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Imperfect maintenance policies for warranted products under stochastic performance degradation

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ABSTRACT

Both customers and manufacturers benefit from warranties that effectively guarantee products' performance with reasonable price and cost. Reliability-oriented degradation analysis has boosted the accuracy of lifetime prediction in the past decades and prompted additional maintenance and warranty options to manufacturers. In this paper, we present a systematic framework that aims to optimize imperfect maintenance policies for warranted products subject to stochastic performance degradation. We propose a novel objective-oriented imperfect repair scheme, and consider two scenarios corresponding to different types of products/systems. In the first one, warranties are claimed by customers in a random manner driven by performance degradation, whereas in the second one, the manufacturer offers periodic inspection services to customers and implements imperfect repairs if necessary. In both scenarios, we derive critical closed-form results to facilitate computation and optimization, from which managerial insights are gleaned. We find that (i) the randomness in customers' claiming behaviors leads to remarkably more conservative warranty policies in which repairs are suggested to be more substantial; and (ii) the variable objective repair level in the imperfect repair exercise can effectively reduce the total maintenance cost over a finite warranty period. Overall, this work will equip practitioners with quantitative tools to prescribe optimal maintenance policies for warranted degrading products in various practical scenarios.

1. Introduction

1.1. Background and motivation

A warranty is an assurance offered by a manufacturer who agrees to replace or repair failed products within the warranty coverage (Ye & Murthy, 2016). As a free-of-charge attachment to a sold product, warranty plays important roles in marketing. On the one hand, it indicates a manufacturer's confidence in product quality and reliability. On the other hand, a warranty immediately offers preferable insurance to risk-averse customers (Wu & Akbarov, 2011) and thereby encourages their purchases. These both contribute to the attractiveness and competitiveness of products and brands. Under a (base) warranty policy, the manufacturer oftentimes commits to free-of-charge repair or replacement upon any product failure under normal usage. From the perspective of manufacturers, the cost of warranties usually accounts for a remarkable proportion of total sales (Wang, Li, & Xie, 2020b). How to determine a cost-effective warranty policy and its accompanying maintenance obligation has become a vital part of the manufacturers' after-sales service strategy.

Warranty issues are inevitably connected to product reliability in the sense that reliability governs the number of prospective warranty claims probabilistically (Murthy, 2006). Most existing studies have assumed conventional models that hinge on lifetime distributions (e.g., exponential, Weibull, or lognormal) and utilized the associated failure rate function to evaluate the number of warranty claims within the protection period (Wu, 2014). Practically, several concerns on the underlying assumption may arise. First, not all types of failures can be adequately described by conventional lifetime models. Lu & Meeker (1993) argue to utilize degradation measures to estimate the time-to-failure distribution. In recent decades, degradation analysis has prevailed both in the literature and in industrial applications, where product degradation levels, such as the loss of battery health, reduction in LED lamp brightness, and mechanical wear, are devoted to reliability modeling and maintenance optimization (Alaswad & Xiang, 2017; Ye & Xie, 2015). However, maintenance optimization problems for warranted

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degrading products are still underexplored. For such problems, a drawback of lifetime models is that they usually fail to characterize different types of maintenance models concerning performance degradation. A common exercise to repair such products is to partially restore their degraded performance characteristics, which is usually called "imperfect maintenance/repair." Second, for products with performance degradation, a customer does not necessarily claim a warranty exactly upon the failure time, and the perception of a failure can vary among different customers. In fact, the degradation levels of claimed products are more likely to be random. For example, a rechargeable battery system under warranty is commonly deemed as failed and eligible to claim a free warranty if its capacity is below 80% or less of the original capacity. Within the warranty coverage, customers may not return the product once the capacity reaches the threshold of 80%. More realistically, a product may be returned to the manufacturer if the customer subjectively considers it as has failed. Due to the randomness in customer behavior, modeling warranty claims becomes more complicated and thus cannot be well fulfilled by conventional lifetime models. In contrast, for some industrial facilities, the users do not claim warranties by themselves. Instead, the manufacturers provide regular inspection services as well as mandatory maintenance based on performance degradation revealed by scheduled inspections. In both scenarios, how to optimize the maintenance policies for degrading products/systems protected by warranties becomes an inevitable and critical problem for manufacturers. Two typical examples of warranty under performance degradation are presented as follows.

Warranty of batteries in electric/hybrid vehicles. With the increasing popularity of new energy vehicles, reliability and performance concerns of built-in Lithium-ion batteries are drawing increasing attentions. A typical cause of vehicle battery failure is the degradation in capacity of certain battery cells, which gradually leads to defects of the whole system. To attract consumers' purchases, electric/hybrid vehicle manufacturers usually provide longer warranties for the battery system than other components in vehicles. For example, Lexus offers a 10-year/150,000-mile warranty for batteries of its hybrid vehicle models, which is remarkably longer than those of regular models.[1] Under such warranty policies, manufacturers usually commit to guarantee the capacity of batteries, via repair or replacement if necessary upon customer claims.

Warranty of machine tools. Key components in machine tools generally degrade with usage. For example, the spindles in machine tools rotate in very high speeds during operations, and the wear accumulated in the process results in gradual performance degradation. On the one hand, degradation adversely affects the quality of work pieces, and on the other, the whole system is subject to higher risks of traumatic failure under severe conditions of degradation. Therefore, the performance of key components equipped in machine tools is oftentimes warranted by the manufacturer. For example, DMG MORI offers a 36-month warranty for spindles in a series of machine tools regardless of spindle hours.[2] The maintenance of such systems is usually scheduled and on-site carried out by the manufacturer in a periodic manner.

We can notice that for the electric vehicle example, the warranties are claimed randomly by customers, whereas for the example of machine tools, the warranties are generally entertained by manufacturers' planned maintenance services. By and large, it is of great application value for manufacturers to incorporate performance degradation into the modeling and optimization of warranty and maintenance policies. Moreover, involving imperfect repair can remarkably enhance the generality of the proposed model by offering decision makers more maintenance options. To this end, we establish a novel framework to prescribe optimal imperfect maintenance policies for warranted products subject to stochastic performance degradation.

1.2. Related literature

This paper intends to build a systematic bridge between degradation analysis and maintenance-warranty policies and fill the research gap that is of theoretical and practical interests. In the literature, plenty of attention has been drawn by either of the two topics. Degradation analysis has prevailed in the field of reliability engineering since the seminal work of Lu & Meeker (1993). A variety of statistical methods have been proposed to model degradation data; see Ye & Xie (2015) for an overview. In the literature review, we omit the studies that are merely concerned with degradation analysis and focus on the following aspects.

Maintenance policies under degradation models. Maintenance modeling plays a pivotal role in warranty studies. Owing to the advent of degradation models, a collection of maintenance models and policies have been proposed for degrading systems. Condition-based maintenance that utilizes degradation models is one of the most sought-after topics in reliability engineering in the recent two decades (Alaswad & Xiang, 2017; de Jonge & Scarf, 2020; Grall, Bérenguer, & Dieulle, 2002a; Grall, Dieulle, Berenguer, & Roussignol, 2002b). In an earlier phase, most studies only consider replacement or perfect maintenance on systems. In practice, maintenance actions do not always restore systems to the as-good-as-new state. Some recent studies aim at modeling the effect of imperfect repair based on degradation models. Mercier & Castro (2013) borrow the idea of age reduction from the lifetime models (Doyen & Gaudoin, 2004) to describe the reduction of degradation level induced by maintenance. In a followup work by Mercier & Castro (2019), the degradation-reduction and age-reduction models are compared stochastically under the gamma degradation process. Further, these proposed methods have been widely extended to optimize maintenance policies. Wu, Niknam, & Kobza (2015) propose a cost model based on the expected degradation reduction and optimize the imperfect maintenance policy. In Zhang, Gaudoin, & Xie (2015), beta distribution is used to characterize the effect of imperfect repairs, and stochastic filtering facilitates the maintenance decisions. Liu, Xie, Xu, & Kuo (2016) consider a mixed failure model with both shocks and degradation failures and minimize the long-run maintenance cost rate. Liu, Wu, Xie, & Kuo (2017) incorporate both system age and state into the cost model and optimize the maintenance policies under various repair models via the Markov decision process (MDP). In Omshi, Grall, & Shemehsavar (2020), remaining useful life is used to adaptively optimize the maintenance policy in the presence of unknown degradation models. It can be witnessed that degradation-based maintenance has prevailed in both literature and real applications due to its flexibility and cost-effectiveness. Despite the emergence of warranty studies for degradation models in recent years, the existing literature mainly focuses on utilizing degradation models to extend the lifetime-based models, and neglects the realistic difference in customer behaviors, maintenance outcomes, and related cost characteristics between lifetime models and performance degradation models.

Warranty-oriented maintenance policies. Various warranty-oriented maintenance models have been studied in the literature. In general, different combinations of maintenance types (such as replacement, imperfect repair, and minimal repair) and warranty policies (such as base warranty, extended warranty, and two-

[1] Lexus. https://pressroom.lexus.com/lexus-extends-hybrid-battery-warranty-for-model-year-2020-vehicles/.

[2] DMG MORI. https://en.dmgmori.com/news-and-media/magazine/technology-excellence-01-2018/36-months-warranty-for-master-spindles.

Table 1
Difference between the proposed model and the existing literature on warranty.

Literature	Degrading warranted product	Random return level	Imperfect maintenance
Yang et al. (2016)			✓
Lim et al. (2019)			✓
Wang et al. (2017)	✓		
Oh et al. (2015)	✓		
Shang et al. (2018)	✓		
Wang et al. (2021)	✓		
Cha & Finkelstein (2022)	✓		
Huang et al. (2022)	✓		
Zhao et al. (2018)	✓		✓
Li et al. (2022)	✓	✓	
This paper	✓	✓	✓

dimensional warranty) are considered in these studies. Shafiee & Chukova (2013) provide a comprehensive overview for earlier contributions on related issues. Specifically, Yun, Murthy, & Jack (2008) and Yeo & Yuan (2009) argue to model and optimize warranty policies under imperfect repair scenarios. In Yang, He, & He (2016), a general imperfect repair model is proposed and used for warranty claim prediction by involving the product usage rate. Wang, Zhou, & Peng (2017) apply a game-theoretic approach to warranty decision making for systems subject to periodic preventive maintenance. Products are assumed to be attached with maintenance service contracts in Darghouth, Ait-kadi, & Chelbi (2017), where product design, sale price, and warranty period are jointly optimized. Chien, Zhang, & Yin (2019) model product failures by the generalized Polya process to optimize preventive maintenance policies under warranty. By relaxing the restrictions upon maintenance execution under warranty, Wang et al. (2020b) optimize an unpunctual preventive maintenance policy in which maintenance actions can be mildly advanced or postponed. Lim, Kim, & Park (2019) optimize the imperfect maintenance policies for warranted second-hand products.

Warranty analysis for products subject to degradation. In comparison to the literature regarding the aforementioned two topics, the research on warranty analysis for degrading products is relatively scant. Oh, Choi, Kim, Youn, & Pecht (2015) discuss a two-phase degradation model for electronic devices and employ the model for warranty abuse detection. Zhao, He, & Xie (2018) develop a degradation-based warranty prediction framework utilizing accelerated degradation data. Shang, Si, Sun, & Jin (2018) study warranty design and post-warranty maintenance both in a monopoly market and in a competitive market. Wang, Yang, Zhu, & Liu (2020a) presume a multi-state failure model and optimize a condition-based renewable warranty policy. Cha, Finkelstein, & Levitin (2021) discuss a renewable warranty policy for heterogeneous products subject to inverse Gaussian degradation processes. Recently, Wang, Liu, & Zhao (2021) propose a performance-based warranty for products subject to competing hard and soft failure processes. Similar assumptions are made in Huang, Fang, Lu, & (Bill) Tseng (2022) to optimize warranty policies for consumer electronics. Cha & Finkelstein (2022) suggest a new warranty policy for heterogeneous degrading products by screening at certain time epochs. Random failure threshold is proposed in Li, He, & Zhao (2022) to characterize the heterogeneity in warranted products. Broadly speaking, the amount of research on warranties for degrading products is surprisingly small despite the growing popularity in degradation-related research in reliability engineering.

Table 1 summarizes the most related works concerning warranty and compares them with our work. We fill the gap in the literature by simultaneously incorporating random return level and imperfect repair into maintenance models for warranted degrading products and systematically optimizing the policies.

1.3. Overview

Our work is among the earliest to consider the relationship between performance degradation and warranty claims, and optimize imperfect maintenance policies under fixed-length warranties for degrading products. We tackle the problem of maintenance optimization for warranted degrading products/systems under two scenarios. In the first one, products are returned by customers based on their subjective judgments on the degradation levels; the scenario mainly covers consumer durables such as personal electronic devices. Renewal theory-based methods are employed to efficiently evaluate the number of warranty claims under imperfect repair, which prompts minimization of the warranty cost via selecting an optimal objective repair level. In the second, the manufacturer implements periodic inspections for the product/system of interest and maintenance decisions are made upon the inspection epochs; the scenario mainly deals with industrial systems or facilities subject to periodic maintenance visits. The problem is formulated as an MDP. Key structural properties of the optimal maintenance policy are obtained through analyzing the stochastic monotonicity of the value function. To solve the MDP, backward dynamic programming algorithms are put forward.

The paper proceeds as follows. The first part of Sections 2 describes the degradation model and imperfect repair model. Next, Sections 2 and 3 deal with the maintenance policy optimization problems under the aforementioned two scenarios, respectively. Sections 4 presents numerical examples to demonstrate the proposed models. Concluding remarks are given in Sections 5. Technical proofs are relegated to the Appendix.

2. Degradation modeling and optimal policies under customer claiming

2.1. Stochastic degradation

Performance degradation of many systems can be well modeled by the Wiener process (Cui, Huang, & Li, 2016; Ye, Chen, & Shen, 2015), of which the form can be a partial derivative equation as follows:

$$\mathrm{d}X(t) = \mu \mathrm{d}\Lambda(t) + \sigma \mathrm{d}\mathcal{B}(\Lambda(t)), \tag{1}$$

where $\mu > 0$ and σ are the drift parameter and the diffusion parameter, respectively, and $\mathcal{B}(\cdot)$ denotes the standard Brownian motion. The function $\Lambda(\cdot)$ captures the non-linearity in the expected degradation path. Specifically, $\Lambda(t) = t$ indicates a linear expected path. The current setting yields the degradation model as $X(t) = \mu\Lambda(t) + \sigma\mathcal{B}(\Lambda(t))$. We are interested in when the degradation path will first exceed a threshold L, and the first passage time (FPT) to L, denoted by T, is proved to be an inverse Gaussian distributed random variable, of which the distribution function is

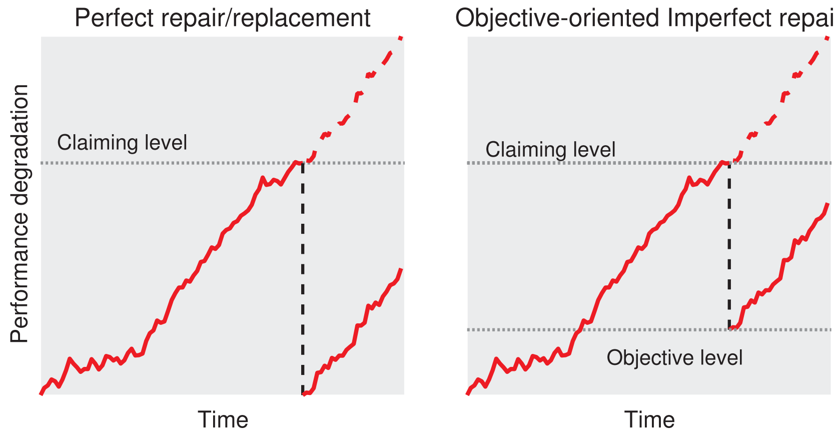

Fig. 1. Two maintenance schemes: replacement and objective-oriented imperfect repair (OOIR).

given by

$$F_T(t;\mu,\sigma,L) = \Phi\left[\sqrt{\frac{1}{\sigma^2\Lambda(t)}}(\mu\Lambda(t)-L)\right] + \exp\left(\frac{2\mu L}{\sigma^2}\right)\Phi\left[-\sqrt{\frac{1}{\sigma^2\Lambda(t)}}(\mu\Lambda(t)+L)\right], \quad (2)$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution. Without loss of generality, we simply assume $\Lambda(t)=t$ for mathematical convenience.

2.2. Warranty claim modeling

We first elaborate warranty claim modeling for degrading products under the pure corrective maintenance policy. In other words, no preventive interventions are considered at this stage. The assumption is reasonable for various consumer durables, such as laptops and cellphones. For such products, the warranties are claimed by customers, and corrective maintenance takes place when the degradation level reaches some threshold, which can be either deterministic or random according to customer behaviors. We suppose that the warranty period is deterministic and denoted by t_W, and the manufacturer sustains all the cost induced by maintenance actions.

2.2.1. Description of maintenance policies and cost

Two types of maintenance actions are considered in the model: replacement and imperfect repair. Replacements are also referred to as perfect repairs, which restore the system to the as-good-as-new state. Then, we introduce an objective-oriented imperfect repair (OOIR) scheme to model the imperfect maintenance. In the OOIR scheme, a repair renovates the system to a required objective degradation level. In other words, the degradation level after a repair is deterministic, by which we call the "objective level." Fig. 1 illustrates the degradation paths under the two maintenance schemes. It is natural to assume that a lower objective level leads to a higher repair cost.

To characterize the cost induced by different types of repairs, we regard the replacement cost as a benchmark and apply a reduction factor to imperfect repair cases. Let C_r be the constant cost of a replacement, and we denote by $D(L_c,l_o)$ the reduction factor, where L_c and l_o are the claiming level and the objective level, respectively, and we assume that $L_c > l_o$. Intuitively, either a replacement or a repair induces a fixed cost C_f caused by the minimum consumption of manpower, maintenance facility, and time. Thus, the cost for an imperfect repair given L_c and l_o is

$$C(L_c,l_o) = (C_r - C_f)D(L_c,l_o) + C_f. \quad (3)$$

The construction of an appropriate form of $D(L_c,l_o)$ depends on the product of interest. An easy yet effective way is to simply assume that $D(L_c,l_o)$ is a nondecreasing function of $L_c - l_o$. That is, the cost reduction in comparison to replacement only depends on the absolute amount of performance restoration. Alternative functional forms can be adopted to reflect the influence of both L_c and l_o. For simplicity, in the following derivations, we assume that $D(L_c,l_o) = \alpha_d(L_c - l_o)$, where $\alpha_d > 0$. Note that we impose no constraint on the upper bound of $D(L_c,l_o)$, implying that it can be greater than one. This is supported by the empirical evidence that repairs can be more costly than replacements under certain circumstances.

We next discuss two scenarios of warranty claiming. In the first one, the claiming threshold L_c is deemed to be deterministic while in the second it is modeled as a random variable under certain constraints on warranty. In either case, we investigate the warranty claim model under schemes of replacement and OOIR. To avoid confusion across the schemes, we let ΔL be the reduction of degradation level under both schemes. Specifically,

$$\Delta L = \begin{cases} L_c, & \text{under replacement,} \\ L_c - l_o, & \text{under OOIR,} \end{cases} \quad (4)$$

which makes the difference between the deterministic and random scenarios only lies in whether ΔL is a fixed value or a random variable.

2.2.2. Model 1: Deterministic warranty claiming threshold

When L_c is deterministic, we use a constant l_c to denote it instead, which also implies that ΔL is deterministic, and we denote it by Δl. Let ΔT_i, $i \ge 1$, be the time interval between the $(i-1)$th and the ith maintenance actions. The property of the Wiener degradation process guarantees the following result:

$$\Delta T_i \sim \begin{cases} \mathcal{IG}(l_c/\mu,\, l_c^2/\sigma^2), & \text{for } i=1, \\ \mathcal{IG}(\Delta l/\mu,\, \Delta l^2/\sigma^2), & \text{for } i \ge 2, \end{cases} \quad (5)$$

where $\mathcal{IG}(a,b)$ represents an inverse Gaussian (IG) distribution with mean a and shape b. The following proposition establishes the random distribution of time until the kth failure, denoted by $T_k = \sum_{i=1}^{k}\Delta T_i$:

Proposition 1. *T_k follows an IG distribution given as follows:*

$$T_k \sim \begin{cases} \mathcal{IG}(l_c/\mu,\, l_c^2/\sigma^2), & \text{for } k=1, \\ \mathcal{IG}\big([l_c+(k-1)\Delta l]/\mu,\, [l_c+(k-1)\Delta l]^2/\sigma^2\big), & \text{for } k>1. \end{cases} \quad (6)$$

Proposition 1 enables a convenient evaluation of the density functions and distribution functions of T_k, which are then employed to derive the mass function of the number of warranty claims for a single sold product within the warranty period t_W, denoted by N_W. Owing to the fact that T_k and ΔT_{k+1} are independent, we can give the mass function of N_W as follows:

$$\Pr(N_W = k) = \begin{cases} 1 - F_{T_1}(t_W), & k = 0, \\ \int_0^{t_W} f_{T_k}(t)\left[1 - F_{\Delta T_{k+1}}(t_W - t)\right]\mathrm{d}t, & k \geq 1. \end{cases} \tag{7}$$

Then, the expected warranty cost for a single sold product, comprising all the maintenance costs over the warranty period, denoted by $\mathbb{E}[C_W(l_c, l_o)]$, is given by

$$\begin{aligned} \mathbb{E}[C_W(l_c, l_o)] &= C(l_c, l_o)\sum_{k=1}^{\infty}\Pr(N_W = k)k + \sum_{k=0}^{\infty} C_P(k)\Pr(N_W = k) \\ &= C(l_c, l_o)\mathbb{E}(N_W) + \mathbb{E}(C_P(N_W)), \end{aligned} \tag{8}$$

where $C_P(k)$ is a nondecreasing penalty cost function associated with the number of warranty claims, with $C_P(0) = 0$. An optimal objective level l_o^* can be found via numerical methods to minimize the expected warranty cost $\mathbb{E}[C_W(l_c, l_o)]$; that is,

$$l_o^* = \arg\min_{l_o > 0} \mathbb{E}[C_W(l_c, l_o)].$$

Remark 1 (Rationale of. $C_P(k)$.) The penalty term in the model implies that the total warranty cost does not increase proportionally in the number of claims within t_W, and an additional claim induces higher marginal cost to the manufacturer. The idea originates from the fact that customers tend to believe that more warranty claims indicate lower product quality, which will have indirect negative influences on the manufacturer's reputation and sales of its products. These influences are characterized by the added cost term $C_P(k)$. The form of $C_P(k)$ is implicitly assumed to be discrete and nondecreasing in k.

2.2.3. Model 2: Heterogeneous warranty claiming threshold

Following the discussions in Section 2.2.2, we relax the assumption that L_c is deterministic. Instead, we treat L_c as a random variable to reflect the heterogeneity in customer behaviors. In practice, manufacturers impose restrictions on the claiming levels of products. In other words, a claim with a too low degradation level would not be entertained as this implies only mild performance degradation and a considerable length of remaining life. Thus, we use a deterministic level $l_{\min}$ to represent the minimum degradation level eligible for warranty claiming. Naturally, we assume that $L_c \geq l_{\min}$. Rather than working on L_c, we model $L_c - l_{\min}$ as a nonnegative random variable whose probability distribution can be chosen from a number of those with supports on $[0, \infty)$. We define $L_d = L_c - l_{\min}$ and presume that it can be well characterized by a random distribution F_{L_d}. Under the assumption of random claiming levels, the expected warranty cost for a single sold product under warranty satisfies:

$$\begin{aligned} &\mathbb{E}[C_W(L_d, l_o, l_{\min})] \\ &= \mathbb{E}_{L_d}\left[C(L_d + l_{\min}, l_o)\sum_{k=1}^{\infty}\Pr(N_W = k|L_d)k + \sum_{k=1}^{\infty}\Pr(N_W = k|L_d)C_P(k)\right] \\ &= \int_0^{\infty}\left[C(l + l_{\min}, l_o)\sum_{k=1}^{\infty}\Pr(N_W = k|L_d = l)k \right. \\ &\left. + \sum_{k=1}^{\infty}\Pr(N_W = k|L_d = l)C_P(k)\right] f_{L_d}(l)\mathrm{d}l, \end{aligned} \tag{9}$$

where

$$\Pr(N_W = k|L_d = l) = \begin{cases} 1 - F_{T_1'}(t_W), & k = 0, \\ \int_0^{t_W} f_{T_k'}(t)\left[1 - F_{\Delta T_{k+1}'}(t_W - t)\right]\mathrm{d}t, & k \geq 1, \end{cases} \tag{10}$$

and the distribution of T_k', $\forall k$, is given by

$$T_k' \sim \begin{cases} \mathcal{IG}((l_{\min} + l)/\mu, (l_{\min} + l))^2/\sigma^2), & k = 1, \\ \mathcal{IG}\begin{pmatrix} [l_{\min} + l + (k-1)(l_{\min} + l - l_o)]/\mu, \\ [l_{\min} + l + (k-1)(l_{\min} + l - l_o)]^2/\sigma^2 \end{pmatrix}, & k > 1. \end{cases} \tag{11}$$

Based on the aforementioned elaborations, an optimal objective level l_o^* that minimizes the expected cost $\mathbb{E}[C_W(L_d, l_o, l_{\min})]$, can be found via numerical methods; that is,

$$l_o^* = \arg\min_{l_o > 0} \mathbb{E}[C_W(L_d, l_o, l_{\min})]. \tag{12}$$

3. Optimal policies under periodic decisions

The previous section focuses on consumer durables whose warranties are claimed by customers per se. Nowadays, industrial machines and facilities are increasingly sold with warranties as well. As in the DMG MORI example in Section 1, such systems are usually sold with packages of professional inspection and maintenance services provided by the manufacturers. In other words, the user of such a system cannot be informed of complete system health condition until an inspection is carried out. If the degradation level exceeds a certain threshold, a penalty cost will be incurred to the manufacturer due to the user's loss in machine performance. Under a fixed-length warranty, the manufacturer seeks for an optimal maintenance policy under pre-determined inspection schedules. For continuously degrading systems subject to discrete inspections, the manufacturer can execute OOIR with different objective levels upon different inspection epochs to save cost within the warranty period. In what follows, we construct the cost function under the preventive maintenance policy and formulate the problem as an MDP. It is noteworthy that MDP-based methods have been widely adopted in various maintenance problems (see, e.g., Chen, Ye, Xiang, & Zhang, 2015; uit het Broek, Teunter, de Jonge, Veldman, & Van Foreest, 2020; van Oosterom, Peng, & van Houtum, 2017).

3.1. Preliminaries

Suppose again that the system of interest is protected by a warranty of fixed length t_W. The manufacturer provides K times of inspection services in a periodic manner with a constant interval of δ, i.e., the system is inspected at $k\delta$, where $k = 1, \cdots, K$. Specifically, we assume that $K\delta = t_W$, implying that upon the termination of warranty period there is still an inspection and decision epoch. The manufacturer is liable for all the maintenance and inspection costs before and upon t_W, as well as additional penalty costs caused by degradation. Moreover, upon t_W, the manufacturer commits to guarantee that the system degradation is no greater than $l_{\max}$.

The cost terms imposed to the manufacturer can be classified into the following types:

1. *Inspection cost.* Each time the system is inspected, a fixed cost of C_I is brought to the manufacturer. In practice, if the inspection schedule is pre-determined within a fixed warranty period, then the inspection cost can be merged into system-wise fixed cost and thus be eliminated from the decision models.
2. *Maintenance cost.* We stick to the OOIR cost models in Section 2.2.1. The manufacturer can either carry out the repair with a particular objective level or leave the system as it is.
3. *Degradation-induced penalty cost*. Since the system is presumed to be inspected at certain epochs, no corrective maintenance is carried out until an upcoming inspection. Degradation causes an additional cost to the system users as a consequence of performance loss. More often than not, the penalty cost emerges

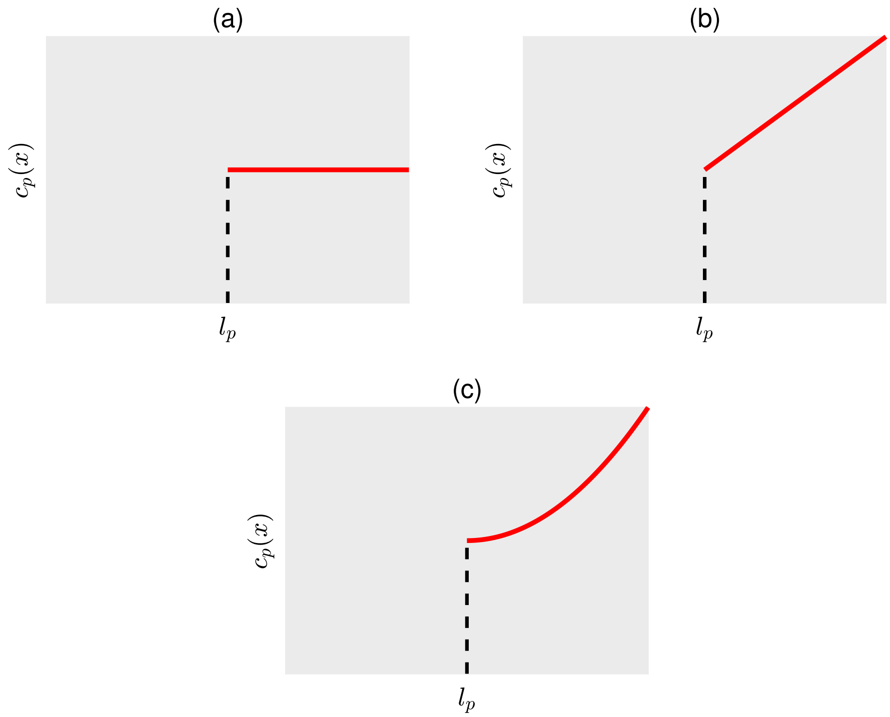

Fig. 2. Three special schemes of $c_p(x)$: (a) constant rate; (b) linearly increasing rate; and (c) quadratically increasing rate.

when the degradation level exceeds a certain threshold, denoted by l_p. It is natural to assume that the degradation-induced penalty cost rate $c_p(X(t))$ is a convex nondecreasing function of degradation level $X(t)$. Then, the expected penalty cost $W_{k\delta}(x)$ within the decision period following the kth inspection is given by

$$W_{k\delta}(x) = \mathbb{E}\left[\int_{k\delta}^{(k+1)\delta} c_p(X(t))\mathrm{d}t \middle| X_k = x\right], \quad \forall k \le K-1, \tag{13}$$

where X_k is the degradation level at $t = k\delta$, and $W_{K\delta}(\cdot) \equiv 0$. In particular, Fig. 2 shows three special forms of $c_p(X(t))$. Specifically, if $c_p(x) = c_p$ when $x > l_p$ (see Fig. 2(a)), then $W_{k\delta}(x)$ in (13) can be simplified as:

$$W_{k\delta}(x) = \mathbb{E}\left[\int_{k\delta}^{(k+1)\delta} c_p \mathbb{1}_{\{X(t)>l_p\}}\mathrm{d}t \middle| X_k = x\right], \quad \forall k \le K-1, \tag{14}$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function that equals 1 if the argument is true and 0 otherwise. If $c_p(x) = a + bx$ for $x > l_p$, i.e, $c_p(x)$ is linearly increasing in x (see Fig. 2(b)), then (13) can be rewritten as

$$W_{k\delta}(x) = \mathbb{E}\left[\int_{k\delta}^{(k+1)\delta} (a + bX(t))\mathbb{1}_{\{X(t)>l_p\}}\mathrm{d}t \middle| X_k = x\right], \quad \forall k \le K-1. \tag{15}$$

The integral in the two equations above can be regarded as a case in the Itô calculus (Bichteler & Klaus, 2002) and be calculated through manipulations on Itô processes. Likewise, the case of quadratic or higher orders of polynomial models can also be entertained via similar procedures.

Regarding the general form in (13), since $c_p(X(t))$ is also a random process, its integral can be rather complicated to derive under general settings of $c_p(\cdot)$. We thus resort to simulation methods to evaluate $W_{k\delta}(x)$. By (13), we find that $W_{k\delta}(x)$ does not change with k, thus it is adequate to compute all the values of $W_0(x)$ for all possible x. The following lemma is presented to facilitate subsequent problem formulation:

Lemma 1. *The expected degradation-induced penalty cost within one period, $W_{k\delta}(x_k)$, is nondecreasing in the observed system degradation level x_k.*

Lemma 1 indicates that a worse system state would induce a higher degradation-induced penalty cost. Next, we formulate the problem as an MDP and consider the following two possible scenarios: In the first one, the manufacturer and customer have an agreement to fix the objective repair level at a specific value y, and the manufacturer can only decide, upon each inspection, whether to repair the system or not. In the second, the manufacturer can not only decide whether to repair or not, but also choose an appropriate objective level upon each decision of repair.

3.2. MDP formulation and properties for constant objective value

Let us first assume that the objective level is fixed at y. In other words, after each repair, the degradation level of the system is always reduced to y. We use the value function in the form of Bellman equation, denoted by $V_{k\delta}(x_k)$, to represent the minimum expected cost from the current period k to the terminating time t_W:

$$V_{k\delta}(x) = \begin{cases} W_{k\delta}(x) + U_{k\delta}(x), & x \le y, \\ \min\left\{W_{k\delta}(x) + U_{k\delta}(x), C(x,y) + W_{k\delta}(y) + U_{k\delta}(y)\right\}, & x > y, \end{cases} \tag{16}$$

where $U_{k\delta}(x) = \mathbb{E}[V_{(k+1)\delta}|X_k = x]$ is the expected value function at the $(k+1)$th epoch. The term $W_{k\delta}(x) + U_{k\delta}(x)$ is the cost of "doing nothing" at epoch $k\delta$, while the term $C(x,y) + W_{k\delta}(y) + U_{k\delta}(y)$ is the cost of carrying out a repair with an objective level y at epoch $k\delta$. Note that $U_{K\delta}(x) \equiv 0$ and $W_{K\delta}(y) \equiv 0$ upon the terminating epoch K. Next, we present some essential properties of the value function and the optimal decisions.

Proposition 2. *The value function $V_{k\delta}(x)$ is nondecreasing in x for any $k = 0, \cdots, K$.*

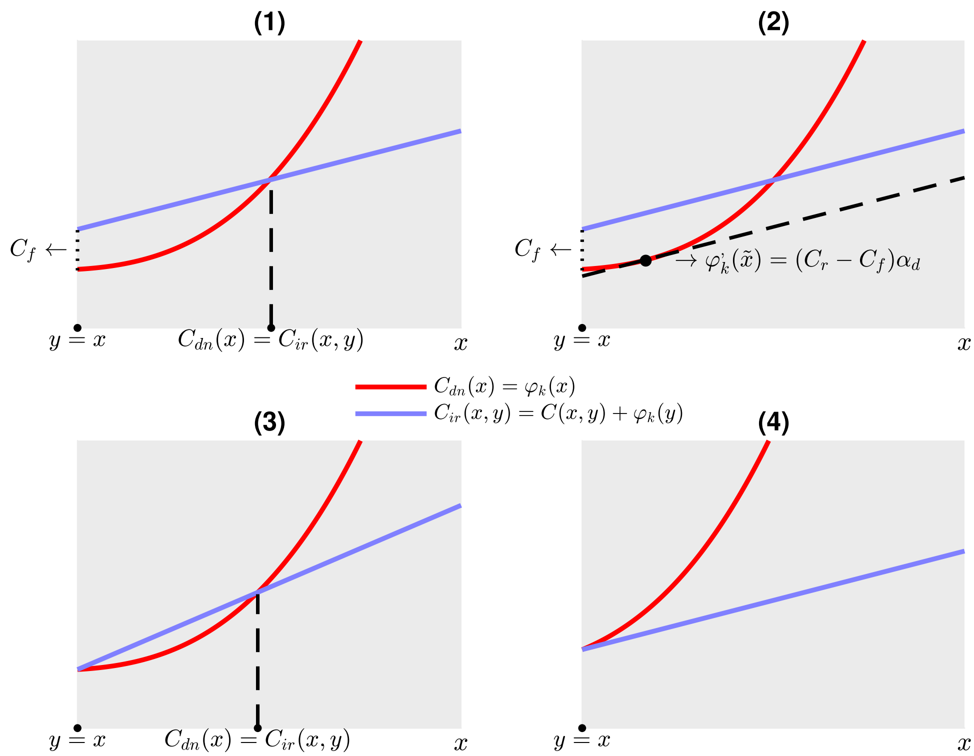

Fig. 3. Illustration of the four circumstances described in Corollary 1.

Proposition 2 states that the value function is higher for a worse system state, implying that a more severe state of degradation would result in a higher cost regardless of selected maintenance decisions. Suppose again that $C(x, y) = (C_r - C_f)\alpha_d(x - y) + C_f$ as discussed in Section 2.2.1; in other words, the cost of an imperfect repair increases linearly in degradation-reduction amount $x - y$. We have the following insights under mild assumptions.

Corollary 1. *Suppose that $x \geq y$ and $\varphi_k(x) = W_{k\delta}(x) + U_{k\delta}(x)$ is convex, continuous, and second-order differentiable, then the optimal decision between repair and doing nothing is a control limit policy if any one of the following conditions is satisfied:*

(i) *$C_f > 0$, and $\varphi_k(x)$ is strictly convex;*
(ii) *$C_f > 0$, and there exists some point $\tilde{x} > 0$ such that $\varphi_k'(\tilde{x}) > (C_r - C_f)\alpha_d$;*
(iii) *$C_f = 0$, $\varphi_k'(y) < (C_r - C_f)\alpha_d$, and there exists some point $\tilde{x} > y$ such that $\varphi_k'(\tilde{x}) = (C_r - C_f)\alpha_d$ and $\varphi_k'(x) > (C_r - C_f)\alpha_d$ for $x > \tilde{x}$;*
(iv) *$C_f = 0$, and $\varphi_k'(y) > (C_r - C_f)\alpha_d$. Moreover, the optimal control limit is $x^* = y$.*

In essence, Corollary 1 guarantees the optimality of the control limit policy under certain circumstances. In addition to the proof provided in the appendix, we illustrate the four circumstances with specific examples in Fig. 3. Next, the following corollary further reveals the relationship between the value function and $l_{\max}$.

Corollary 2. *The value function $V_{k\delta}(x)$ is nonincreasing in $l_{\max}$ for $k = 0, \cdots, K$ under any x.*

Corollary 2 shows that the value function becomes smaller when the required maximum level of degradation upon the end of the planning horizon is higher. It is worth mentioning that beyond the decision between repair and doing nothing, the determination of an optimal objective level y can be an alterable agreement between the manufacturer and customers. That is, the manufacturer may alter the value of y in order to attain more cost cutting. We will demonstrate this issue through numerical examples in Section 4.

3.3. MDP formulation and properties for variable objective level

Now let us assume that the manufacturer can implement imperfect repairs with different objective levels upon inspections, and denote by y_k the objective level upon the kth inspection. The value function can thus be revised as

$$\breve{V}_{k\delta}(x) = \min\left\{W_{k\delta}(x) + U_{k\delta}(x), \inf_{0<y_k<x}\left\{C(x, y_k) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\right\}\right\}. \tag{17}$$

Likewise, we zoom into the structural properties of the value function to glean insights through the following propositions and corollaries.

Proposition 3. *For any given x, the inequality $\breve{V}_{k\delta}(x) \leq V_{k\delta}(x)$ holds.*

This result follows directly from the forms of $\breve{V}_{k\delta}(x)$ and $V_{k\delta}(x)$, thus its proof is omitted. Proposition 3 tells us that the maintenance policy with variable objective levels outperforms—or at least identical to—that with a constant objective level in terms of cost cutting at any inspection interval. A special case of the proposition, in which we set $k = 0$, implies that adopting variable objective levels can save more on the total expected costs over the entire horizon. We then investigate the properties of the optimal objective levels.

Proposition 4. *Suppose that $C(x, y) = (C_r - C_f)\alpha_d(x - y) + C_f$ and imperfect repair is the optimal decision upon the kth inspection. Then, the optimal y_k, denoted by y_k^*, is nondecreasing in x.*

Proposition 4 states that the optimal objective level increases if the observed degradation level is higher. Next, we consider the case in which the repair cost depends merely on the objective level y, i.e., $C(x, y) = C(y)$. The practical implication behind this assumption is that rather than repairing a system, the manufacturer

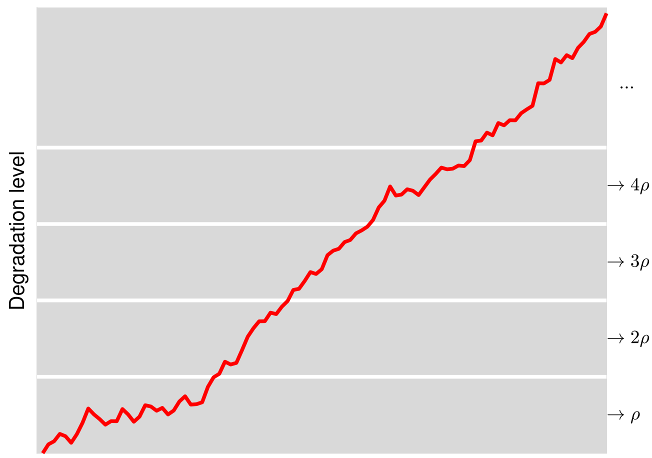

Fig. 4. Discretization of the degradation characteristic.

may implement an "imperfect replacement," that is, to dispose the degraded part and replace it with a part with discounted performance in comparison to a new one. In this situation, we have the following result.

Corollary 3. *Suppose that $C(x, y) = C(y)$, where $C(y)$ is nonincreasing in y, and $U_{k\delta}(x)$ is nondecreasing in x. Then, the optimal decision between repair and doing nothing is a control limit policy.*

In analogy to Corollary 2, a similar result also holds under variable objective levels as follows:

Corollary 4. *The value function $\breve{V}_{k\delta}(x)$ is nonincreasing in $l_{\max}$ for $k = 0, \cdots, K$ under any x.*

3.4. Dynamic programming to solve the MDP

In this subsection, we concretize the resolvent of the maintenance problems discussed above. For this purpose, we first prepare the problem setting by discretizing the state space and discuss the computation of transition probabilities. To ameliorate the computational burden, we assume that the system has M discrete states. Specifically, by the degradation level, the state set is given by $\{\rho, 2\rho, \cdots, M\rho\}$. The value of $M\rho$ needs to be large enough in the sense that the prospective cost of such a state would be overwhelmingly large and undesirable to the manufacturer. A degradation level x that satisfies $(m-1)\rho \le x < m\rho$ is said to be in the state $x = m\rho$, as shown in Fig. 4. Upon the kth inspection, given the current state x, the transition probability of the state at the next inspection can be given by

$$\Pr(X_{k+1} = \tilde{x} | X_k = x) = \Phi\left(\frac{\tilde{x} - x + \rho - \mu\delta}{\sigma\sqrt{\delta}}\right) - \Phi\left(\frac{\tilde{x} - x - \mu\delta}{\sigma\sqrt{\delta}}\right). \tag{18}$$

We note that $\Phi(-\mu\rho/\sigma\sqrt{\delta}) \approx 0$. In words, the degradation increment between two inspections is almost unlikely to be negative. For the problems described in Sections 3.2 and 3.3, we respectively develop Algorithms 1 and 2 to evaluate the value functions for all k. Specifically, in Step 5 of each algorithm, the calculation of $V_{k\delta}(x)$ or $\breve{V}_{k\delta}(x)$ involves the evaluation of $W_{k\delta}(x)$ and $U_{k\delta}(x)$. On the one hand, $W_{k\delta}(x)$ can be calculated via Monte Carlo simulation. On the other hand, $U_{k\delta}(x)$ can be evaluated via the following equation for the first algorithm:

$$U_{k\delta}(x) = \sum_{i=1}^{m} \Pr(X_{k+1} = m\rho | X_k = x) V_{(k+1)\delta}(m\rho),$$

Algorithm 1 Backward dynamic programming algorithm when the objective value is constant.

Require: Degradation parameters; cost factors and functions; inspection interval δ; objective level y; maximum number of inspections K
Ensure: $V_{k\delta}(x)$, $\forall x, k$
1: Compute $W_{k\delta}(x)$ and $\Pr(X_{k+1} | X_k = x)$, $\forall x$, via Monte Carlo simulation and (18), respectively.
2: For $k = K$, compute $V_{K\delta}(x)$, $\forall x$, through (A.3).
3: **repeat**
4: Set $k = k - 1$.
5: Calculate $V_{k\delta}(x)$, $\forall x$, through (16).
6: **until** $k = 0$.
7: **return** $V_{k\delta}(x)$, $\forall x, k$, and optimal decisions.

Algorithm 2 Backward dynamic programming algorithm when the objective value is variable.

Require: Degradation parameters; cost factors and functions; inspection interval δ; objective level y; maximum number of inspections K
Ensure: $\breve{V}_{k\delta}(x)$, $\forall x, k$
1: Compute $W_{k\delta}(x)$ and $\Pr(X_{k+1} | X_k = x)$, $\forall x$, via Monte Carlo simulation and (18), respectively.
2: For $k = K$, compute $\breve{V}_{K\delta}(x)$, $\forall x$, through (A.3).
3: **repeat**
4: Set $k = k - 1$.
5: Calculate $\breve{V}_{k\delta}(x)$, $\forall x$, through (17) and record y_k^*, $\forall x, k$, if imperfect repair is the decision.
6: **until** $k = 0$.
7: **return** $\breve{V}_{k\delta}(x)$ and y_k^*, $\forall x, k$, for imperfect repair decision epochs.

where $V_{(k+1)\delta}(m\rho)$ is obtained from Step 5 in the previous iteration, and the equation holds for the second algorithm by replacing $V_{k\delta}(x)$ with $\breve{V}_{k\delta}(x)$. Finally, the optimal decision is automatically selected by the arguments in (16) and (17) while computing them. As a side note, we can notice that the algorithm only needs to evaluate $W_{k\delta}(x)$ for all x for one time under a certain set of parameters instead of computing it in every iteration. In this manner, all the computations in the iterations are tractable, which limits the computational burden.

Once the value functions are fully evaluated, the optimal maintenance decisions can be readily obtained for any x and k. Denote by C_{Total} the total expected cost incurred to the manufacturer, and the expression is given by

$$C_{\text{Total}} = W_0(0) + U_0(0) + KC_I, \tag{19}$$

where $U_0(0)$ is the expected value function upon the first inspection given the initial state $x = 0$.

Note that the foregoing methods deal with the problem that minimizes the value function of the MDP, in which the inspection policy is predetermined. Apart from optimizing decisions upon fixed epochs, the inspection interval can also be a pivotal factor that affects the manufacturer's overall cost. For both models in Sections 3.2 and 3.3, the inspection interval δ can be regarded as an additional decision variable. For mathematical convenience and practical concerns, we assume that δ can be selected from a finite set of possible time spans, denoted by $\Omega = \{\delta_1, \cdots, \delta_{\max}\}$. We further suppose that there exist a series of integers satisfying $K_i = t_W/\delta_i$, $\forall \delta_i$. In general, more frequent inspections can reduce failure and penalty risks thereby save maintenance cost; however, inspections per se induce additional expenses. Therefore, the saving in maintenance cost and the additional inspection cost incurred form a trade-off. Considering the manufacturer's expected total cost, we can minimize (19) subject to $\delta \in \Omega$ in addition to the original decision variables in the MDP.

3.5. Extension to other stochastic degradation processes

Throughout the previous elaborations, we have been focusing on degradation paths characterized by the Wiener process. In many existing studies, the gamma process and the inverse Gaussian process are prevailingly used for monotonically increasing degradation paths, for which the Wiener process loses its appropriateness. Although the analyses under these models can be implemented in a similar fashion, unique properties of different stochastic processes may result in more specific insights in addition to the aforementioned ones. To this end, we elaborate an extension for Section 3.2, where we start from (16) and illustrate with the gamma process. The propositions and corollaries in Section 3.2 can be readily obtained for the gamma degradation model. The degradation increment over a period of δ, denoted by ΔX, follows a gamma distribution with shape and rate given by $\alpha\delta$ and β, respectively, i.e., the density function of ΔX is given by

$$f_{\Delta X}(x) = \frac{\beta^{\alpha\delta}}{\Gamma(\alpha\delta)} x^{\alpha\delta-1} \exp(-\beta x),\ \forall x \geq 0.$$

Then, the following lemmas hold:

Lemma 2. *If $\alpha \leq 1/\delta$, and denote by $\tilde{F}_{\Delta X}(x)$ the distribution function of degradation increment over a period of $\tilde{\delta} < \delta$, then $\tilde{F}_{\Delta X}(x)$ is concave.*

Lemma 3. *If $\tilde{F}_{\Delta X}(x)$ is concave and $c_p(\cdot)$ is convex, then $W_{k\delta}(x)$ is convex and nondecreasing in x.*

In light of Lemma 2 and 3, the following result can be readily obtained.

Proposition 5. *If $\tilde{F}_{\Delta X}(x)$ is concave, $c_p(\cdot)$ is convex, then $\varphi_k(x) = W_{k\delta}(x) + U_{k\delta}(x)$ is convex and nondecreasing in x for $k = 1, \cdots, K$.*

Proposition 5 guarantees an important condition in Corollary 1. For the inverse Gaussian process, the condition in Lemma 2 that $f_{\Delta X}(x)$ is decreasing in x cannot be satisfied as inverse Gaussian distribution has a positive mode, implying that at values close enough to zero, the density function is increasing in x. Unfortunately, this violates the condition in Lemma 2 and thus making it onerous to investigate the convexity of $\varphi_k(x)$. On the bright side, if the mode of an inverse Gaussian distribution is very close to zero, the density function can be regarded as approximately decreasing in x, which could justify Proposition 5 in an approximate sense. Detailed analysis is left for future investigation.

4. Numerical examples

In this section, we present numerical examples to demonstrate the proposed maintenance decision making models. Following the structure of Sections 2 and 3, this section is split into two subsections. Section 4.1 focuses on the first scenario in which customers claim warranties by themselves, while Section 4.2 deals with the second scenario in which the manufacturer implements periodic inspections and performs imperfect maintenance if necessary.

4.1. Example 1: Customers claim warranties

Consider a manufacturer that sells a certain type of product and offers a 24-month warranty to customers. A repair would be provided by the manufacturer if customer claims a warranty and the degradation threshold is eligible. Reliability tests show that the product's degradation path can be characterized by a Wiener process with $\mu = 0.3$ and $\sigma = 0.6$. The repair cost as a function of L_c and l_o is given by $C(L_c, l_o) = 1.5(L_c - l_o) + 3$, and the penalty cost function follows $C_P(k) = 2k^2$. In what follows, we present the numerical examples for constant and random claiming thresholds, respectively.

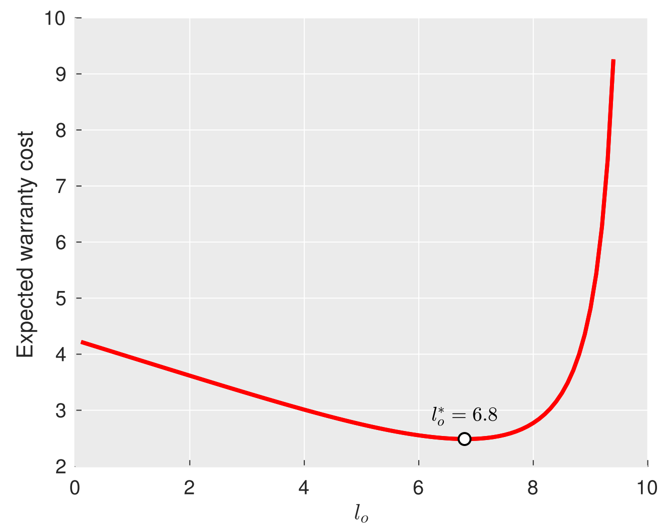

Fig. 5. Expected warranty cost under different l_o.

When the Claiming Threshold is Constant

When the customer claiming threshold is constant, the framework in Section 2.2.2 is adopted. Assume that the claiming threshold $L_c \equiv l_c = 10$. The curve of expected warranty cost against l_o is shown in Fig. 5. As can be seen, the expected warranty cost first decreases then increases in l_o, and reaches its minimum at $l_o^* = 6.8$. The corresponding minimum expected cost is 2.4867. Under the current optimal setting, the probability that a sold product does not fail within the warranty period is 0.7875, while the probabilities of 1–3 failures within $[0, t_W]$ are 0.1849, 0.0264, and 0.0012, respectively. As a side note, an optimal objective level of 6.8 is relatively large and this only repairs the product to a level that is worse than the "half-degraded" state upon each claim. Next, we tune the input parameters to adapt to the case of heterogeneous thresholds.

When the Claiming Threshold is Heterogeneous

To retain the consistency with the foregoing example, the mean claiming threshold is fixed at 10, i.e., $\mathbb{E}(l_{\min} + L_d) = 10$. The minimum eligible degradation level for warranty claiming is assumed to be 8, implying that $l_{\min} = 8$ and $\mathbb{E}(L_d) = 2$. Then, we assume that L_d follows a gamma distribution with shape and scale being 0.5 and 4, respectively. Figure 6 shows the probability density curve of the claiming threshold $l_{\min} + l_d$ and the curve of the expected cost per unit sold against l_o. The optimal objective level is given by $l_o^* = 4.8$, and the corresponding minimum expected cost is 3.81. In the presence of uncertainty in the claiming level, the objective level is significantly lower than that when the claiming threshold is constant. Meanwhile, the minimum expected cost becomes higher. The potential reason behind this is that the distribution has a very high variance of 8, and higher inherent uncertainty results in a more conservative maintenance decision (lower objective level) and higher cost. To this end, we show another example in which L_d follows a gamma distribution with shape and scale being 4 and 0.5, respectively. The setting still constrains the mean of L_d to be 2 while the variance reduces to 1, and the results under this setting are demonstrated in Fig. 7. The optimal objective level is $l_o^* = 6.0$, which results in an expected cost of 2.83. Evidently, the reduction in uncertainty of L_d encourages a less conservative opti-

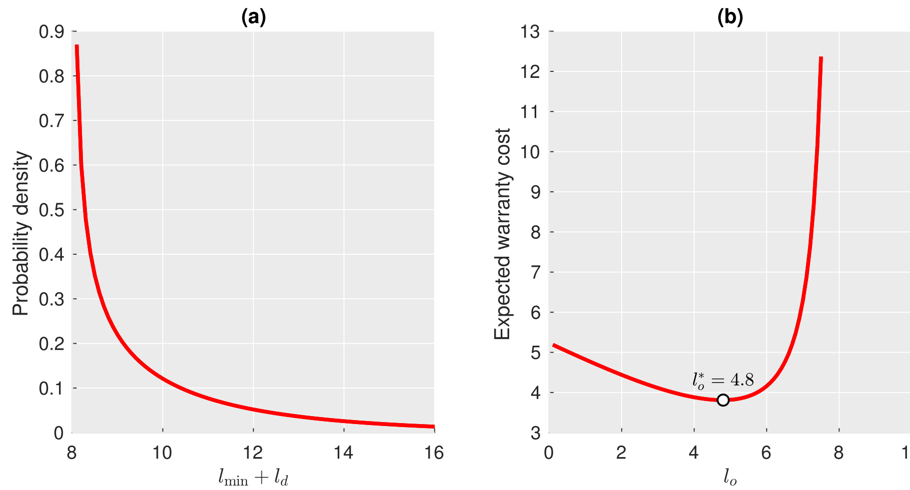

Fig. 6. (a) Probability density of $l_{\min} + l_d$. (b) Expected warranty cost against l_o.

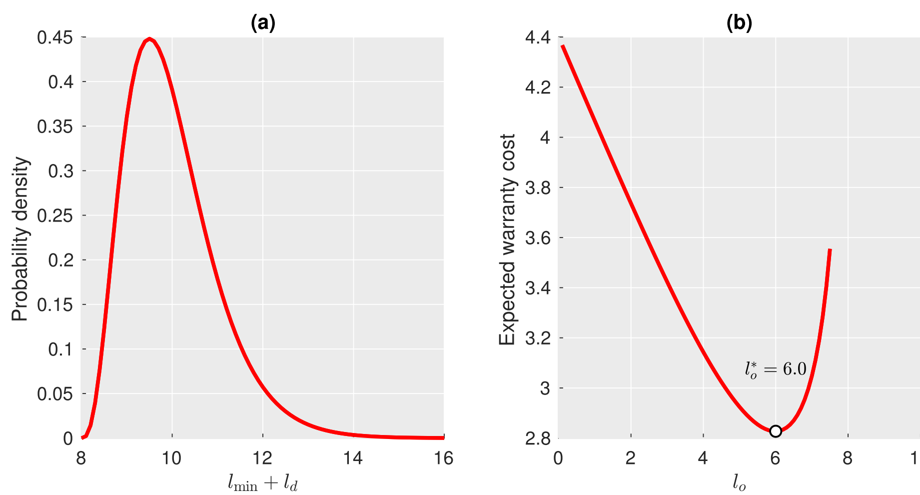

Fig. 7. The plots analogous to Fig. 6 under $L_d \sim \text{Gamma}(4, 0.5)$.

mal decision in which the objective level is higher. Nevertheless, l_o^* is still lower than that under the constant claiming threshold.

Influence of Form of $C(L_c, l_o)$

It is of interest to investigate how the form of $C(L_c, l_o)$ influences the optimal objective level and minimum expected cost. Since one may find it onerous to obtain related analytical results, a numerical sensitivity analysis is carried out. By assuming the model $C(L_c, l_o) = a(L_c - l_o) + b$ and letting $b \equiv 3$, we change the value of a, i.e, the marginal repair cost for a unit of reduction in degradation level, from 0 to 10 and observe the change in l_o^* and the corresponding expected cost. Figure 8 displays the results for the cases of constant and heterogeneous claiming levels. As a increases, the optimal objective level increases due to the higher marginal cost for imperfect repair. The increasing rate in l_o^* becomes lower when a is larger. The minimum expected cost increases roughly in a linear manner as a increases. By comparing the results when the claiming threshold is either constant or heterogeneous, l_o^* is higher and the consequent cost is lower when the threshold is constant for any a. Further, the increase in the optimal expected cost is faster when the claiming threshold is heterogeneous. In other words, when a increases, the heterogeneity in the claiming threshold incurs higher additional cost in comparison with the cost when the threshold is constant.

Influence of C_P

As a complement to the foregoing analyses, the influence of penalty cost C_P is studied. We stick to the quadratic form of C_P and assume that $C_P(k) = ck^2$. We change the value of c from 0 to 8 to study its influence on the optimal decisions, of which the results are shown in Fig. 9. The increase in c results in a lower l_o^* and a higher consequent cost, across the constant and heterogeneous claiming cases. Likewise, l_o^* is higher and the optimal expected cost is lower under the constant claiming case for any c.

Influence of Distribution of L_d

As implied in Figs. 6 and 7, the distribution of L_d may exert a remarkable influence on the optimal decisions. This motivates us to carry out a comprehensive sensitivity analysis with respect to the distribution of L_d. Table 2 lists l_o^* and the optimal expected cost under a couple of candidate probability distributions of L_d. For the sake of consistency, the mean values of these distributions are fixed at 2. The result showcases a general phenomenon that l_o^* is lower and the expected cost is higher in the presence of higher variability in L_d for a particular distribution. However, the optimal decisions can vary considerably across different distributions, even under the same standard deviation (e.g., see the results under Gamma (0.5,4) and IG(2,2)). Evidently, identifying an appropriate probability distribution for L_d is of importance to decision makers to effectively optimize the maintenance policies.

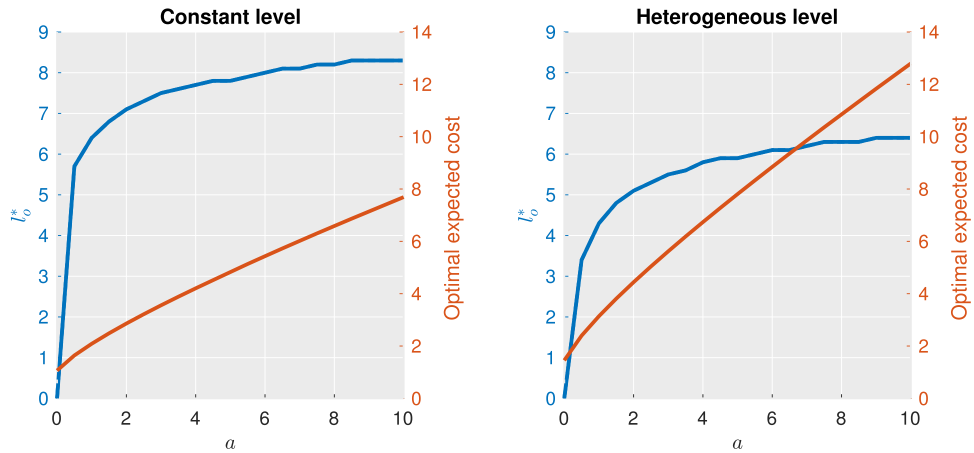

Fig. 8. Curves of l_o^* and optimal expected cost against varying a.

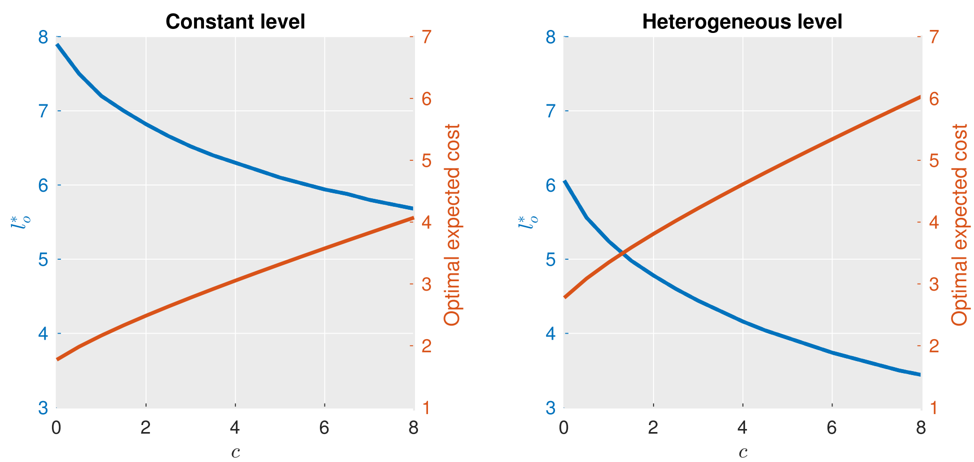

Fig. 9. Curves of l_o^* and optimal expected cost against varying c.

Table 2
l_o^* and optimal expected cost.

Distribution of L_D	Standard deviation	Optimal objective level l_o^*	Expected cost
Gamma (0.5,4)	2.83	4.78	3.81
Gamma (4,0.5)	1.00	5.98	2.83
Exponential (0.5)	2.00	5.16	3.38
Gamma (2,1)	1.41	5.58	3.06
IG (2,0.2)	6.32	4.60	4.52
IG (2,0.5)	4.00	4.86	4.00
IG (2,1)	2.00	5.12	3.61
IG (2,2)	2.83	5.42	3.27

From the foregoing analyses, we can observe a remarkable difference between the results under constant and heterogeneous claiming cases. In general, the uncertainty embedded in customer claiming behaviours leads to more conservative decisions.

4.2. Example 2: Products are periodically inspected

We proceed to the demonstration of the methods proposed in Section 3 that deal with periodically inspected products. To start with, we formulate the problem and list the model input. Since the products of interest for the method are most likely to be industrial systems, we set a longer warranty period of $t_W = 5$ years and the inspection interval is assumed to be 0.5 year. The drift and diffusion parameters of the degradation process are set as 1.0 and 0.6, respectively. The degradation-induced penalty cost rate c_p is specified as follows:

$$c_p(x) = \begin{cases} 0, & x \le 3, \\ 15(x-3), & x > 3. \end{cases} \tag{20}$$

In other words, no penalty cost is induced when the degradation level is below 3. Moreover, the imperfect repair cost is given by $C(L_c, l_o) = 5(L_c - l_o) + 4$. Upon the last inspection, the manufacturer commits to guarantee the degradation level to be 1.5 or smaller, i.e., $l_{\max} = 1.5$. The state of the product is discretized as $\{0.1, 0.2, \cdots, 15.0\}$.

When the Objective Level is Constant

First, we demonstrate the case in which the objective level is fixed at a constant level. For the purpose of illustration, the objective level is fixed at $y = 0.5$. Figure 10 exhibits the control limit of the degradation level for the optimal decisions. The corresponding optimal cost is 24.02. As can be observed, the control limit is

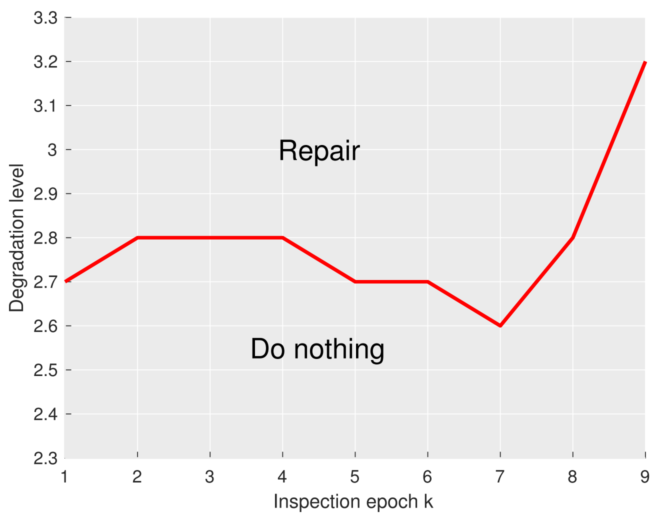

Fig. 10. Optimal decisions for all inspection epochs and degradation levels under the constant objective level $y = 0.5$.

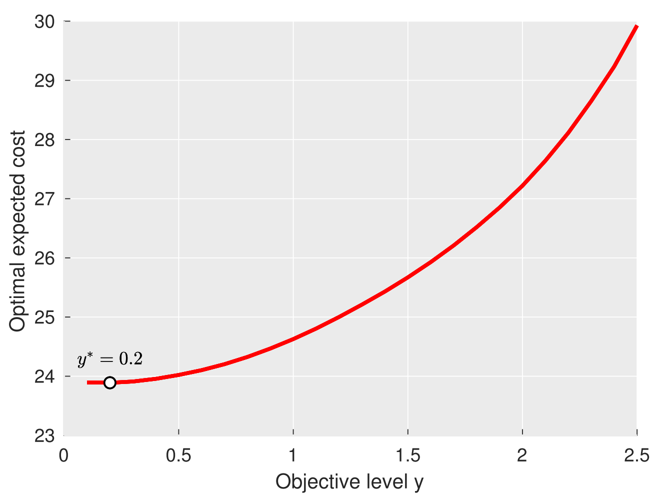

Fig. 11. Optimal expected cost against varying y.

non-monotone in the number of inspection epochs. To be specific, the limit first increases slightly and then remains almost constant for a couple of inspection epochs; afterwards, the limit decreases a bit and finally increases sharply towards the end of the horizon. We notice that upon the second last inspection ($k = K - 1 = 9$), the control limit is even higher than the triggering level of the penalty cost, which is previously set as 3.

As mentioned previously, the objective level can also be a decision variable. In other words, the manufacturer may flexibly choose an objective level that is below the maximum threshold specified by customers as it is natural that customers do not mind if the product is repaired to a better condition upon each inspection. Therefore, we carry out an optimization on the expected cost over different selections of objective level y. Figure 11 shows the curve of the optimal expected cost under varying y. We observe that the unique optimal objective is given by $y^* = 0.2$. When y increases within the range $[0.2, 2.5]$, the optimal expected cost increases in a convex manner from an intuitive observation. This implies that objective levels that are higher than $y^* = 0.2$ are normally unrecommended for the sake of cost saving and risk reduction for both the manufacturer and customers. In practice, although the optimal objective level of $y^* = 0.2$ costs least to the manufacturer, customers may request an even lower objective level. In the current example, as $y^* = 0.2$ is already a very low objective level, to repair the product to the perfect state ($y = 0.1$ under the model settings) can

Table 3
Optimal control limits for each inspection under different $l_{\max}$ under the constant objective level $y = 0.5$.

Inspection epoch k	Guaranteed level upon inspection K, $l_{\max}$					
	0.1	0.5	1.0	1.5	2.0	2.5
1	2.7	2.7	2.7	2.7	2.7	2.7
2	2.8	2.8	2.8	2.8	2.7	2.7
3	2.8	2.8	2.8	2.8	2.8	2.8
4	2.8	2.8	2.8	2.8	2.8	2.8
5	2.7	2.7	2.7	2.7	2.8	2.7
6	2.6	2.6	2.7	2.7	2.6	2.3
7	2.6	2.6	2.7	2.6	2.4	2.4
8	2.8	2.8	2.8	2.8	2.7	2.9
9	3.3	3.2	3.1	3.2	3.5	3.8
Optimal expected cost	**31.31**	**29.24**	**26.53**	**24.02**	**21.44**	**19.47**

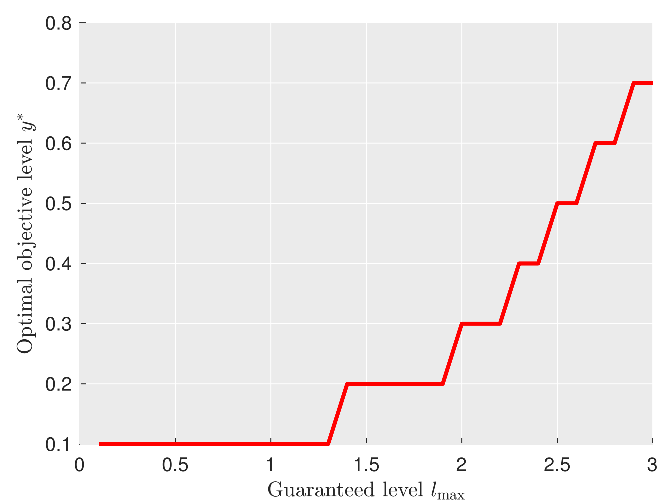

Fig. 12. Optimal objective levels under different guaranteed levels $l_{\max}$.

be a good choice. This only induces a slightly higher cost while repairing to the perfect state is a sought-after approach to improve customers' satisfaction.

Next, we explore the influence of the guaranteed level upon the last inspection, i.e., $l_{\max}$. Table 3 lists the optimal control limits and respective expected cost under different $l_{\max}$. In consistency with Corollary 2, the optimal expected cost decreases as $l_{\max}$ gets larger. Moreover, the control limits vary slightly with respect to $l_{\max}$ for the earlier inspection epochs. When $l_{\max}$ becomes larger, the optimal control limit drops and rises to a greater extent for the later inspection epochs, especially for $k \geq 5$. It is also of interest to explore the relationship between the optimal objective level $y*$ and the guaranteed level $l_{\max}$, of which we illustrate in Fig. 12. We notice a remarkable positive correlation between y^* and $l_{\max}$.

When the Objective Level is Variable

Following the model in Section 3.3, we remove the restriction that the objective level has to be constant. In Fig. 13, the optimal decisions under the variable objective level are compared to those when the objective level is fixed at $y = 0.5$. The control limits between repair and doing nothing becomes lower when the objective level is variable, especially when k is small. When $k \geq 7$, the control limits under the two cases coincide. In addition, the optimal objective level upon each inspection turns out to be independent of the degradation level if the optimal decision is to repair. To be specific, the optimal objective level is 0.1 for $k = 1$ to 7, while for $k = 8$ and $k = 9$, the optimal levels are given by 0.5 and 0.8, respectively. The optimal expected cost is 18.82, in contrast to 24.02 when the objective level is fixed at $y = 0.5$. In light of the result, the variable objective levels ameliorate the optimal policy

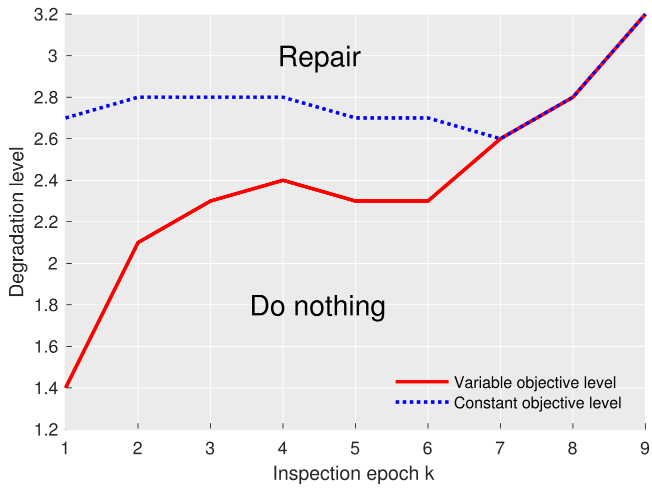

Fig. 13. Optimal decisions under variable objective level and constant objective level $y = 0.5$.

Table 4
Optimal control limits for each inspection under different $l_{\max}$ under variable objective level.

Inspection epoch k	Guaranteed level upon inspection K, $l_{\max}$					
	0.1	0.5	1.0	1.5	2.0	2.5
1	0.7	0.8	0.9	1.4	2.0	2.2
2	0.8	1.0	1.6	2.1	2.3	2.4
3	1.1	1.7	2.2	2.3	2.5	2.5
4	1.9	2.2	2.3	2.4	2.5	2.4
5	1.7	1.9	2.2	2.3	2.3	2.5
6	1.9	2.1	2.3	2.3	2.5	2.6
7	2.3	2.4	2.5	2.6	2.7	2.7
8	2.7	2.8	2.8	2.8	2.8	2.8
9	3.2	3.1	3.1	3.2	3.1	3.2
Optimal expected cost	**23.02**	**21.93**	**20.39**	**18.82**	**17.00**	**15.48**

by saving more cost, which validates the result in Proposition 3. In analogy to Table 3, we list the optimal decisions under different selections of $l_{\max}$ in Table 4, which then confirms the result in Corollary 4. The control limits are significantly lower than those in Table 3. Meanwhile, the optimal expected cost is also lower, especially when $l_{\max}$ is small. Taken together, the merit of the policy with variable objective level lies in substantial cost saving, although the application of the policy may be subject to practical constraints, such as maintenance flexibility and customers' preferences.

Additional Analyses and Discussions

The foregoing elaborations have not incorporated the influence of cost functions, specifically, the marginal costs induced by imperfect repair and the penalty cost. We thus shift our focus to the sensitivity analysis with respect to the cost functions. First, we recall the repair cost model $C(L_c, l_o) = a(L_c - l_o) + b$ and change the value of a from 0.5 to 20; the optimal maintenance policies are presented in Table 5. By comparison, the minimum expected cost when the objective level is constant is more sensitive to the change in a, and the optimal control limits vary little for the first several inspection epochs. When a gets larger, the control limit decreases faster upon the 5th and 6th inspection epochs and then increases faster towards the terminating inspection. In contrast, when the objective level is variable, the optimal control limit gets quite low when a is larger for $1 \le k \le 5$. In general, variable objective level provides more flexibility in the decision model and thus can save more cost than the model with constant objective level by avoiding high degradation levels during the decision horizon when the marginal repair cost is higher.

Table 5
Optimal control limits for each inspection under different a under the constant objective level $y = 0.5$ and under variable objective level (in parentheses).

Inspection epoch k	Marginal repair cost a				
	0.5	1	5	10	20
1	2.8 (2.6)	2.8 (2.5)	2.7 (1.4)	2.7 (0.5)	2.7 (0.3)
2	2.8 (2.6)	2.8 (2.6)	2.8 (2.1)	2.8 (0.7)	2.8 (0.4)
3	2.8 (2.6)	2.8 (2.6)	2.8 (2.3)	2.8 (0.9)	2.8 (0.5)
4	2.7 (2.6)	2.7 (2.6)	2.8 (2.4)	2.8 (1.4)	2.8 (0.7)
5	2.7 (2.6)	2.7 (2.6)	2.7 (2.3)	2.7 (1.8)	2.7 (1.6)
6	2.7 (2.7)	2.7 (2.7)	2.7 (2.3)	2.5 (2.3)	2.2 (2.6)
7	2.8 (2.8)	2.8 (2.7)	2.6 (2.6)	2.5 (2.7)	2.5 (2.7)
8	2.9 (2.7)	2.8 (2.7)	2.8 (2.8)	2.8 (2.8)	3.1 (2.9)
9	2.7 (2.6)	2.8 (2.8)	3.2 (3.2)	3.6 (3.3)	4.4 (3.3)
Optimal expected cost	**9.12 (6.64)**	**10.80 (8.03)**	**24.02 (18.82)**	**40.32 (30.83)**	**72.61 (52.29)**

Table 6
Optimal control limits for each inspection under different c under the constant objective level $y = 0.5$ and under variable objective level (in parentheses).

Inspection epoch k	Marginal penalty cost c				
	5	10	15	20	50
1	3.1 (1.6)	2.9 (1.4)	2.7 (1.4)	2.7 (1.3)	2.5 (1.2)
2	3.1 (2.4)	2.9 (2.2)	2.8 (2.1)	2.7 (2.0)	2.5 (1.8)
3	3.1 (2.5)	2.9 (2.4)	2.8 (2.3)	2.7 (2.3)	2.5 (2.1)
4	3.1 (2.4)	2.9 (2.4)	2.8 (2.4)	2.7 (2.4)	2.5 (2.2)
5	3.0 (2.3)	2.8 (2.3)	2.7 (2.3)	2.7 (2.3)	2.5 (2.2)
6	2.9 (2.6)	2.7 (2.4)	2.7 (2.3)	2.6 (2.3)	2.4 (2.1)
7	3.0 (2.9)	2.7 (2.7)	2.6 (2.6)	2.6 (2.5)	2.4 (2.3)
8	3.2 (3.3)	2.9 (2.9)	2.8 (2.8)	2.7 (2.7)	2.4 (2.5)
9	4.1 (3.9)	3.5 (3.4)	3.2 (3.2)	3.1 (3.0)	2.8 (2.8)
Optimal expected cost	**23.26 (18.48)**	**23.75 (18.70)**	**24.02 (18.82)**	**24.21 (18.90)**	**24.83 (19.14)**

By generalizing the penalty cost model in (20) as follows:

$$c_p(x) = \begin{cases} 0, & x \le 3, \\ c(x-3), & x > 3, \end{cases} \tag{21}$$

we change the value of c to characterize the fluctuation of the marginal penalty cost per unit time for degradation level exceeding 3. Table 6 summarizes the optimal decisions against c, in which we can observe similar results to those in Table 5. The variable objective level allows for more flexibility in the control limit and thereby results in lower expected cost. An observation is that as c gets very large, the optimal control limits and the expected cost do not change drastically due to the fact that the optimal decisions lead to extremely low probabilities that penalty cost is incurred.

By systematically comparing the results under constant and variable objective levels, we demonstrate the merit in unchaining the restriction on the objective level. In essence, the exercise using the variable objective level is recommended to a manufacturer that seeks for more cost cutting, especially when the marginal repair cost is high.

5. Concluding remarks

In the paper, we systematically explore imperfect maintenance policies for warranted products subject to stochastic performance degradation. Two scenarios that cover different types of products are studied and two models, one utilizing the renewal theory and the other based on the MDP, are proposed and investigated. In the first scenario, in which customers return their items in a random manner, we find that the randomness in the return degradation level induces more conservative maintenance policies in which the product needs to be repaired to a better status. In the second scenario, in which repairs are dynamically optimized upon fixed inspection epochs, the advantage of variable objective repair levels is proved and some structural proprieties of the optimal policy are put forward. The proposed methods are demonstrated with extensive numerical studies and sensitivity analyses, from which some case-specific insights have been obtained.

The proposed framework can be extended in several promising aspects. First, the randomness in customer claiming behaviors can be investigated by data-driven approaches and dynamic decision making framework can be studied accordingly. Moreover, similar approaches that accommodate more complex warranty schemes, such as two-dimensional warranties and extended warranties, are underexplored. More considerations in operations management, such as product/warranty pricing as well as product marketing, can also be incorporated into the current framework.

Acknowledgments

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Appendix: Technical Proofs

Proof of Proposition 1. When $k = 1$, $T_1 \sim \mathcal{IG}(l_c/\mu, l_c^2/\sigma^2)$ immediately follows (5). When $K \geq 2$, the proof can be entertained by using the moment generating functions (MGF). To be specific, since ΔT_i, $i \geq 1$ are independent random variables, the MGF of T_k is given by

$$\begin{aligned} M_{T_k}(z) &= \prod_{i=1}^{k} M_{\Delta T_i}(z) \\ &= \exp\left\{ \frac{l_c\mu}{\sigma^2}\left(1-\sqrt{1-\frac{2\sigma^2 z}{\mu^2}}\right) + \sum_{i=2}^{k}\left[\frac{\Delta l\mu}{\sigma^2}\left(1-\sqrt{1-\frac{2\sigma^2 z}{\mu^2}}\right)\right]\right\} \\ &= \exp\left[\frac{\left(L_c + \sum_{i=2}^{k}\Delta l\right)\mu}{\sigma^2}\left(1-\sqrt{1-\frac{2\sigma^2 z}{\mu^2}}\right)\right], \end{aligned}$$

which is also in a form of the MGF of the IG distribution and thus proves the result. □

Proof of Lemma 1. To start with, $W_{K\delta}(x)$ is nondecreasing for any x since it always equals to zero. Next, the definition of stochastic order is introduced.

Definition A.1 A random variable X is smaller than a random variable Y in the stochastic order if $\Pr(X \leq a) \geq \Pr(Y \leq a)$ for any a, and this is denoted by $X \prec Y$. If $X \prec Y$, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$ provided the expectations exist.

According to the property of the Wiener process, given $x_k = x$, $X(t)$ follows a normal distribution with mean $x + \mu(t - k\delta)$ and variance $\sigma^2(t - k\delta)$, thus we can conclude that for any $\tilde{x}_k \leq x_k$, the following inequality holds:

$$\Pr(X(t|\tilde{x}_k) \leq a) \geq \Pr(X(t|x_k) \leq a), \ \forall a, t \tag{A.1}$$

due to the fact that $X(t|\tilde{x}_k)$ follows a normal distribution with a smaller mean and the same variance in comparison to $X(t|x_k)$, implying that $X(t|\tilde{x}_k) \prec X(t|x_k)$, $\forall t$. Switching to the the multivariate case, it can be easily shown that for any sets of time points $(t_1, \cdots, t_n)$, we have $(X(t_1|\tilde{x}_k), \cdots, X(t_n|\tilde{x}_k)) \prec (X(t_1|x_k), \cdots, X(t_n|x_k))$. This satisfies the definition of the stochastic order of random processes, i.e., $\{X(t|\tilde{x}_k, t \in \mathcal{T})\} \prec \{X(t|x_k), t \in \mathcal{T}\}$, where $\mathcal{T}$ is the time parameter space and we assume that $\mathcal{T} = [0, \infty)$. According to Shaked & Shanthikumar (2007, Eq. (6.B.29)), $\{X(t|\tilde{x}_k), t \in \mathcal{T})\} \prec \{X(t|x_k), t \in \mathcal{T}\}$ if and only if

$$\mathbb{E}\{g(X(t|x_k), t \in \mathcal{T})\} \leq \mathbb{E}\{g(X(t|\tilde{x}_k), t \in \mathcal{T})\}, \tag{A.2}$$

for any nondecreasing functional g for which the expectations exist. It is natural that

$$\int_{k\delta}^{(k+1)\delta} c_p(X(t))\mathrm{d}t \bigg| X_k = x_k$$

is a nondecreasing function of $X(t)$ as c_p is positive and nondecreasing in $X(t)$. Thus, for $\tilde{x}_k \leq x_k$, $W_{k\delta}(\tilde{x}_k) \leq W_{k\delta}(x_k)$ holds for any k. □

Proof of Proposition 2. The proof is done by backward induction on k. First, we consider the period of termination. When $k = K$, the value function becomes

$$V_{K\delta}(x) = \begin{cases} 0, & \text{if } x \leq l_{\max}, \\ C(x, l_{\max}), & \text{if } x > l_{\max}. \end{cases} \tag{A.3}$$

Because $C(x, y_K)$ is nondecreasing in x, it is clear that $V_{K\delta}(x)$ is nondecreasing in x. Supposing that the nondecreasing property holds for $(k+1)$th period where $0 \leq k \leq K-1$, we need to show that it also holds for the kth period. For any $x^- < x^+$, by the definition of the stochastic order, it is straightforward that $\langle X_{k+1}|X_k = x^-\rangle \prec \langle X_{k+1}|X_k = x^+\rangle$. Further, considering the value function at $k\delta$, recall that we have $U_{k\delta}(x) = \mathbb{E}\left[V_{(k+1)\delta}|X_k = x\right]$. By Shaked & Shanthikumar (2007), combining the fact that $V_{(k+1)\delta}(x)$ is nondecreasing in x, we obtain

$$\begin{aligned} U_{k\delta}(x^-) &\equiv \mathbb{E}\left[V_{(k+1)\delta}(X_{k+1})|X_k = x^-\right] \\ &\leq \mathbb{E}\left[V_{(k+1)\delta}(X_{k+1})|X_k = x^+\right] \equiv U_{k\delta}(x^+), \end{aligned}$$

indicating that $U_{k\delta}(x)$ is nondecreasing in x. Given the above results, for the kth period, in (16), both $W_{k\delta}(x) + U_{k\delta}(x)$ and $C(x, y) + W_{k\delta}(y) + U_{k\delta}(y)$ are nondecreasing in x. We can readily conclude that the value function is nondecreasing in x and the hypothesis of induction holds, which completes the proof. □

Proof of Corollary 1. Given that $\varphi_k(x) = W_{k\delta}(x) + U_{k\delta}(x)$ is convex, we have the fact that $\varphi_k'(x)$ is nondecreasing with x. The proof proceeds as follows:

1. If $\varphi_k(x)$ is strictly convex, then $\varphi_k''(x) > 0$ for all x, implying that $\varphi_k'(x)$ keeps increasing in x. When $x > y$, it is optimal to repair the system if the cost of doing nothing $C_{dn}(x) = \varphi_k(x)$ is larger than the imperfect repair cost $C_{ir}(x, y) = (C_r - C_f)\alpha_d(x - y) + C_f + \varphi_k(y)$. As shown in Fig. 3-(1), it is obvious that $C_{dn}(y) < C_{ir}(y, y)$, indicating that the optimal decision under $x = y$ is doing nothing. As x increases, strictly convex function $C_{dn} = \varphi_k(x)$ increases monotonically with an increasing rate $\varphi_k'(x)$, while $C_{ir}(y, y)$ increases linearly with a constant rate α_d. Then there must exist some point x^* that satisfies $C_{dn}(x^*) = C_{ir}(x^*, y)$ and $C_{dn}(x) > C_{ir}(x, y)$ for any $x > x^*$. That is, when $x > x^*$, the optimal decision is to repair. Taken together, x^* is the optimal control limit for the decisions between doing nothing and imperfect repair, and it is apparent that $x^* > y$.
2. If the assumption of strong convexity is dropped, we still have $C_{dn}(y) < C_{ir}(y, y)$. Further, we have the condition that there exists some point $\tilde{x} > 0$ such that $\varphi_k'(\tilde{x}) = (C_r - C_f)\alpha_d$ and for any $x > \tilde{x}$, we have $\varphi_k'(x) > (C_r - C_f)\alpha_d$. Likewise, there must exist some point x^* that satisfies that $C_{dn}(x^*) = C_{ir}(x^*, y)$ and $C_{dn}(x) > C_{ir}(x, y)$ for any $x > x^*$, which forms a control limit.
3. If the fixed cost for repair is set to 0, i.e., $C_f = 0$, then $C_{dn}(y) = C_{ir}(y, y)$. Additionally, since C_{dn} and $C_{ir}(y, y)$ as well as $\varphi_k'(x)$ all increase in x, inequalities $\varphi_k'(y) < \alpha_d$ and $\varphi_k'(\tilde{x}) = (C_r - C_f)\alpha_d$ imply that when $x \geq \tilde{x}$, the increase rate of $C_{dn}(x)$ is higher and therefore $C_{dn}(x) > C_{ir}(x, y)$. Then, since $\varphi_k'(x) > (C_r - C_f)\alpha_d$ for $x > \tilde{x}$, the increase rate of $C_{ir}(x, y)$ is higher than $C_{dn}(x)$,

thus likewise, there must exist some point x^* that satisfies that $C_{dn}(x^*) = C_{ir}(x^*, y)$ and $C_{dn}(x) > C_{ir}(x, y)$ for any $x > x^*$, which forms a control limit.

4. Conditions $C_f = 0$, and $\varphi_k'(y) > (C_r - C_f)\alpha_d$ guarantee that $C_{dn}(x) > C_{ir}(x, y), \forall x > y$. Therefore, once the system degradation level reaches y, it is more cost-effective to repair the system.

□

Proof of Corollary 2. The proof is entertained by mathematical induction. Recall that the value function upon the last inspection in (A.3), which is clearly nonincreasing in $l_{\max}$. By supposing that the property holds for the $(k+1)$th period where $0 \le k \le K-1$, then under the kth period, we have $U_{k\delta}(x)$ is nonincreasing in $l_{\max}$ for any x. Moreover, $W_{k\delta}(x)$ is independent of $l_{\max}$. Thus, $V_{k\delta}(x)$ is nonincreasing in $l_{\max}$, which completes the proof. □

Proof of Proposition 4. If imperfect repair is the optimal choice, we have

$$\begin{aligned}\check{V}_{k\delta}(x) &= \inf_{0\le y_k<x}\left\{C(x, y_k) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\right\}\\ &= \inf_{0\le y_k<x}\left\{(C_r - C_f)\alpha_d(x-y) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\right\} + C_f.\end{aligned}$$

Let us consider two system states x and $x+\epsilon$, where ϵ is an arbitrary positive number that makes $x+\epsilon > x$. The value function at $x+\epsilon$ is

$$\begin{aligned}\check{V}_{k\delta}(x+\epsilon) &= \inf_{0\le y_k<x+\epsilon}\left\{C(x+\epsilon, y_k) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\right\}\\ &= \inf_{0\le y_k<x+\epsilon}\left\{(C_r - C_f)\alpha_d(x-y_k) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\right\} + (C_r - C_f)\alpha_d\epsilon + C_f\\ &= \min\left\{\begin{array}{l}\inf_{0\le y_k<x}\left\{(C_r - C_f)\alpha_d(x-y_k) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\right\} + (C_r - C_f)\alpha_d\epsilon + C_f,\\ \inf_{x\le y_k<x+\epsilon}\left\{(C_r - C_f)\alpha_d(x-y_k) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\right\} + (C_r - C_f)\alpha_d\epsilon + C_f\end{array}\right\}.\end{aligned}$$

Let y_k^* and $\tilde{y}_k^*$ be the optimal objective values when the current system states are x and $x+\epsilon$, respectively. Then there are two possibilities of $\tilde{y}_k^*$. The first one is $\tilde{y}_k^* \le x$, under which case the value of function is given by

$$\begin{aligned}\check{V}_{k\delta}(x+\epsilon) &= \inf_{0\le y_k<x}\left\{(C_r - C_f)\alpha_d(x-y_k) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\right\}\\ &\quad + (C_r - C_f)\alpha_d\epsilon + C_f\\ &= \check{V}_{k\delta}(x+\epsilon) + (C_r - C_f)\alpha_d\epsilon,\end{aligned}$$

and this indicates the equivalence of $\arg\min_{y_k}\check{V}_{k\delta}(x+\epsilon)$ and $\arg\min_{y_k}V_{k\delta}(x)$, which gives $y_k^* = \tilde{y}_k^*$. The second possibility is that $\tilde{y}_k^* > x$, which is naturally followed by $\tilde{y}_k^* > y_k$. Taken together, we have $\tilde{y}_k^* \ge y_k^*$, thus y_k^* is nondecreasing in x. □

Proof of Corollary 3. Recall the value function under $C(x, y) = C(y)$, we have

$$\check{V}_{k\delta}(x) = \min\left\{W_{k\delta}(x) + U_{k\delta}(x), \inf_{0<y_k<x}\left\{C(y_k) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\right\}\right\}.$$

Given the condition that $U_{k\delta}(x)$ is nondecreasing in x, $W_{k\delta}(x) + U_{k\delta}(x)$ is nondecreasing in x due to the fact that $W_{k\delta}(x)$ has the same property, which has been proved in Lemma 1. On the other hand, $\inf_{0<y_k<x}\{C(y_k) + W_{k\delta}(y_k) + U_{k\delta}(y_k)\}$ is nonincreasing in x. Thus, the decision between doing nothing and imperfect repair is a control limit policy. □

Proof of Corollary 4. The proof is analogous to that of Corollary 2, and thus is omitted. □

Proof of Lemma 2. The first derivative of $f_{\Delta X}(x)$, denoted by $f'_{\Delta X}(x) < 0$, is given by

$$\begin{aligned}f'_{\Delta X}(x) &= (\alpha\delta - 1)x^{\alpha\delta-2}\exp(-\beta x) - \beta\exp(-\beta x)x^{\alpha\delta-1}\\ &= x^{\alpha\delta-2}\exp(-\beta x)(\alpha\delta - 1 - \beta x).\end{aligned}$$

It is obvious that $x^{\alpha\delta-2}\exp(-\beta x)$ is positive. To ensure that $f_{\Delta X}(x)$ decreases in x, the term $\alpha\delta - 1 - \beta x$ needs to be negative. Under the condition that $\alpha \le 1/\delta$, $\alpha\delta - 1$ is nonpositive, which implies $\alpha\delta - 1 - \beta x < 0$. Thus for any $\tilde{\delta} < \delta$, since $\alpha > 0$, we have $\alpha\tilde{\delta} - 1 < \beta x$, which implies that the corresponding density function $\tilde{f}_{\Delta X}(x)$ also decreases in x and $\tilde{f}'_{\Delta X}(x) < 0$. This readily gives $\tilde{F}_{\Delta X}(x)$ is concave in x. □

Proof of Lemma 3. First, the definition of convexity in the usual stochastic order is introduced:

Definition A.2 Let $\{Y(\theta), \theta \in \Theta\}$ be a family of random variables with survival functions $\bar{F}_\theta(y)$. The family $\{Y(\theta), \theta \in \Theta\}$ is said to be stochastically nondecreasing and convex in the usual stochastic ordering if $\mathbb{E}[g(Y(\theta))]$ is nondecreasing and convex for all nondecreasing functions g. We denote this by $\{Y(\theta), \theta \in \Theta\} \in \mathrm{SICX(st)}$.

It has been stated in Lemma 1 that

$$\int_{k\delta}^{(k+1)\delta} c_p(X(t))\mathrm{d}t\,\Big|\,X_k = x_k$$

is a nondecreasing function of $X(t)$, and taking the fact that $c_p(\cdot)$ is convex into consideration, $\int_{k\delta}^{(k+1)\delta} c_p(X(t))dt|X_k = x_k$ is also convex, as the integration can be regarded as a limit of positive linear combination. Next, we model $X(t)$ as a random variable with parameter x of at any t. It has been given that $F_{\Delta X}(\cdot)$ is concave, then $F_{X(t)}(\cdot)$ is increasing and concave for any $t < (k+1)\delta$ according to Lemma 2, and this implies $\bar{F}_{X(t)}(\cdot)$ is decreasing and convex in $x(t)$. Then, as $x_k = x$ increases, $\bar{F}_{X(t)}(\cdot)$ increases in a linear (convex) manner in x for any argument z, as $\bar{F}_{X(t)|x}(z) = \bar{F}_{X(t)|0}(z-x)$ where $F_{X(t)|x}(\cdot)$ is the distribution of $X(t)$ given $X_k = x$. By Theorem 8.C.1 in Shaked & Shanthikumar (2007), we can obtain the SICX(st) property of $X(t)$ in x. By Definition 1, the proof is complete. □

Proof of Proposition 5. The proof of the proposition is similar to that of Proposition 2. Note that we need to prove the convexity given the results in Proposition 2. We begin the mathematical induction with the period of termination as in (A.3), in which $\varphi_k(x)$ is obviously convex and increasing in x. Supposing that the nondecreasing properties hold for the $(k+1)$th period where $0 \le k \le K-1$, then we need to show them for the kth period. First, we focus on the convexity in $U_{k\delta}(x)$. Borrowing the idea from the proof of Lemma 3, we find that $V_{(k+1)\delta}(X_{k+1})$ is an increasing function in X_{k+1}. Random variable X_{k+1} can be treated as a random variable with parameter x. A similar elaboration to that in Lemma 3 shows that $\bar{F}_{X_{k+1}}$ is increasing and convex in x. Again, according to Theorem 8.C.1 in Shaked & Shanthikumar (2007), $U_{k\delta}(x)$ is convex and nondecreasing in x. In light of the above results and Lemma 3, the convexity of $\varphi_k(x)$ is proved. □

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