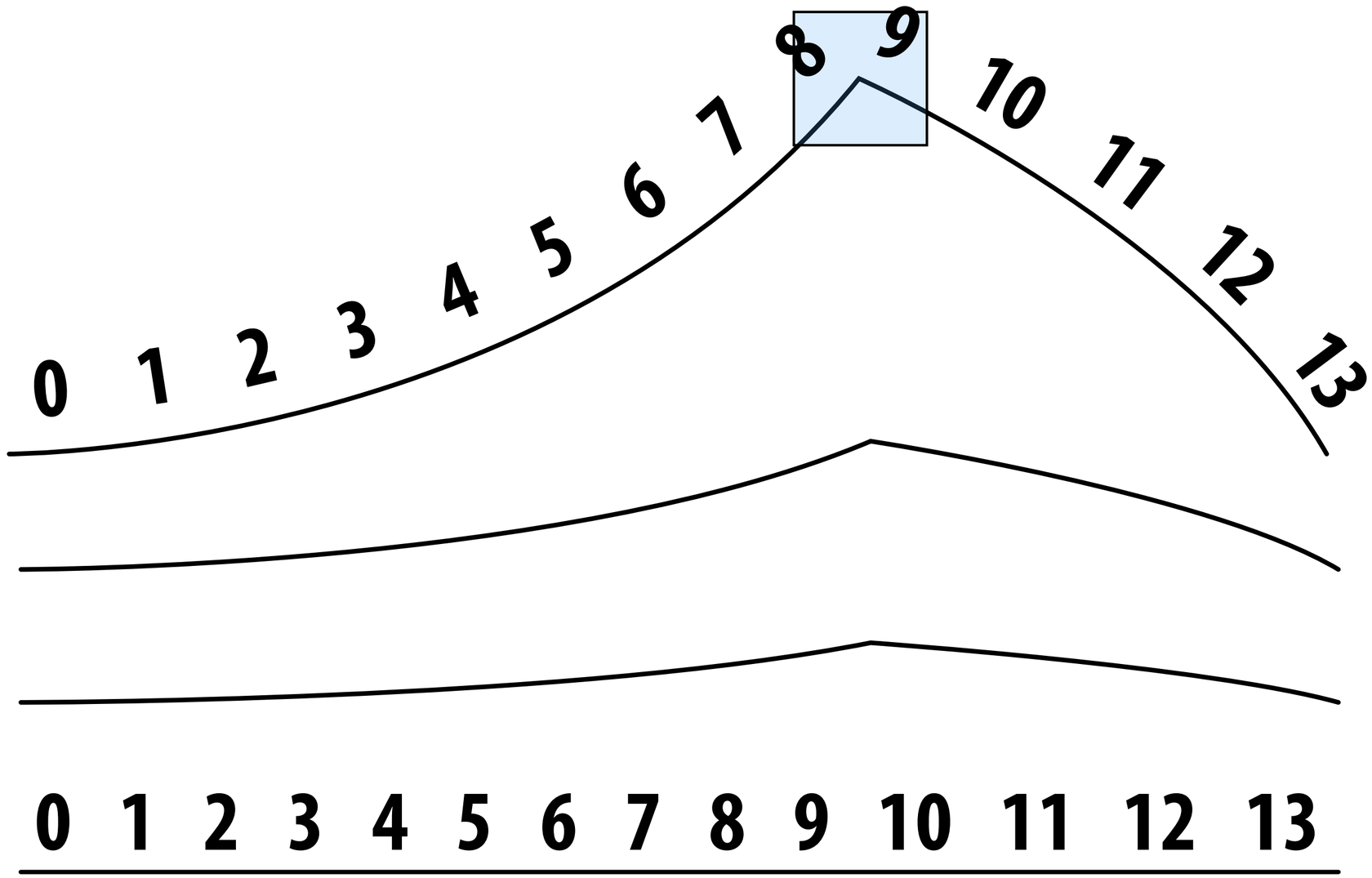


Not all curves are smooth manifolds



No matter how close we get, doesn't look like a single line!

What about sharp corners?



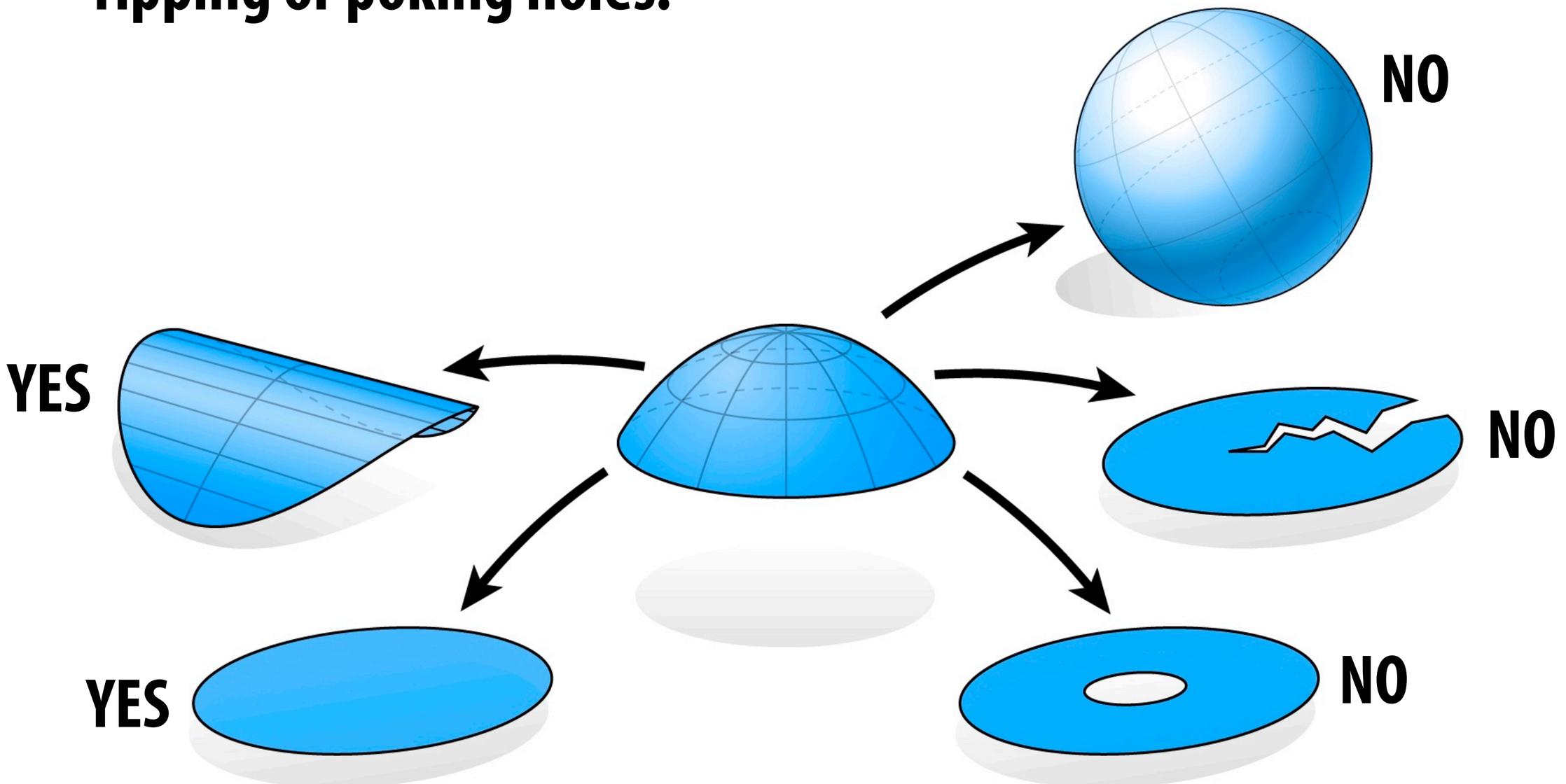
Can easily be flattened *into* a line.

Can still assign coordinates (just like Manhattan!)

...But is it a manifold?

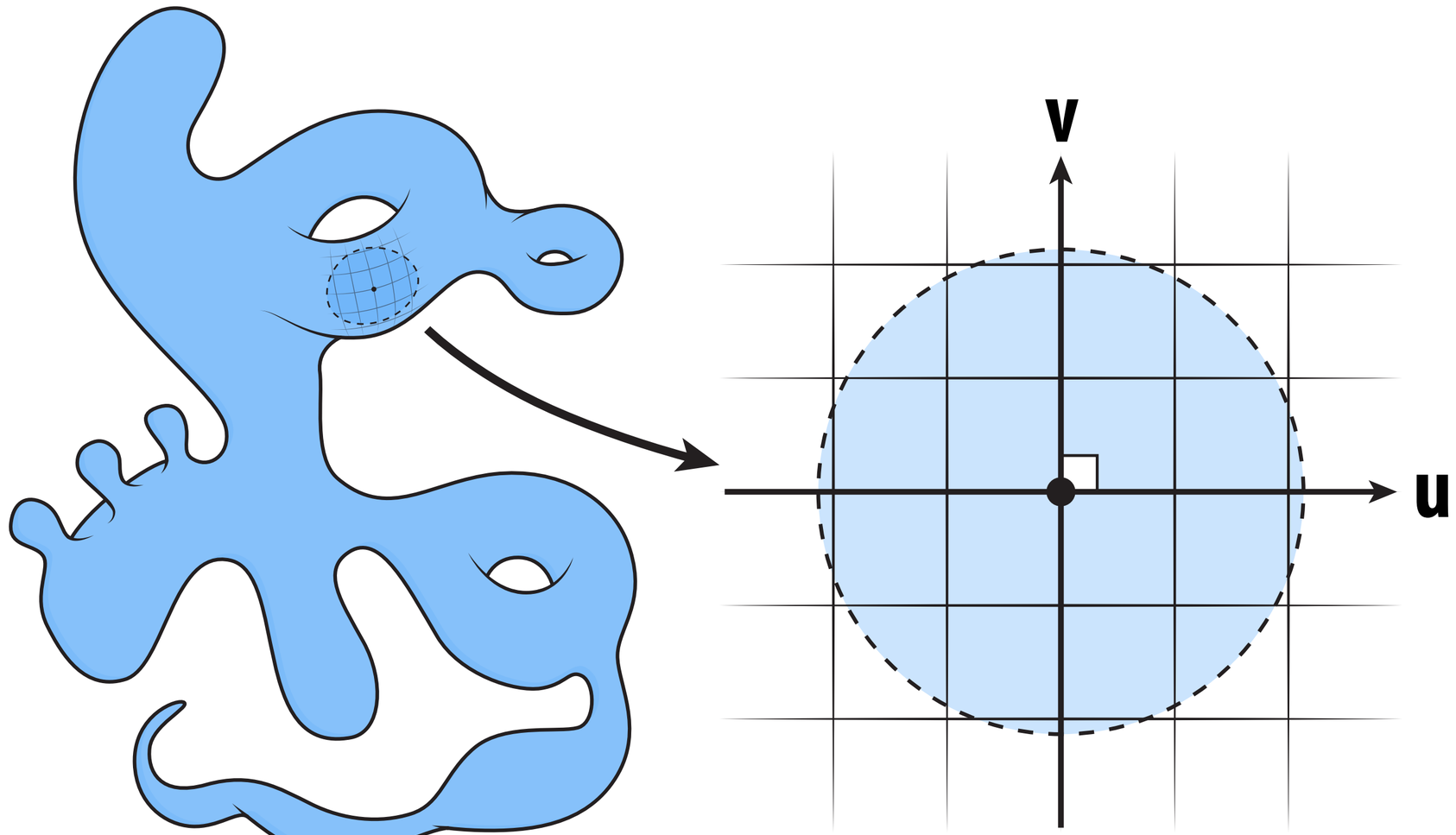
Definition of a manifold

- *"A subset S of R^m is an n-manifold if every point p in S is contained in a neighborhood that can be mapped bijectively and continuously (both ways) to the open ball in R^n."*
- **In other words: each little piece can be made flat without "ripping or poking holes."**



Why is the manifold property valuable?

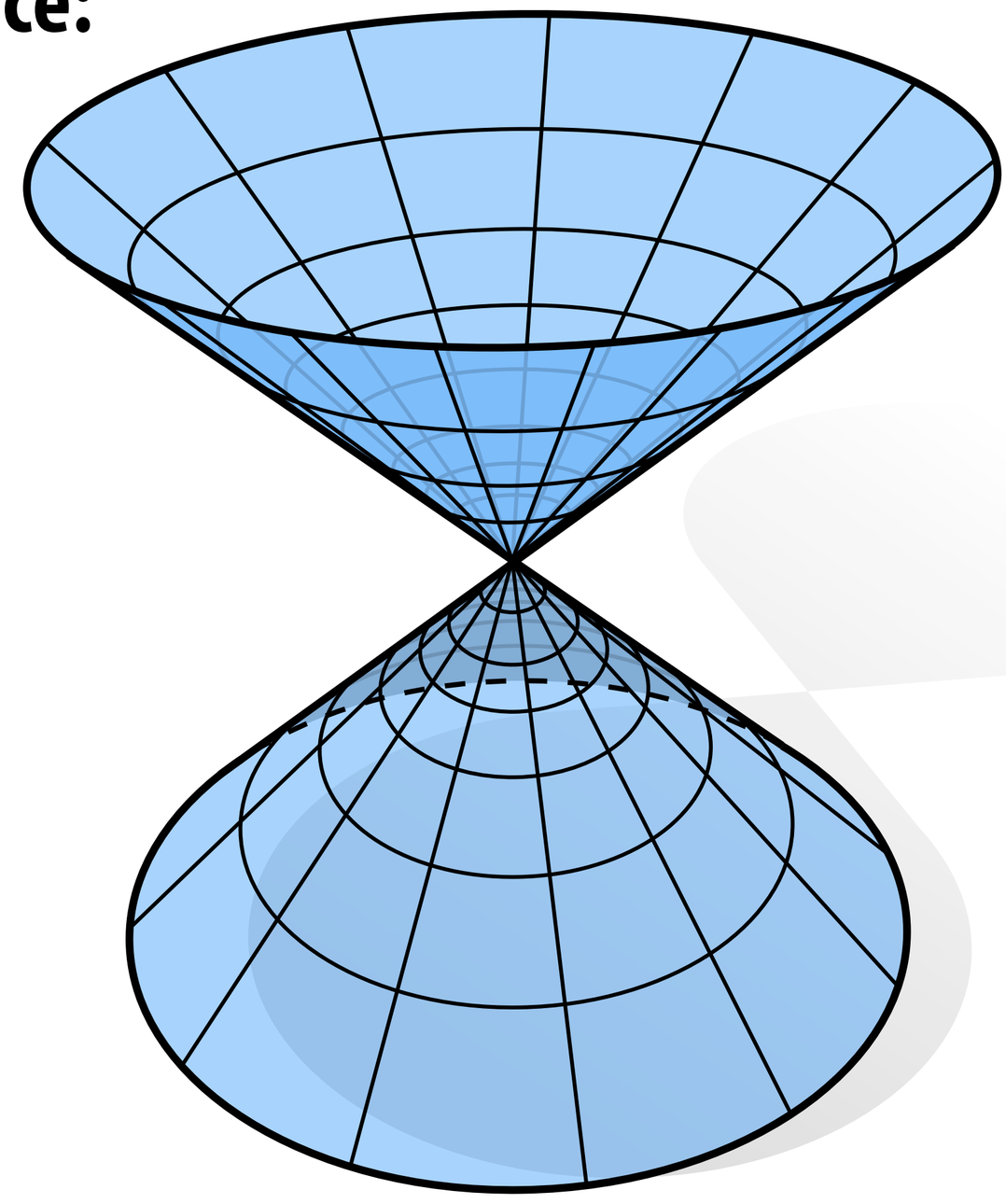
- Makes life simple: all surfaces look the same (at least locally).
- Gives us coordinates! (At least locally.)



- More abstractly, lets us talk about curved surfaces in terms of familiar tools: vector calculus & linear algebra.

Isn’t every shape manifold?

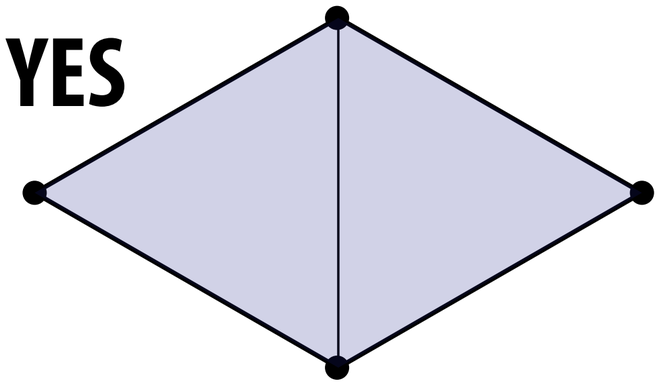
- No, for instance:

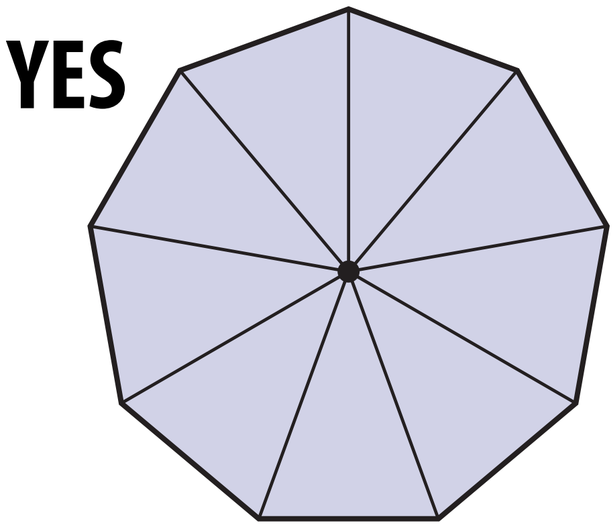


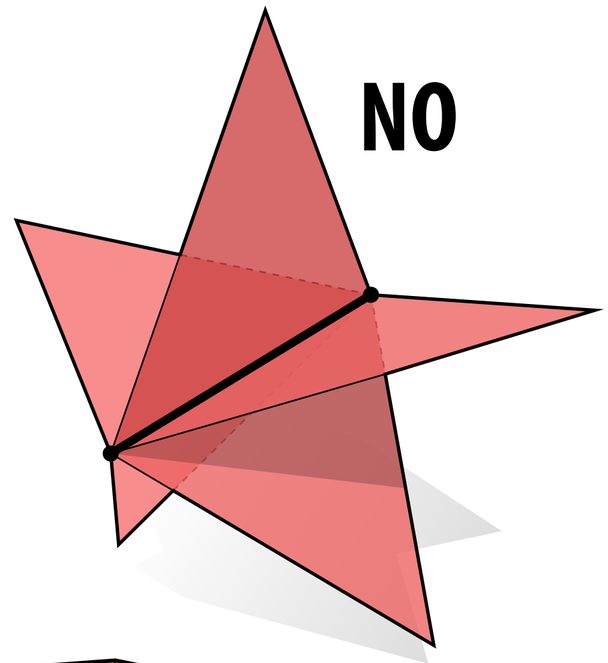
No way to put a (simple) coordinate system on the center point!

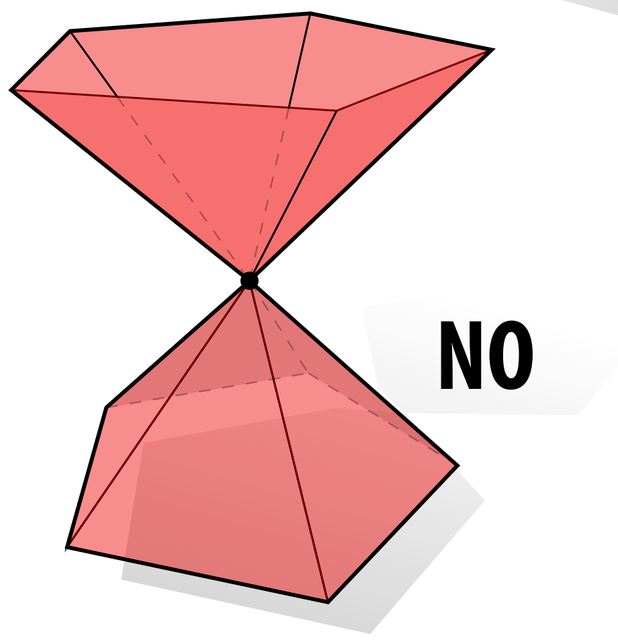
What about discrete surfaces?

- Surfaces made of, e.g., triangle are no longer *smooth*.
- But they can still be *manifold:*
  - two triangles per edge (no "fins")
  - every vertex looks like a "fan"





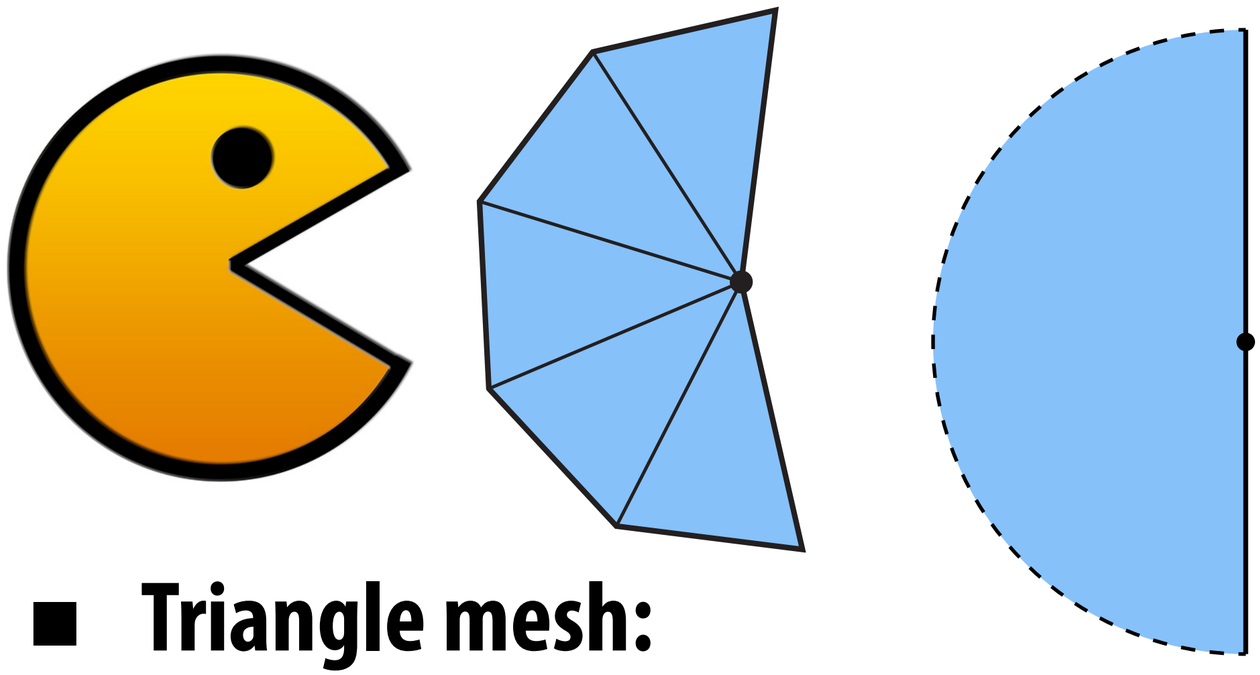




- Why? *Simplicity.*
  - no special cases to handle
  - keeps data structures (reasonably) simple)

What about boundary?

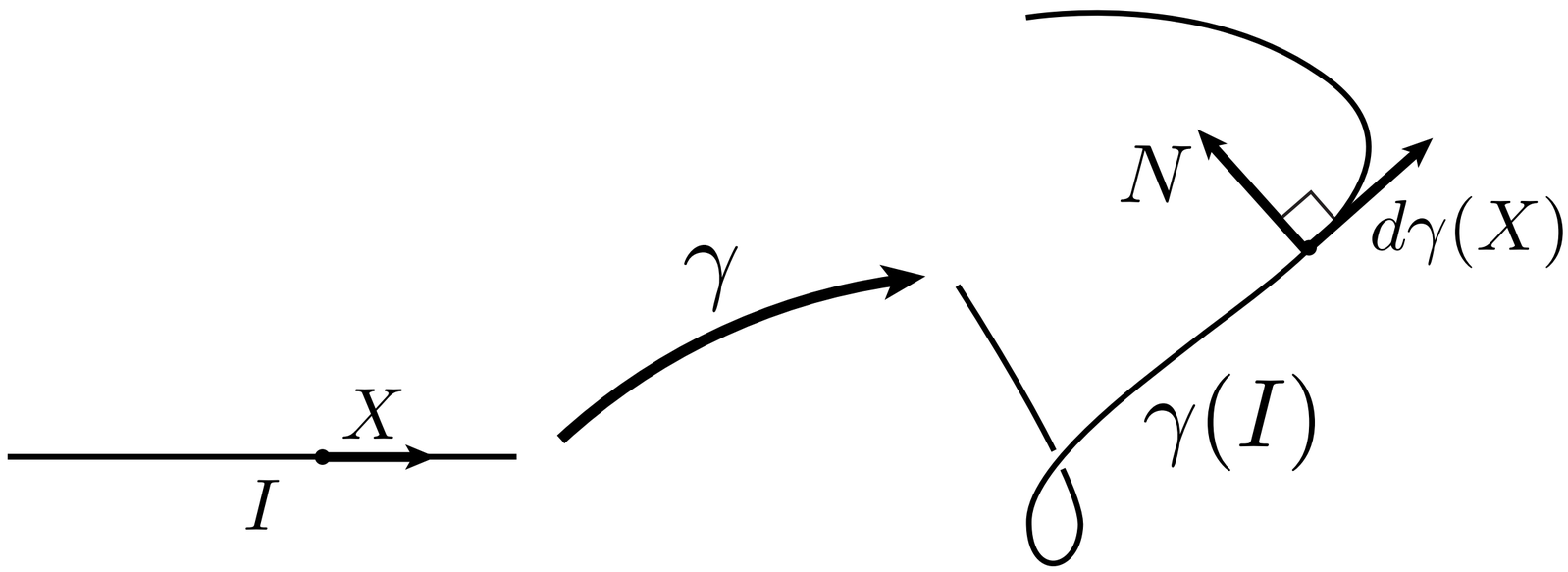
- The boundary is where the surface "ends."
- E.g., waist & ankles on a pair of pants.
- Locally, looks like a *half* disk
- Globally, each boundary forms a loop

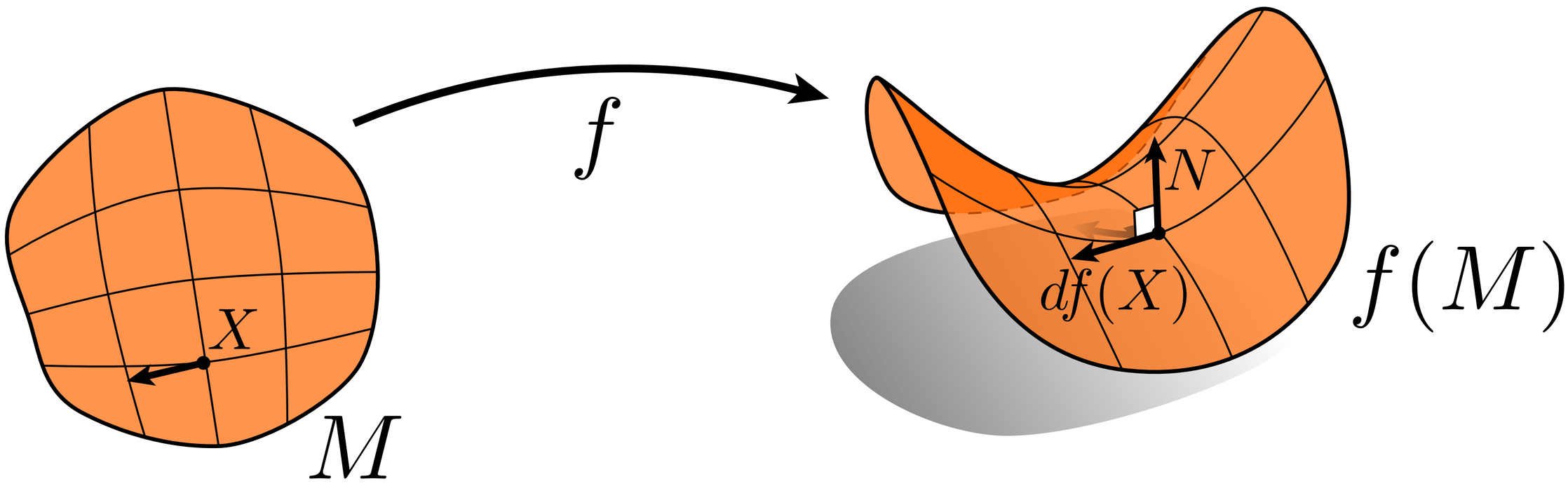


YES

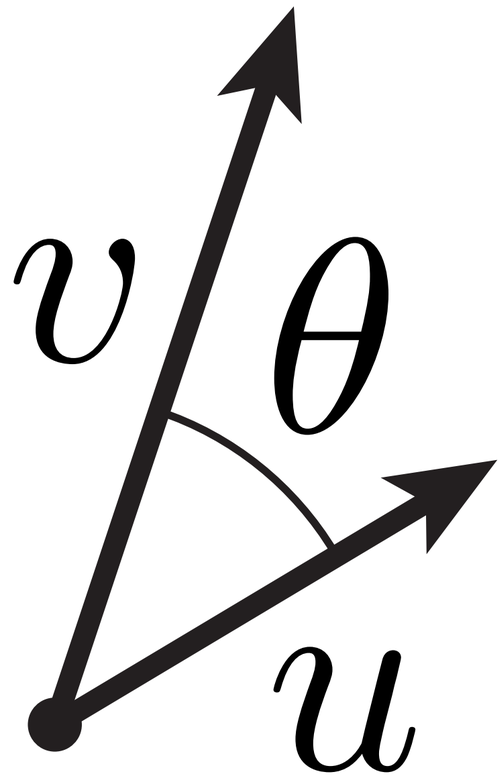
- Triangle mesh:
  - one triangle per boundary edge
  - boundary vertex looks like "pacman"

Anatomy of a manifold (in 2D and 3D)



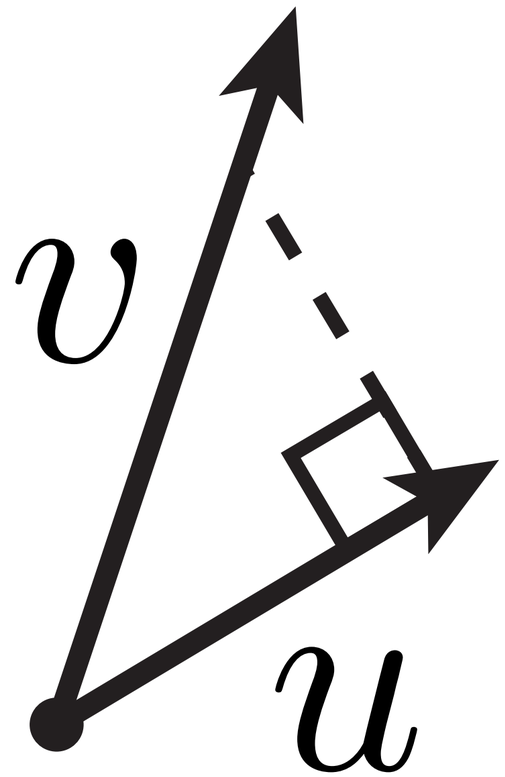


What can we measure about vectors?



$$u \cdot v = |u||v| \cos\theta$$

What can we measure about vectors?



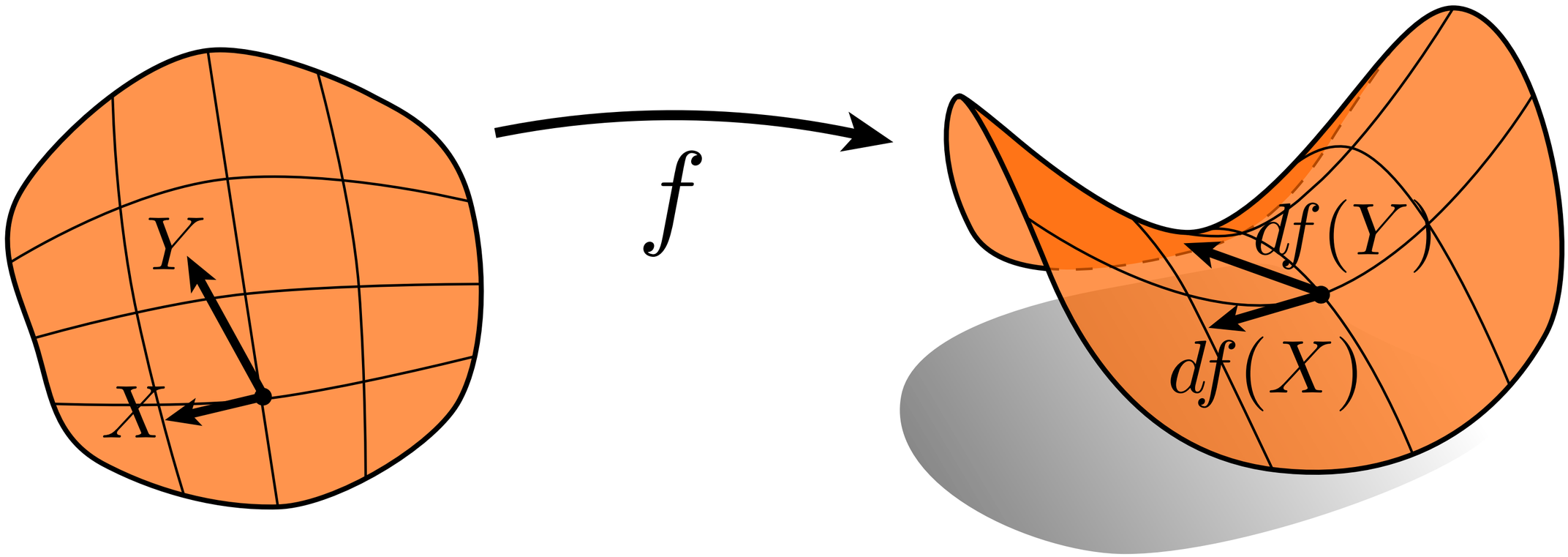
$$v \cdot (u/|u|)$$

Inner product of *tangent* vectors is "metric"

$$g(X,Y) = df(X) \cdot df(Y)$$

metric

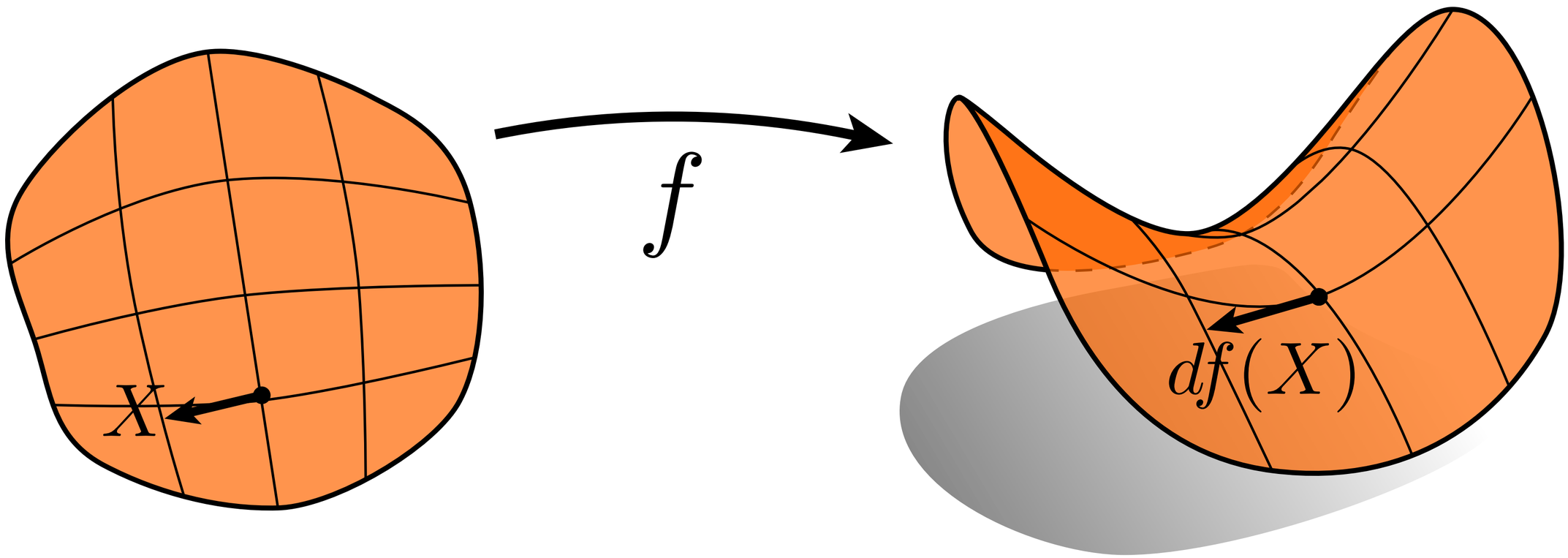
("first fundamental form")



Q: What's the *length* of a tangent vector?

$$|X| = \sqrt{df(X) \cdot df(X)}$$

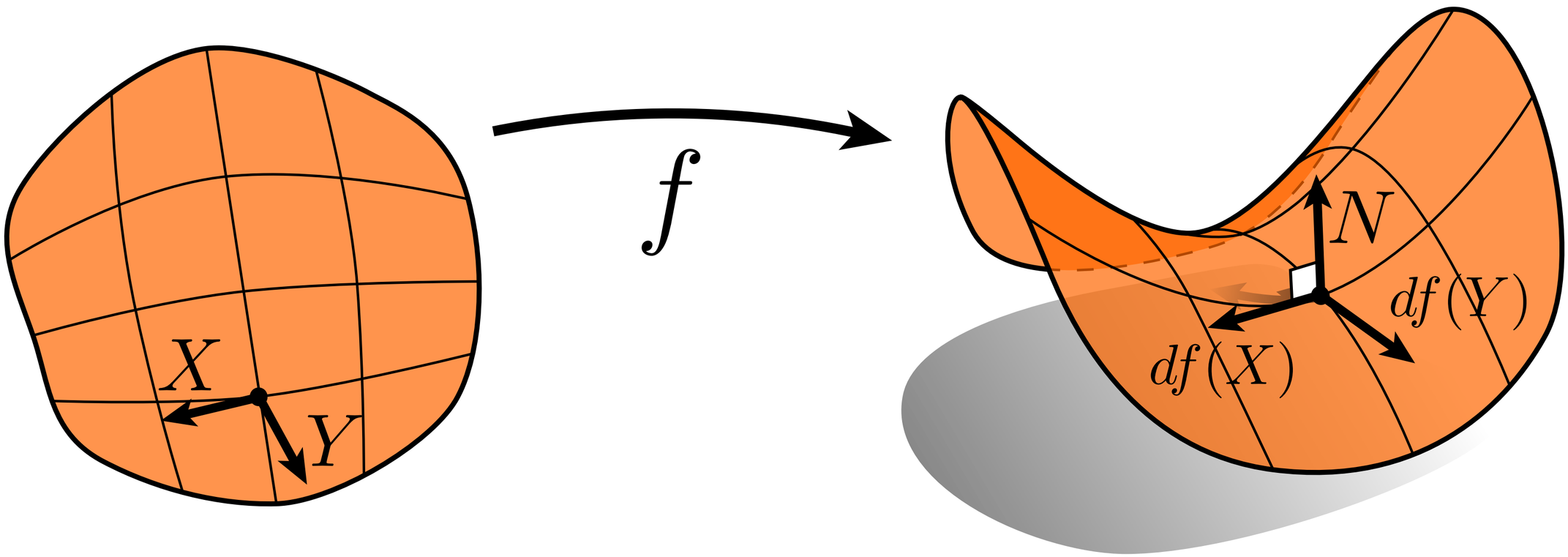
$$= \sqrt{g(X, X)}$$



Normal is vector orthogonal to all tangents

$$N$$

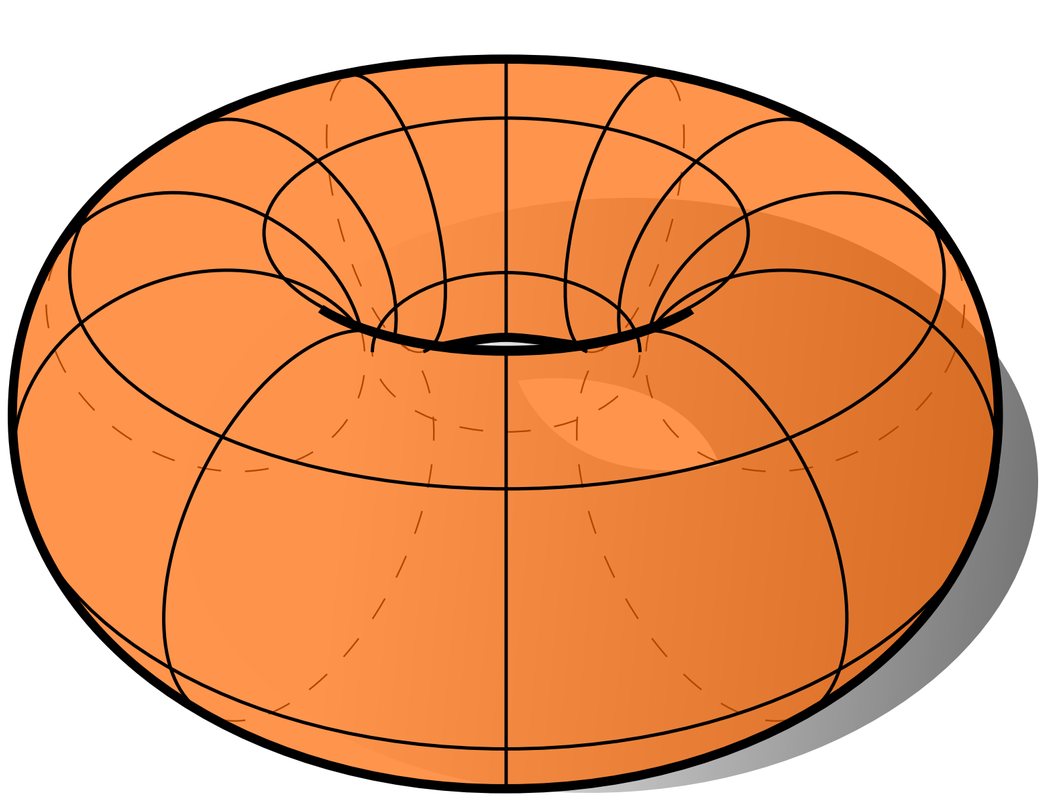
$$N \cdot df(X) = 0 \quad \forall X$$



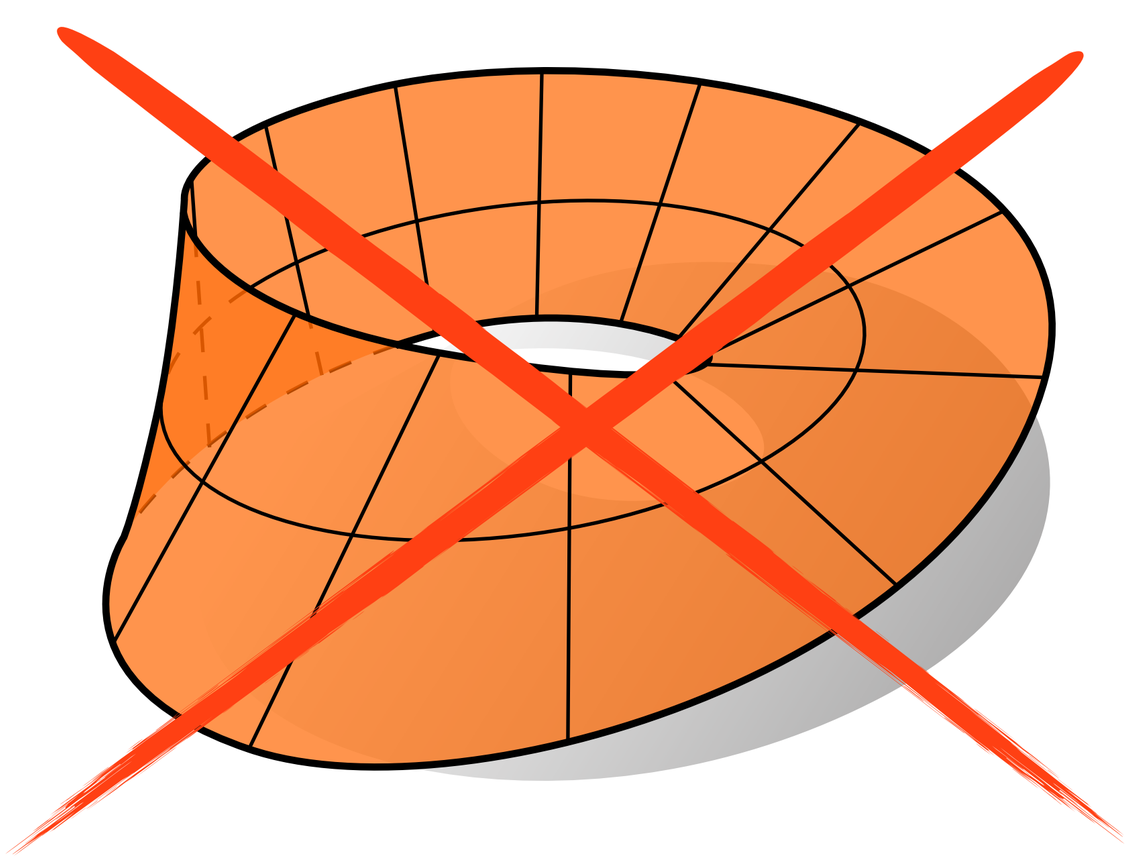
Which direction does the normal point?

$$|N| = 1$$

$$N \cdot df(X) = 0 \quad \forall X$$

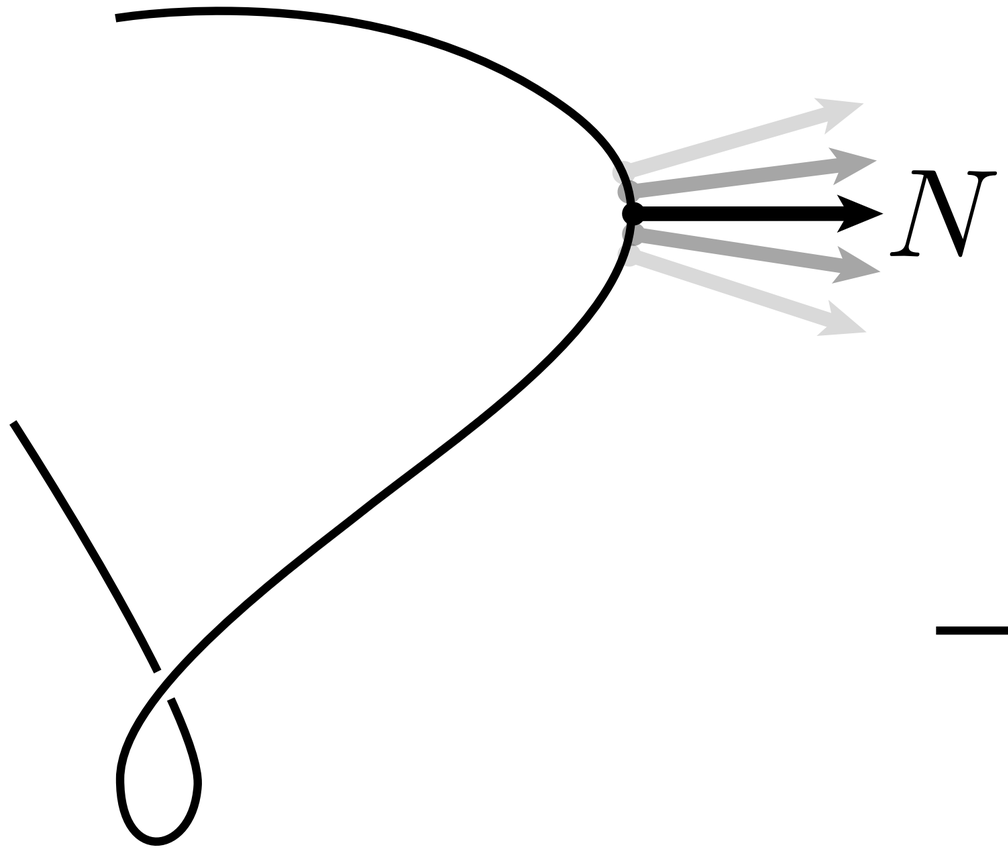


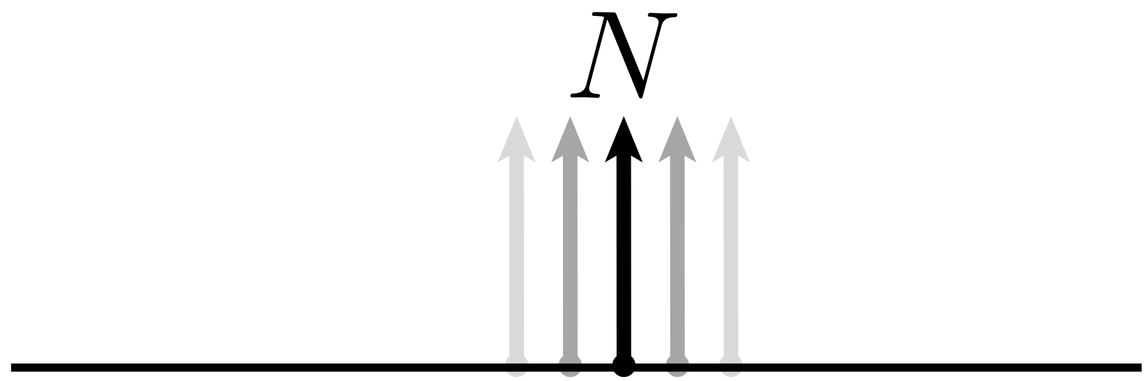
orientable



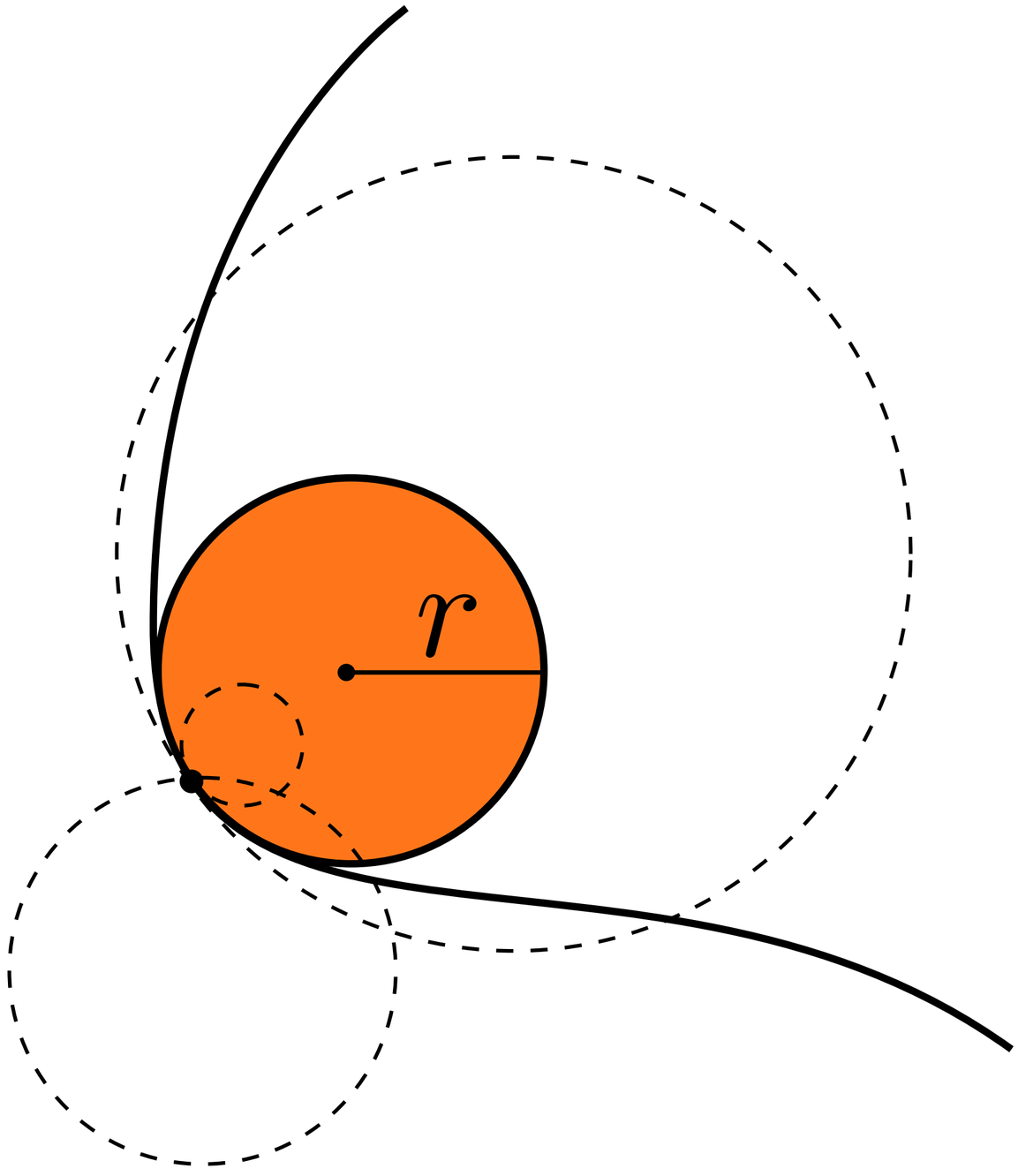
nonorientable

Curvature is *change* in normal





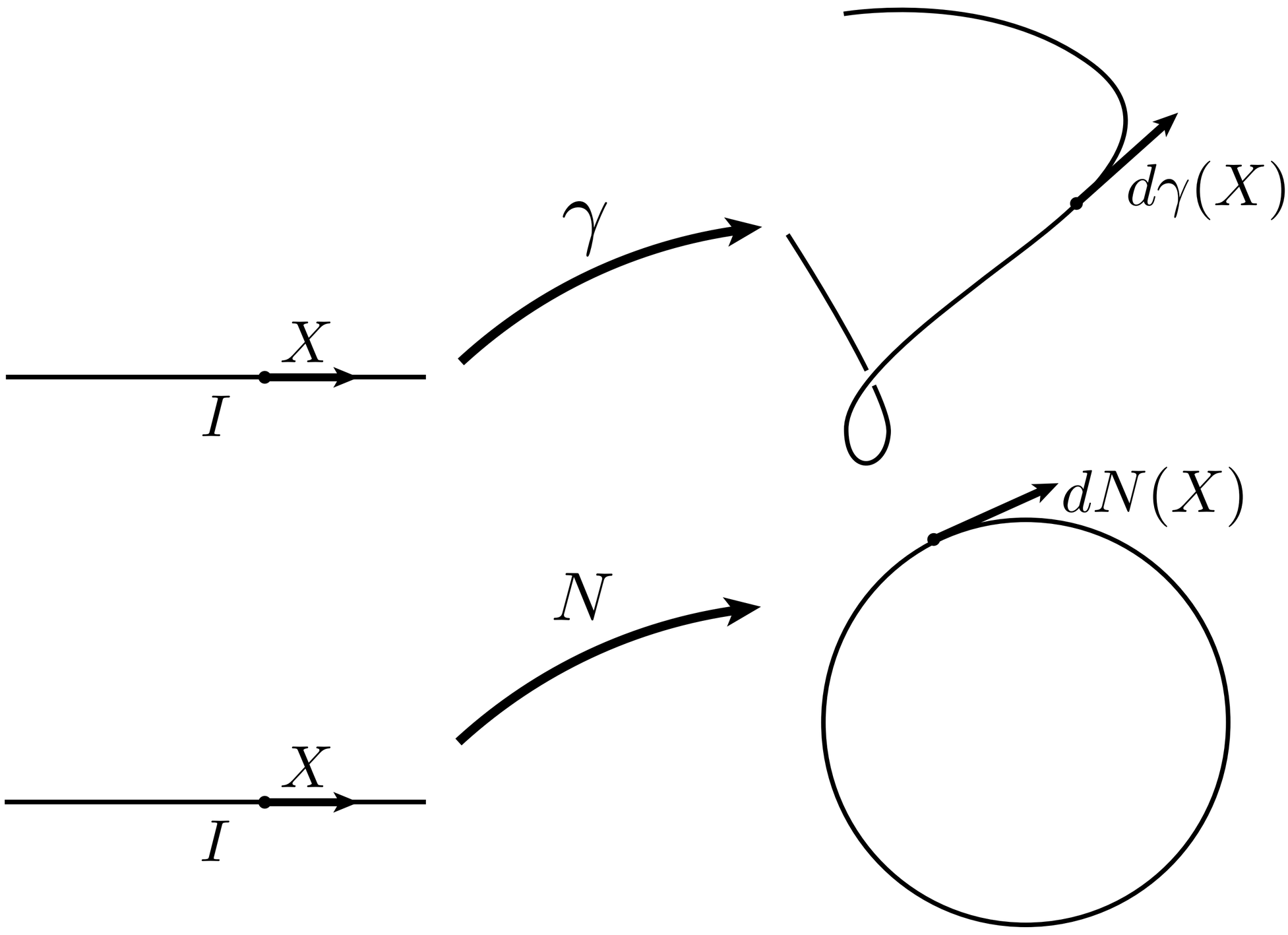
Standard definition: radius of curvature



$$\kappa = \frac{1}{r}$$

curvature

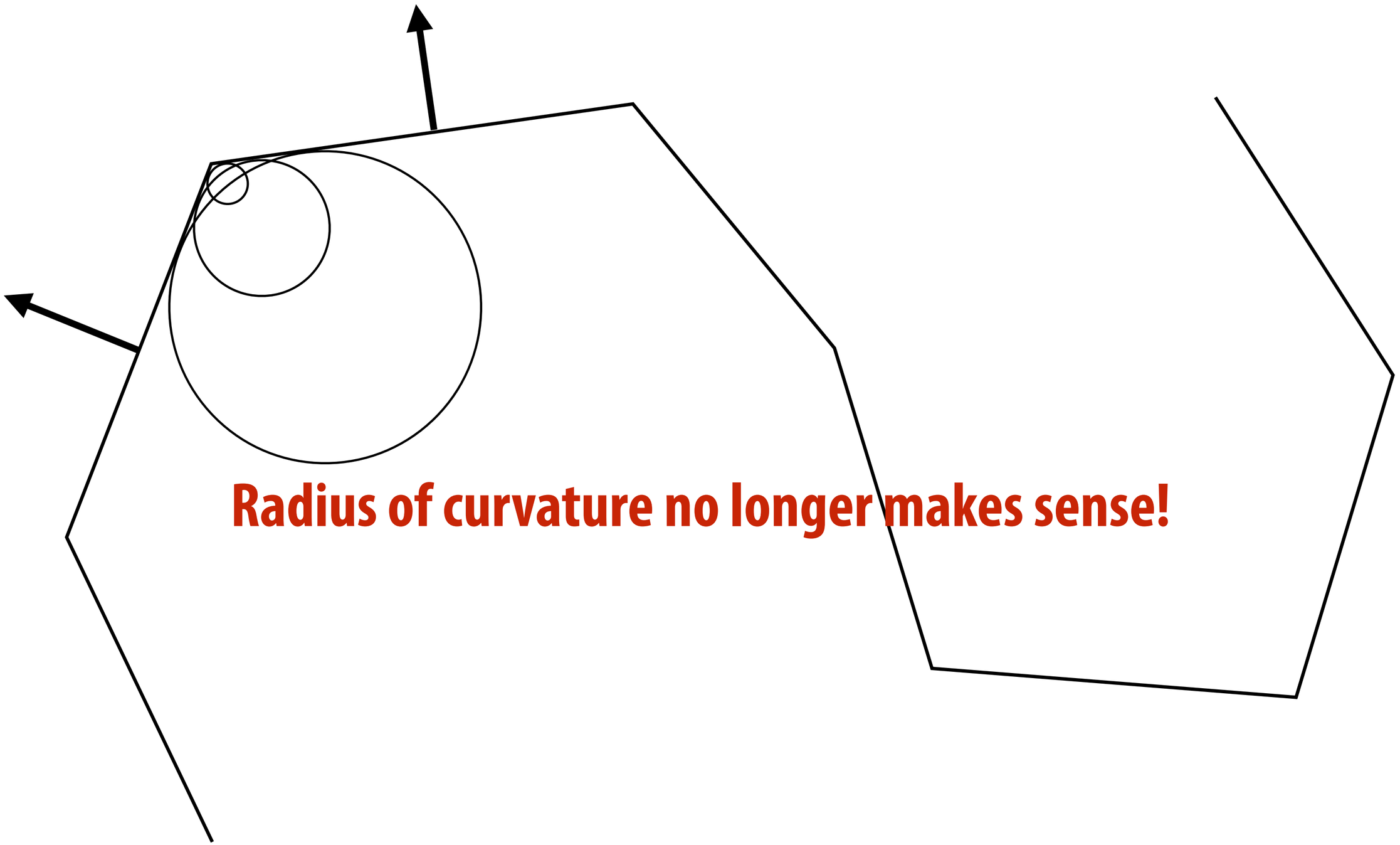
Alternative: normals as map to unit circle



Key idea: size of swept-out piece gives total curvature.

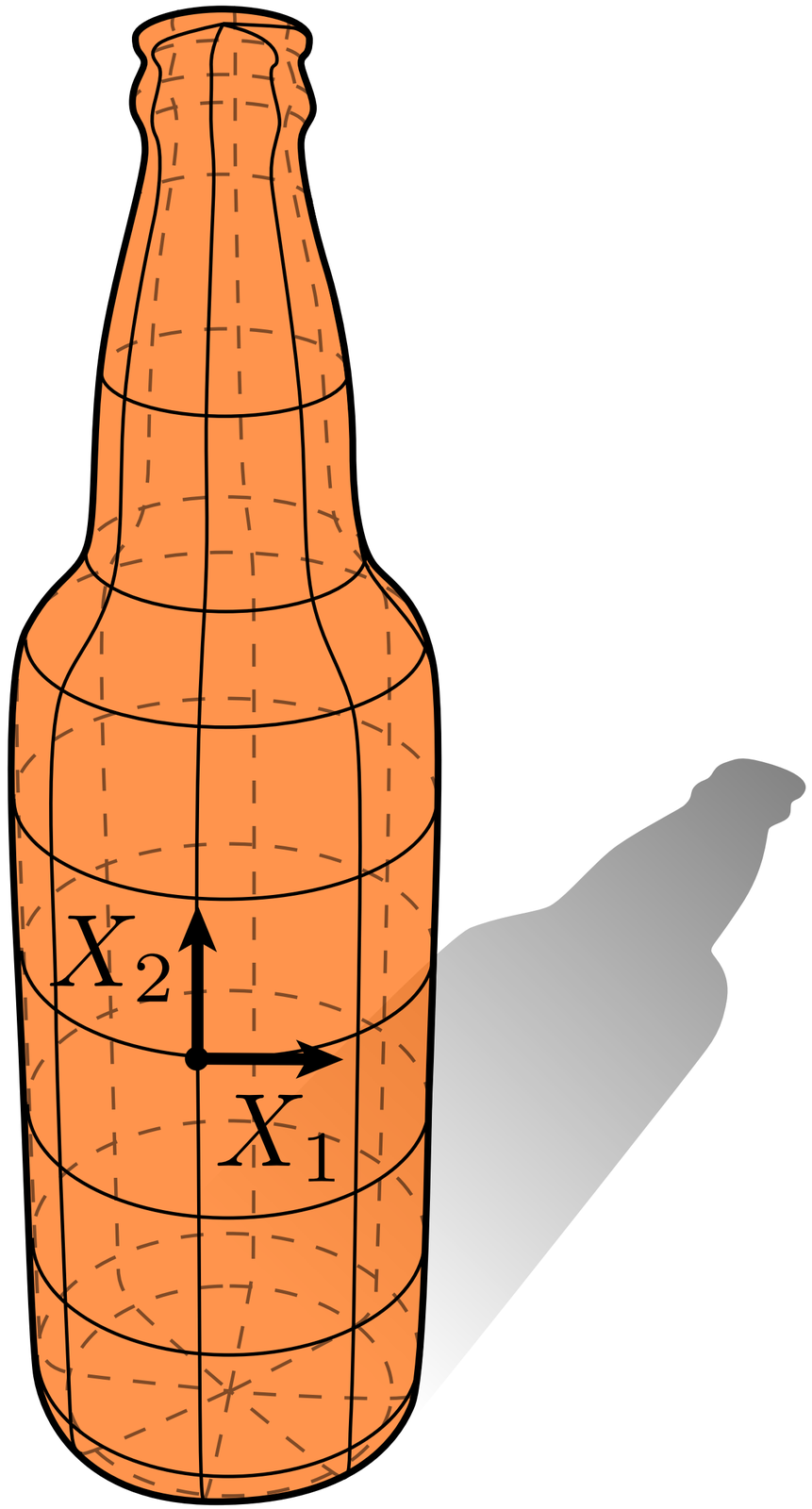
Discrete curvature as change in normal

...can still talk about change in normal.

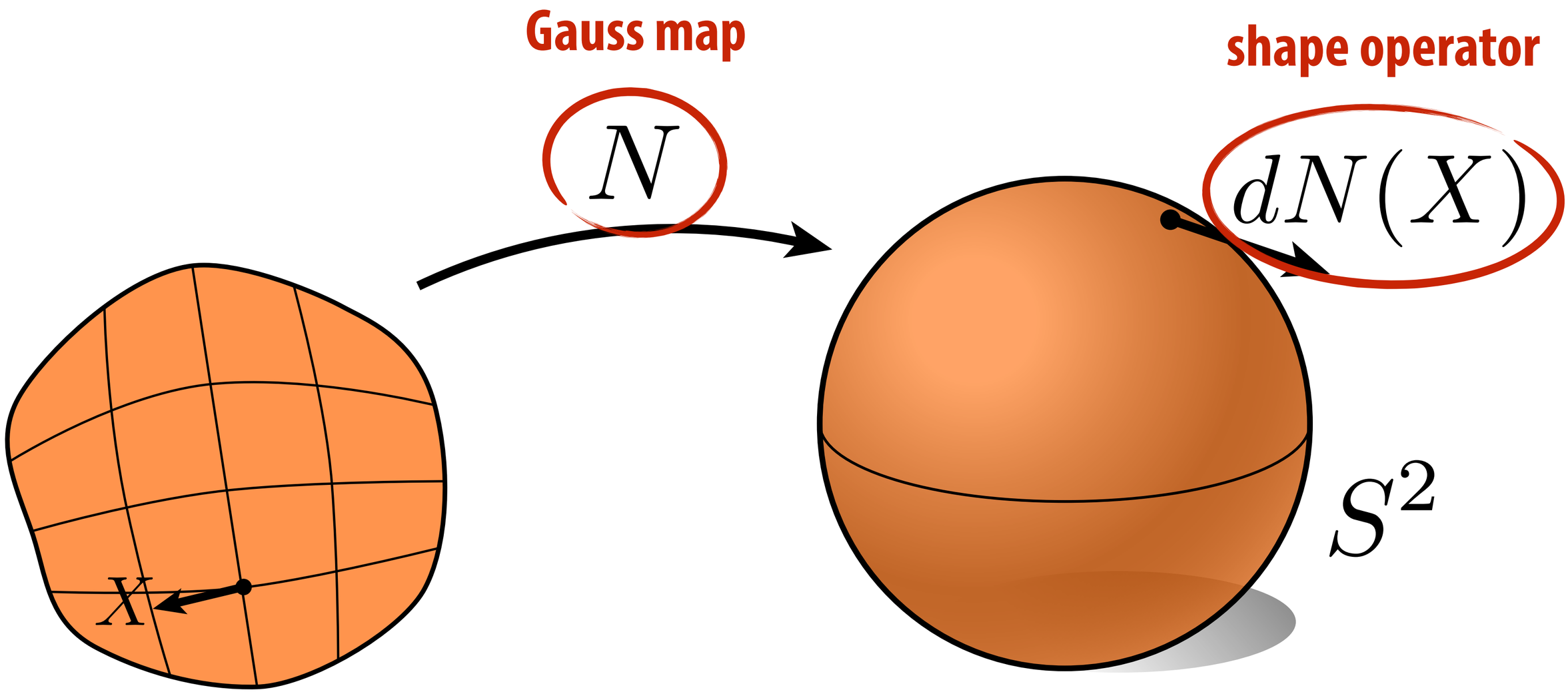


Radius of curvature no longer makes sense!

What about surfaces?



Normal is now map to the *sphere*

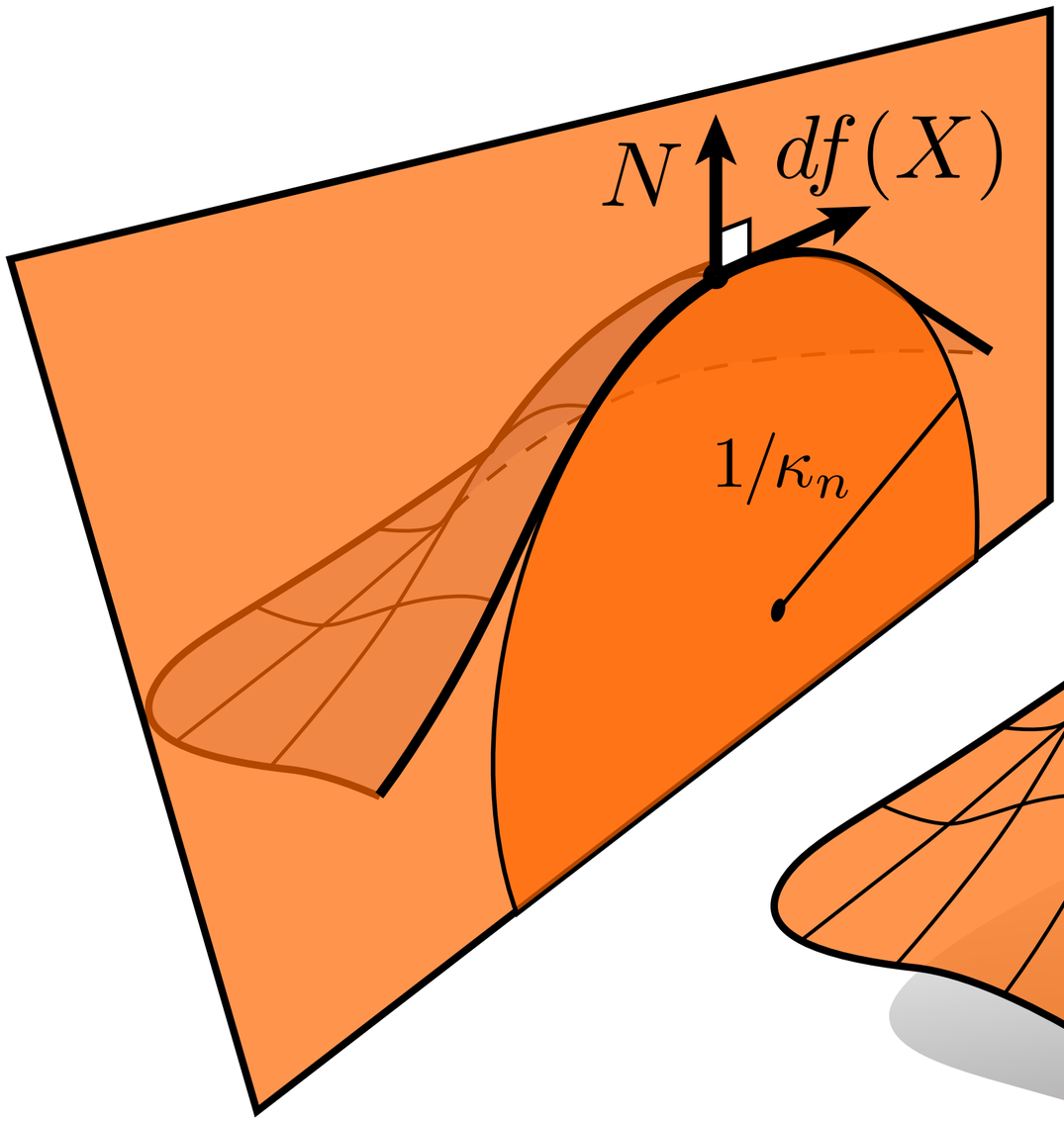


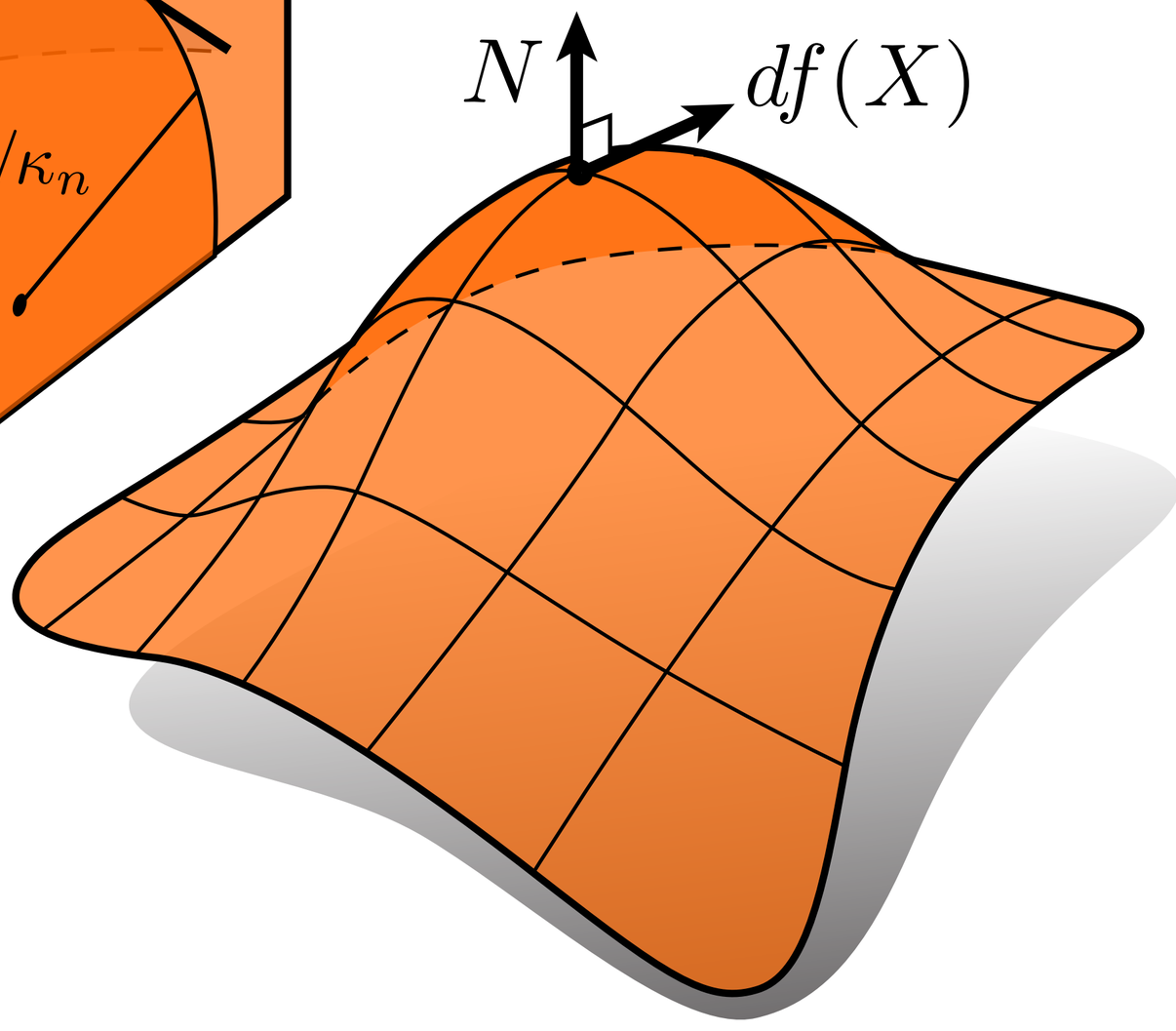
Normal curvature

$$\kappa_n(X) = -df(X) \cdot dN(X)$$

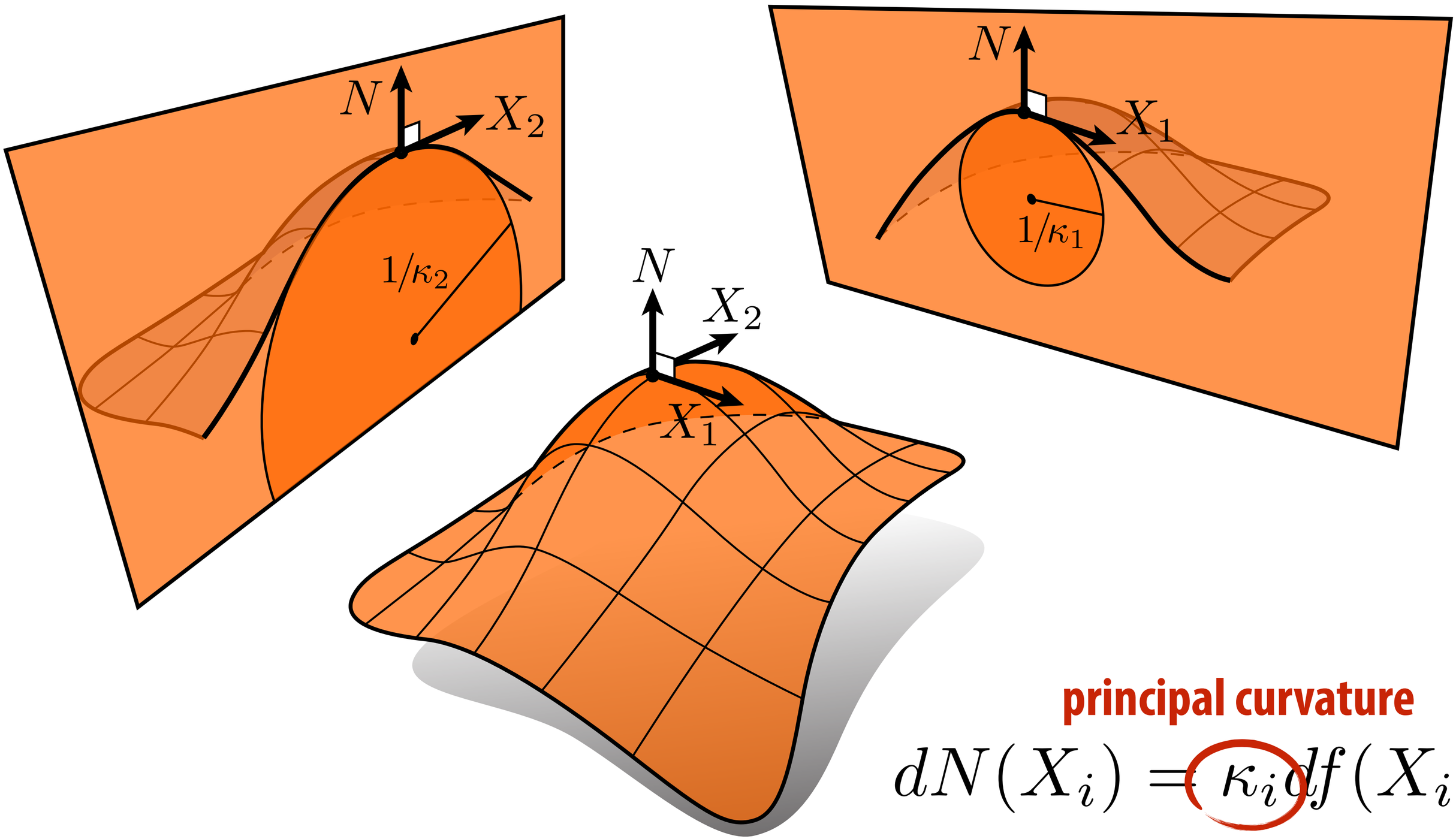
$$-df(X) \cdot dN(Y)$$

("second fundamental form")





Principal Curvatures

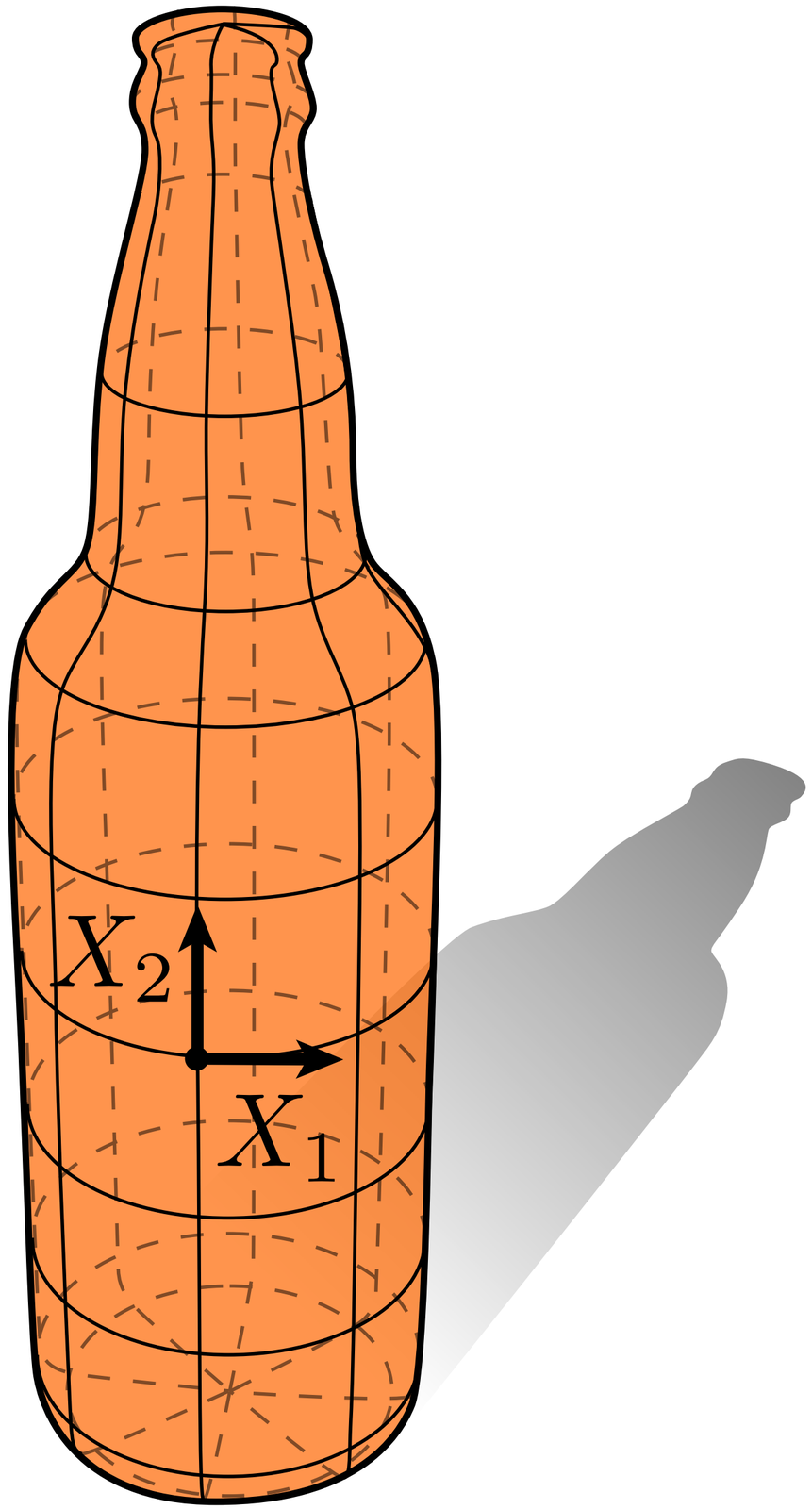


principal curvature

$$dN(X_i) = \kappa_i df(X_i)$$

Fact: principal curvature directions are orthogonal.

Q: What are the principal curvatures?

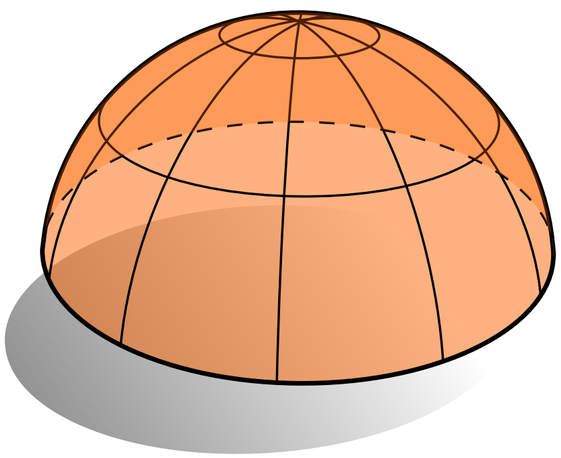


Mean & Gaussian Curvature

mean $H = \frac{1}{2}(\kappa_1 + \kappa_2)$

Gaussian $K = \kappa_1 \kappa_2$

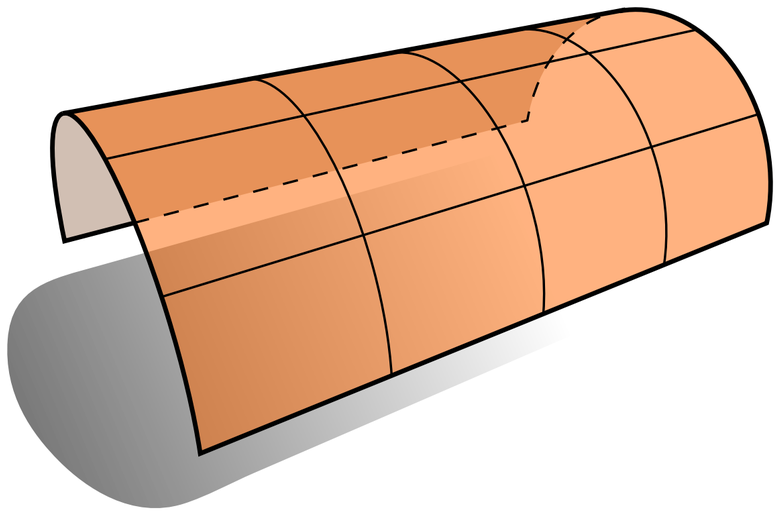
$\kappa_1 > 0, \kappa_2 > 0$



$H > 0$

$K > 0$

$\kappa_1 > 0, \kappa_2 = 0$

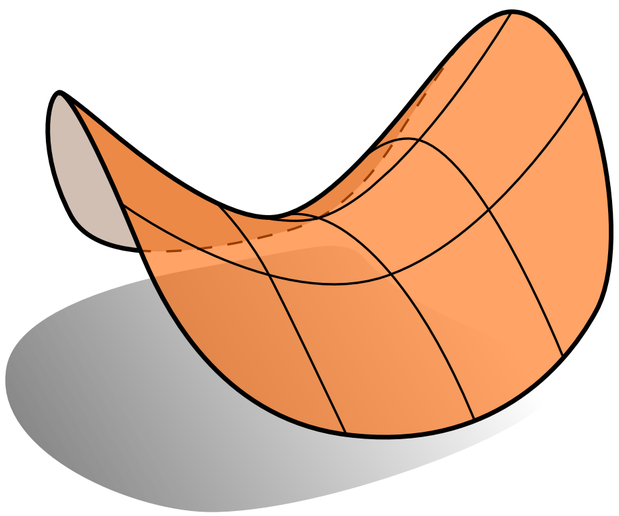


$H \neq 0$

$K = 0$

developable

$\kappa_1 = 1, \kappa_2 = -1$



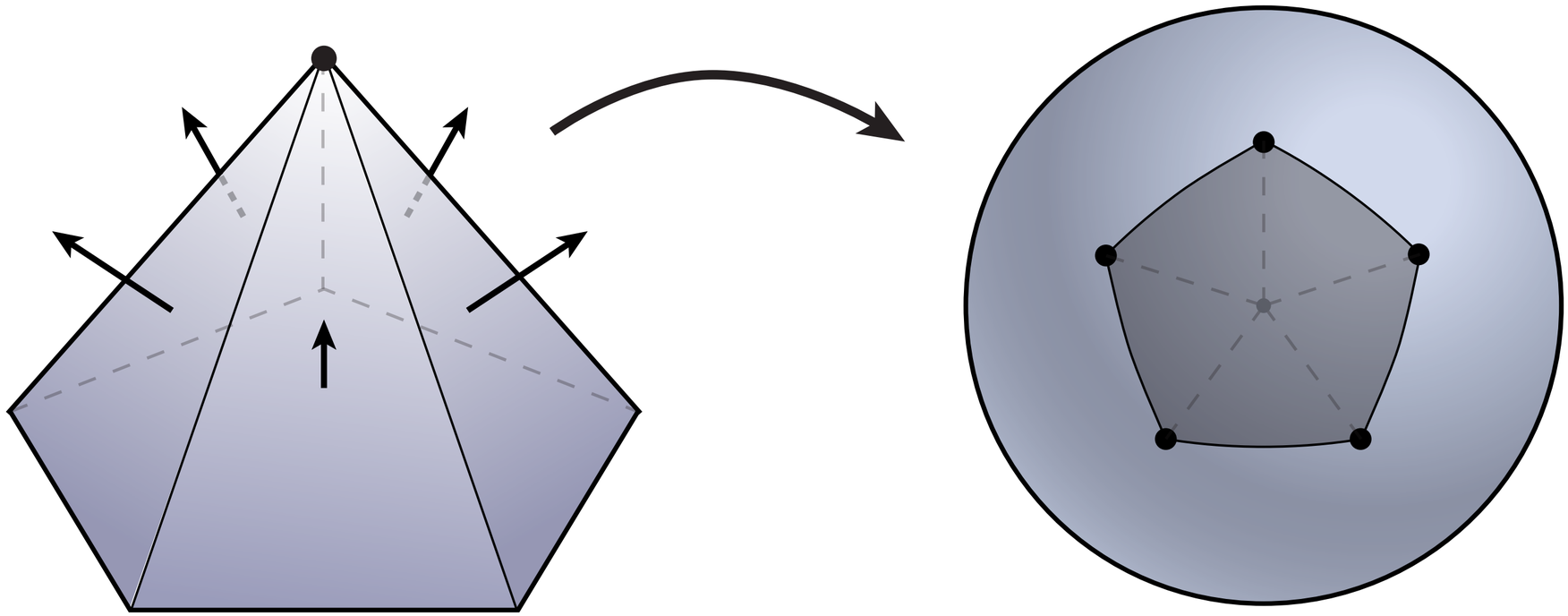
minimal

$H = 0$

$K < 0$

Discrete Gaussian Curvature?

- Once again, use area on Gauss sphere:

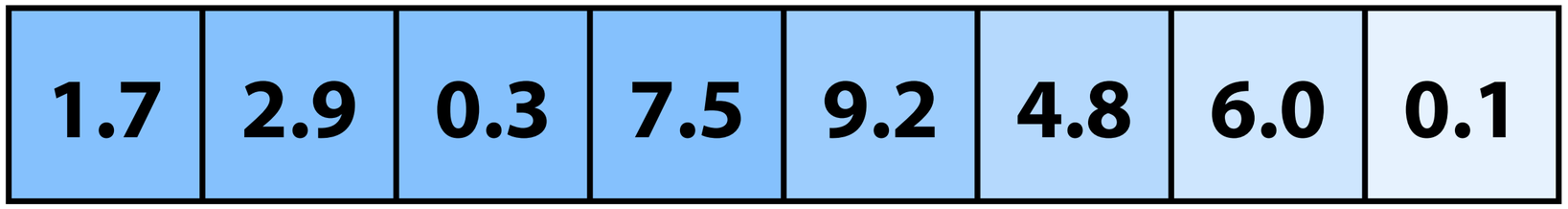


Long story from there! See http://keenan.is/dgpdec for more.

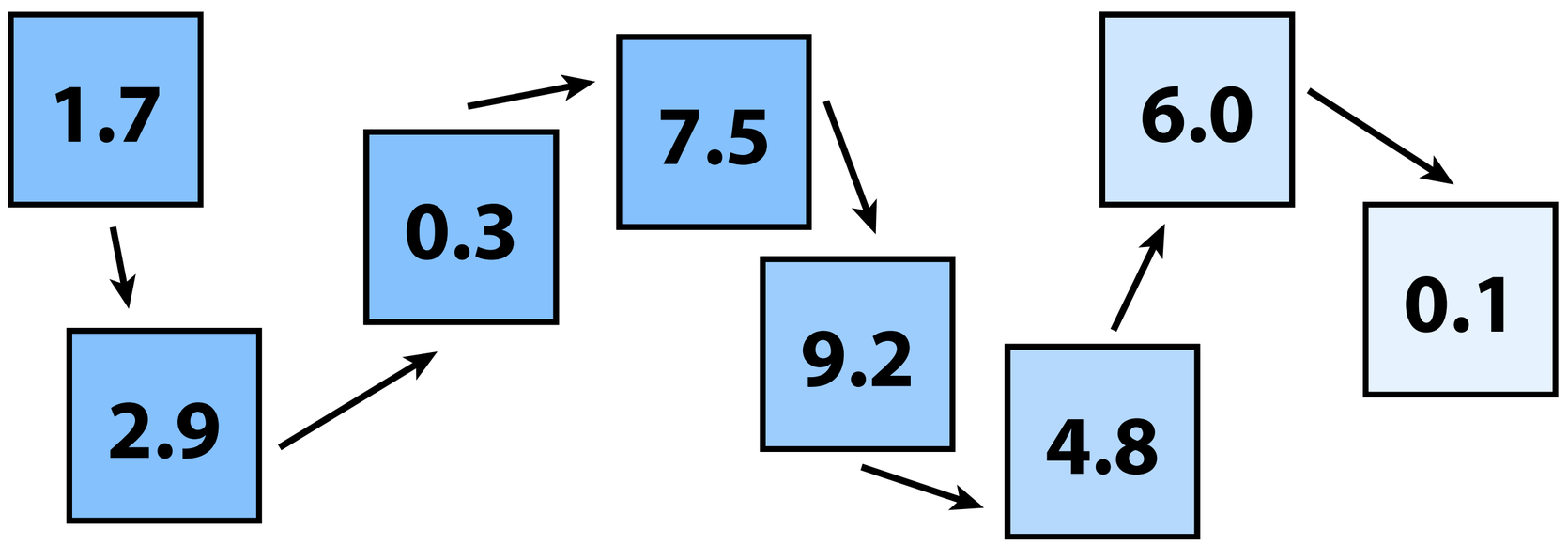
How do we actually encode all this data?

Warm up: arrays vs. linked lists

- Want to store a list of numbers
- One idea: use an *array* (constant time lookup, coherent access)



- Alternative: use a linked list (linear lookup, incoherent access)



- Q: Why bother with the linked list?
- A: For one, we can easily insert numbers wherever we like…

Polygon soup, revisited

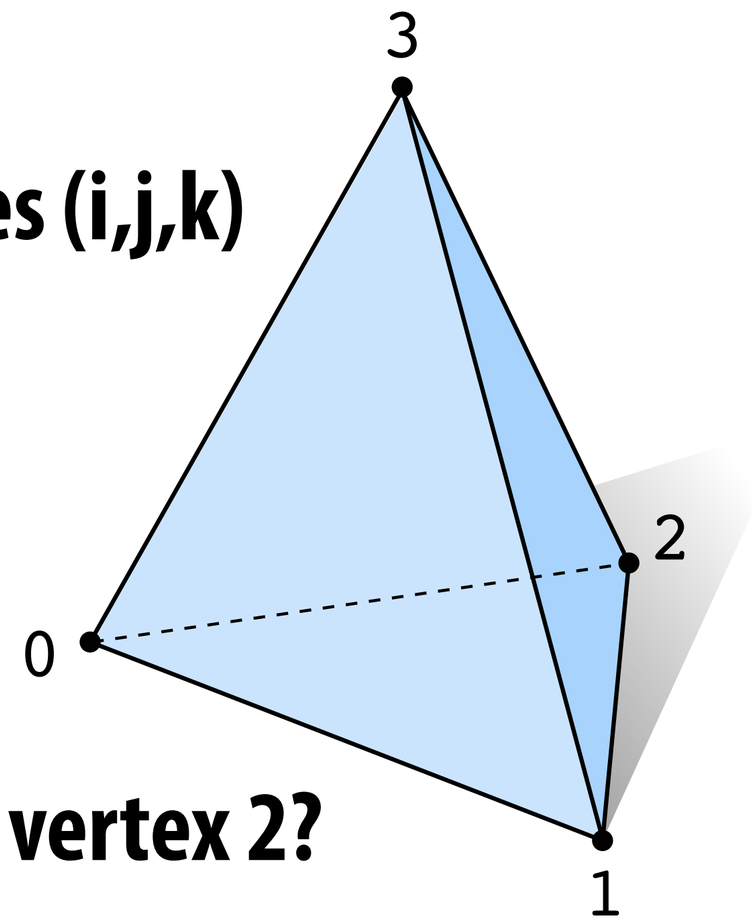
- Store triples of coordinates (x,y,z) and indices (i,j,k)
- E.g., tetrahedron:

VERTICES

	x	y	z
0:	-1	-1	-1
1:	1	-1	1
2:	1	1	-1
3:	-1	1	1

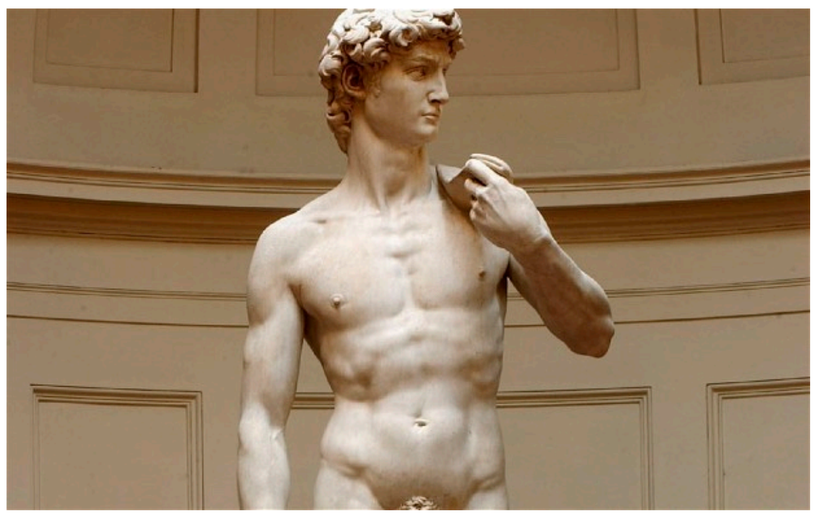
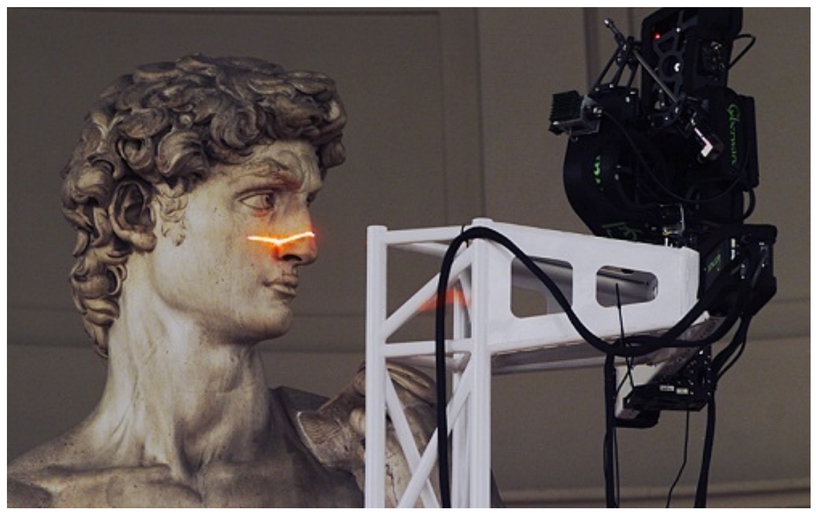
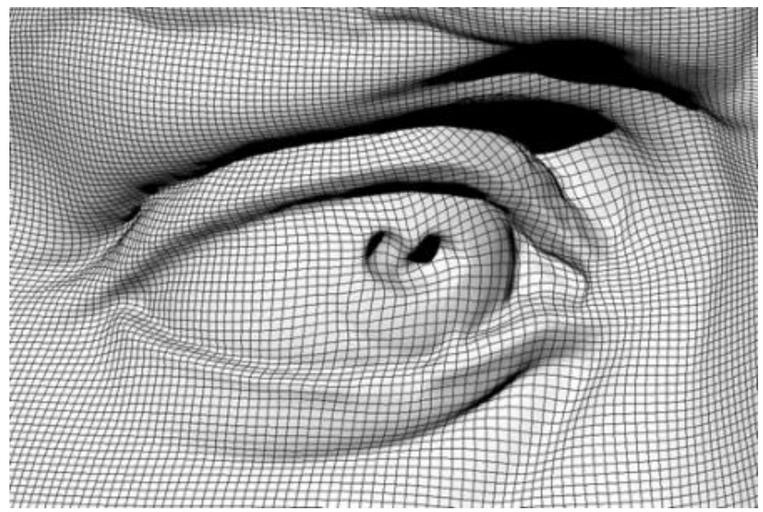
TRIANGLES

i	j	k
0	2	1
0	3	2
3	0	1
3	1	2



- Q: How do we find all the triangles touching vertex 2?
- Ok, now consider a more complicated mesh:

~1 *billion* polygons

Very **expensive to find the neighboring triangles! (What's the cost?)**

Alternative: Incidence Matrices

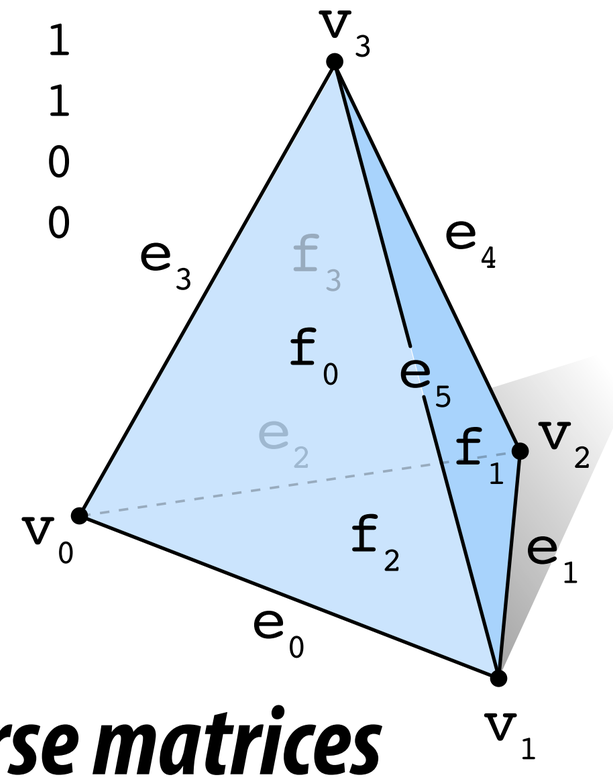
- If we want to answer neighborhood queries, why not simply store a list of neighbors?
- Can encode all neighbor information via *incidence matrices*
- E.g., tetrahedron:

VERTEX↔EDGE

	v0	v1	v2	v3
e0	1	1	0	0
e1	0	1	1	0
e2	1	0	1	0
e3	1	0	0	1
e4	0	0	1	1
e5	0	1	0	1

EDGE↔FACE

	e0	e1	e2	e3	e4	e5
f0	1	0	0	1	0	1
f1	0	1	0	0	1	1
f2	1	1	1	0	0	0
f3	0	0	1	1	1	0

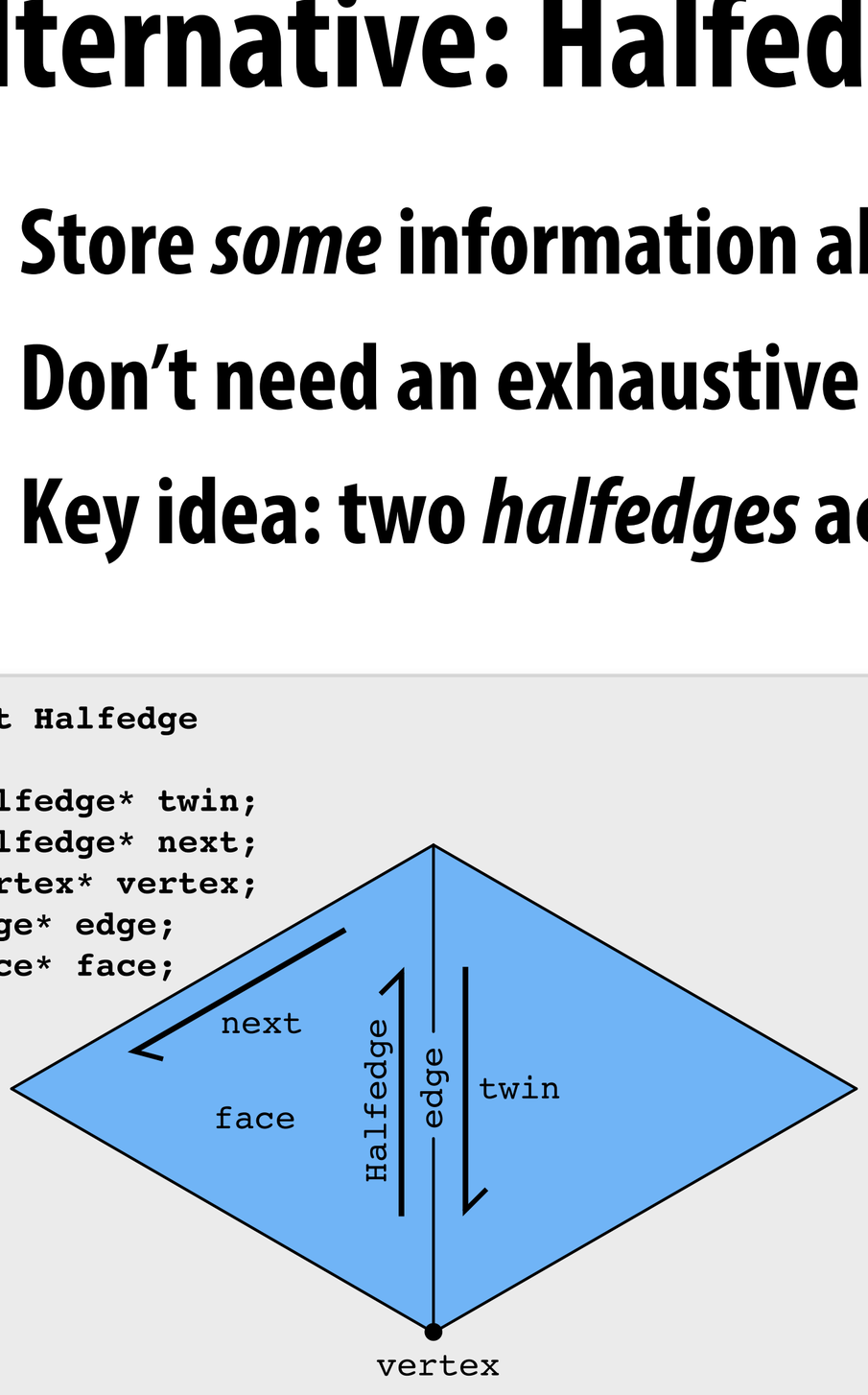


- 1 means "touches"; 0 means "does not touch"
- For large meshes, *most entries will be zero!*
- Can dramatically reduce storage cost using *sparse matrices*
- Still large storage cost, but finding neighbors is now O(1)
- (Bonus feature: mesh does not have to be manifold)

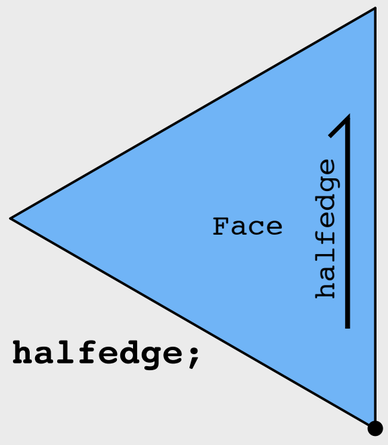
Alternative: Halfedge Data Structure

- Store *some* information about neighbors
- Don't need an exhaustive list; just a few key pointers
- Key idea: two *halfedges* act as "glue" between mesh elements:

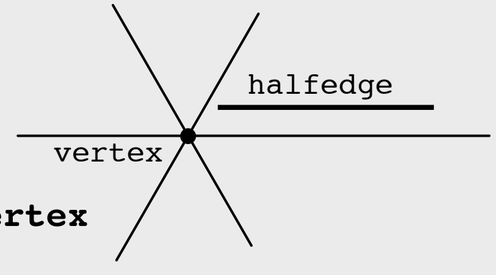
```
struct Halfedge
{
   Halfedge* twin;
   Halfedge* next;
   Vertex* vertex;
   Edge* edge;
   Face* face;
};
```




```
struct Edge
{
   Halfedge* halfedge;
};
```




```
struct Face
{
   Halfedge* halfedge;
};
```



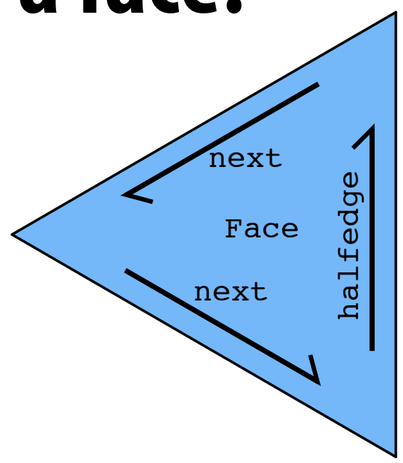

```
struct Vertex
{
   Halfedge* halfedge;
};
```

- Each vertex, edge face points to just *one* of its halfedges.

Halfedge makes mesh traversal easy

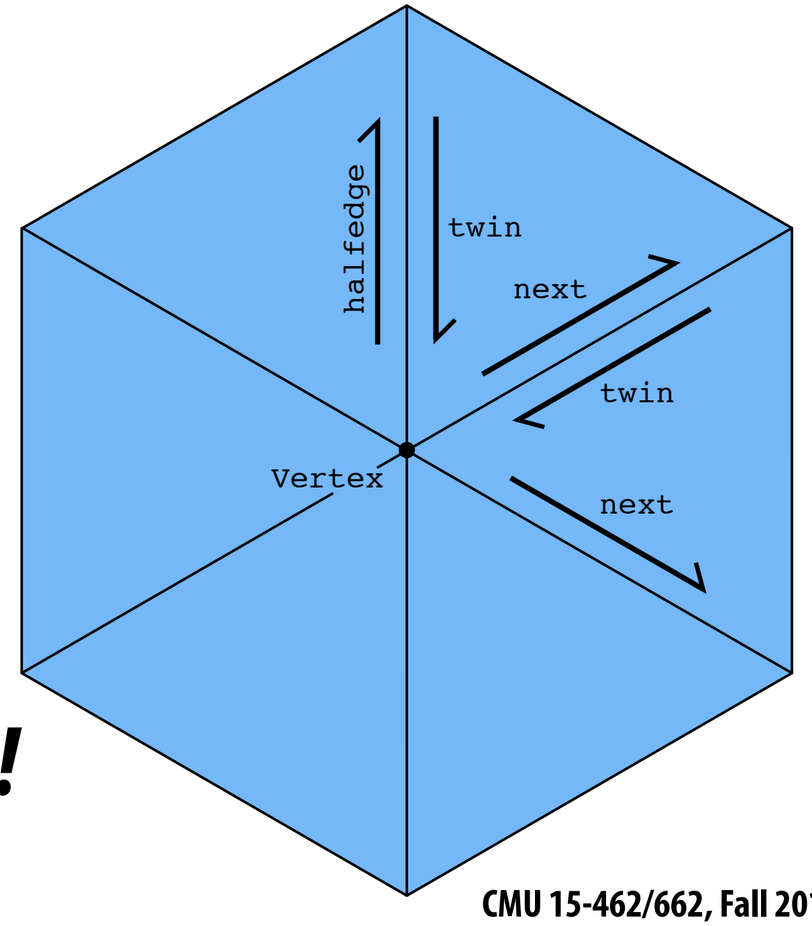
- Use “twin” and “next” pointers to move around mesh
- Use “vertex”, “edge”, and “face” pointers to grab element
- Example: visit all vertices of a face:

```
Halfedge* h = f->halfedge;
do {
   h = h->next;
}
while( h != f->halfedge );
```



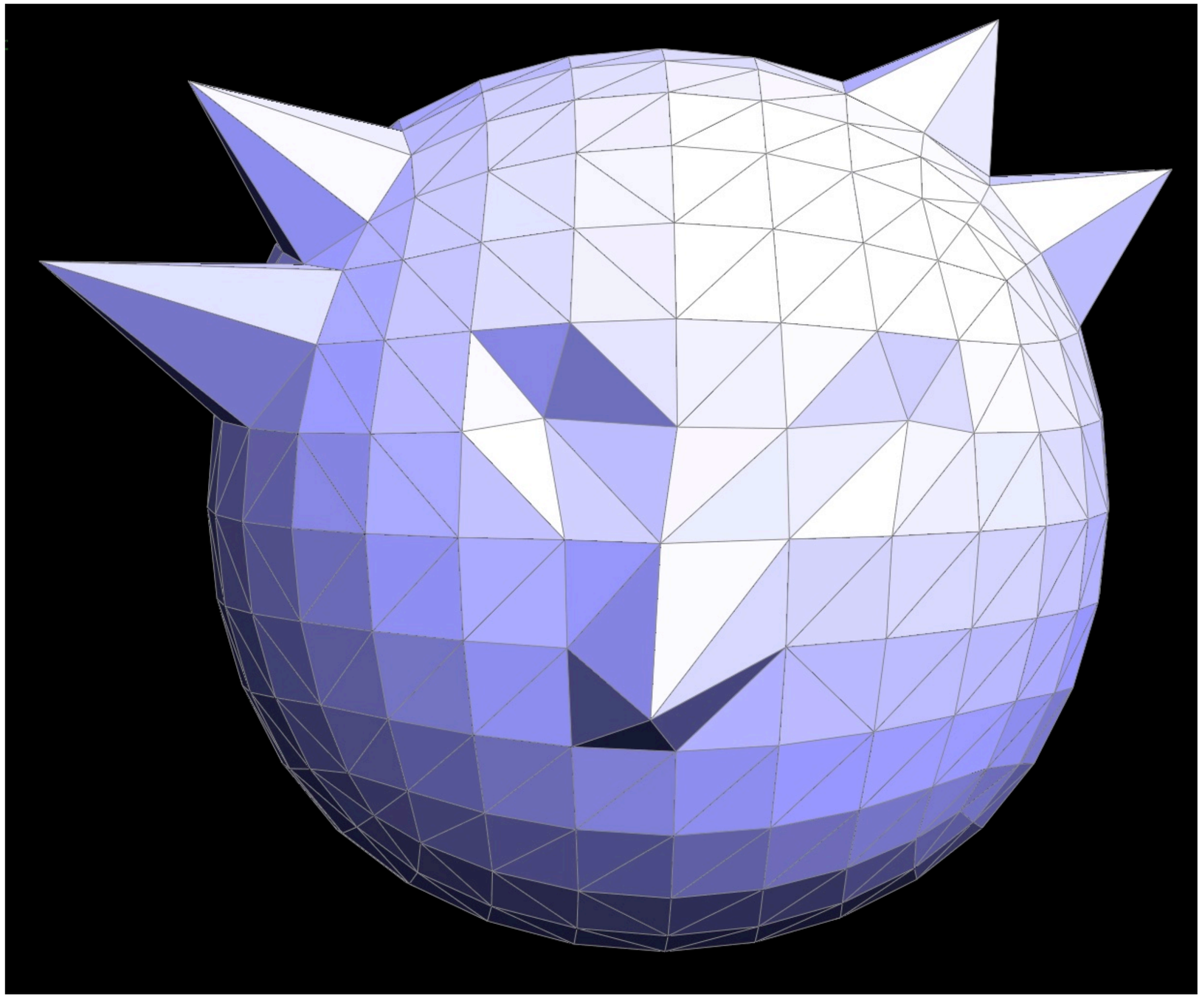
- Example: visit all neighbors of a vertex:

```
Halfedge* h = v->halfedge;
do {
   h = h->twin->next;
}
while( h != v->halfedge );
```



- [DEMO]
- Note: only makes sense if mesh is *manifold!*

Next time: Geometry Processing!



(How do we make this mesh look more like Kayvon?)