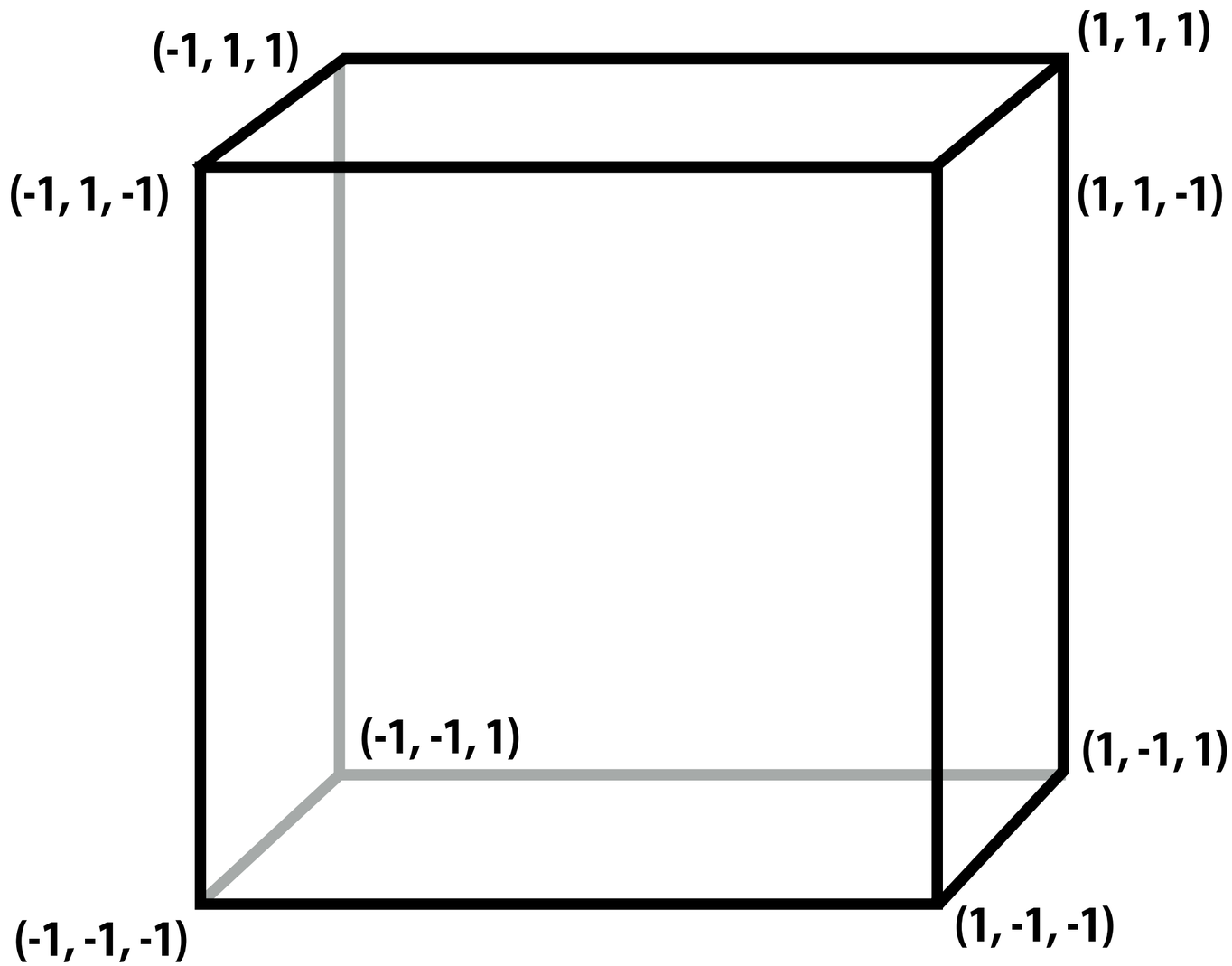
Lecture 3:

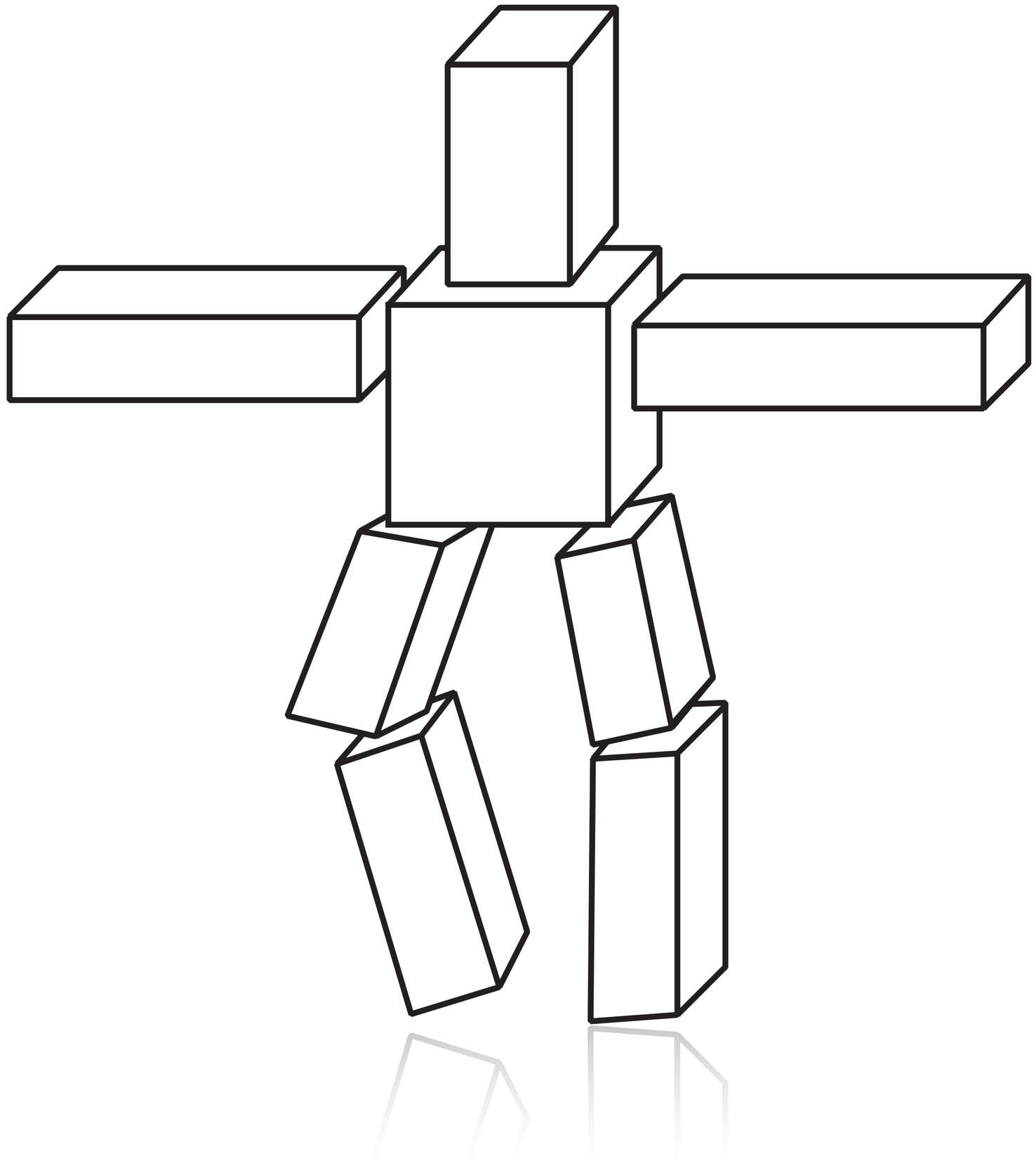
Transforms

Computer Graphics
CMU 15-462/15-662, Fall 2015

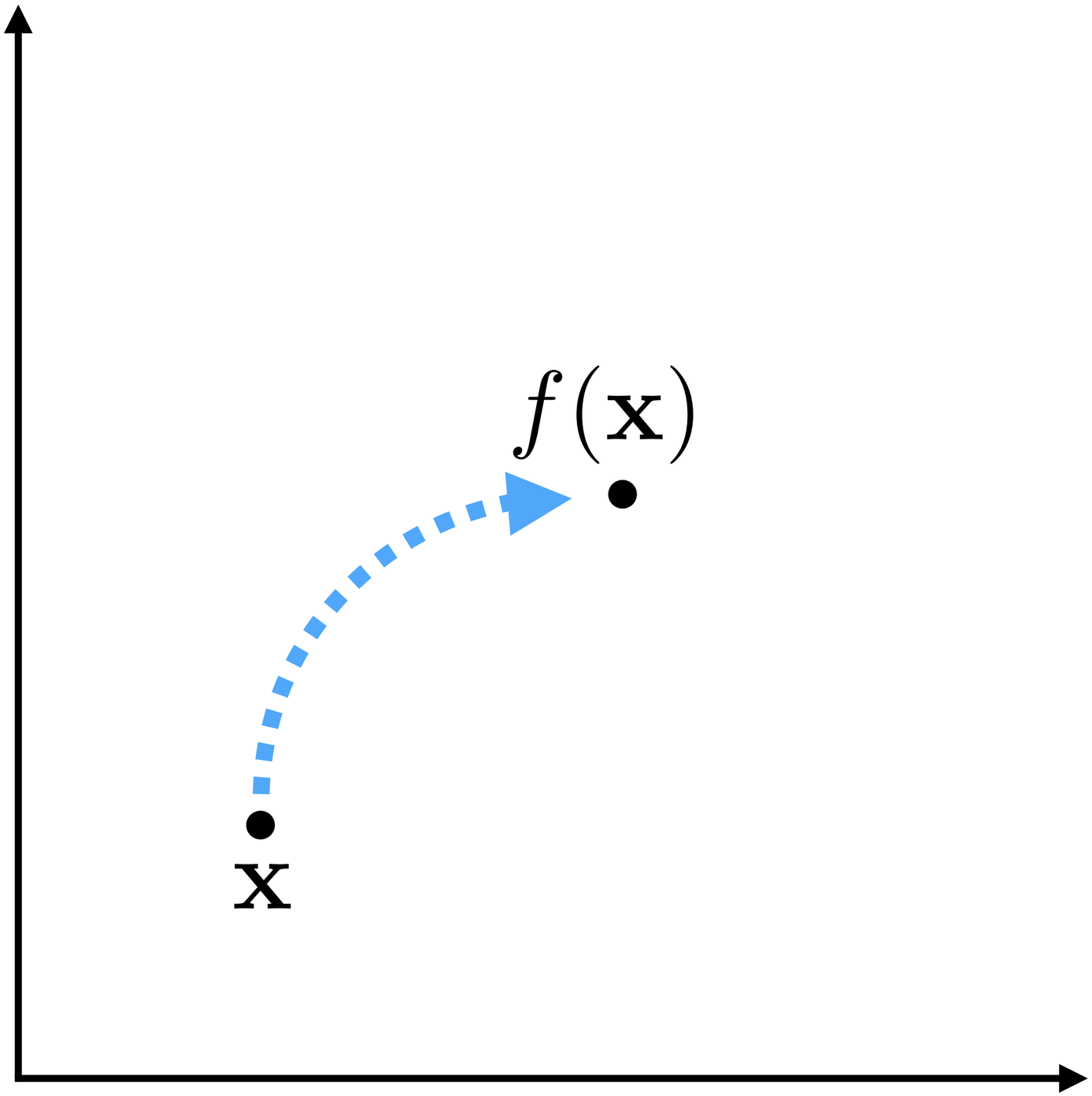
Cube



Cube man



f transforms $\mathbf{x}$ to $f(\mathbf{x})$

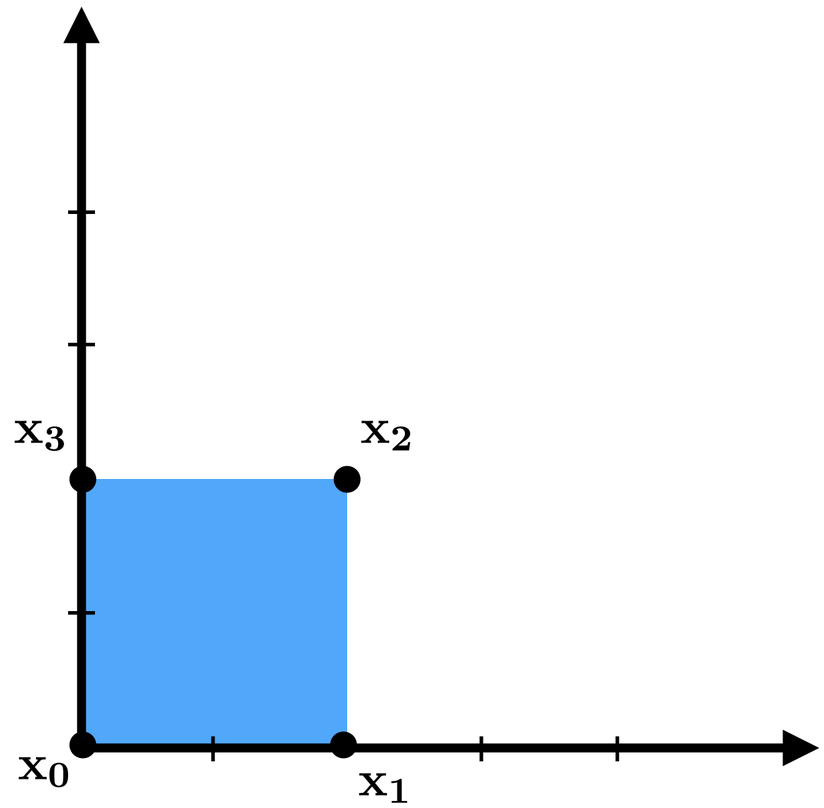


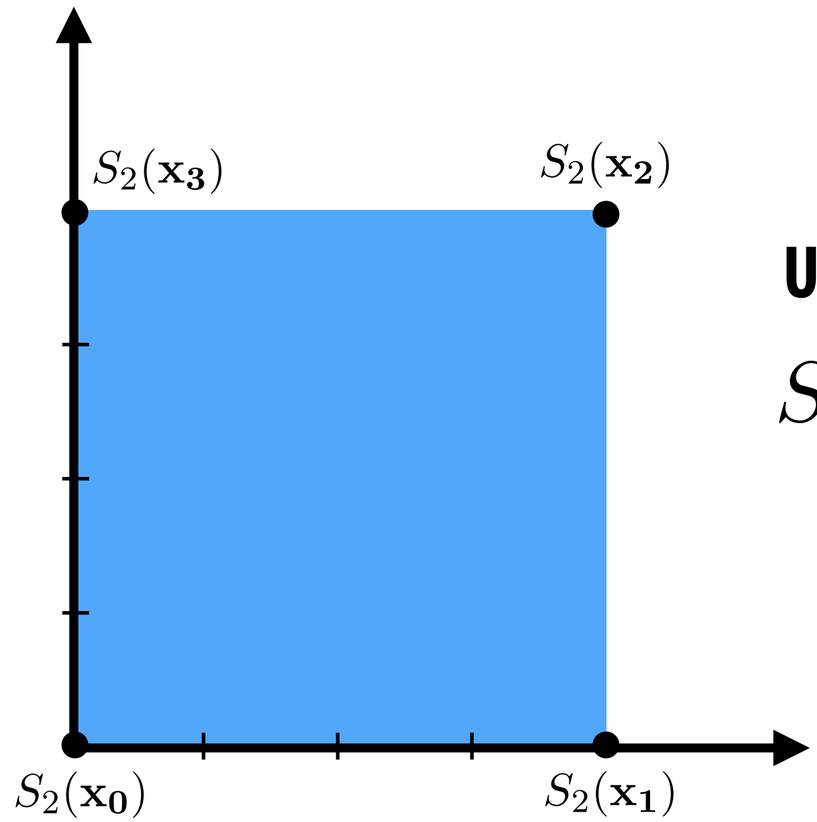
Linear transforms

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$

$$f(a\mathbf{x}) = af(\mathbf{x})$$

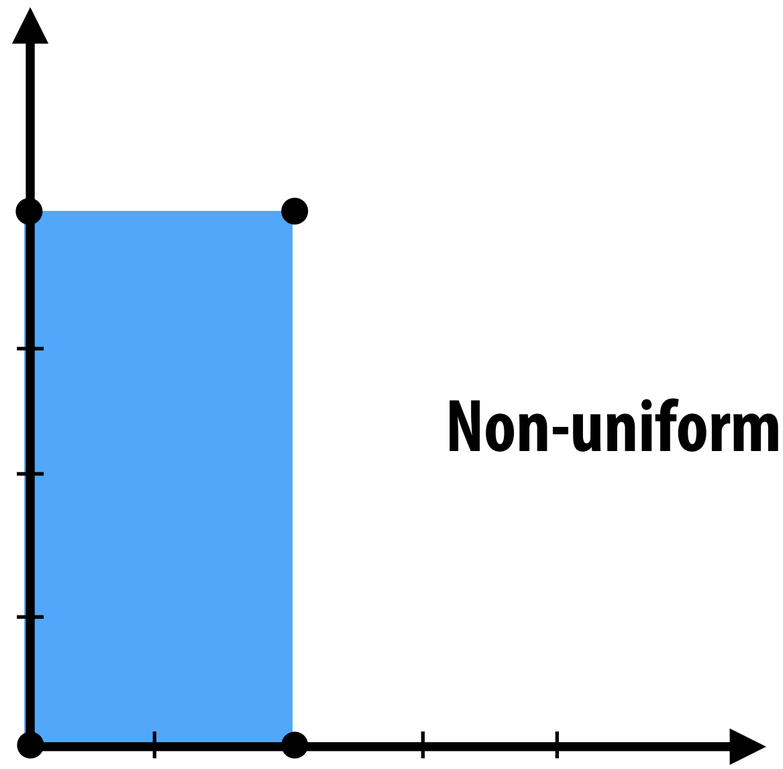
Scale





Uniform scale:

$$S_a(\mathbf{x}) = a\mathbf{x}$$



Non-uniform scale??

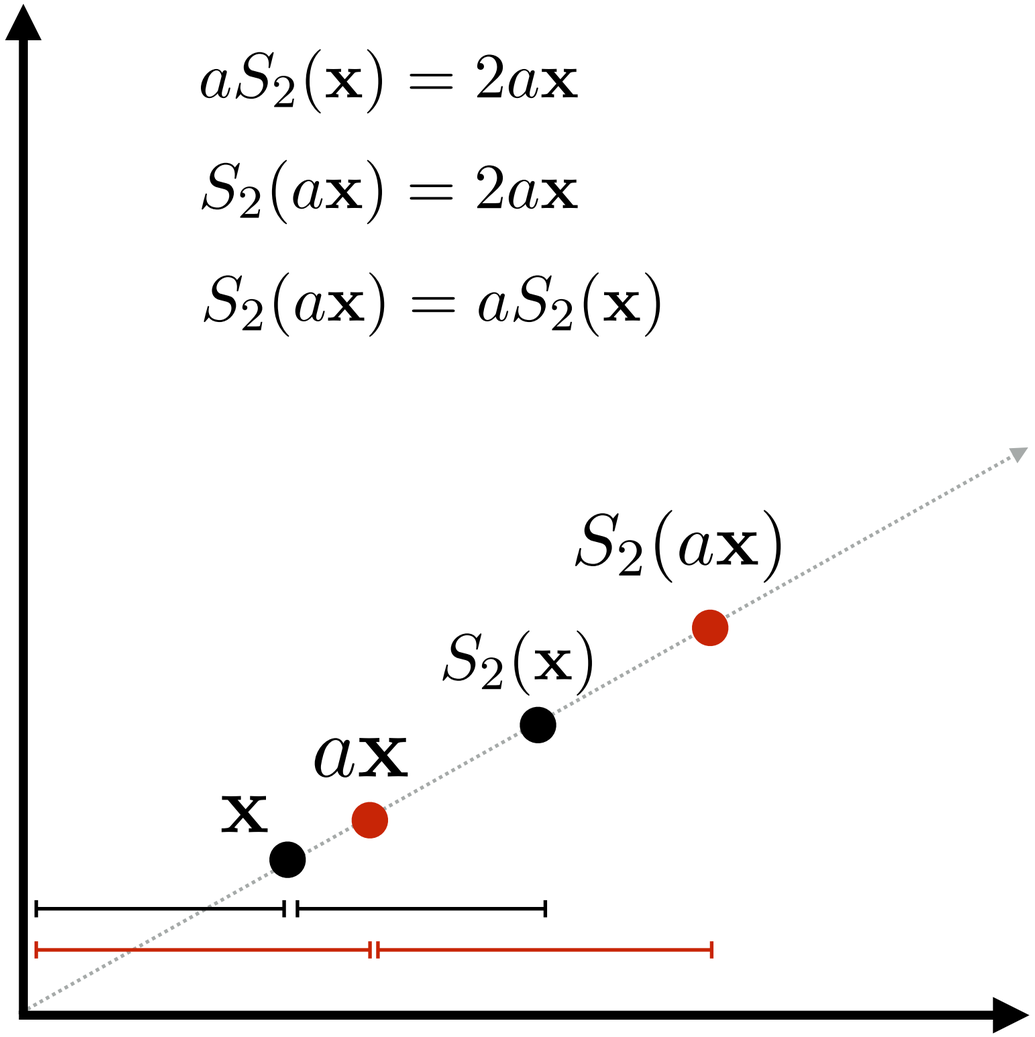
Is scale a linear transform?

$$S_2(\mathbf{x}) = 2\mathbf{x}$$

$$aS_2(\mathbf{x}) = 2a\mathbf{x}$$

$$S_2(a\mathbf{x}) = 2a\mathbf{x}$$

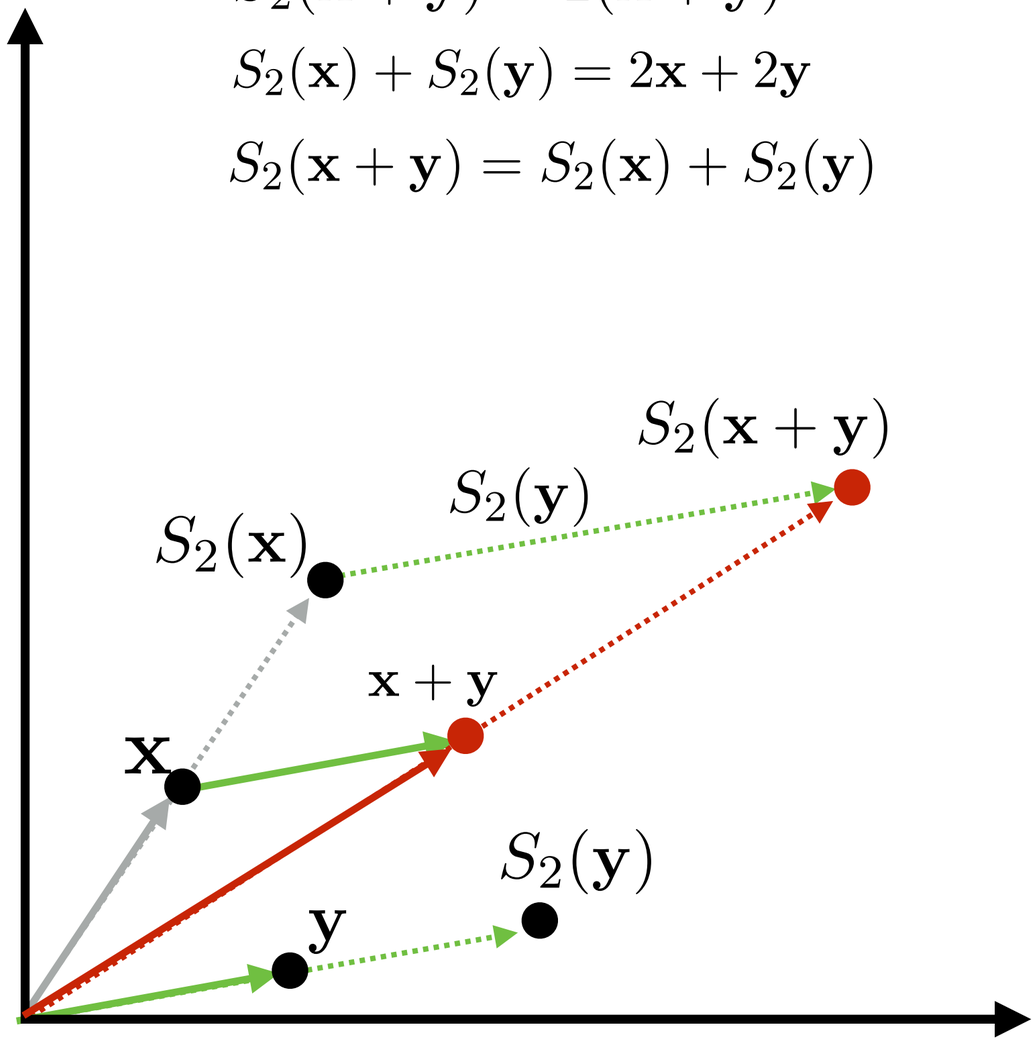
$$S_2(a\mathbf{x}) = aS_2(\mathbf{x})$$



$$S_2(\mathbf{x}+\mathbf{y}) = 2(\mathbf{x}+\mathbf{y})$$

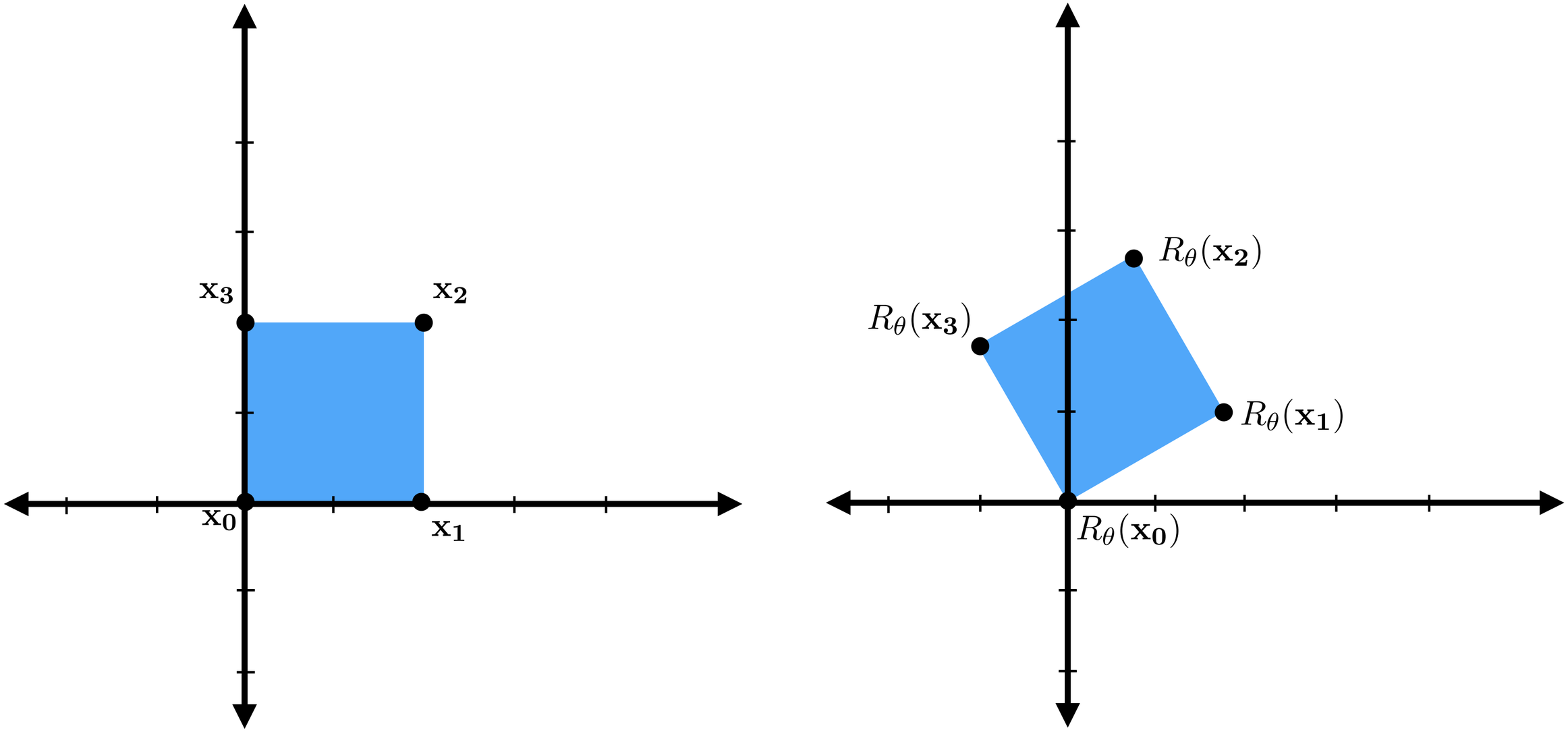
$$S_2(\mathbf{x}) + S_2(\mathbf{y}) = 2\mathbf{x} + 2\mathbf{y}$$

$$S_2(\mathbf{x}+\mathbf{y}) = S_2(\mathbf{x}) + S_2(\mathbf{y})$$



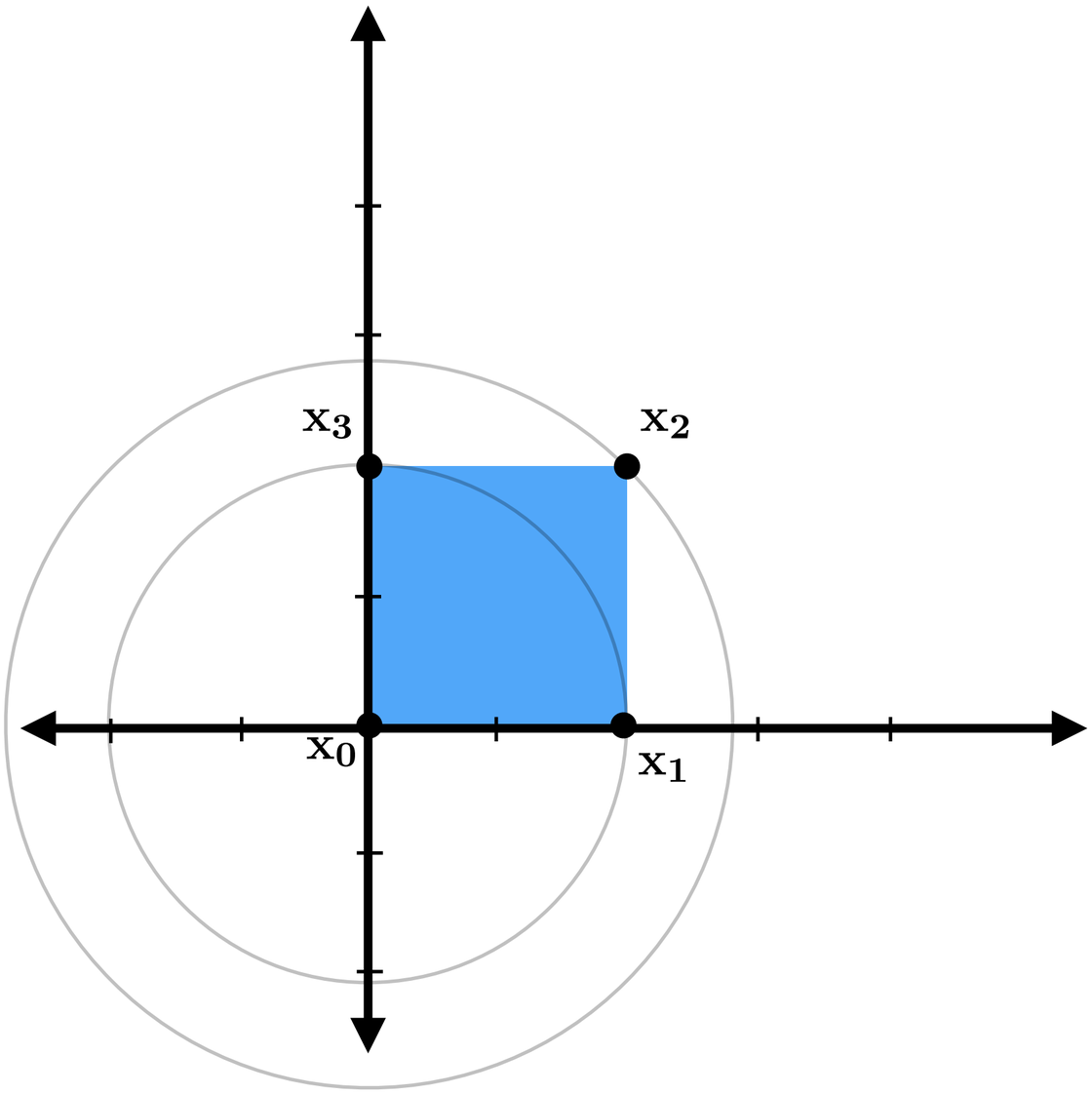
Yes!

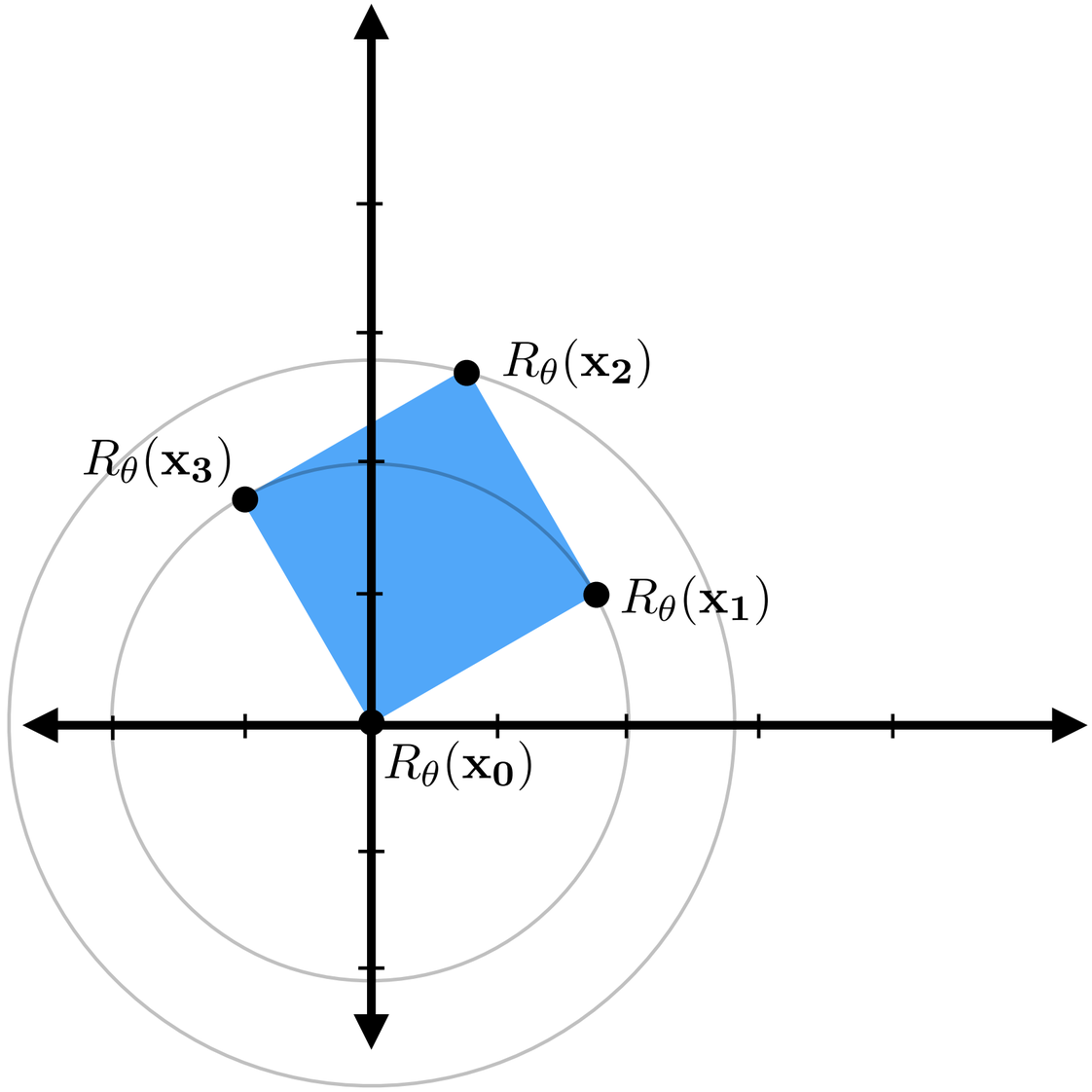
Rotation



R_θ = **rotate counter-clockwise by** θ

Rotation as Circular Motion



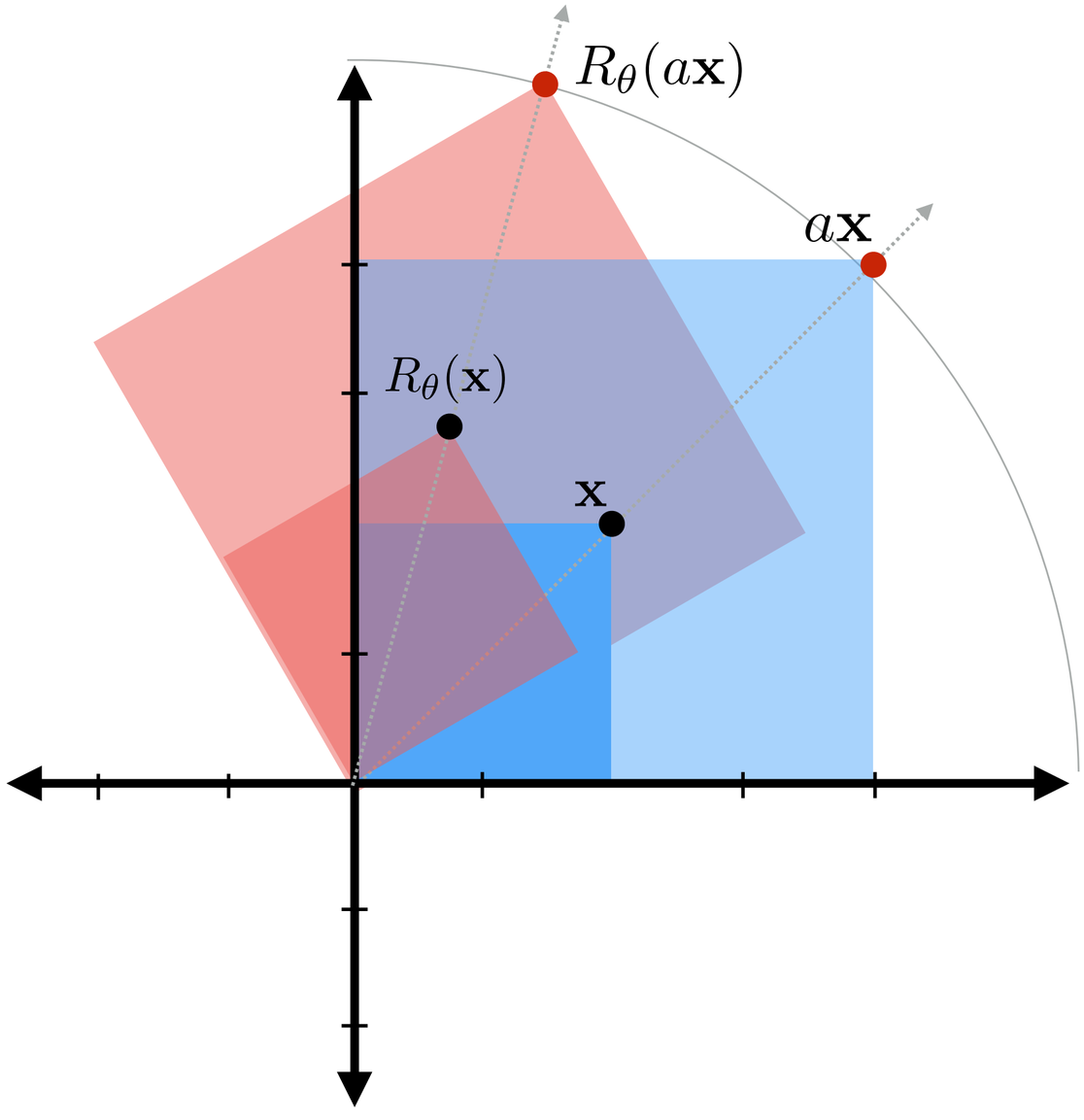


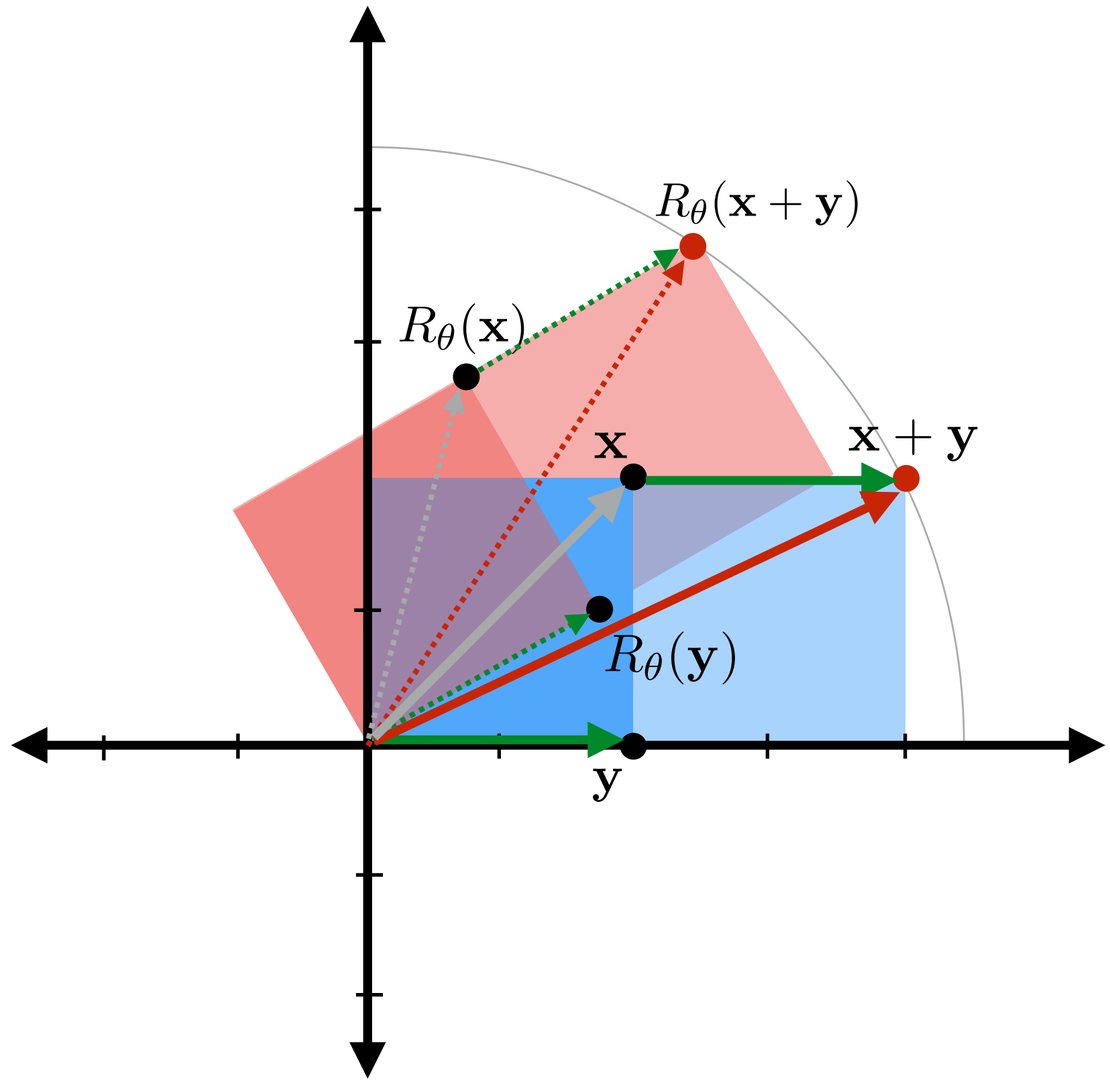
R_θ = **rotate counter-clockwise by** θ

As angle changes, points move along *circular* trajectories.

Hence, rotations preserve length of vectors: $\left|R_\theta(\mathbf{x})\right| = \left|\mathbf{x}\right|$

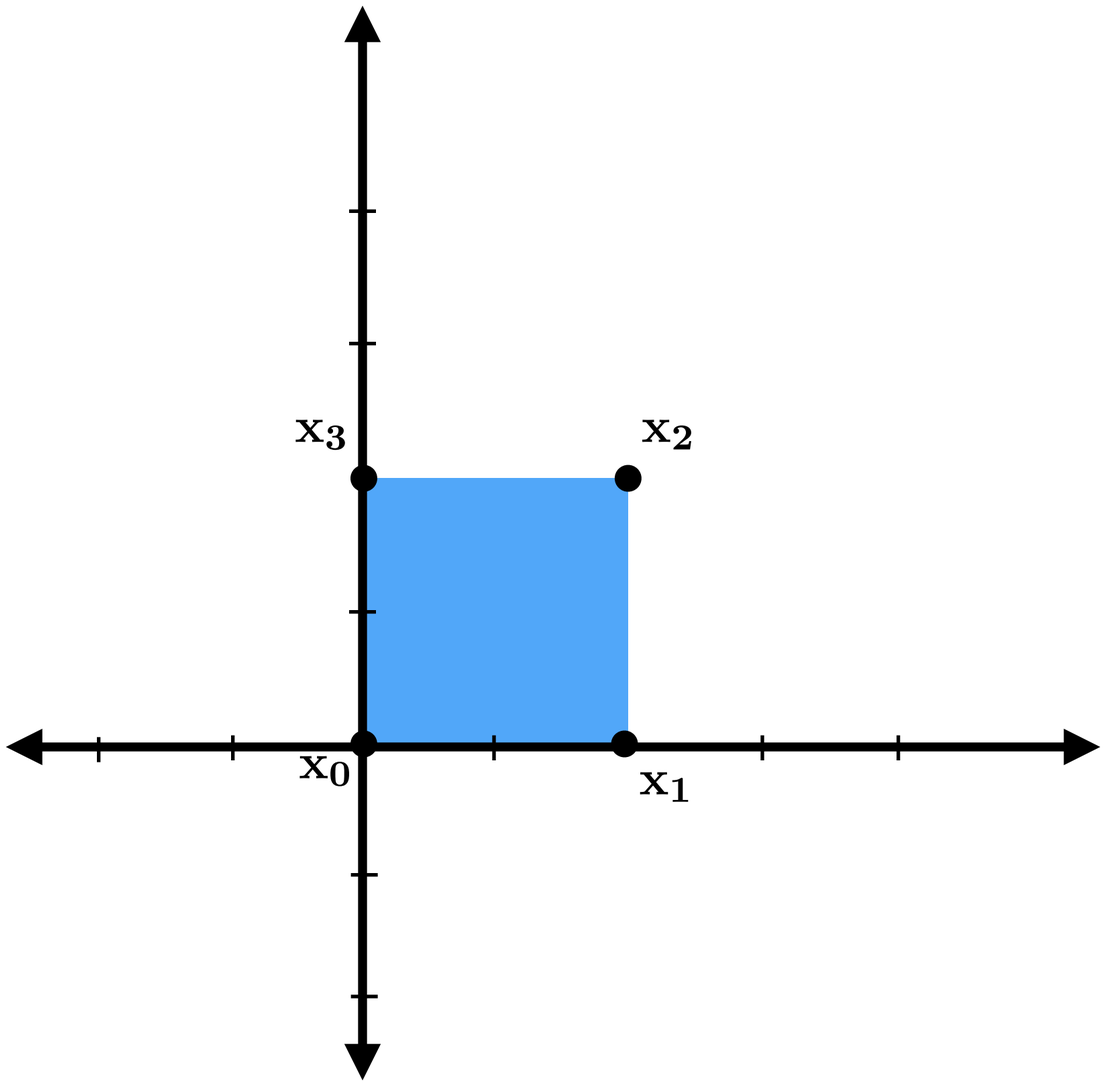
Is rotation linear?

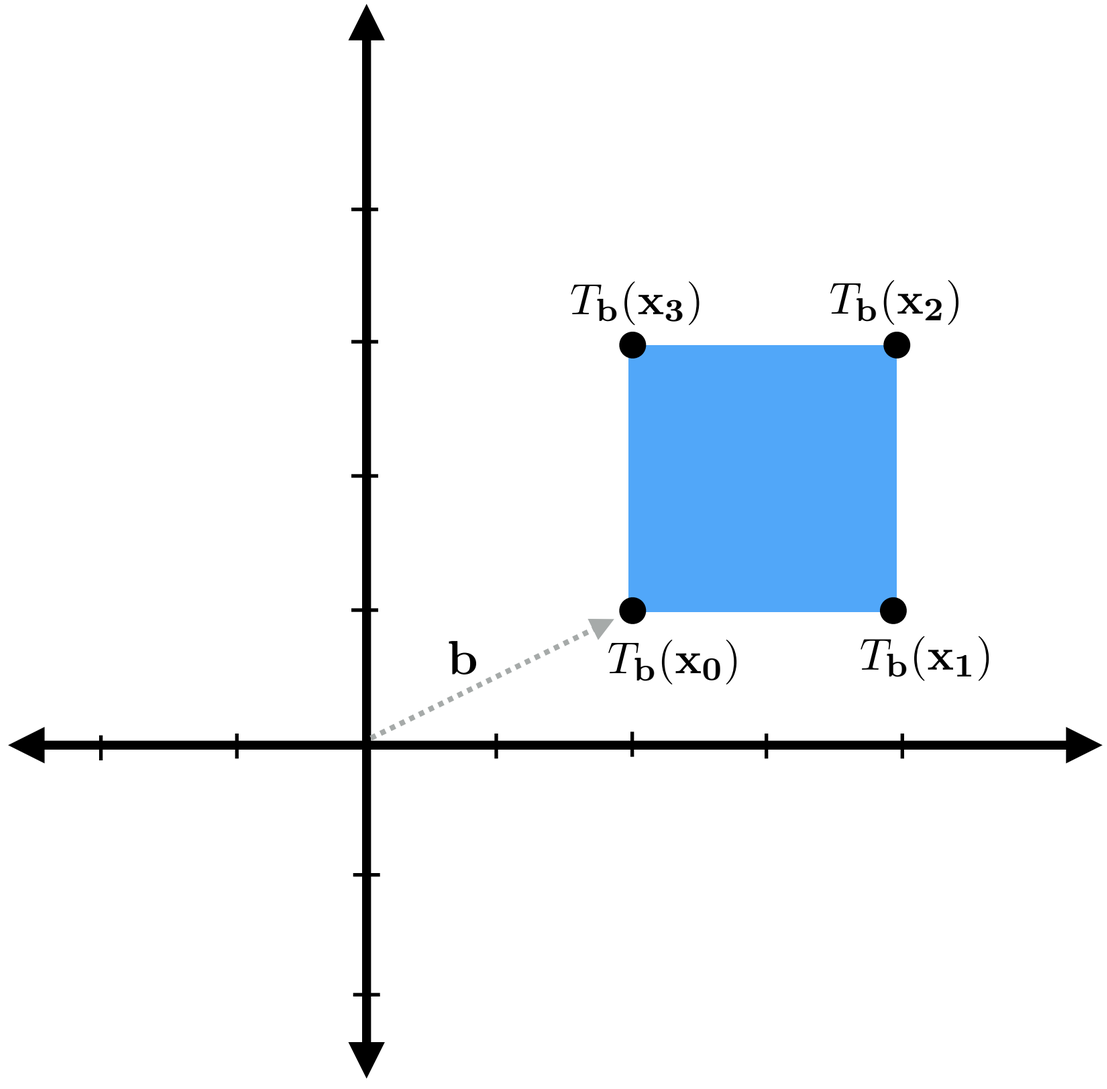




Yes!

Translation

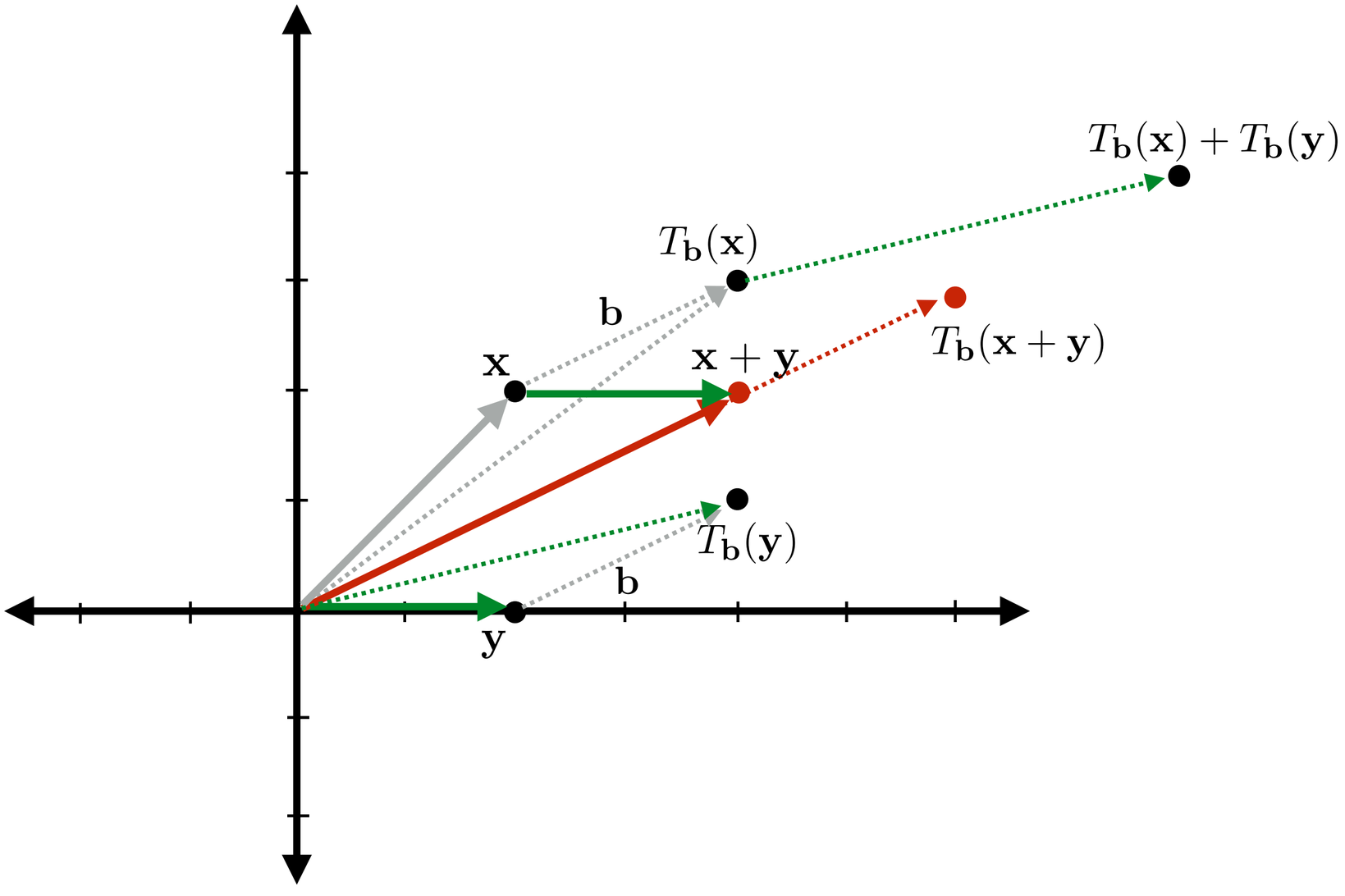




$T_\mathbf{b}(\mathbf{x})$ = **translate by** $\mathbf{b}$

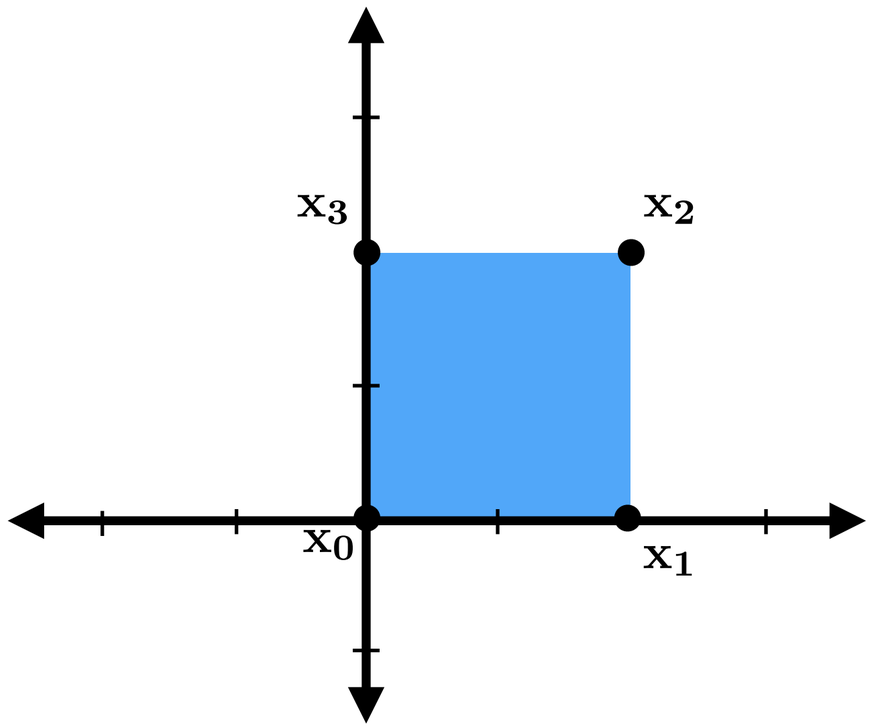
$$T_\mathbf{b}(\mathbf{x}) = \mathbf{x} + \mathbf{b}$$

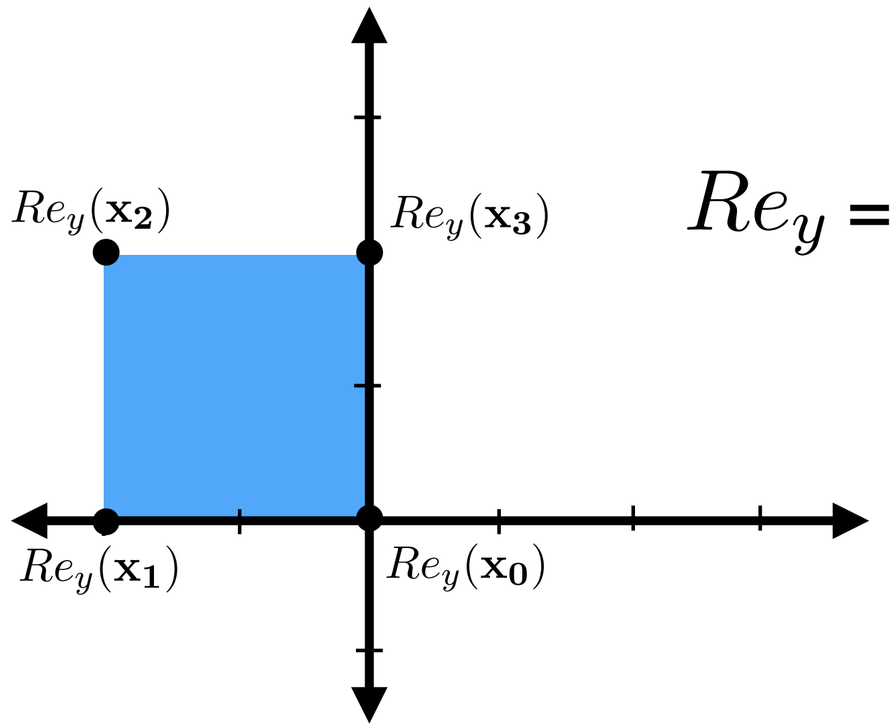
Is translation linear?



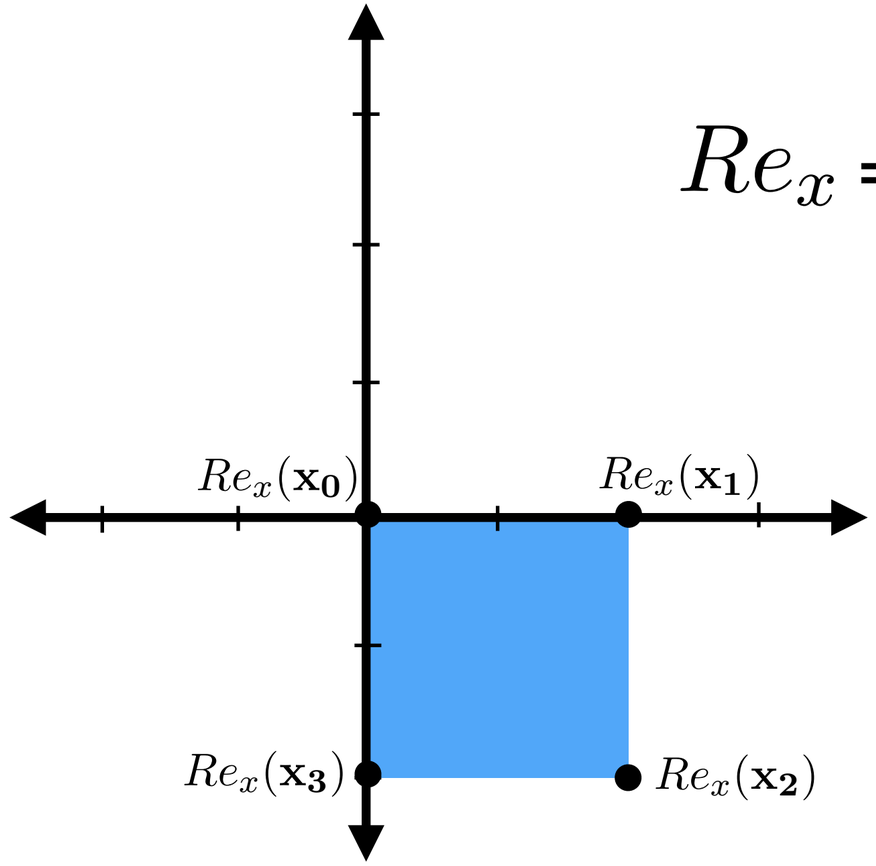
No. Translation is affine.

Reflection



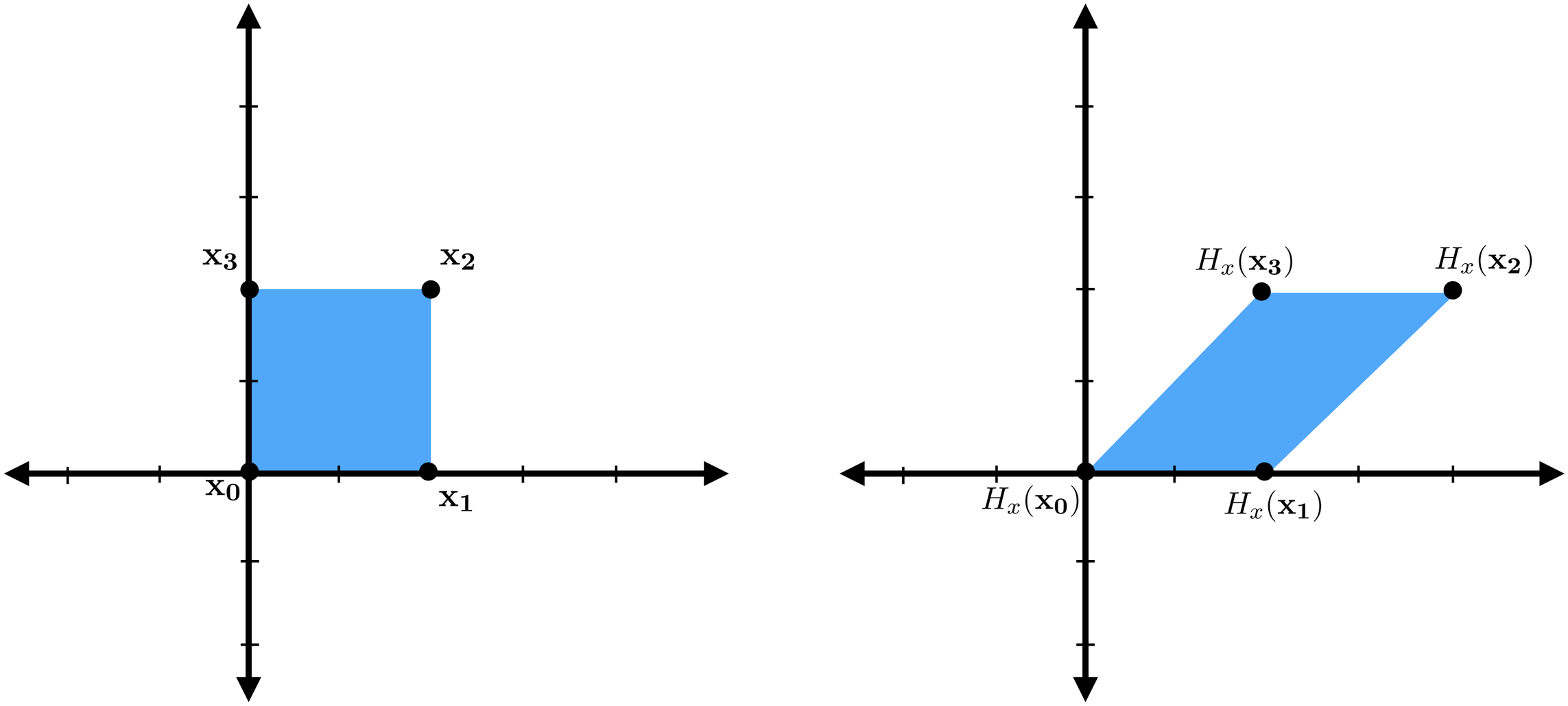


Re_y = **reflection about *y***

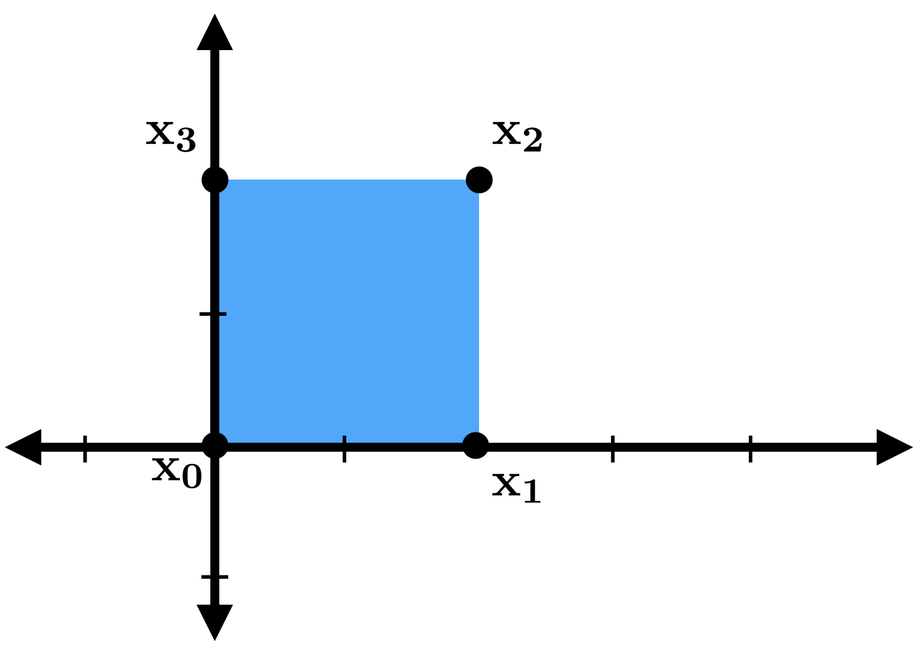


Re_x = **reflection about *x***

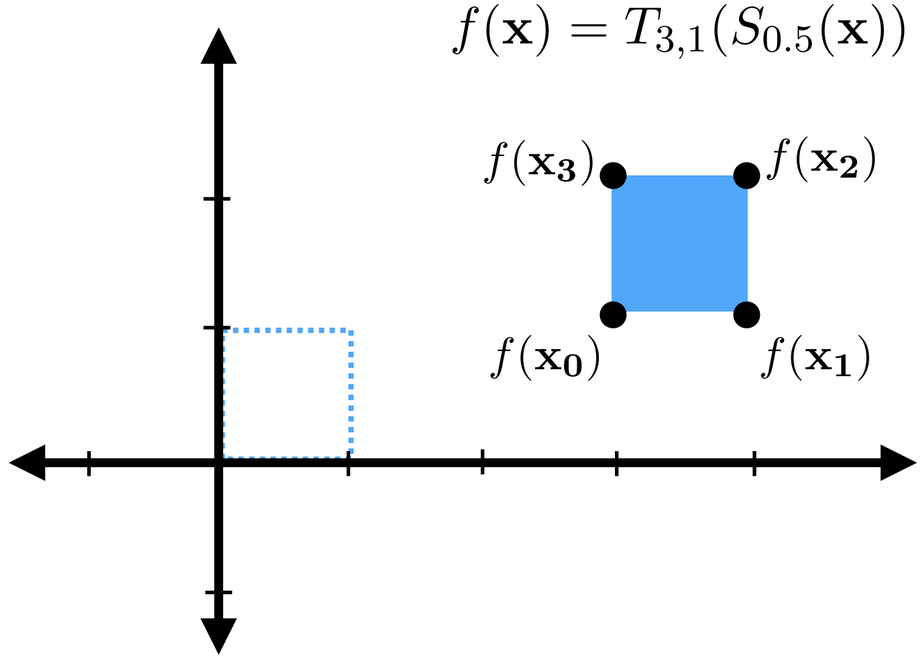
Shear (in x direction)



Compose basic transforms to construct more complex transforms



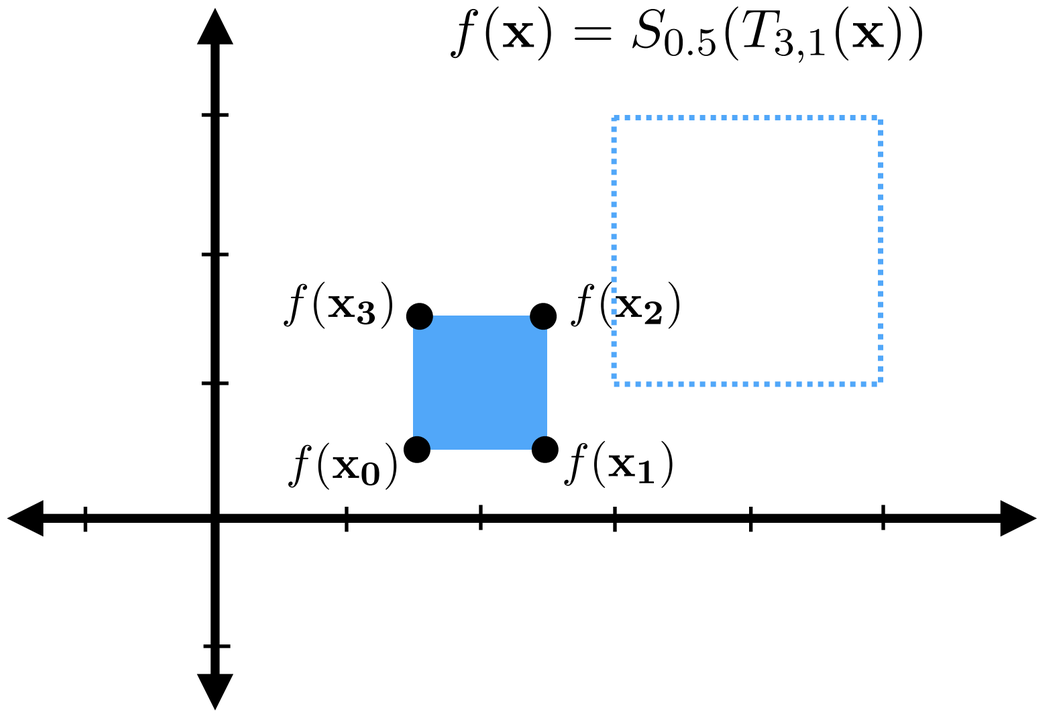
$$f(\mathbf{x}) = T_{3,1}(S_{0.5}(\mathbf{x}))$$



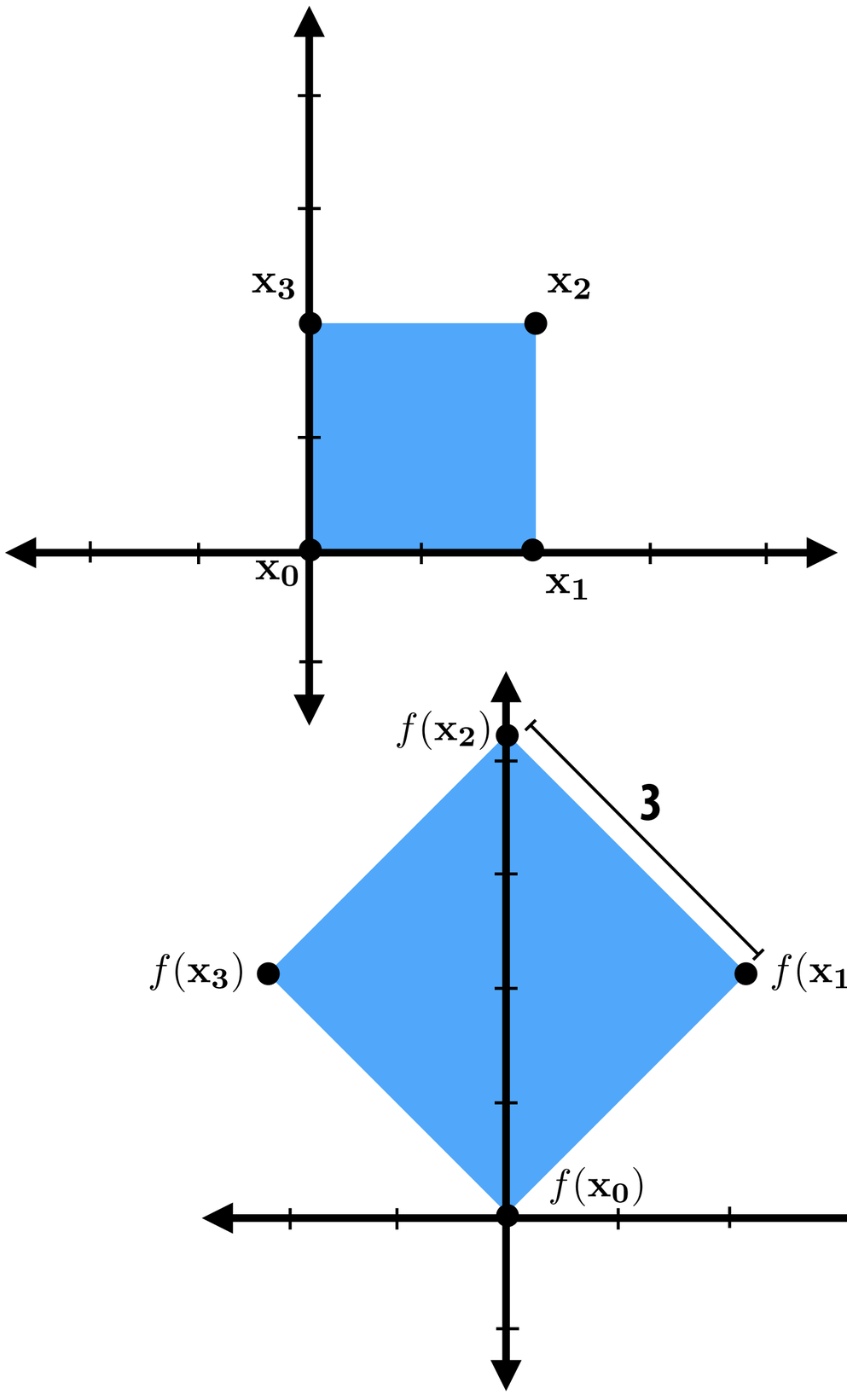
Note: order of composition matters

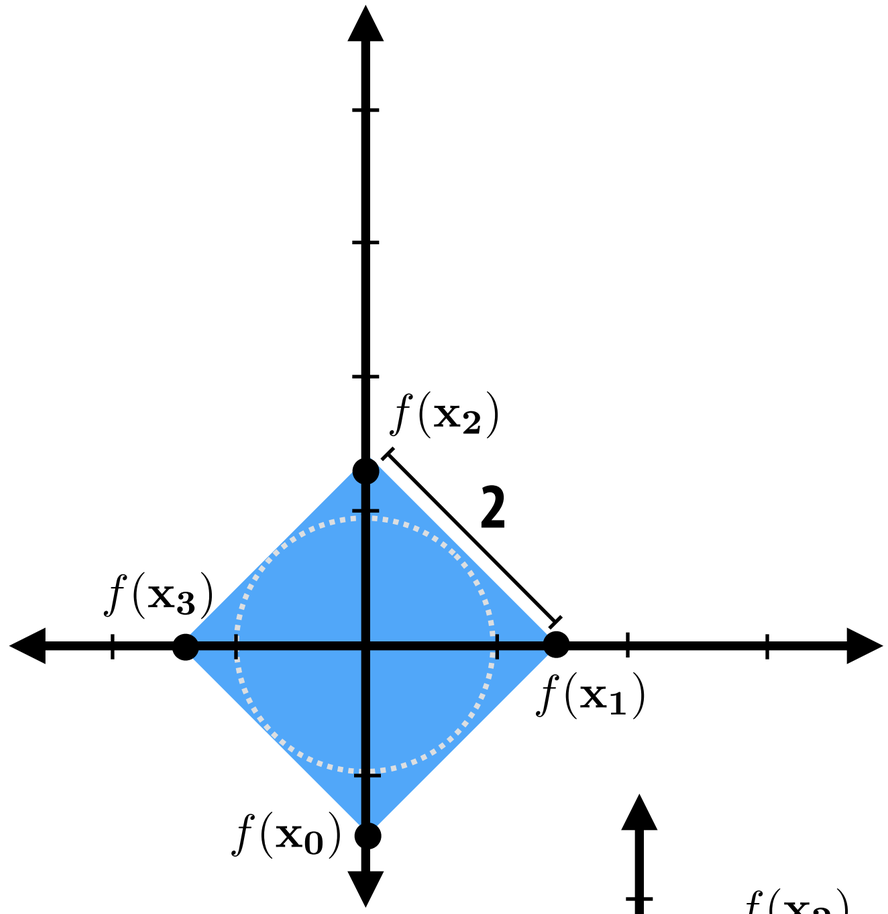
Top-right: scale, then translate
Bottom-right: translate, then scale

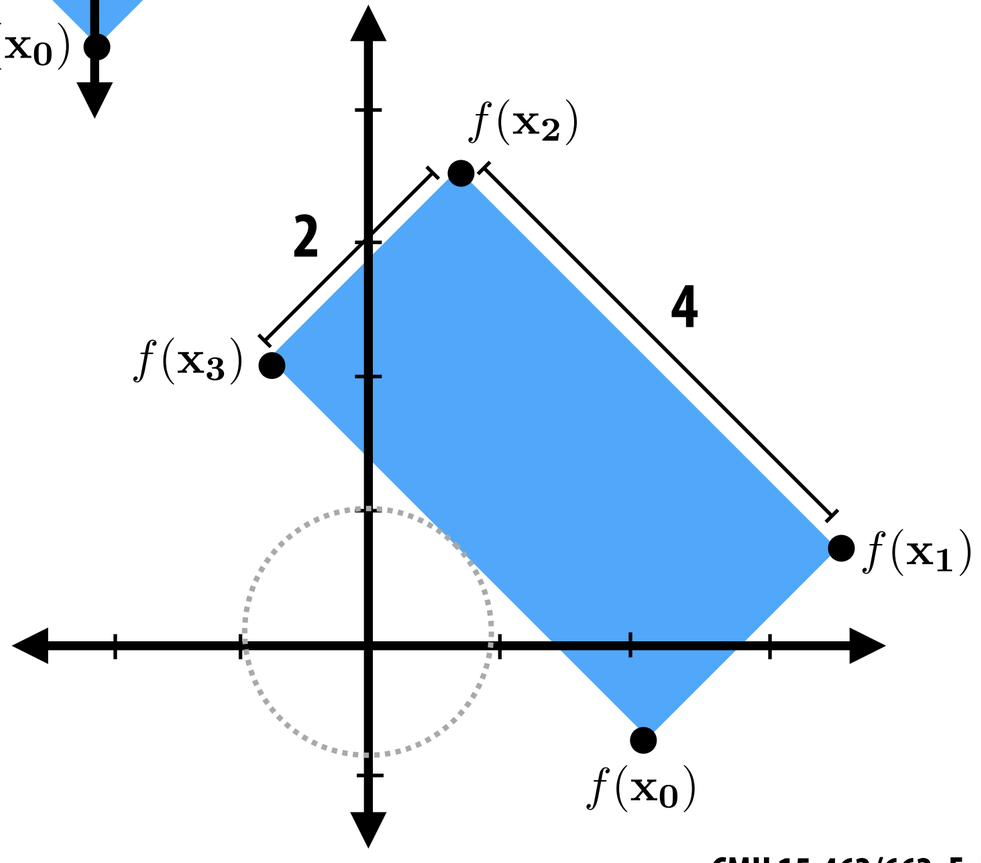
$$f(\mathbf{x}) = S_{0.5}(T_{3,1}(\mathbf{x}))$$



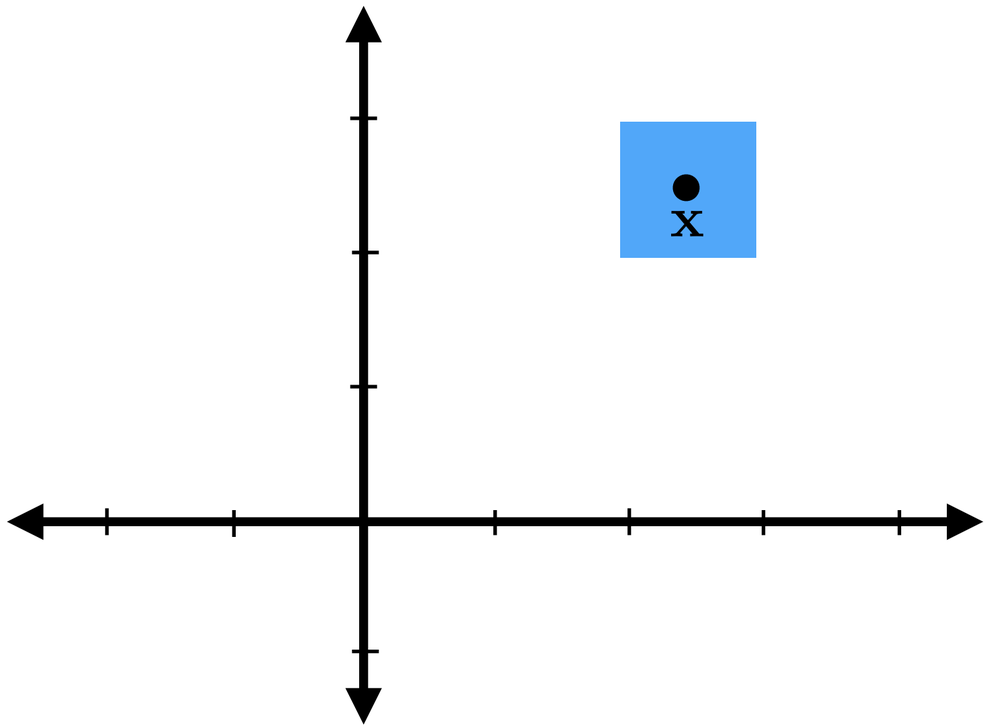
How would you perform these transformations?

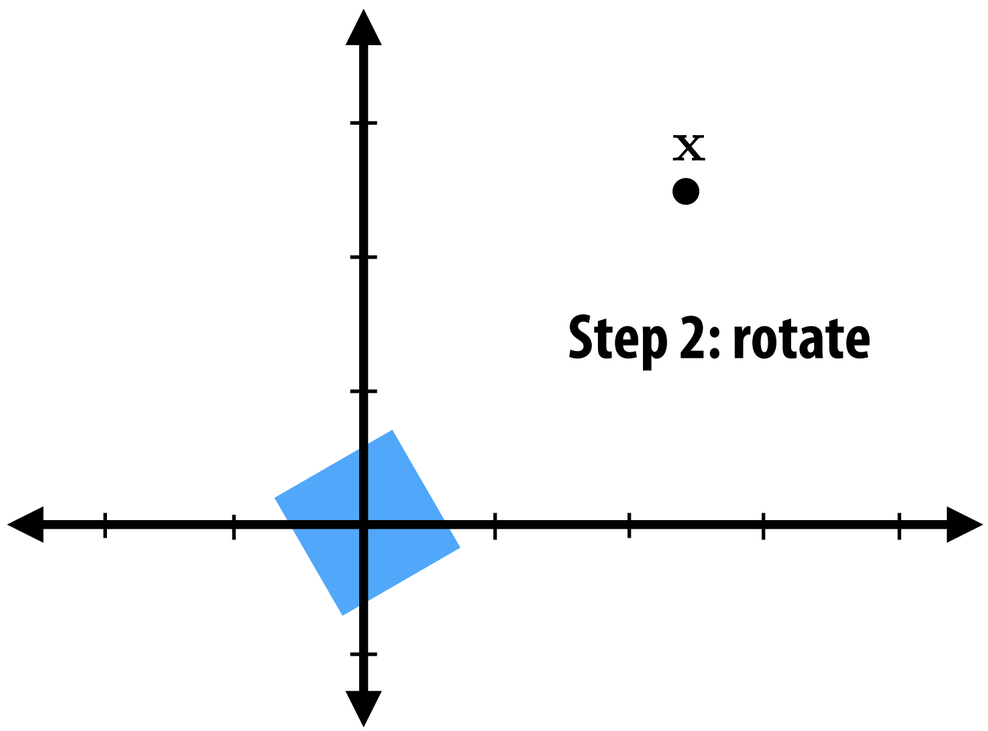


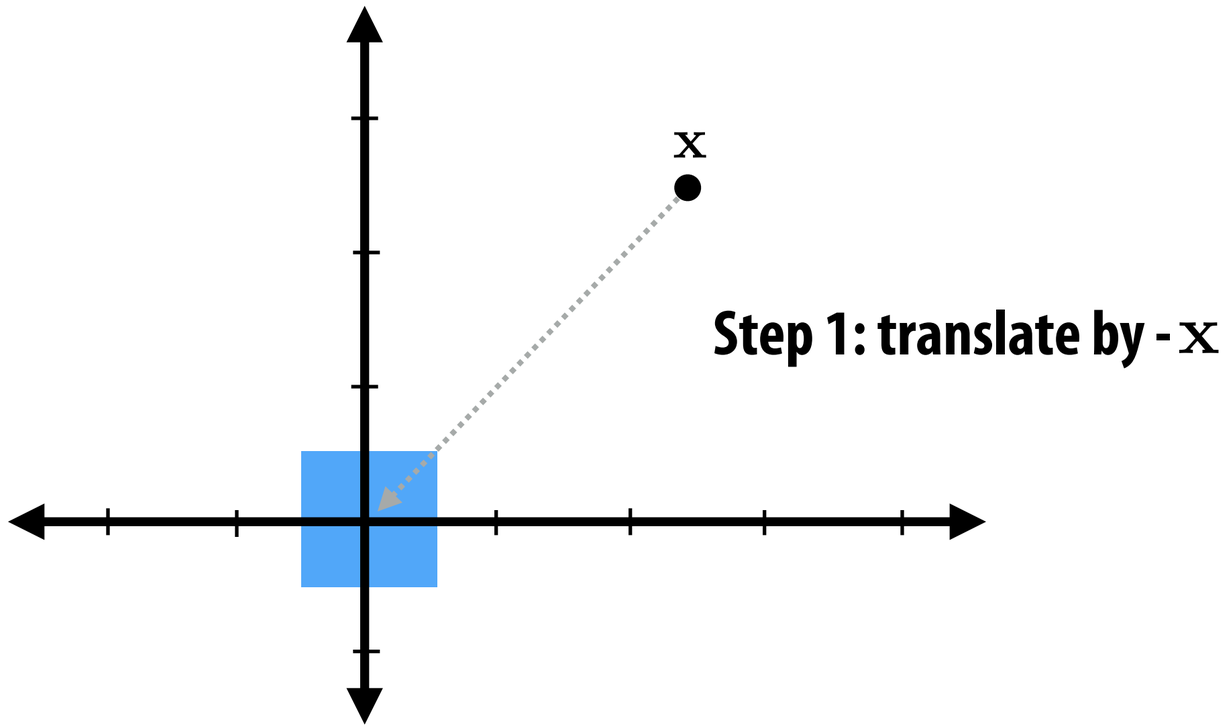


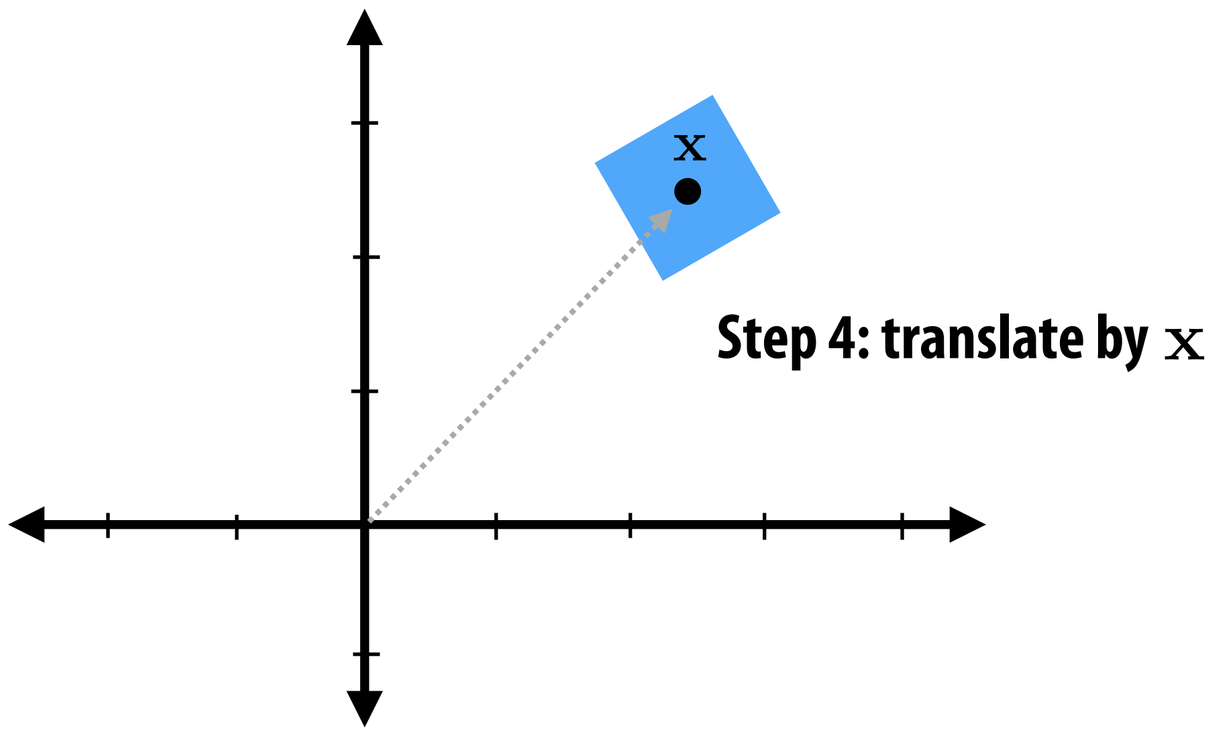


Common pattern: rotation about point x









Summary of basic transforms

Linear:

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$
$$f(a\mathbf{x}) = af(\mathbf{x})$$

Scale
Rotation
Reflection
Shear

Not linear:

Translation

Affine:

Composition of linear transform + translation
(all examples on previous two slides)

$$f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{b}$$

Not affine: perspective projection (will discuss later)

Euclidean: (Isometries)

Preserve distance between points (preserves length)

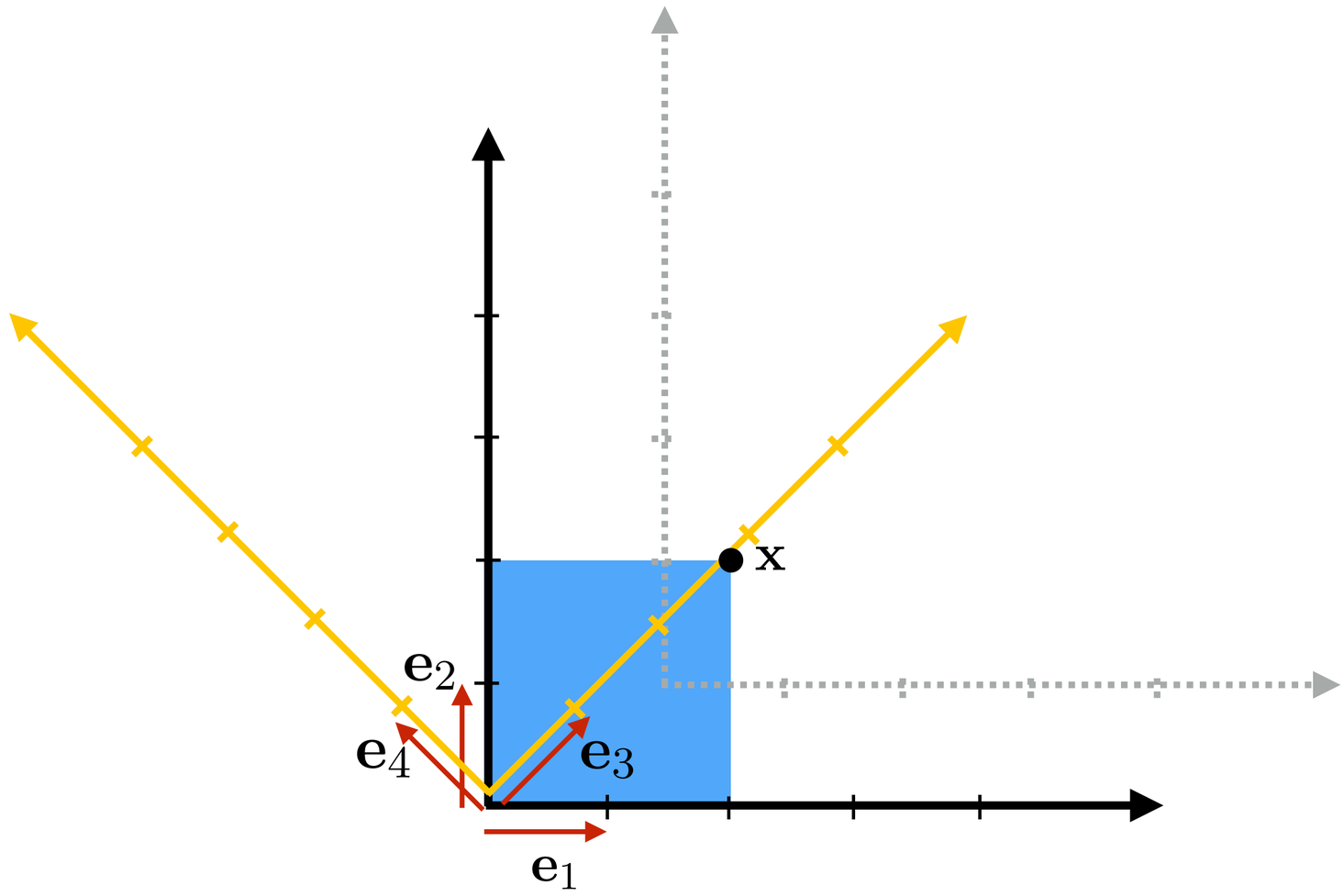
$$|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$$

Translation
Rotation
Reflection

"Rigid body" transforms are Euclidean transforms that also preserve "winding" (does not include reflection)

Representing Transforms

Review: representing points in a coordinate space



Consider coordinate space defined by orthogonal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$

$$\mathbf{x} = 2\mathbf{e}_1 + 2\mathbf{e}_2$$

$$\mathbf{x} = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

$\mathbf{x} = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$ **in coordinate space defined by $\mathbf{e}_1$ and $\mathbf{e}_2$, with origin at (1.5, 1)**

$\mathbf{x} = \begin{bmatrix} \sqrt{8} & 0 \end{bmatrix}$ **in coordinate space defined by $\mathbf{e}_3$ and $\mathbf{e}_4$, with origin at (0, 0)**

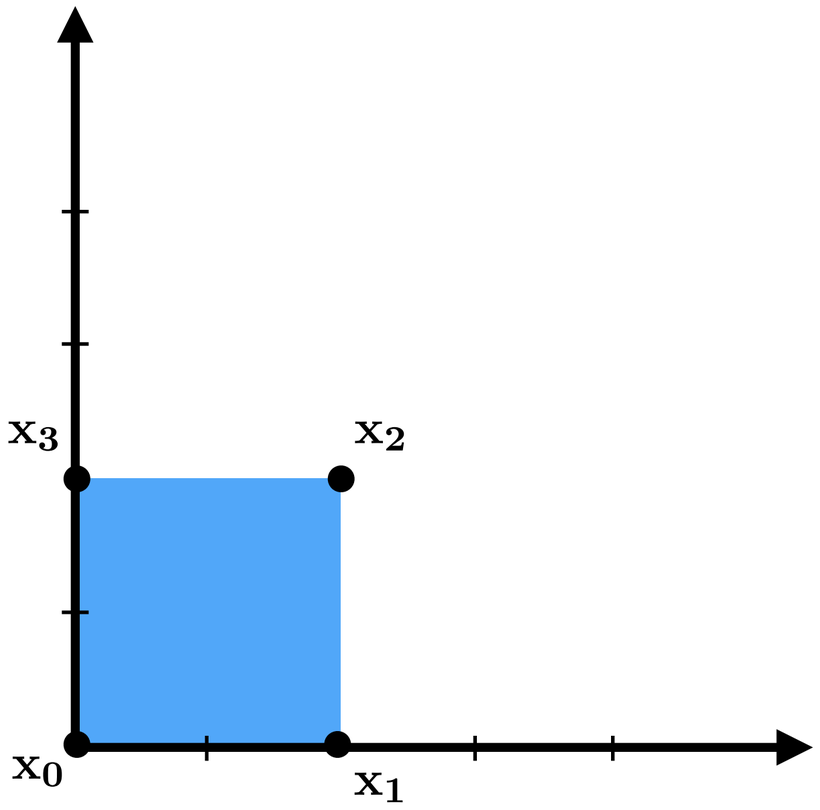
Review: 2D matrix multiplication

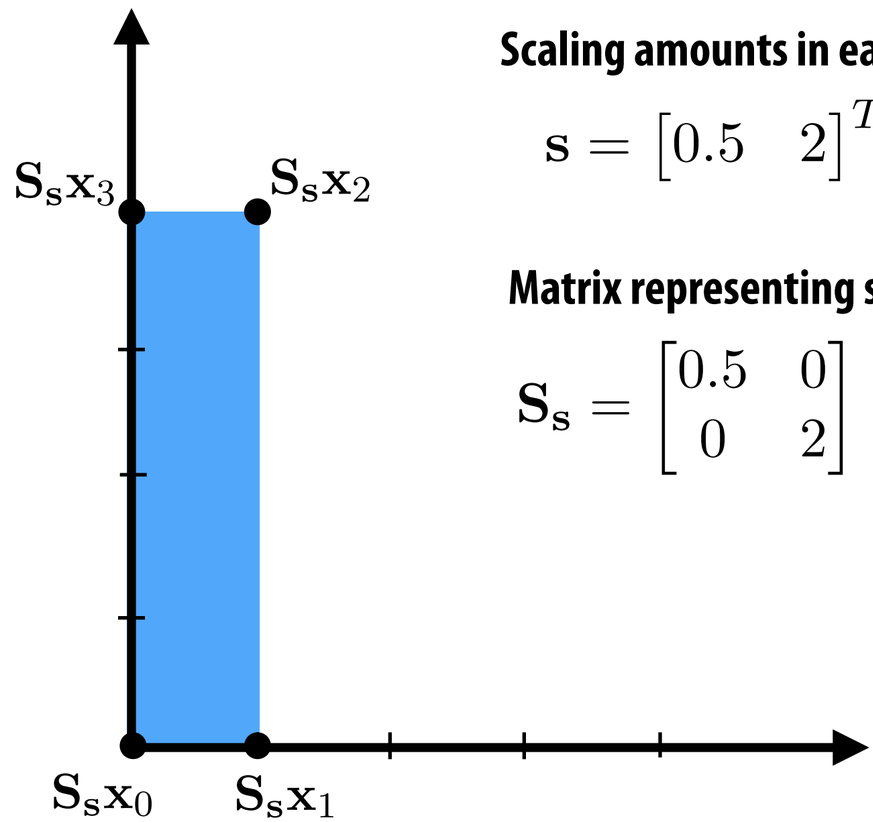
$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} =$$

Linear transforms in 2D can be represented as 2x2 matrices

Consider non-uniform scale: $\mathbf{S_s} = \begin{bmatrix} \mathbf{s}_x & 0 \\ 0 & \mathbf{s}_y \end{bmatrix}$





Scaling amounts in each direction:

$$\mathbf{s} = \begin{bmatrix} 0.5 & 2 \end{bmatrix}^T$$

Matrix representing scale transform:

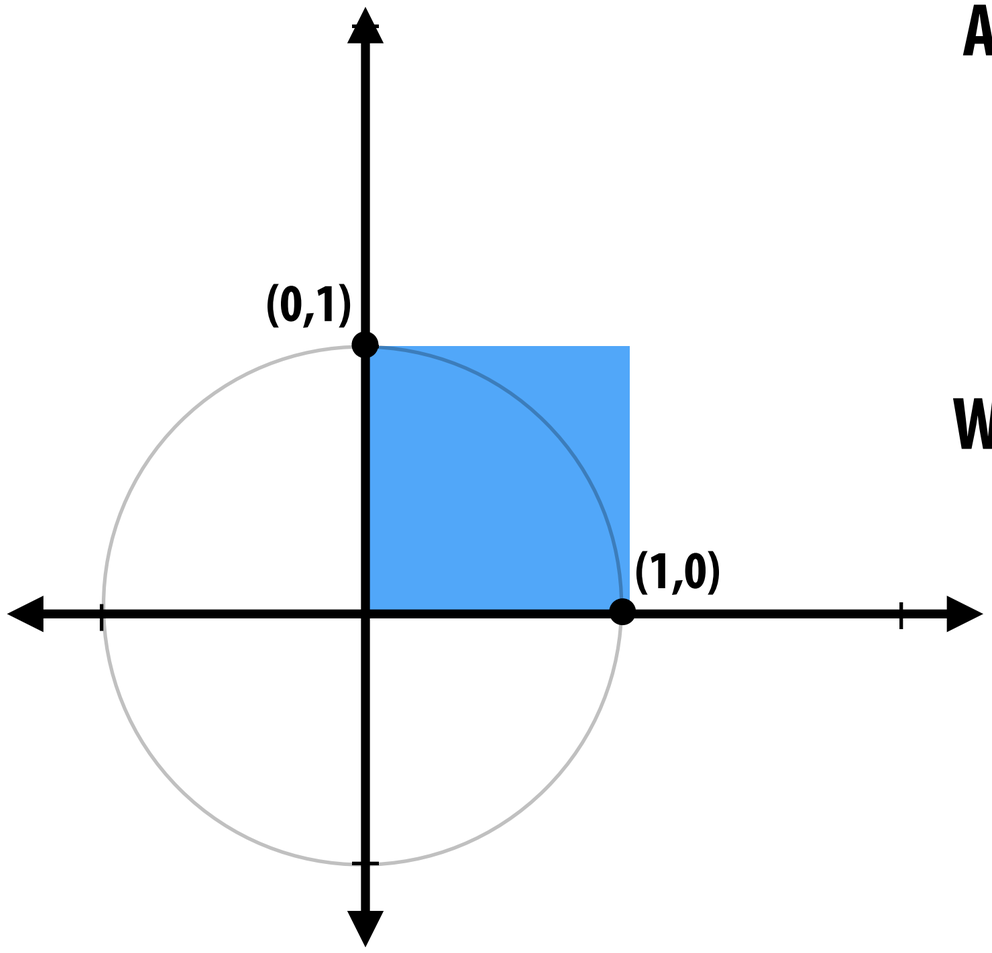
$$\mathbf{S_s} = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}$$

Rotation matrix (2D)

Question: what happens to (1, 0) and (0,1) after rotation by θ?

Reminder: rotation moves points along circular trajectories.

(Recall that $\cos\theta$ and $\sin\theta$ are the coordinates of a point on the unit circle.)



Answer:

$$R_\theta(1,0) = (\cos(\theta), \sin(\theta))$$

$$R_\theta(0,1) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2))$$

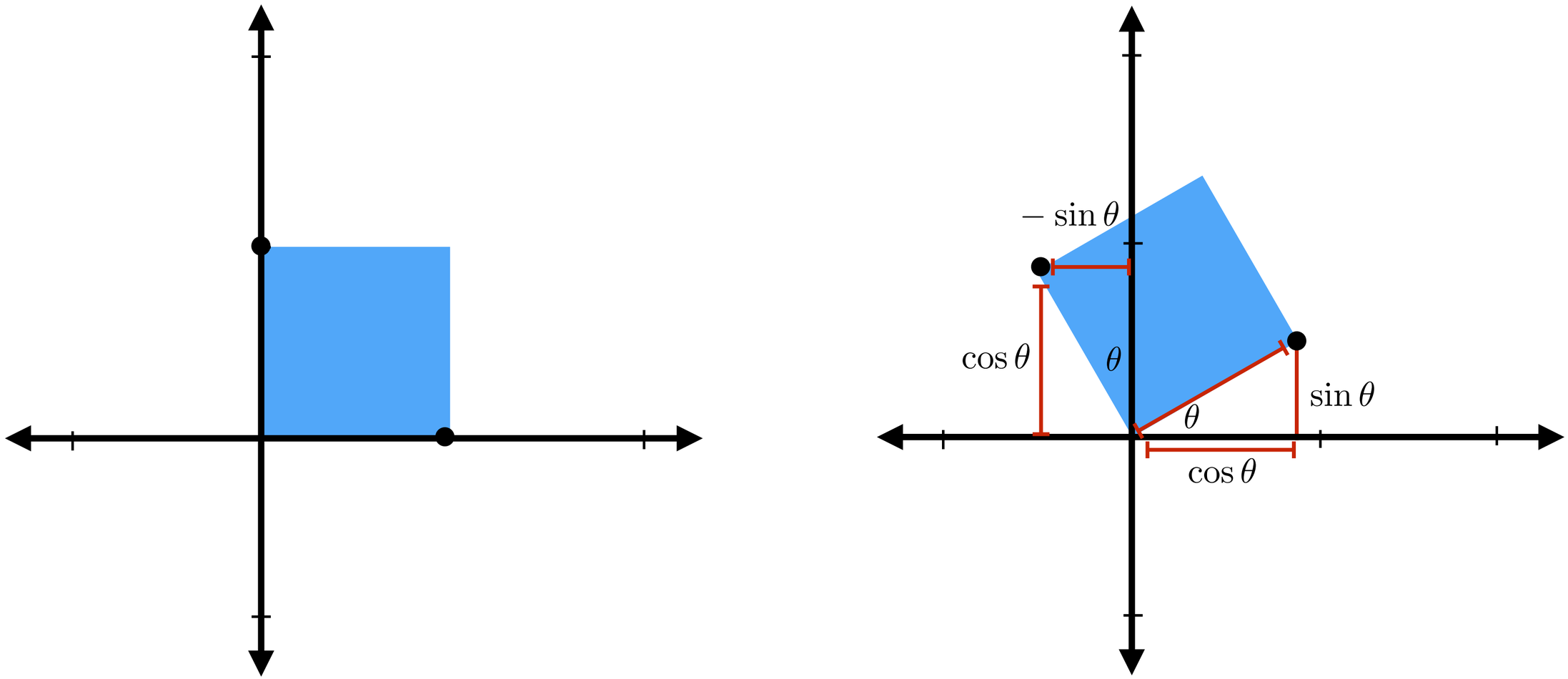
Which means the matrix must look like:

$$R_\theta = \begin{bmatrix} \cos(\theta) & \cos(\theta + \pi/2) \\ \sin(\theta) & \sin(\theta + \pi/2) \end{bmatrix}$$

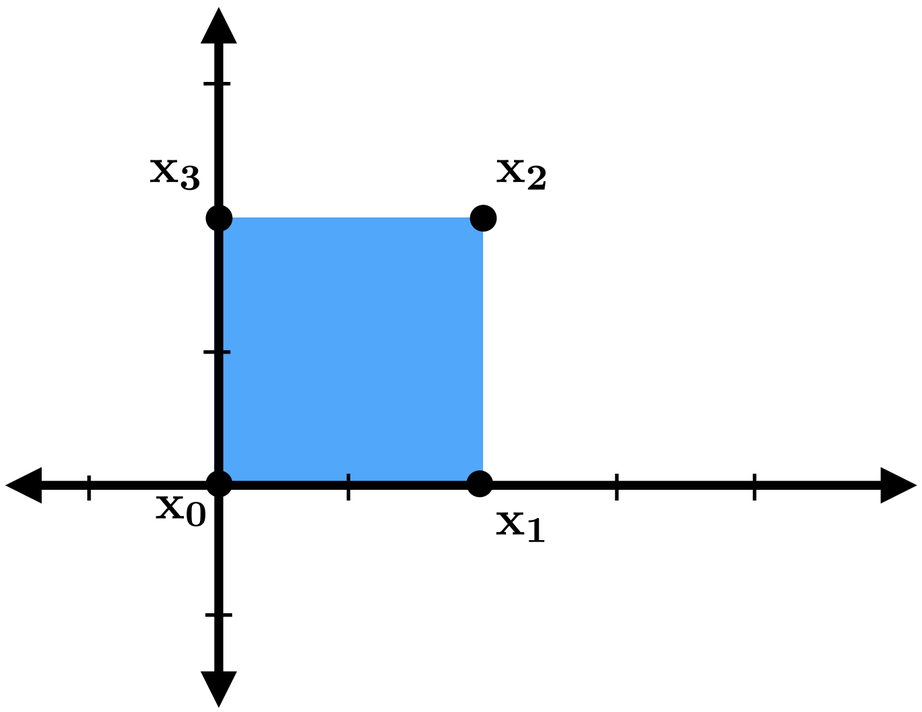
$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

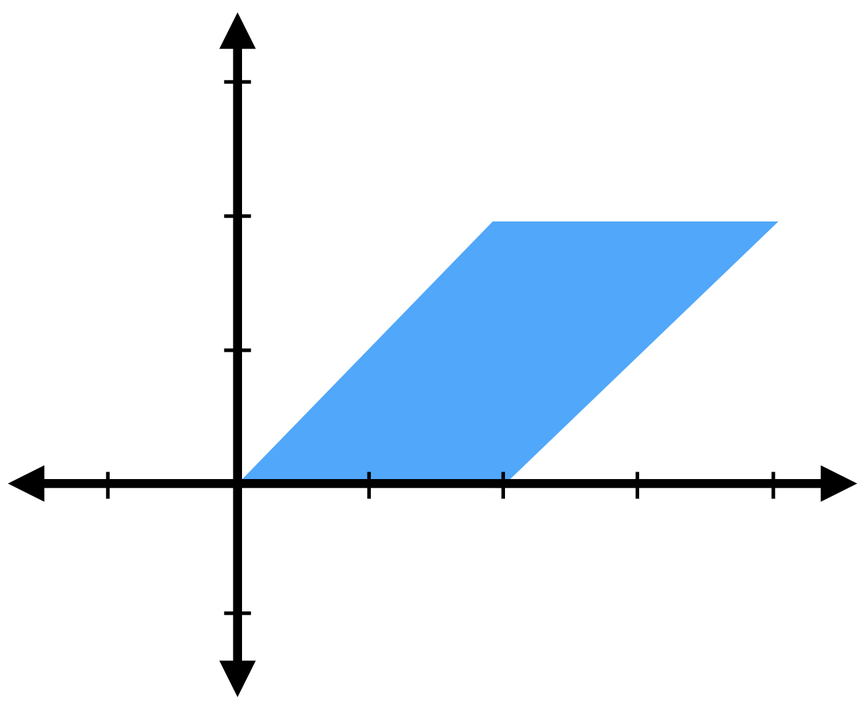
Rotation matrix (2D): another way…

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



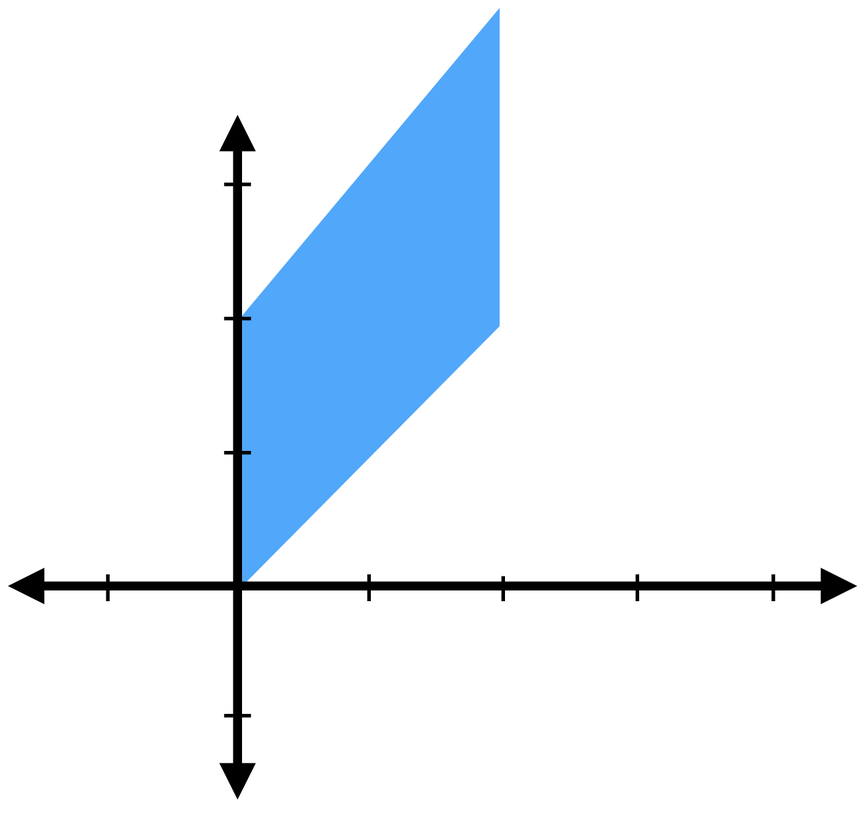
Shear





Shear in x:

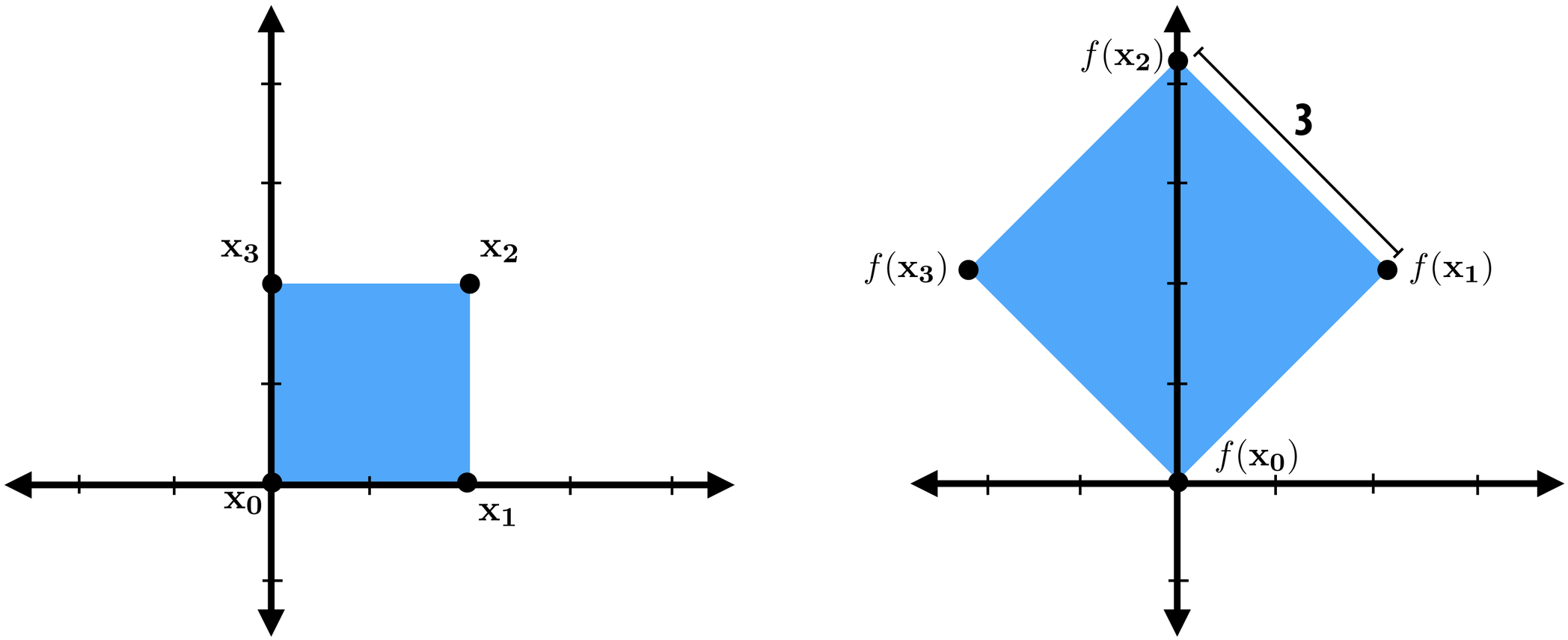
$$\mathbf{H}_{xs} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$



Shear in y:

$$\mathbf{H}_{ys} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

How do we compose linear transforms?



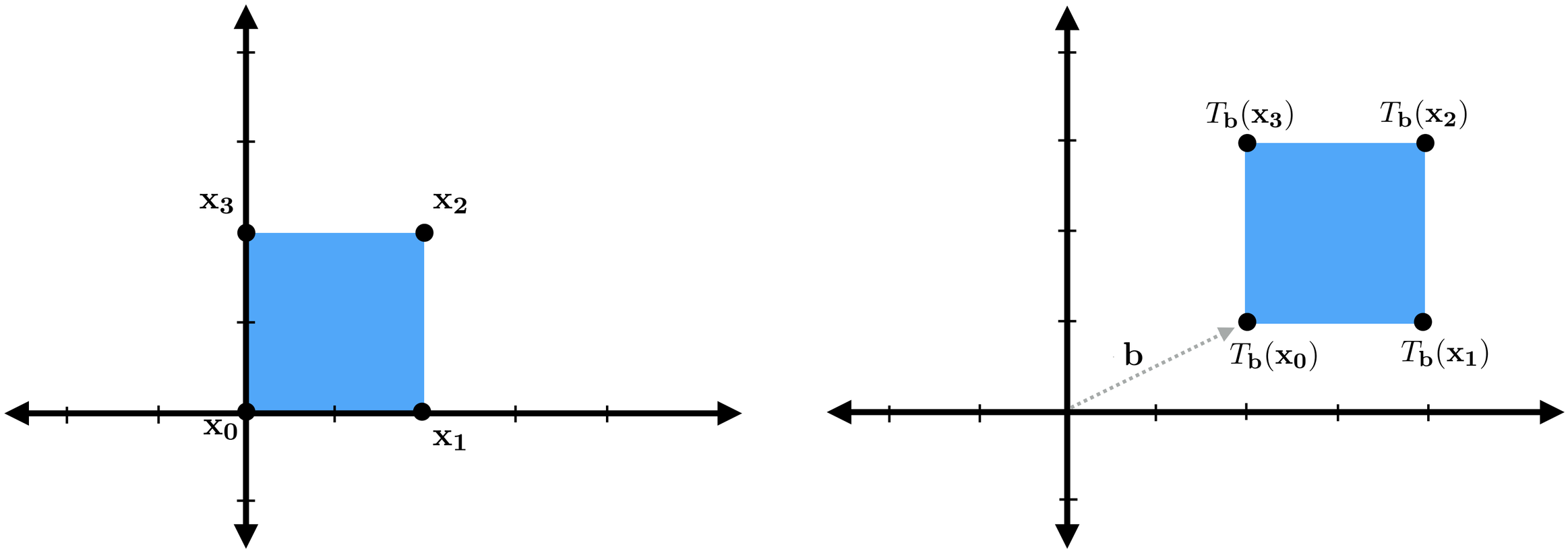
Compose linear transforms via matrix multiplication.
This example: uniform scale, followed by rotation

$$f(\mathbf{x}) = R_{\pi/4}\mathbf{S}_{[1.5,1.5]}\mathbf{x}$$

Enables simple, efficient implementation: reduce complex chain of transforms to a single matrix multiplication.

Translation?

$$T_{\mathbf{b}}(\mathbf{x}) = \mathbf{x} + \mathbf{b}$$



Recall: translation is not a linear transform

→ Output coefficients are not a linear combination of input coefficients

→ Translation operation cannot be represented by a 2x2 matrix

$$\mathbf{x}_{\mathbf{out}\,x} = \mathbf{x}_x + \mathbf{b}_x$$

$$\mathbf{x}_{\mathbf{out}\,y} = \mathbf{x}_y + \mathbf{b}_y$$

Translation math

2D homogeneous coordinates (2D-H)

Key idea: represent 2D points in 3D coordinate space

So the point (x, y) is represented as the 3-vector: $\begin{bmatrix} x & y & 1 \end{bmatrix}^T$

And transforms are represented a 3x3 matrices that transform these vectors.

For example: here are 2D scale and rotation transforms written in 2D homogeneous form:

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 \\ 0 & \mathbf{S}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe:

In these examples, the last row just propagates third coordinate of input to output.

Expressing transformations in 2D-H coords

Translation expressed as 3x3 matrix multiplication:

$$\mathbf{T_b} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix}$$

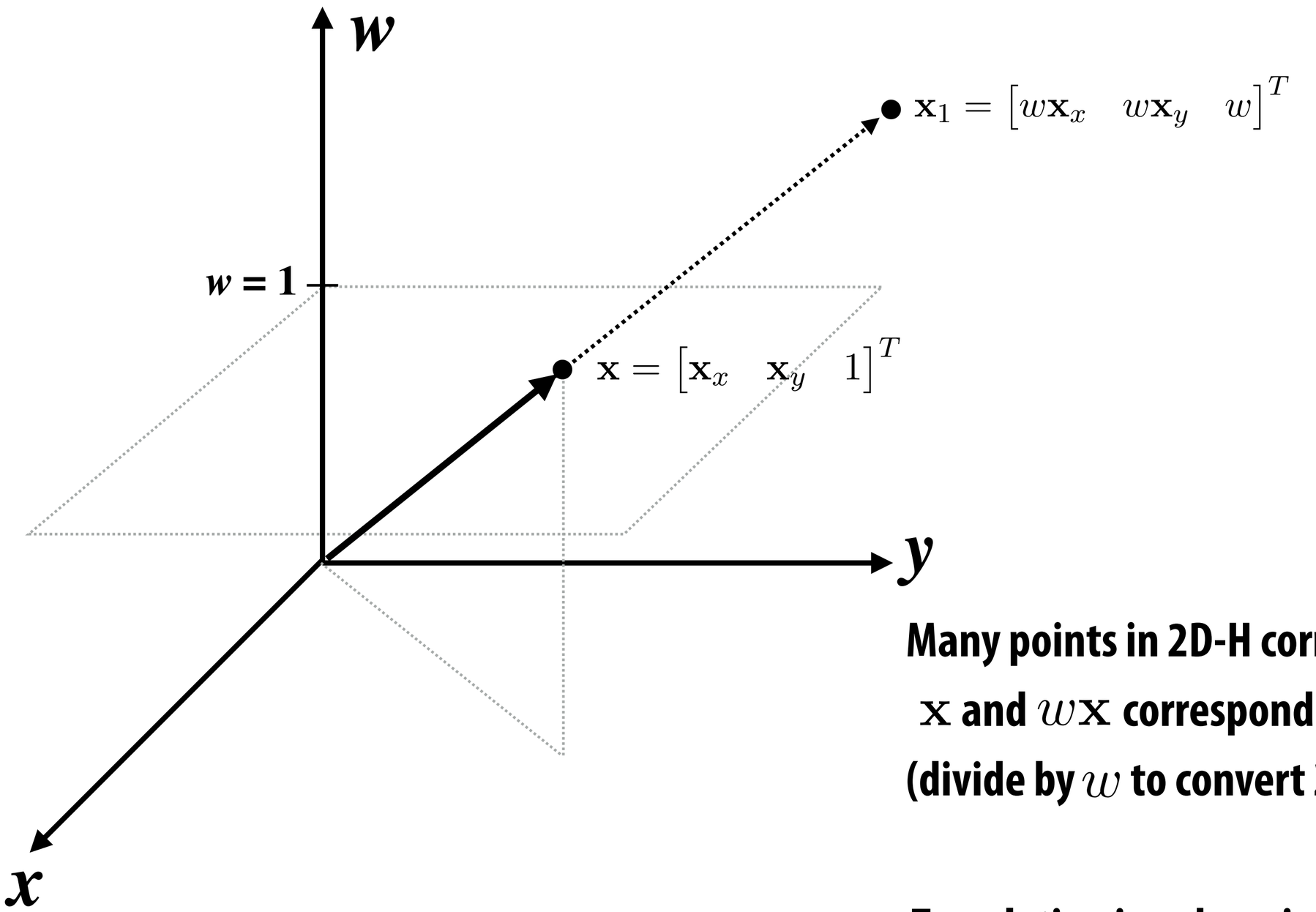
$$\mathbf{T_b x} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_x \\ \mathbf{x}_y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_x + \mathbf{b}_x \\ \mathbf{x}_y + \mathbf{b}_y \\ 1 \end{bmatrix}$$

Homogeneous representation enables composition of affine transforms!

Example: rotation about point $\mathbf{b}$

$$\mathbf{T_b R}_\theta \mathbf{T}_{-\mathbf{b}}$$

Homogeneous coordinates: some intuition



Many points in 2D-H correspond to same point in 2D

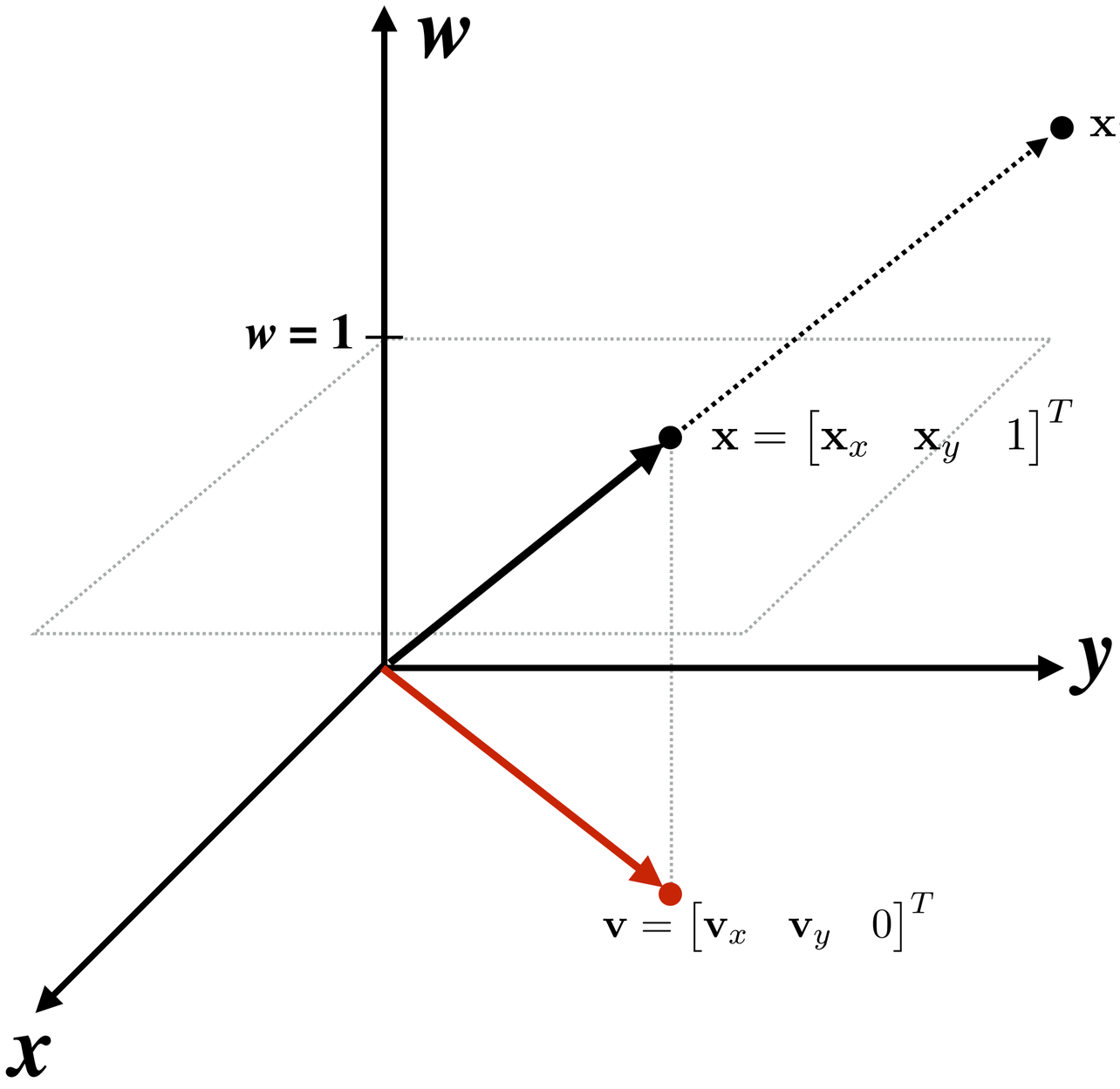
$\mathbf{x}$ and $w\mathbf{x}$ correspond to the same 2D point

(divide by w to convert 2D-H back to 2D)

Translation is a shear in x and y in 2D-H space

$$\mathbf{T}_\mathbf{b}\mathbf{x} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w\mathbf{x}_x \\ w\mathbf{x}_y \\ w \end{bmatrix} = \begin{bmatrix} w\mathbf{x}_x + w\mathbf{b}_x \\ w\mathbf{x}_y + w\mathbf{b}_y \\ w \end{bmatrix}$$

Homogeneous coordinates: points vs. vectors



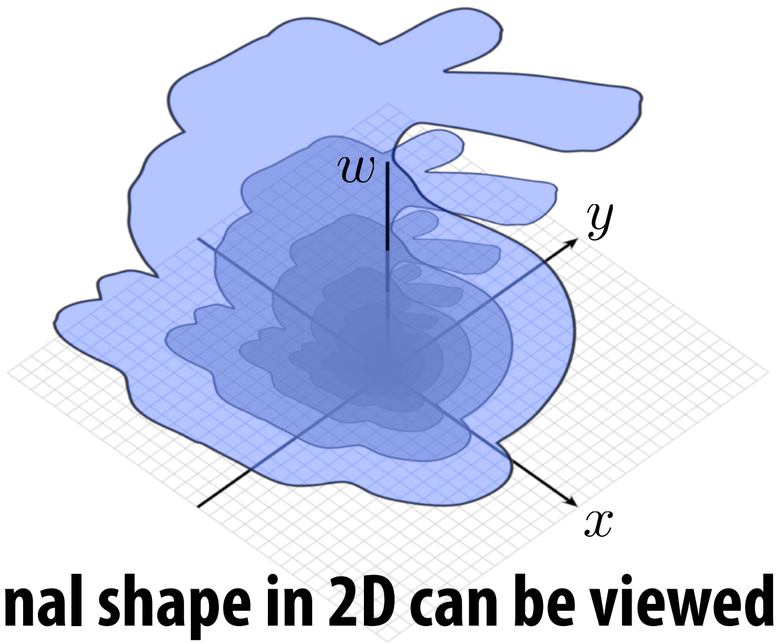
2D-H points with w=0 represent 2D vectors (think: directions are points at infinity)

Unlike 2D, points and directions are distinguishable by their representation in 2D-H

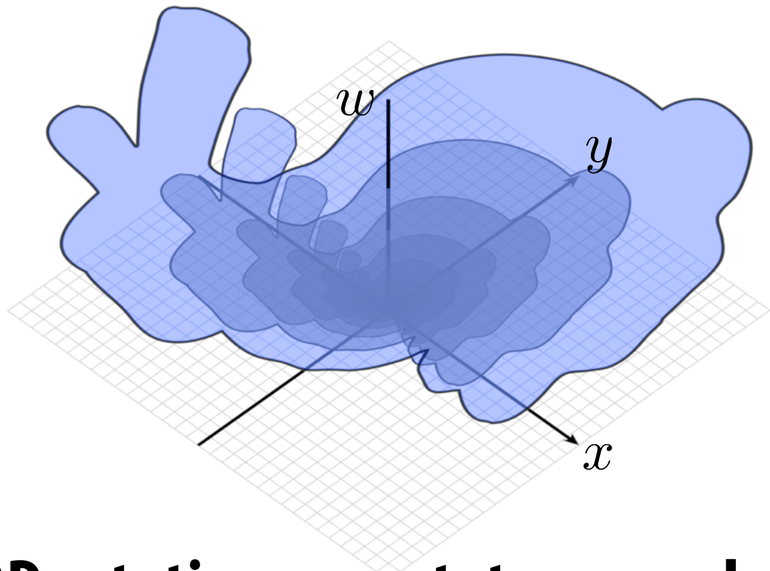
Note: translation does not modify directions:

$$\mathbf{T_b v} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix}$$

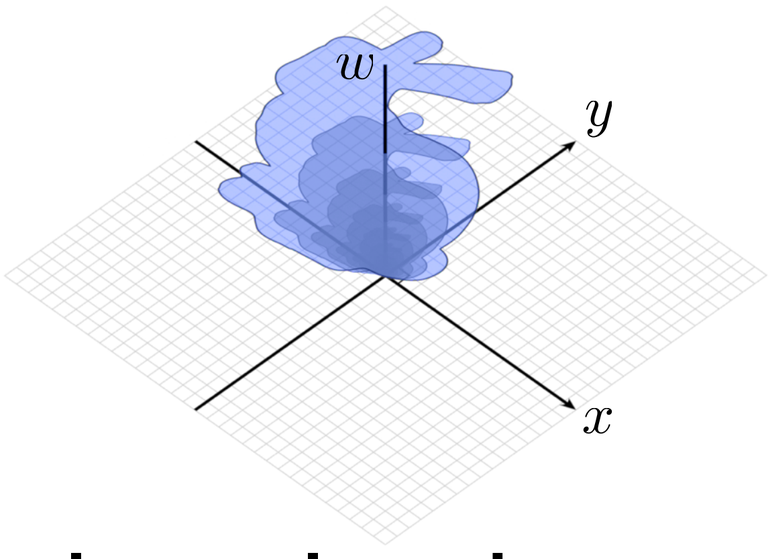
Visualizing 2D transformations in 2D-H



Original shape in 2D can be viewed as many copies, uniformly scaled by w.

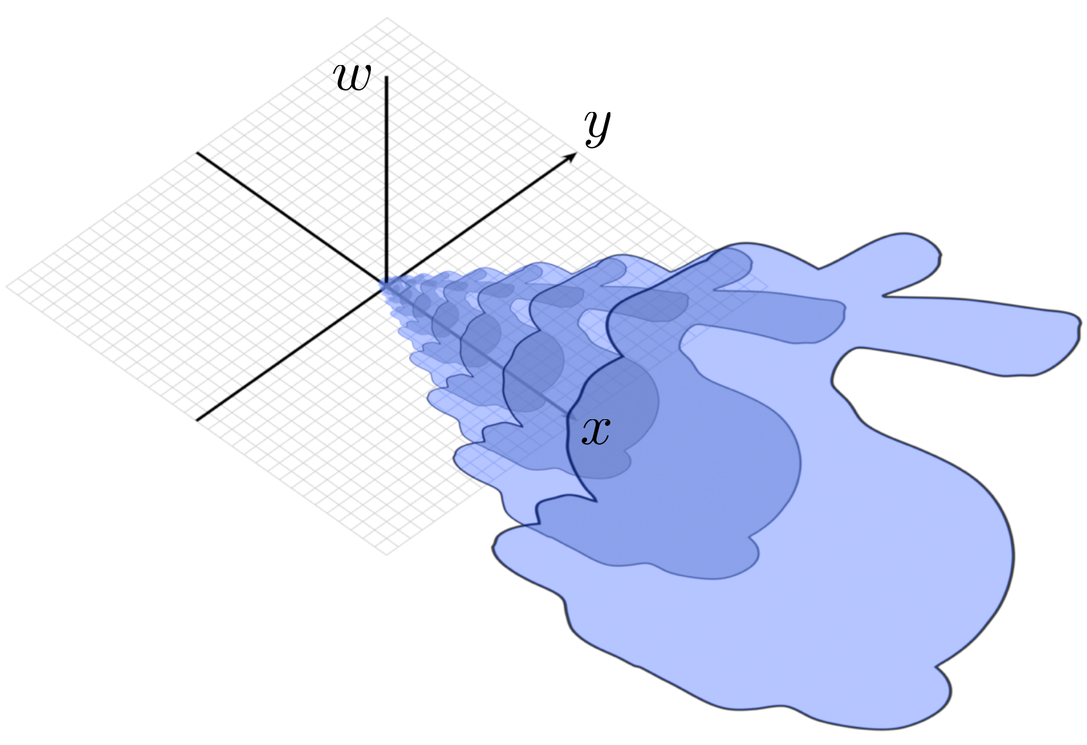


2D rotation ↔ rotate around w



2D scale ↔ scale x and y; preserve w

(Question: what happens to 2D shape if you scale x, y, and w uniformly?)



2D translate ↔ shear in xy

Moving to 3D (and 3D-H)

Represent 3D transforms as 3x3 matrices and 3D-H transforms as 4x4 matrices

Scale:

3D

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 \\ 0 & \mathbf{S}_y & 0 \\ 0 & 0 & \mathbf{S}_z \end{bmatrix}$$

3D-H

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 & 0 \\ 0 & \mathbf{S}_y & 0 & 0 \\ 0 & 0 & \mathbf{S}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear (in x, based on y,z position):

$$\mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translate:

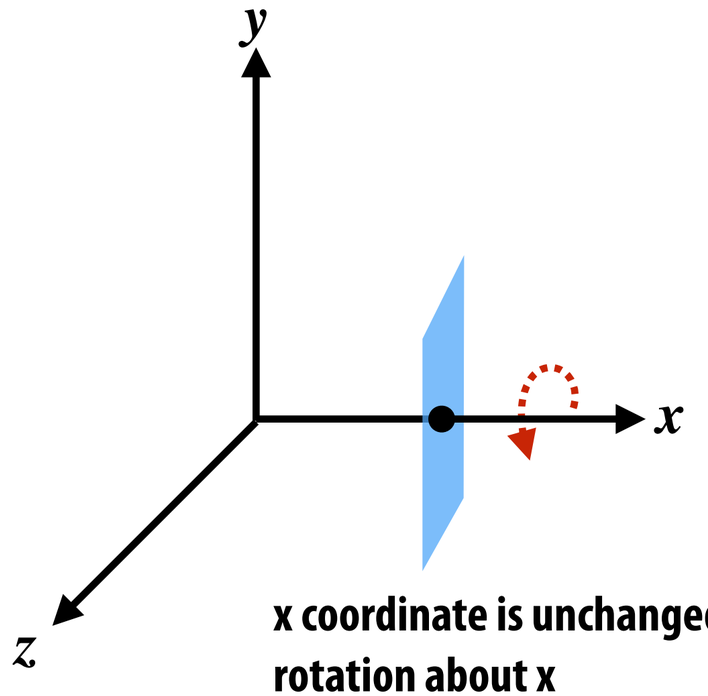
3D-H

$$\mathbf{T_b} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{b}_x \\ 0 & 1 & 0 & \mathbf{b}_y \\ 0 & 0 & 1 & \mathbf{b}_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations in 3D

Rotation about x axis:

$$\mathbf{R}_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

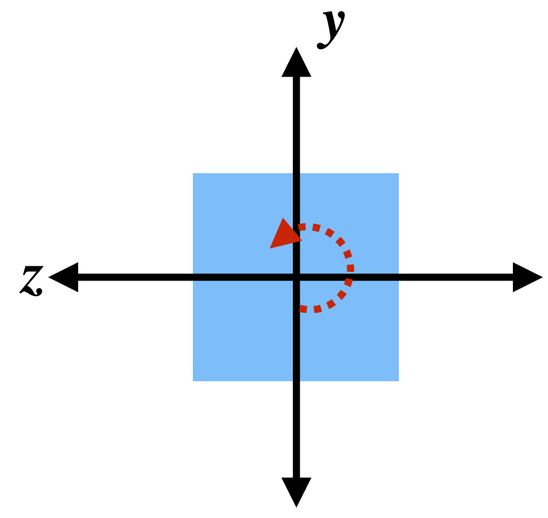


x coordinate is unchanged by rotation about x

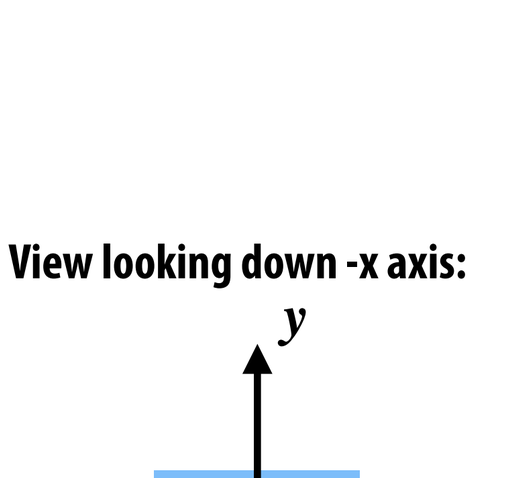
Rotation about y axis:

$$\mathbf{R}_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

View looking down -x axis:

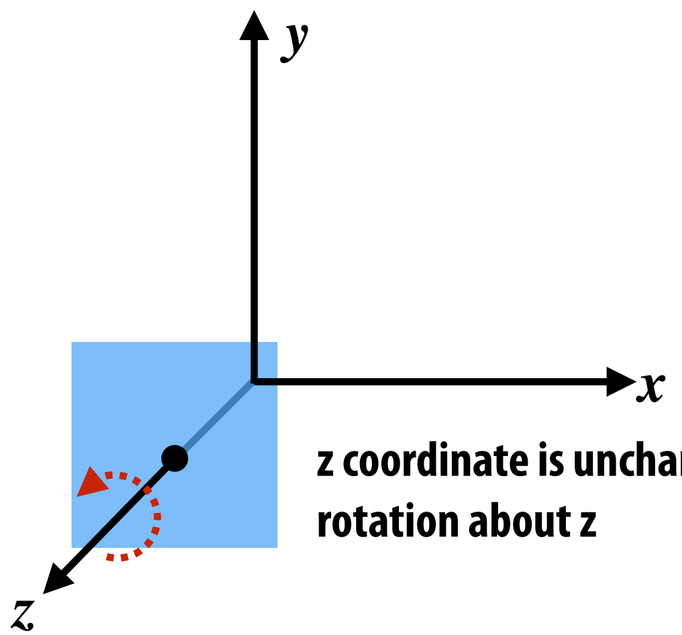


View looking down -y axis:



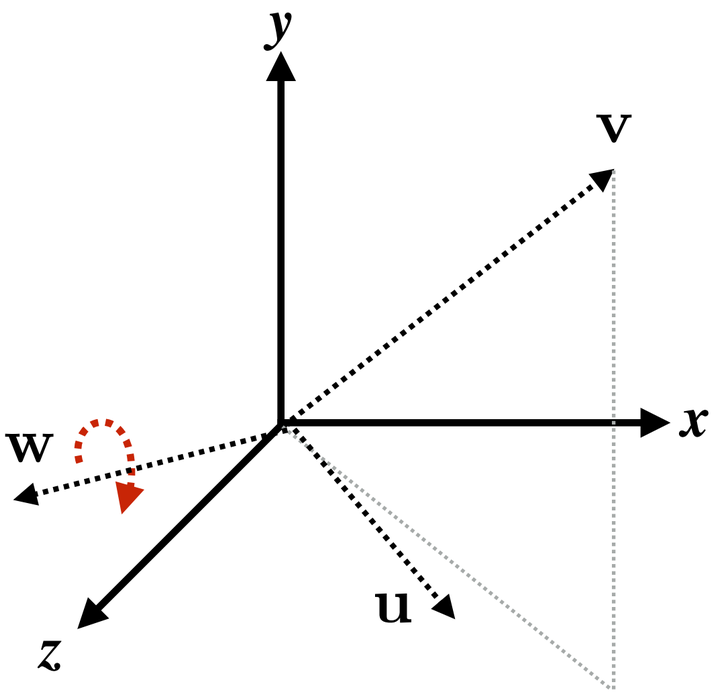
Rotation about z axis:

$$\mathbf{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



z coordinate is unchanged by rotation about z

Rotation about an arbitrary axis



To rotate by θ about $\mathbf{w}$:

1. **Form orthonormal basis around $\mathbf{w}$ (see $\mathbf{u}$ and $\mathbf{v}$ in figure)**
2. **Rotate to map $\mathbf{w}$ to [0 0 1] (change in coordinate space)**

$$\mathbf{R}_{uvw} = \begin{bmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \mathbf{v}_x & \mathbf{v}_y & \mathbf{v}_z \\ \mathbf{w}_x & \mathbf{w}_y & \mathbf{w}_z \end{bmatrix}$$

$$\mathbf{R}_{uvw}\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{R}_{uvw}\mathbf{v} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

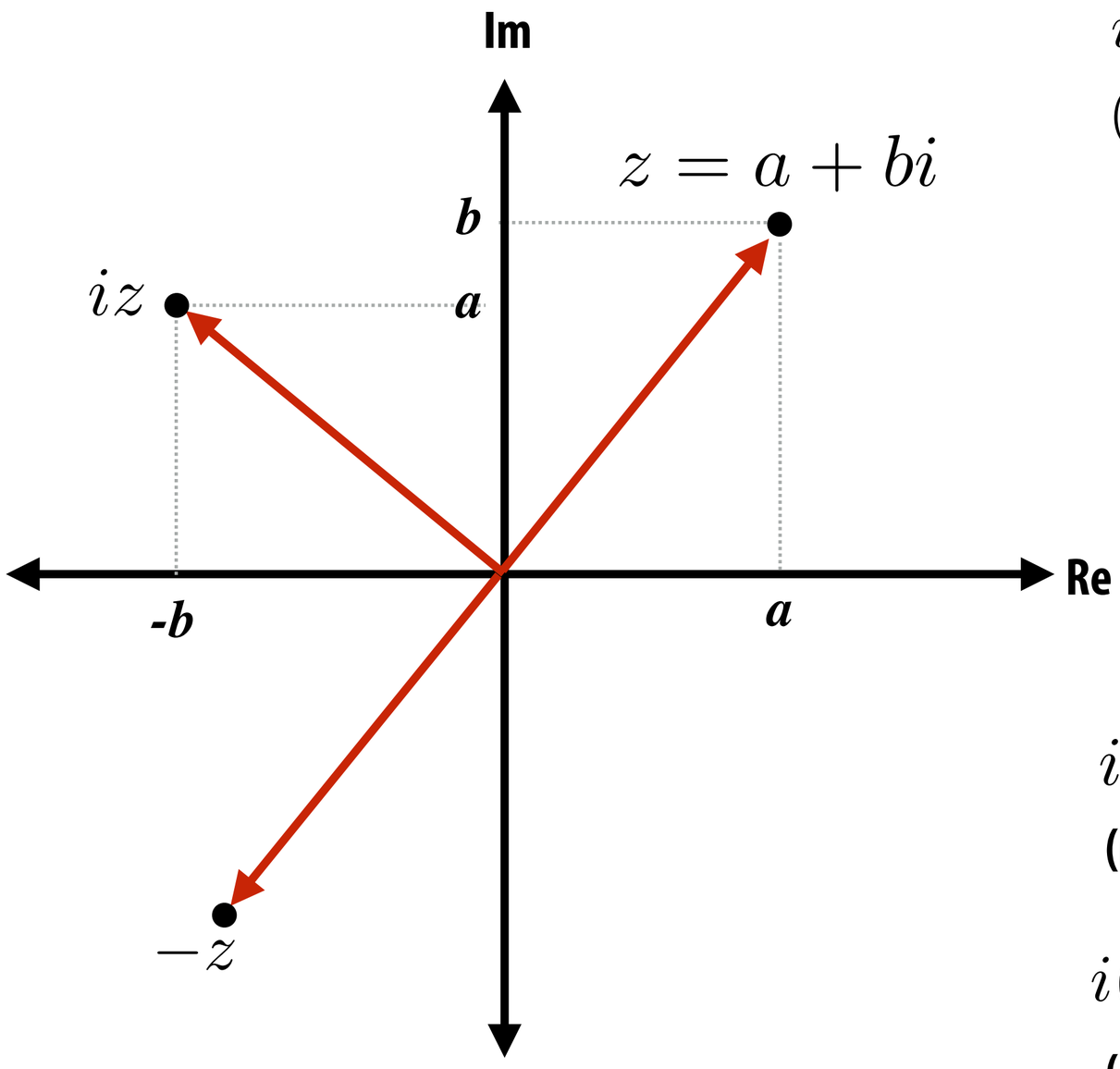
$$\mathbf{R}_{uvw}\mathbf{w} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

3. **Perform rotation about z: $\mathbf{R}_{z,\theta}$**
4. **Rotate back to original coordinate space: $\mathbf{R}^{\mathbf{T}}_{\mathbf{uvw}}$**

$$\mathbf{R}^{-1}_{uvw} = \mathbf{R}^{T}_{uvw} = \begin{bmatrix} \mathbf{u}_x & \mathbf{v}_x & \mathbf{w}_x \\ \mathbf{u}_y & \mathbf{v}_y & \mathbf{w}_y \\ \mathbf{u}_z & \mathbf{v}_x & \mathbf{w}_z \end{bmatrix}$$

$$\mathbf{R}_{\mathbf{w},\theta} = \mathbf{R}^{\mathbf{T}}_{\mathbf{uvw}}\mathbf{R}_{z,\theta}\mathbf{R}_{\mathbf{uvw}}$$

Alternative representation for rotations: complex numbers



$$i^2 = -1$$

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i$$

$$iz = i(a + bi) = -b + ai$$

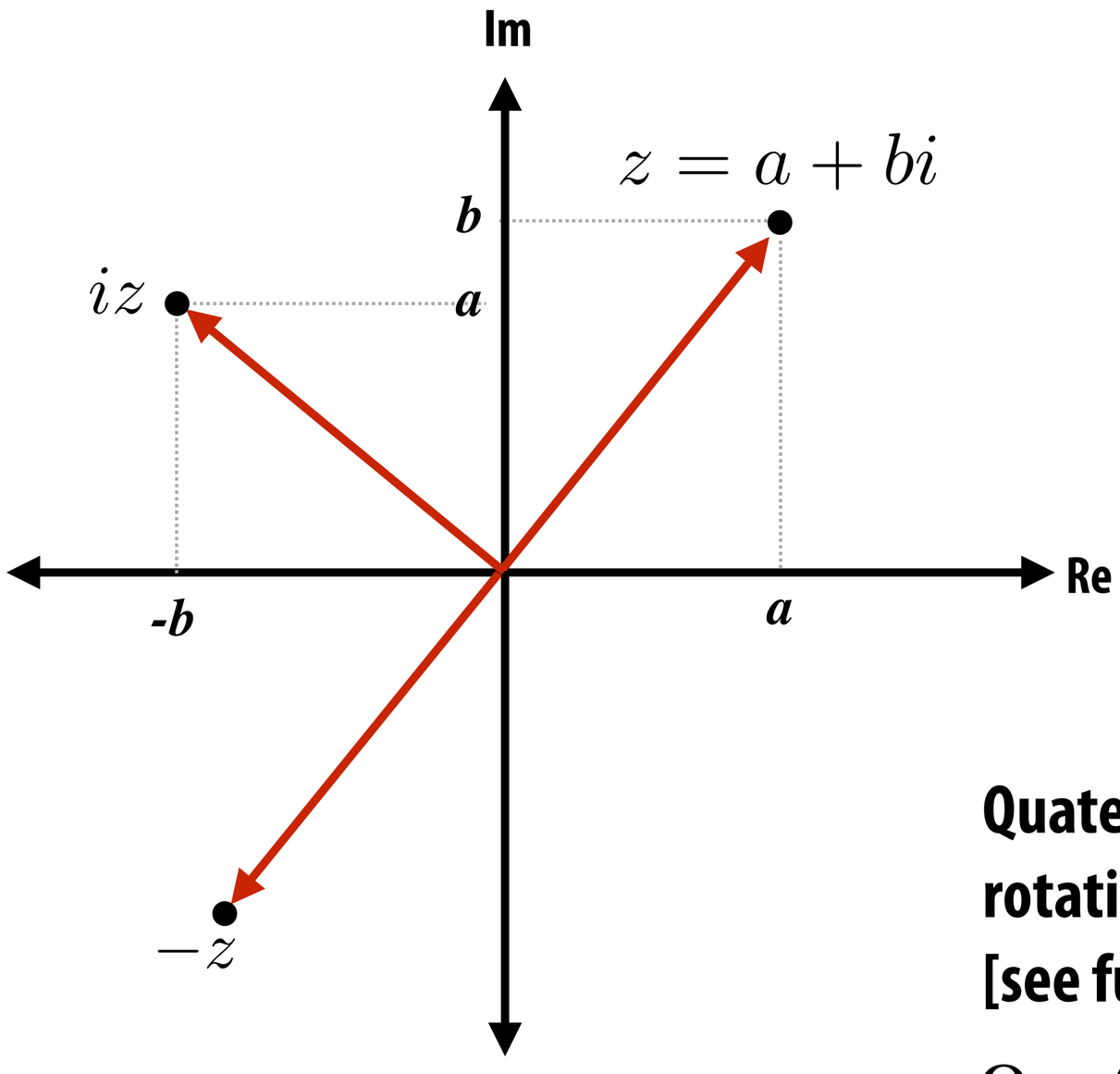
(multiplication by i → rotation by $\pi/2$)

$$i(iz) = -a - bi = -z$$

(multiplication by i^2 → rotation by π)

$$\mathbf{R}_\theta = e^{i\theta} = \cos\theta + i\sin\theta$$

Alternative representation for rotations: complex numbers



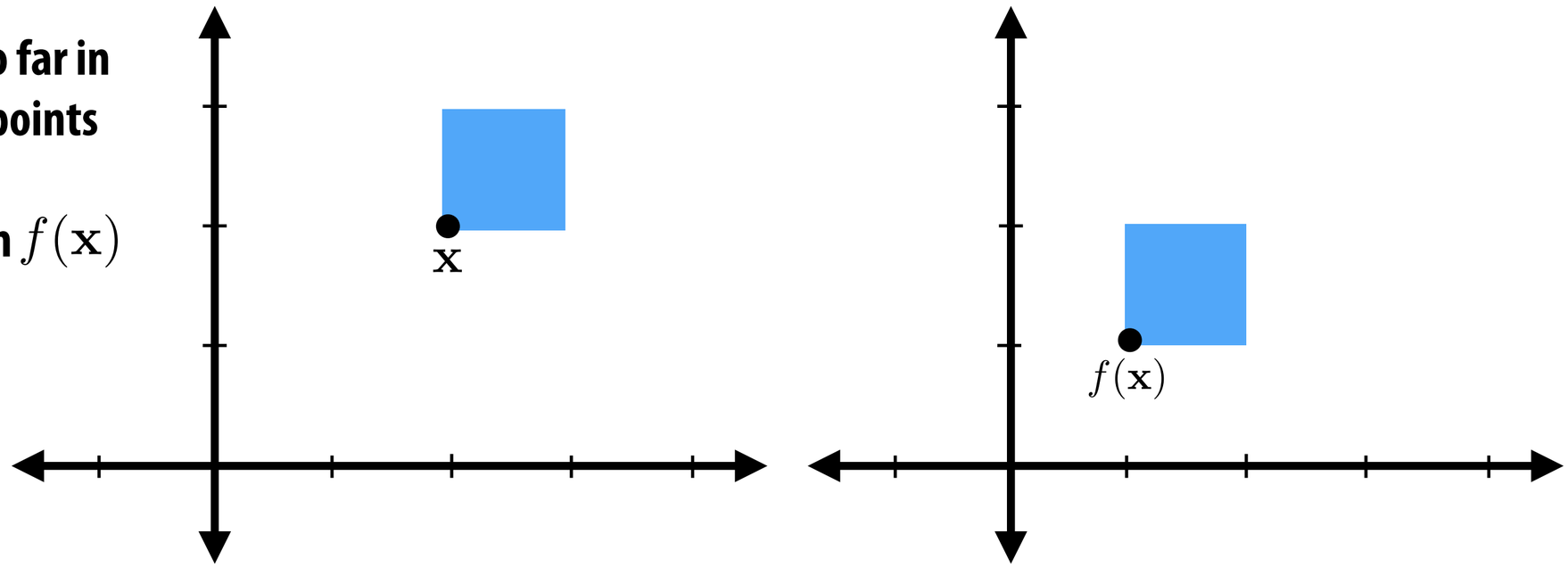
Quaternions are a representation of 3D rotations based on complex numbers [see further reading on web site]

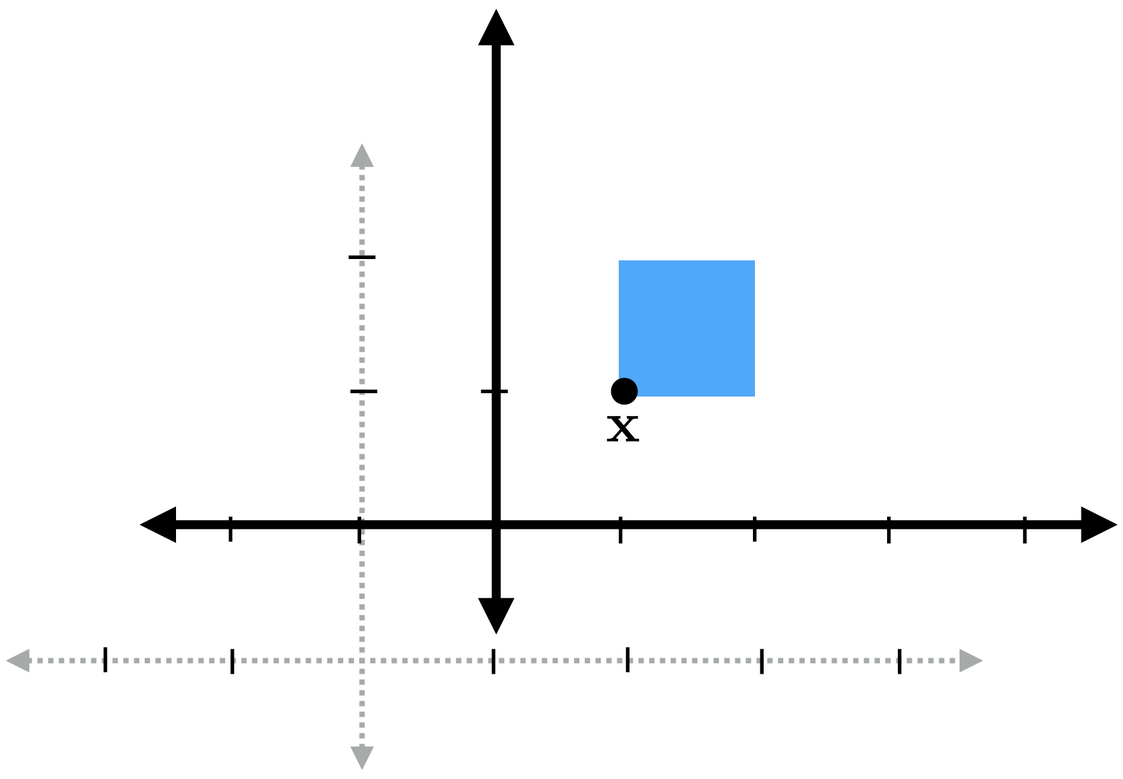
$$\mathbf{Q} = (\mathbf{q}_v, \mathbf{q}_w) = i\mathbf{q}_x + j\mathbf{q}_y + k\mathbf{q}_z + \mathbf{q}_w$$

Another way to think about transformations: change in coordinate space

Interpretation of transforms so far in this lecture: transforms move points

Point $\mathbf{x}$ moved to new position $f(\mathbf{x})$





Alternative interpretation:

Transformations induce of change of coordinate space: Representation of $\mathbf{x}$ changes since point is now described in a new coordinate space

Review from last time: screen transform *

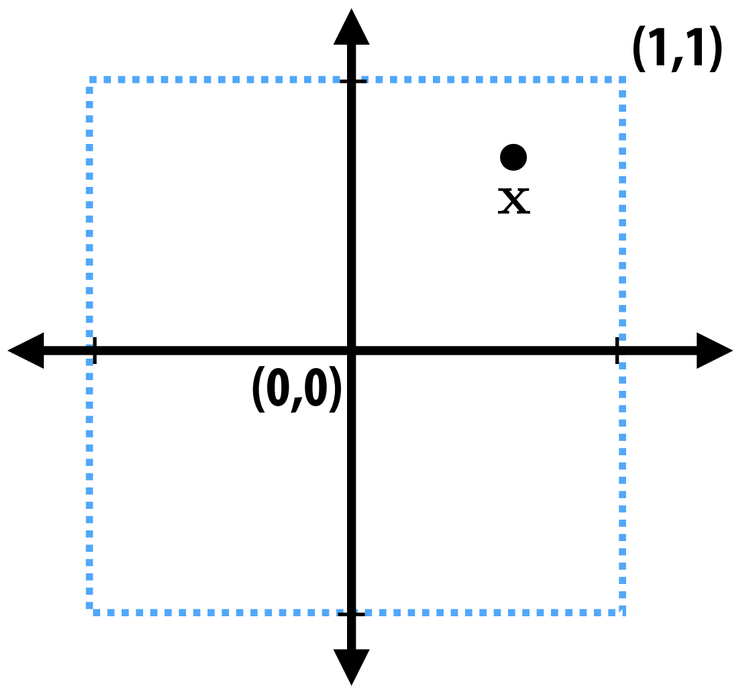
Convert points in normalized coordinate space to screen pixel coordinates

Example:

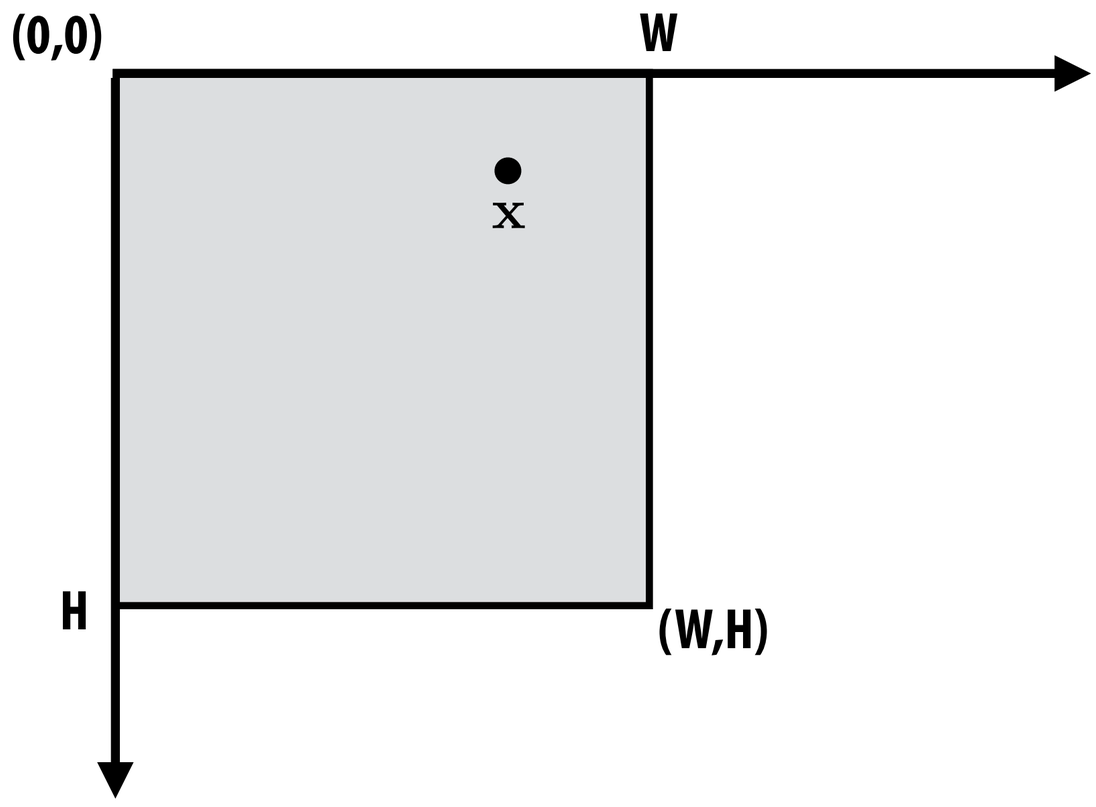
All points within (-1,1) to (1,1) region are on screen

(1,1) in normalized space maps to (W,0) in screen

Normalized coordinate space:



Screen (W x H output image) coordinate space:



Step 1: reflect about x

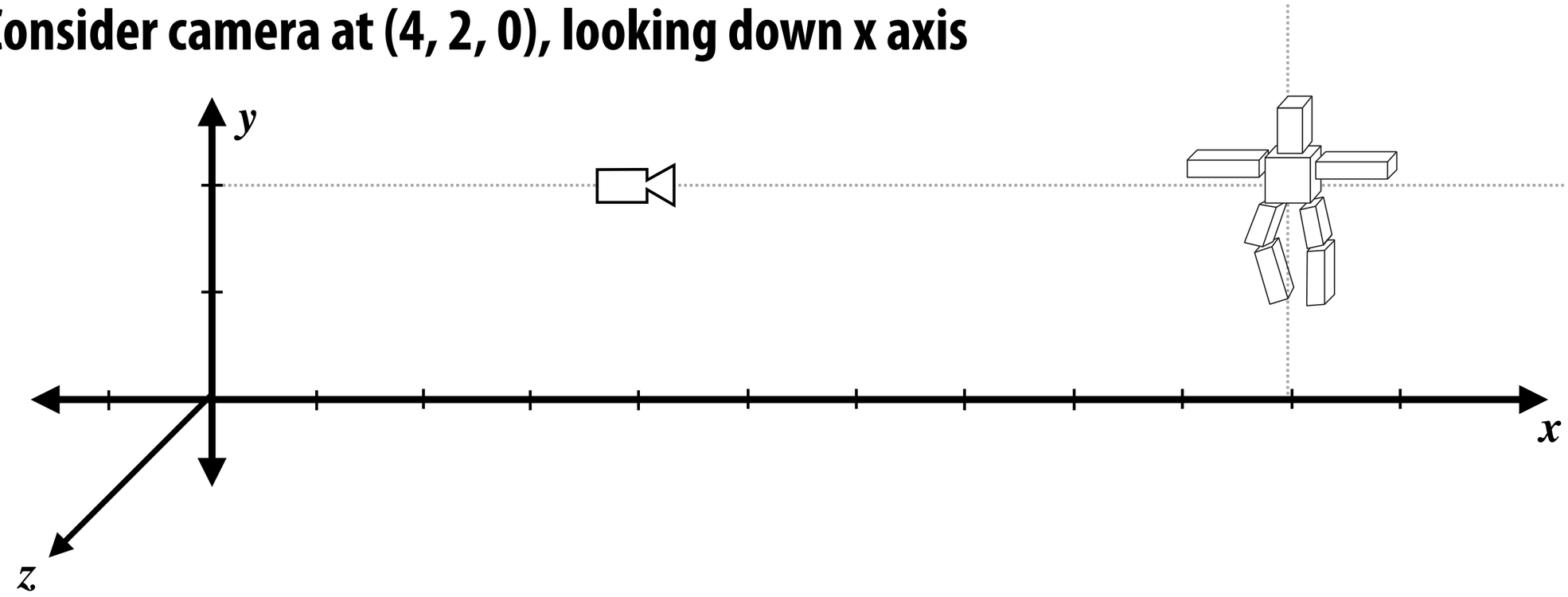
Step 2: translate by (1,1)

Step 3: scale by (W/2,H/2)

*** Adopting convention that top-left of screen is (0,0) to match SVG convention in Assignment 1.**
Many 3D graphics systems like OpenGL place (0,0) in bottom-left. In this case what would the transform be?

Example: simple camera transform

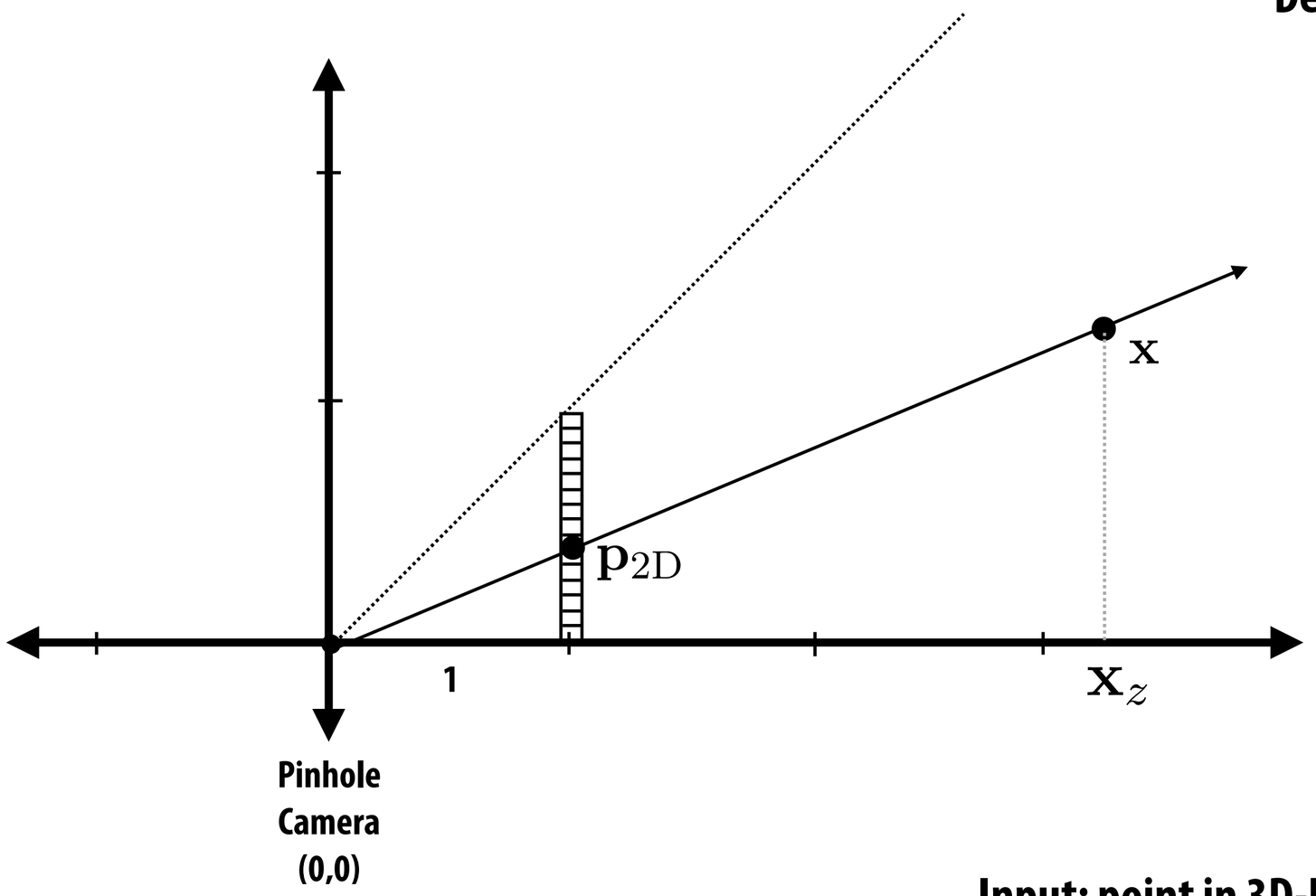
- Consider object in world at (10, 2, 0)
- Consider camera at (4, 2, 0), looking down x axis



- Translating object vertex positions by (-4, -2, 0) yields position relative to camera.
- Rotation about y by $-\pi/2$ gives position of object in coordinate system where camera's view direction is aligned with the z axis *

* The convenience of such a coordinate system will become clear on the next slide!

Basic perspective projection



Desired perspective projected result (2D point):

$$\mathbf{p}_{\text{2D}} = \begin{bmatrix} \mathbf{x}_x/\mathbf{x}_z & \mathbf{x}_y/\mathbf{x}_z \end{bmatrix}^T$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Input: point in 3D-H $\mathbf{x} = \begin{bmatrix} \mathbf{x}_x & \mathbf{x}_y & \mathbf{x}_z & 1 \end{bmatrix}$

After applying $\mathbf{P}$: point in 3D-H $\mathbf{P}\mathbf{x} = \begin{bmatrix} \mathbf{x}_x & \mathbf{x}_y & \mathbf{x}_z & \mathbf{x}_z \end{bmatrix}^T$

Point in 2D-H (drop z coord) $\mathbf{p}_{\text{2D-H}} = \begin{bmatrix} \mathbf{x}_x & \mathbf{x}_y & \mathbf{x}_z \end{bmatrix}^T$

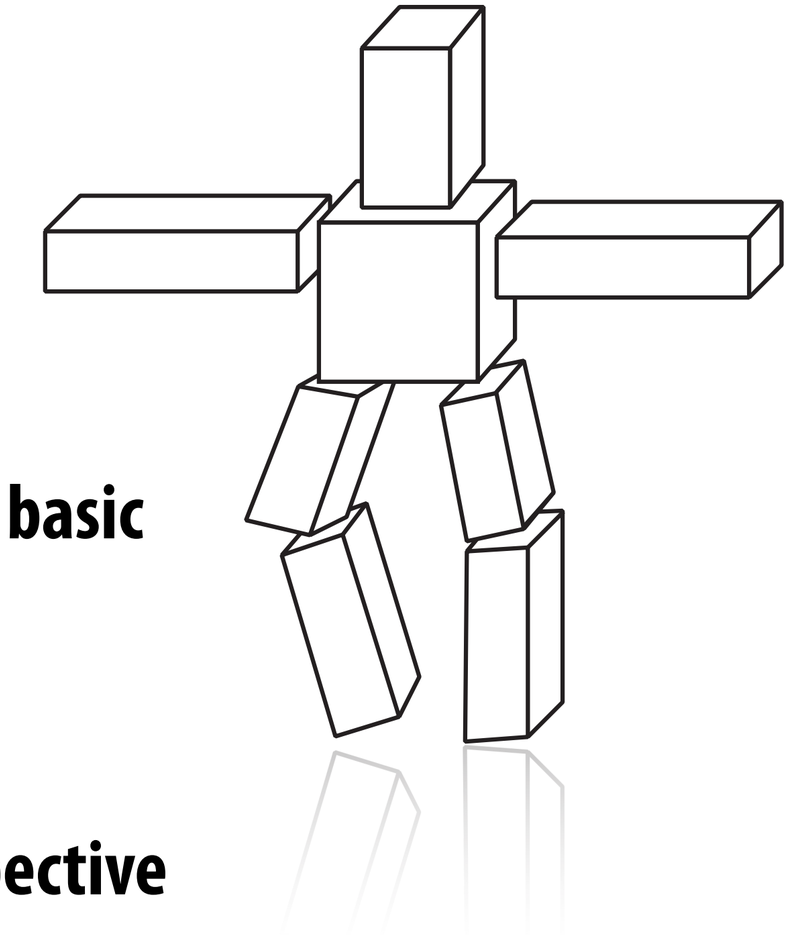
Point in 2D (homogeneous divide) $\mathbf{p}_{\text{2D}} = \begin{bmatrix} \mathbf{x}_x/\mathbf{x}_z & \mathbf{x}_y/\mathbf{x}_z \end{bmatrix}^T$

Assumption:

Pinhole camera at (0,0) looking down z

Tranformations summary

- Transformations can be interpreted as operations that move points in space
  - e.g., for modeling, animation



- Or as a change of coordinate system
  - e.g., screen and view transforms

- Construct complex transformations as compositions of basic transforms

- Homogeneous coordinate representation allows for expression of non-linear transforms (e.g., affine, perspective projection) as matrix operations (linear transforms) in higher-dimensional space
  - Matrix representation affords simple implementation and efficient composition