Simultaneous or Sequential? Retail Strategy of a Durable Product and an Extended Warranty

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***Abstract*—Extended warranties are highly profitable after-sales service contracts that consumers can buy with durable products. In practice, a consumer's purchase decision on an extended warranty can be either conditional on or joint with a product purchase, leading to a sequential or simultaneous consumer purchase pattern. This article considers a monopolistic firm offering a durable product and an accompanying extended warranty directly to end consumers, and develops a stylized model to examine the impact of consumer purchase pattern on the firm's optimal retail decisions and profitability. We find that the optimal warranty lengths are identical across consumer purchase patterns; the warranty should be sold at its marginal cost under the simultaneous pattern, whereas at a premium under the sequential one. Moreover, we show that either retail strategy can result in a higher expected profit depending on the consumers' price sensitivities, although the firm is generally better off inducing a simultaneous consideration of products and warranties. This is attained by setting a lower warranty price to attract more consumers to purchase the product-warranty combination, and charging a higher markup on the product to make more profit. In addition, we show that these findings are robust to an extension to the base model that incorporates the consumers' failure probability distortion behavior caused by the lack of precise product reliability information. The primary findings and insights provide key guidance to practitioners in managing product and extended warranty businesses.**

***Index Terms*—Consumer purchase pattern, extended warranty, pricing, retail operations.**

I. Introduction

An EXTENDED warranty is an after-sales service contract under which the provider guarantees to repair or replace a failed item free of charge or at a reduced price over a certain period of time—usually following, sometimes overlapping with, the manufacturer's base warranty [1]. In general, an extended warranty contract specifies the price, the length of the contract period (during which the item is protected), and the compensation type (repair, replacement, or refund) when the item fails to perform its intended functions in the protection period. Unlike base warranties, which are "free-of-charge" and bundled with product sales, extended warranties are optional and have to be bought separately [1]. This means that an extended warranty is essentially an add-on—it is meaningless to buy an ancillary extended warranty without the main product it protects.

In today's highly competitive market, profit margins on most consumer durables are constantly declining. In contrast, selling extended warranties[1] on consumer durables is highly profitable, especially when compared with product sales. For example, Bošković et al. [2] estimated, using a data set from a nationwide Canadian retailer, that profit margins on extended warranties for appliances and consumer electronics are 36.2% on average, dwarfing the 12.3% margin for the base goods. According to the National Automobile Dealers Association, gross profits from vehicle extended warranties are over 20%, which is larger than the margins on vehicles alone [3]. Despite such a high profit margin, Warranty Week [4] finds that the fraction of product buyers who also purchase extended warranties (referred to as the *attach rate*) ranges from 16% to 54%, even higher, depending on product categories. Recent years have witnessed a growing market of extended warranties. The global extended warranty market size reached US$139.1 billion in 2023, and is projected to reach US$232.8 billion by 2032, exhibiting a growth rate of 5.8%.[2]

The ancillary nature of extended warranties makes the retail-strategy choice regarding durable products and accompanying warranties an important question. In practice, we might observe the following two product-warranty retail scenarios [5], [6]: 1) Consumers first make their product purchase decisions based on the product's features, base warranty, and retail price and, if the product is bought, they are then confronted with the warranty information; and 2) consumers are offered, and thus consider, product and warranty characteristics at the same time. In essence, the former scenario implies a sequential purchase pattern, whereas the latter induces a simultaneous pattern. Whereas firms such as Best Buy and Tmall once adopted the sequential retail

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[1]Henceforth, we will be using the terms "warranties" and "extended warranties" interchangeably. Any reference to manufacturer's base warranties will be explicitly stated.

[2][Online]. Available: https://www.imarcgroup.com/extended-warranty-market (accessed on December 30, 2023).

strategy,[3] many others such as Abt Electronics, JD.com, and Vmall are embracing the simultaneous strategy—they provide product and warranty information simultaneously (displayed on the same webpage) to facilitate a joint consideration.

How consumers make their purchase decisions about products and warranties may impose a substantial impact on the firm's retail decisions regarding, for example, product price, warranty length, and warranty price. This is because in different consumer purchase patterns, firms have different levers to affect consumers' purchase decisions. For example, when consumers jointly consider products and extended warranties, the warranty characteristics might affect their product purchase decisions; this is certainly not the case, if consumers consider warranties only after they have bought main products. Although the aforementioned retail strategies have been studied in the extant literature [5], [6], little is known about their impacts on the firm's optimal retail decisions and profitability through an analytical lens. In this article, we aim to tackle this issue from the perspective of a monopolistic firm that sells a durable product and an ancillary warranty directly to end consumers. An example of such a monopoly is Huawei, which establishes its website (vmall.com) on which consumers can choose warranties while buying smart phones, laptops, or other goods. Our research contributes to the literature in two aspects: 1) We investigate—from a monopoly's perspective—the optimal retail decisions on product price, warranty length, and warranty price under both simultaneous and sequential consumer purchase patterns. 2) We are the first to examine the impact of different consumer purchase patterns on optimal retail decisions and profitability. Even though there seems an ongoing strategy transition from sequential to simultaneous in the current practice, it is important to uncover the mechanism underlying this transition and the associated impact on retail decisions. This would have important implications for product-warranty retail operations.

To this end, we first develop two stylized choice models regarding product and/or warranty under simultaneous and sequential consumer purchase patterns, respectively, and formulate the corresponding expected total profit models. Then, we derive optimal product price, warranty length, and warranty price for each pattern such that the corresponding expected total profit is maximized. Comparing these results shows that the optimal warranty lengths are invariant to consumer purchase patterns; the warranty should be sold at its marginal cost under the simultaneous pattern, whereas at a premium under the sequential one. This means that the role of warranty under the former pattern is to promote product sales, whereas that under the latter is to make profits by itself. This result is consistent with the empirical finding by Jindal [6] and supplements his insights through an analytical lens. Moreover, we show that though either retail strategy can yield a higher expected profit depending on the consumers' price sensitivities, inducing a simultaneous purchase pattern generally results in a higher profit. This is consistent with the recent trend that mainstream e-commerce platforms start to display product and warranty information together, including Best Buy and Tmall that advocated the sequential retail strategy previously. In essence, under the simultaneous strategy, the warranty is priced at a lower level to attract more consumers to buy the product-warranty combination, and a higher markup is charged for the product to extract more profits from product sales.

We then generalize the base model by considering the consumers' failure probability distortion behavior caused by the lack of enough product reliability information. We show that the optimal retail decisions have indeed changed; however, our main findings and insights are robust to the extension. The rest of this article is organized as follows. Section II reviews the relevant literature. Section III formulates two choice models under both simultaneous and sequential consumer purchase patterns, respectively. Section IV derives optimal retail decisions in a profit-maximizing fashion, compares the two retail strategies, and performs comparative statics on several key model parameters. Section V extends the base model by considering the consumers' failure probability distortion. Finally, Section VI concludes this article and suggests some future research topics. All proofs are relegated to the Appendix.

II. Literature Review

There is a rich body of literature on the interplay of product- and warranty-related decision makings. Early research focuses mainly on the determination of optimal product price, base warranty period, and/or product reliability for profit-maximizing purposes. Glickman and Berger [9] made an early contribution by studying optimal product price and (base) warranty period under a log-linear demand model. Huang et al. [10], Cheong et al. [11], Li et al. [12], and Qiao et al. [13] are good examples of follow-up research that takes into account various product and/or warranty characteristics. In particular, Huang et al. [10] investigated optimal product reliability, retail price, and warranty policy for new durable products in both stable and dynamic markets. Qiao et al. [13] studied joint optimization of warranty options and post-warranty maintenance strategy under a renewable warranty menu.

In recent years, design and pricing of extended warranties have aroused growing interest from many researchers. Lam and Lam [14] examined an extended warranty policy that offers renewal and repair options to consumers at the end of a free repair period. Jack and Murthy [15] discussed a flexible policy in which consumers can decide when to start an extended warranty and how long it lasts. Hartman and Laksana [16] studied the design and pricing of extended warranty menus that include restrictions on deferrals and renewals. Gallego et al. [17] proposed residual value warranties that can refund a proportion of up-front price to consumers who have zero or few claims. Lei et al. [18] studied a dynamic product and warranty pricing problem in which consumers can learn about product reliability based on warranty prices. Wang et al. [8] and Wang [19] addressed the design and pricing of extended warranty menus that offer multiple options with differentiated protection lengths and prices. Wang and

[3]Best Buy trained its sales people to recommend warranties to consumers only after they make their decisions to buy the main products (see Abito and Salant [7] and Jindal [6]); when people shopped for durable goods at Tmall, the warranty information was shown only when they proceed to the "Confirm Order Information" page (see Wang et al. [8]).

Ye [20] focused on customization of 2-D extended warranties with heterogeneities in consumer usage rate and failure history. Cao et al. [21] explored whether it is wise to offer rebates for unused extended warranties under trade-in programs. Our work deviates from these studies in the sense that they focus only on extended warranty decisions, whereas we also consider product-related decision.

In practice, it is not uncommon that multiple players (possibly with conflicting interests) participate in an extended warranty business. This stimulates the study of extended warranties from a supply chain perspective. Desai and Padmanabhan [22] made an early attempt to explore the channel coordination role of extended warranties. Jiang and Zhang [23] and Heese [5] examined the impact of a retailer's extended warranty offering on the manufacturers' base warranties in different supply chain settings. Both find that the retailer's extended warranty generally exerts downward pressure on the manufacturers' base warranties. Li et al. [24] studied the design of extended warranties in various supply chain structures, where warranties can be offered by the manufacturer or resold by the retailer. The supply chain structures in Li et al. [24] have been extensively explored by subsequent research from different perspectives [25], [26], [27]. Furthermore, recent years have witnessed the prevalence of online retailing and e-commerce. An online platform can act either as an *agency* to facilitate product transactions or as a product *reseller*; it might also offer extended warranties to end consumers. Li et al. [28] and Zhang et al. [29] studied optimal extended warranty strategies for online platforms under agency selling and/or reselling modes. In our research, we consider a monopoly offering a durable product and an extended warranty directly to end consumers; unlike this stream of literature, our focus is on the interaction of optimal retail decisions induced by different consumer purchase patterns.

To the best of the authors' knowledge, only two publications have explicitly considered simultaneous and sequential product-warranty retail strategies. Heese [5] studied optimal manufacturer and retailer strategies on base and extended warranty sales, by taking into account simultaneous and sequential consumer purchase patterns. Our research differs from Heese's in three main aspects. First, Heese considers a competition scenario where two manufacturers sell their products through the same retailer and the retailer offers extended warranties, whereas we focus on a monopoly scenario where both product and extended warranty are sold by a single firm. Second, Heese adopts a linear model (derived from the 2-D spatial model) to characterize product and/or warranty sales, whereas we adopt a logit-type model for this purpose. Third, and more importantly, the product's retail price and extended warranty's length and price are assumed to be exogenous in Heese's research, whereas they are key decision variables in our research. Our focus on these important retail decisions enables us to examine the retail-strategy choice problem from an *optimal pricing* viewpoint; moreover, it is more aligned with the current practice—warranty lengths offered by different firms are fairly homogeneous, yet there is substantial room to control warranty prices. In addition, Jindal [6] designed two conjoint surveys that are consistent with simultaneous and sequential purchase processes about products (precisely, washing machines) and extended warranties, respectively. His aim is to explore the key factor in explaining extended warranty purchases. Whereas Jindal uses estimates from his models to understand pricing implications, we explore the impact of consumer purchase patterns on optimal product/warranty pricing strategies analytically.

Finally, our work builds upon the economics/marketing literature that examines the role of various influencing factors in explaining extended warranty purchases and the high premia paid for them. The influencing factors examined include usage heterogeneity, risk aversion, loss aversion, and probability distortion/weighting [6], [7], [30], [31], [32], [33], [34]. In particular, Jindal [6] found that loss aversion is significantly more important than risk aversion and probability weighting in explaining warranty purchases. By contrast, Abito and Salant [7] found that the strong demands for extended warranties are mainly driven by consumers distorting product failure probabilities. This finding is quite close to that obtained by Huysentruyt and Read [33], which highlighted a cognitive explanation—probability neglect. Despite that the existing empirical studies are still at odds with which factor is the dominant one in explaining extended warranty purchases, they indeed lay an important foundation for consumer purchase modeling in our research.

III. Model

In real retail environments, as discussed earlier, consumer purchase processes on durable products and accompanying warranties could be either simultaneous or sequential, depending on the firm's retail strategy. In this section, we first develop two stylized choice models for a durable product and/or an accompanying warranty under simultaneous and sequential consumer purchase patterns, respectively, and then formulate a warranty servicing cost model.

Before presenting the models, we clarify two important settings in our modeling framework. First, the product is sold with a base warranty whose length is normalized to 0, as is typically assumed in the literature [18], [24], [25], [26], [27].[4] This allows us to focus exclusively on modeling and analysis of the extended warranty. We note that relaxing this assumption by considering a fixed, nonzero base warranty would not change our findings (Detailed analysis can be found in the Appendix). Second, throughout this work, the purchase of an extended warranty is assumed to occur solely at the point of product sale, irrespective of whether it is joint with or conditional on product purchase. This setting is supported by the following empirical observations. 1) Over 80% of consumers in the U.K. purchase extended warranties on domestic electrical goods at the point of sale [35]. 2) Despite that consumers are generally allowed to buy extended warranties for their cars either before or after base warranties terminate, around 86% of extended warranty purchases are made at the point of sale [36].

[4]Base warranty is essentially an integrated element of product sale (namely, consumers cannot buy a product without base warranty) and its price has already been factored into product price. In this sense, the valuation of base warranty can be considered as a fixed part of the product's intrinsic valuation.

A. Simultaneous Consumer Purchase Pattern

Under the simultaneous purchase pattern, consumers make their purchase decisions on products and warranties at the same time. We consider the following setting: A durable product of retail price p and an optional warranty of price p_e and length w_e are offered to each individual consumer, simultaneously. In essence, there are three options from which each consumer can choose: buy the product and the warranty, buy the product without the warranty, and buy nothing. Note that consumers cannot buy an ancillary warranty without buying the main product.

If a consumer only buys the product, then the net utility she gets can be defined as follows:

$$u_{01} = \varphi_0 - \gamma_1 p + \epsilon_{01} \tag{1}$$

where $\varphi_0 > 0$ is the product's intrinsic valuation, $\gamma_1 > 0$ is the consumer's sensitivity to prices, and ϵ_{01} captures the random utility shock unobserved to the firm.

On the other hand, if a consumer chooses to purchase both the main product and the accompanying warranty, then her net utility can be defined as follows:

$$\begin{aligned} u_{11} &= f_e(\varphi_0 - \gamma_1 p + \gamma_1 c_m - \gamma_1 p_e) \\ &\quad + (1 - f_e)(\varphi_0 - \gamma_1 p - \gamma_1 p_e) + \epsilon_{11} \\ &= \varphi_0 - \gamma_1 p + \gamma_1(\varphi_e - p_e) + \epsilon_{11} \end{aligned} \tag{2}$$

where $\varphi_e := f_e c_m$ is the warranty's gross valuation, c_m is the average amount of money the consumer pays for an out-of-warranty repair, f_e is the product's failure probability during the warranty period $[0, w_e]$,[5] and ϵ_{11} is a random utility shock as well. In addition, the consumer utility from the outside option (i.e., buy nothing) is defined as $u_{00} := \epsilon_{00}$.

The utility function in (2) can be explained as follows. From a consumer's point of view, the warranty's valuation resolves only when the product unit fails over the warranty period $[0, w_e]$ and the incurred repair cost is waived. Specifically, if a product unit fails in the warranty period (with probability f_e), then the consumer's mean utility from the product-warranty combination is given by $\varphi_0 - \gamma_1 p + \gamma_1 c_m - \gamma_1 p_e$; otherwise, the consumer's mean utility would be $\varphi_0 - \gamma_1 p - \gamma_1 p_e$. In particular, $\gamma_1(\varphi_e - p_e)$ can be regarded as the mean net utility, adjusted by γ_1, the consumer obtains from the warranty. In this sense, consumers make their warranty purchase decision by evaluating the perceived product repair expenses, if the warranty is not bought, against the warranty price [8], [35].

We assume that the random shocks ϵ_{01}, ϵ_{11}, and ϵ_{00} are independent, identically, and type I extreme value distributed with location parameter 0 and scale parameter 1, as is typically done in the literature [6], [7], [8]. Each rational consumer acts as a utility maximizer; that is, she would choose the option with the highest realized utility after ϵ_{01}, ϵ_{11}, and ϵ_{00} are resolved. Then, under the *random utility maximization* framework [37], the probability that a consumer only purchases the main product is of the following logit form:

$$\begin{aligned} &d_{01}(p, w_e, p_e) \\ &\quad = \frac{\exp(\varphi_0 - \gamma_1 p)}{1 + \exp(\varphi_0 - \gamma_1 p) + \exp(\varphi_0 - \gamma_1 p + \gamma_1(\varphi_e - p_e))} \end{aligned} \tag{3}$$

and the probability that a consumer buys both the product and the warranty is given as follows:

$$\begin{aligned} &d_{11}(p, w_e, p_e) \\ &\quad = \frac{\exp(\varphi_0 - \gamma_1 p + \gamma_1(\varphi_e - p_e))}{1 + \exp(\varphi_0 - \gamma_1 p) + \exp(\varphi_0 - \gamma_1 p + \gamma_1(\varphi_e - p_e))}. \end{aligned} \tag{4}$$

The probability for the outside option is thus $d_0(p, w_e, p_e) = 1 - d_{01}(p, w_e, p_e) - d_{11}(p, w_e, p_e)$. A point noteworthy here is that there are two equivalent interpretations for $d_{01}(p, w_e, p_e)$—that is, 1) the probability that a consumer only purchases the main product, and 2) the proportion of consumers who only buy the main product. Similar interpretations also apply to $d_{11}(p, w_e, p_e)$ and $d_0(p, w_e, p_e)$.

It is worth noting that *analytical* studies on extended warranties predominately adopt linear demand models [24], [26] or those derived from the expected utility paradigm [20], [27]. Though these stylized demand models are mathematically tractable, they largely lack empirical support (in the context of warranties). By contrast, logit-type models have been adopted in quite a few *empirical* studies of base or extended warranties for various product categories (see, e.g., [6], [7], [38], and [39]). The successful applications in empirical warranty research well support the adoption of a logit-type model in our study.

The overall product purchase probability is thus given by $d_1(p, w_e, p_e) = d_{01}(p, w_e, p_e) + d_{11}(p, w_e, p_e)$, and the conditional warranty purchase probability becomes the following:

$$\begin{aligned} d_{1|1}(p, w_e, p_e) &= \frac{d_{11}(p, w_e, p_e)}{d_{01}(p, w_e, p_e) + d_{11}(p, w_e, p_e)} \\ &= \frac{\exp(\gamma_1 \varphi_e - \gamma_1 p_e)}{1 + \exp(\gamma_1 \varphi_e - \gamma_1 p_e)} \end{aligned}$$

which is independent of product price p.

Lemma 1: Both $d_1(p, w_e, p_e)$ and $d_{1|1}(p, w_e, p_e)$ are decreasing in p_e, and increasing in w_e.

Lemma 1 shows that when consumers consider the product and the accompanying warranty simultaneously, *ceteris paribus*, a lower warranty price or a longer warranty period would lead to not only a higher conditional warranty purchase probability, but also a higher overall product purchase probability. This is consistent with the *signaling* theory of warranty [40]—that is, a better warranty than competitors' would signal superior product reliability to consumers. According to Lemma 1, an extended warranty, if offered along with the main product, can act as a marketing tool to promote product sales.

B. Sequential Consumer Purchase Pattern

Under the sequential purchase pattern, we consider the following two-stage setting: In the first stage, only a durable product

[5] A note to make is that f_e is an increasing function of w_e, although the function argument is suppressed for notational conciseness. A formulation of f_e is given in (11) below.

of retail price p is offered to each consumer; in the second stage, an optional warranty of length w_e and price p_e is further recommended to those who have bought the product. We assume here that consumers do not account for warranty purchase (second-stage decision) while deciding on product purchase (first-stage decision). In essence, this implies that consumers are myopic and, in the first stage, do not anticipate the subsequent warranty offering [6]. This assumption, albeit restrictive at first glance, is appropriate for the sequential purchase pattern. This is because extended warranty is a secondary influencing factor for consumers' product purchase decisions—it only comes into play after price, features, quality, brand reputation, among others. In this sense, it is natural to assume that consumers do not anticipate warranty attributes if they are not offered together with the product.

In the first stage, there are only two options for each consumer: buy the product or not buy it. If a consumer chooses to buy the product, then her utility is given as follows:

$$\hat{u}_1 = \varphi_0 - \gamma_2 p + \epsilon_1 \tag{5}$$

where $\gamma_2 > 0$ is the consumer's sensitivity to prices as well[6]; otherwise, her utility is $\hat{u}_0 := \epsilon_0$. We use a hat (ˆ) to represent quantities related to the sequential purchase pattern. Under the random utility maximization framework, the probability that a consumer purchases the product can be derived as follows:

$$\hat{d}_1(p) = \frac{\exp(\varphi_0 - \gamma_2 p)}{1 + \exp(\varphi_0 - \gamma_2 p)}. \tag{6}$$

In the second stage, if a product buyer would like to purchase the warranty, then the net utility she obtains within the warranty period $[0, w_e]$ is given as follows:

$$\hat{u}_{1|1} = \gamma_2 \varphi_e - \gamma_2 p_e + \epsilon_{1|1} \tag{7}$$

otherwise, her utility is simply $\hat{u}_{0|1} := \epsilon_{0|1}$. Likewise, the conditional probability that each product buyer would further buy the warranty is thus

$$\hat{d}_{1|1}(w_e, p_e) = \frac{\exp(\gamma_2 \varphi_e - \gamma_2 p_e)}{1 + \exp(\gamma_2 \varphi_e - \gamma_2 p_e)}. \tag{8}$$

Then, the unconditional purchase probability of the product alone becomes $\hat{d}_1(p)(1 - \hat{d}_{1|1}(w_e, p_e))$, and the unconditional purchase probability of both the product and the warranty is $\hat{d}_1(p)\hat{d}_{1|1}(w_e, p_e)$.

We can see from (6) and (8) that when consumers purchase the product and the ancillary warranty sequentially, a lower warranty price or a longer warranty period can only increase the conditional warranty purchase probability, yet has no impact on the product purchase probability. This implies that the sequential consumer purchase pattern suppresses the signaling function of an extended warranty, which would affect optimal product and warranty pricing mechanisms in a different way from the simultaneous purchase pattern.

[6]We note that γ_2 is not necessarily equal to γ_1 [6], thus, the comparison of product/warranty purchase probabilities under simultaneous and sequential purchase patterns depends on the values of γ_1 and γ_2.

C. Warranty Cost Model

Honoring warranties is by no means free from the firm's standpoint, as servicing warranty claims generated by consumers is usually costly. Warranty servicing expenses depend on the corrective repair policy adopted to handle warranty claims, in addition to product reliability and warranty policy [41], [42]. In this work, we confine our attention to the minimal repair strategy with negligible duration, which is commonly adopted in the warranty literature (see, e.g., [8], [10], and [43]). A minimal repair restores a failed item to its operational state just before failure. Under this assumption, the occurrence of item failures obeys a nonhomogeneous Poisson process with an intensity function $\lambda(t)$. The expected warranty servicing cost c_e can thus be formulated as follows:

$$c_e = c_e(w_e) = c_r \int_0^{w_e} \lambda(t)\mathrm{d}t \tag{9}$$

where c_r is the unit rectification cost borne by the firm. In general, the unit rectification cost paid by consumers is much larger than that borne by the firm (i.e., $c_m > c_r$).

The warranty cost model in (9) can accommodate any form of intensity function. For instance, if the intensity function is of a commonly adopted power law form with scale parameter α and shape parameter β

$$\lambda(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \tag{10}$$

then (9) can be expressed as $c_e = c_r(w_e/\alpha)^\beta$. In this research, only $\beta \geq 1$ is considered because infant mortality shall be screened by reliability tests before product release. It indicates that c_e is nonnegative and convex increasing in w_e.

Moreover, the failure probability f_e is an increasing function of w_e, which can be formulated as follows:

$$f_e = f_e(w_e) = 1 - \exp\left(-\int_0^{w_e} \lambda(t)\mathrm{d}t\right). \tag{11}$$

According to (9) and (11), it is clear that $c_e = -c_r \log(1 - f_e)$.

IV. Analysis

In this section, we first derive optimal product prices, warranty lengths, and warranty prices under simultaneous and sequential consumer purchase patterns in Sections IV-A and IV-B, respectively, and then compare the two patterns in terms of key performance metrics in Section IV-C. We further conduct comparative-statics analyses in Section IV-D to examine how the change in several key parameters impacts the results.

A. Simultaneous Consumer Purchase Pattern

We first investigate the simultaneous purchase pattern. Without loss of generality, the market mass is normalized to unity for ease of exposition. Then, the firm's expected total profit from product and warranty sales can be expressed as follows:

$$\begin{aligned}\Pi(p, w_e, p_e) = {} & d_{01}(p, w_e, p_e)(p - c_0) \\ & + d_{11}(p, w_e, p_e)(p - c_0 + p_e - c_e)\end{aligned} \tag{12}$$

where $d_{01}(p, w_e, p_e)$ and $d_{11}(p, w_e, p_e)$ are given by (3) and (4), respectively, c_0 is the marginal production cost, and c_e is given by (9). Here, $p - c_0$ and $p_e - c_e$ are *profit markups* on the product and the warranty, respectively.

The firm's problem is to specify an optimal product price, warranty length, and warranty price such that the expected total profit is maximized; that is

$$\max_{p, w_e, p_e} \Pi(p, w_e, p_e). \tag{13}$$

We solve this joint optimization problem through the following procedure: first determining an optimal warranty length $w_e^*(p)$ and warranty price $p_e^*(p)$ for any given p to maximize $\Pi(p, w_e, p_e)$, then deriving an optimal product price p^* to maximize $\Pi(p, w_e^*(p), p_e^*(p))$. Once p^* is known, w_e^* and p_e^* can be obtained by substituting p^* into $w_e^*(p)$ and $p_e^*(p)$, respectively. We note that solving the problem in (13) using standard optimization techniques is not easy, because the objective function is ill-behaved. To overcome this issue, following Wang [44], we introduce an auxiliary variable $\pi = \Pi(p, w_e, p_e)$ and transform the original profit maximization problem into an equivalent root-finding problem [see (17) below]. By doing so, the problem becomes much easier to handle. Details can be found in the proof of Proposition 1 (refer to the Appendix).

The following proposition characterizes the firm's optimal retail decisions under the simultaneous purchase pattern.

Proposition 1: Let $\pi^* = \max_{p, w_e, p_e} \Pi(p, w_e, p_e)$. The optimal product price, warranty length, and warranty price, respectively, under simultaneous consumer purchase are as follows:

$$p^* = c_0 + \pi^* + \frac{1}{\gamma_1} \tag{14}$$

$$w_e^* = f_e^{-1}(1 - c_r/c_m)^7 \tag{15}$$

and

$$p_e^* = c_e \tag{16}$$

where π^* is the unique solution to

$$\pi = \frac{1}{\gamma_1} \exp\left(\varphi_0 - \gamma_1(c_0 + \pi) - 1\right) \left[1 + \exp(\gamma_1 \varphi_e - \gamma_1 c_e)\right]. \tag{17}$$

Moreover, the resulting product purchase probability and conditional warranty purchase probability, respectively, are as follows:

$$d_1(p^*, w_e^*, p_e^*) = \frac{\gamma_1 \pi^*}{1 + \gamma_1 \pi^*} \tag{18}$$

and

$$d_{1|1}(p^*, w_e^*, p_e^*) = \frac{\exp(\gamma_1 \varphi_e - \gamma_1 c_e)}{1 + \exp(\gamma_1 \varphi_e - \gamma_1 c_e)}. \tag{19}$$

When consumers consider the main product and the accompanying warranty simultaneously, Proposition 1 delivers three key messages.

1) The product should be optimally priced according to a *cost-plus-markup* policy, with a markup $\pi^* + 1/\gamma_1$. This policy is quite common in the logit-based product pricing literature (see, e.g., [8], [19], and [44]). It implies that the product price should be lower when consumers are more price-sensitive, which is to be expected.
2) The optimal warranty length w_e^* should be specified in a way that the product failure probability over the warranty period is equal to $1 - c_r/c_m$ (i.e., $f_e(w_e^*) = 1 - c_r/c_m$). In other words, w_e^* can be determined exclusively by the product's failure probability and repair cost characteristics. This result, albeit surprising to some extent, is insightful. In fact, w_e^* is obtained by maximizing the valuation margin $\eta = \varphi_e - c_e$, which is the difference between the warranty's average valuation to consumers and its expected cost to the firm [8]; see the proof of Proposition 1 in the Appendix. This is because—noticing that $\eta = (\varphi_e - p_e) + (p_e - c_e)$—the valuation margin essentially combines the benefits the warranty brings to both consumers and the firm. Clearly, η must be maximized; otherwise, we can find another solution that yields either a higher warranty markup for the firm (without sacrificing consumers' utility) or a higher average utility to consumers (without reducing the firm's warranty markup). Since warranty price p_e is canceled out, the optimal decision on warranty length is independent of warranty price. It turns out that w_e^* is simply related to the ratio of repair cost borne by the firm (i.e., c_r) to that experienced by consumers (i.e., c_m). This implies that as the repair cost paid by consumers becomes larger relative to that borne by the firm (i.e., $1 - c_r/c_m$ becomes larger), the firm should offer a longer warranty, which is consistent with our intuition.
3) Another insightful finding is that the warranty should be priced at its marginal cost (i.e., $p_e^* = c_e$) and thus the warranty's markup, at optimality, is 0. This finding can be explained by the signaling theory of warranty, yet contradicts some analytical insights that support the high profitability of extended warranties. Specifically, according to the signaling mechanism, pricing the warranty at its marginal cost would increase the overall product purchase probability (Lemma 1), which, in turn, raises the expected profit from product sales. On the other hand, analytical studies like [30], [31], and [45] show that warranties are profitable contracts. In their works, the market is segmented as heterogeneous consumers self-select from a menu of product-warranty bundles, so that the generated profits stem mainly from the warranty's *price discrimination*. Without price discrimination, however, a recent analytical study by Lei et al. [18] even shows that warranty should not be priced higher than its marginal cost. Nevertheless, our research contributes another analytical result to the *mixed* insights on whether warranties should be a profit-generating tool.

B. Sequential Consumer Purchase Pattern

We then investigate the pattern in which consumers make their product and warranty purchase decisions sequentially. By

[7]Here, $f_e^{-1}(\cdot)$ is the inverse function of $f_e(\cdot)$.

normalizing the market mass to unity as well, the firm's expected total profit from product and warranty sales, under sequential consumer purchase, can be expressed as follows:

$$\hat{\Pi}(p, w_e, p_e) = \hat{d}_1(p)\left[p - c_0 + \hat{d}_{1|1}(w_e, p_e)(p_e - c_e)\right] \quad (20)$$

where $\hat{d}_1(p)$ and $\hat{d}_{1|1}(w_e, p_e)$ are given by (6) and (8), respectively. Under this pattern, consumers' warranty purchase decision is conditional on product purchase, so the expected profit for each sold unit consists of two parts: $p - c_0$ is the markup for the main product, and $\hat{d}_{1|1}(w_e, p_e)(p_e - c_e)$ is the expected warranty profit for each sold unit.

The firm's problem is again to seek an optimal product price, warranty length, and warranty price such that the expected total profit is maximized; that is

$$\max_{p, w_e, p_e} \hat{\Pi}(p, w_e, p_e). \quad (21)$$

The problem can also be solved through the two-stage procedure mentioned in Section IV-A. However, what differs from the simultaneous pattern is that the product purchase probability $\hat{d}_1(p)$ is independent of warranty attributes (length and price) due to the myopic purchase setting. As a result, the optimal warranty length $\hat{w}_e^*$ and warranty price $\hat{p}_e^*$ can be obtained by simply maximizing the expected warranty profit for each sold unit; that is, $\max_{w_e, p_e} \hat{\Pi}_e(w_e, p_e) = \hat{d}_{1|1}(w_e, p_e)(p_e - c_e)$.

The following proposition characterizes the firm's optimal retail decisions under the sequential purchase pattern.

Proposition 2: Let $\hat{\pi}^* = \max_{p, w_e, p_e} \hat{\Pi}(p, w_e, p_e)$ and $\hat{\pi}_e^* = \max_{w_e, p_e} \hat{\Pi}_e(w_e, p_e)$. The optimal product price, warranty length, and warranty price, respectively, under sequential consumer purchase are as follows:

$$\hat{p}^* = c_0 - \hat{\pi}_e^* + \hat{\pi}^* + \frac{1}{\gamma_2} \quad (22)$$

$$\hat{w}_e^* = f_e^{-1}(1 - c_r/c_m) \quad (23)$$

and

$$\hat{p}_e^* = c_e + \hat{\pi}_e^* + \frac{1}{\gamma_2} \quad (24)$$

where $\hat{\pi}_e^*$ and $\hat{\pi}^*$, respectively, are the unique solutions to

$$\hat{\pi}_e = \frac{1}{\gamma_2}\exp(\gamma_2\varphi_e - \gamma_2(c_e + \hat{\pi}_e) - 1) \quad (25)$$

and

$$\hat{\pi} = \frac{1}{\gamma_2}\exp(\varphi_0 - \gamma_2(c_0 - \hat{\pi}_e^* + \hat{\pi}) - 1). \quad (26)$$

Moreover, the resultant product purchase probability and conditional warranty purchase probability, respectively, are as follows:

$$\hat{d}_1(\hat{p}^*) = \frac{\gamma_2\hat{\pi}^*}{1 + \gamma_2\hat{\pi}^*} \quad (27)$$

and

$$\hat{d}_{1|1}(\hat{w}_e^*, \hat{p}_e^*) = \frac{\gamma_2\hat{\pi}_e^*}{1 + \gamma_2\hat{\pi}_e^*}. \quad (28)$$

Proposition 2 shows that when consumers consider the product and the warranty sequentially, both of them should be optimally priced according to the *cost-plus-markup* policy, with their markups being $\hat{\pi}^* - \hat{\pi}_e^* + 1/\gamma_2$ and $\hat{\pi}_e^* + 1/\gamma_2$, respectively. Interestingly, the optimal warranty length $\hat{w}_e^*$ is the same as its counterpart w_e^* under the simultaneous pattern. This is because under both patterns, the optimal warranty lengths are determined by the same objective of maximizing valuation margin η, which is not impacted by other decision variables. In particular, if the product's failure intensity has a power law form as in (10), then we have $w_e^* = \hat{w}_e^* = \alpha[\log(c_m/c_r)]^{1/\beta}$. It should be pointed out that the optimal values of φ_e and c_e in (17) and (25) should be $\varphi_e = c_m f_e(w_e^*) = c_m - c_r$ and $c_e = -c_r \log(1 - f_e(w_e^*)) = -c_r \log(c_r/c_m)$, respectively. Because $w_e^* = \hat{w}_e^*$, hereafter we will be using φ_e and c_e for the sake of succinctness, if not noted otherwise.

In addition, following two key points distinguish the sequential purchase pattern from its simultaneous counterpart:

1) First, the optimal product price $\hat{p}^*$ is dependent on the expected warranty profit $\hat{\pi}_e^*$, which is not the case under simultaneous consumer purchase.
2) Second, the warranty should be sold at a premium and the expected warranty profit $\hat{\pi}_e^*$—the unique root of (25)—is larger than 0, whereas that under simultaneous consumer purchase is 0.

The second distinction indicates that the warranty should be sold at its marginal cost under the simultaneous purchase pattern, whereas at a premium under the sequential pattern. This confirms the empirical finding by Jindal [6]. The intuition is that consumers consider products and warranties simultaneously under the former pattern, so the firm has an incentive to lower warranty price to attract more consumers to buy the product-warranty combination (Lemma 1). However, this is not the case for the latter pattern, because, due to the consumers' myopic purchase behavior, reducing warranty price would not affect their first-stage decision on product purchase; instead, the firm would raise warranty price and render both product and warranty profit-generating.

An interesting implication of the second distinction is that it possibly explains the mixed claims on the profitability of extended warranties. Specifically, even though some articles or reports argue that profit margins on extended warranties are significant, most are published at least a decade ago (see, e.g., [35], [36], and [46]). The same circumstance also applies to most empirical data used in academic publications. For example, the dataset used in Abito and Salant [7] covered transactions between December 1998 and November 2004, and that in Bošković et al. [2] covered transactions between December 1999 and December 2009. At that time, the sequential retail strategy was prevalent, as argued by Heese [5] in 2012. With the development of e-commerce and online retailing, however, many firms (e.g., Best Buy and Tmall) have shifted their retail strategy from sequential to simultaneous, under which, according to our findings, extended warranties should not be a profit-generating tool. In this sense, the latest empirical data are needed to reveal the profitability of recently sold warranties. Nonetheless, evaluating the net profit of a warranty program is by no means an easy task

TABLE I
NUMERICAL RESULTS FOR THE CASE OF $\gamma_1 = \gamma_2$

Simultaneous purchase pattern		Sequential purchase pattern	
π^*	30.52	$\hat{\pi}^*$	27.56
π_e^*	0	$\hat{\pi}_e^*$	4.82
p^*	311.89	$\hat{p}^*$	304.11
w_e^*	47.52	$\hat{w}_e^*$	47.52
p_e^*	27.09	$\hat{p}_e^*$	33.28
d_1^*	95.70%	$\hat{d}_1^*$	95.26%
$d_{1\|1}^*$	99.69%	$\hat{d}_{1\|1}^*$	77.87%

from an accounting perspective. This is because future liability costs could occur randomly over the entire warranty period, although the revenue is usually recognized at the outset.[8]

C. Comparison of the Two Patterns

We are now in a position to compare key metrics under the simultaneous and sequential consumer purchase patterns. Such comparison can support the firm's retail-strategy choice regarding products and ancillary warranties.

First of all, we have the following analytical result for the special case of $\gamma_1 = \gamma_2$.

Corollary 1: When $\gamma_1 = \gamma_2$, the following relationships hold: $\pi^* > \hat{\pi}^*$, $p^* > \hat{p}^*$, $w_e^* = \hat{w}_e^*$, $p_e^* < \hat{p}_e^*$, $d_1(p^*, w_e^*, p_e^*) > \hat{d}_1(\hat{p}^*)$, and $d_{1|1}(p^*, w_e^*, p_e^*) > \hat{d}_{1|1}(\hat{w}_e^*, \hat{p}_e^*)$.

Corollary 1 informs us that when the consumers' price sensitivities are invariant to purchase patterns (i.e., $\gamma_1 = \gamma_2$), the firm's expected total profit under the simultaneous purchase pattern is higher than that under its sequential counterpart. This implies that the firm benefits from consumers simultaneously considering products and warranties. The optimal product price (respectively, warranty price) under the simultaneous pattern is higher (respectively, lower) than that under the sequential pattern. Moreover, both the product purchase probability and the conditional warranty purchase probability under the simultaneous pattern, at optimality, are higher than those under the sequential pattern. The former is a natural follow-up result of $\pi^* > \hat{\pi}^*$, whereas the latter is due to $p_e^* < \hat{p}_e^*$ and Lemma 1—that is, a lower warranty price leads to a higher warranty purchase probability. In essence, the simultaneous pattern can induce more consumers to purchase the product-warranty combination by pricing the warranty at its marginal cost, and earn more money by setting a higher markup for the product; in this manner, the reduction in the warranty profit is offset by the growth in the product profit. This indicates that the primary role of warranty sales under the simultaneous purchase pattern is stimulating product sales rather than generating profits. Table I confirms the aforementioned results with a numerical example, where the parameters are set to $\alpha = 58.98$ weeks, $\beta = 3.28$, $c_r = \$55$, $c_m = \$90$, $\varphi_0 = \$225$, $c_0 = \$280$, and $\gamma_1 = \gamma_2 = 0.73$. We present in the Appendix a printer failure dataset used to estimate α and β, along with a description of how other parameters are roughly calibrated.

The results in Corollary 1 are derived under the special condition of $\gamma_1 = \gamma_2$. It is difficult, if not impossible, to derive comparison results for general γ_1 and γ_2, except that $w_e^* = \hat{w}_e^*$ and $p_e^* < \hat{p}_e^*$. We thus resort to numerical analysis. Fig. 1 illustrates the optimal product-warranty retail strategy against γ_1 and γ_2, with the values of the other parameters identical to those above. We can see that the firm can be better off under either retail strategy, depending on the relative magnitude of γ_1 and γ_2. More specifically, Fig. 1(a) shows that when γ_1 is significantly smaller (respectively, larger) than γ_2, the firm's maximum expected profit under the simultaneous pattern is higher (respectively, lower) than that of its sequential counterpart. In particular, the red dotted line—representing $\gamma_1 = \gamma_2$—confirms the results in Corollary 1 in that when the price sensitivities are invariant to purchase patterns, the simultaneous pattern would lead to a higher expected profit. Nevertheless, the partitioning line between the white and gray zones in Fig. 1(a) is quite close to the red dotted line. This implies that when γ_1 becomes slightly larger than γ_2, the simultaneous strategy would lose its advantage over the sequential strategy from the profit maximization perspective, but this scenario is uncommon in reality as suggested by [6].

Another point to note is that the behavioral experiment in Jindal [6] reveals that the value of γ_1 is generally smaller than that of γ_2[9]; that is, the (γ_1, γ_2) pair locates in the gray zone in Fig. 1. In this zone, the simultaneous retail strategy, compared with its sequential counterpart, imposes a higher product price, a lower warranty price, yet the same warranty length; this results in higher product and warranty purchase probabilities and ultimately higher expected profit.

In addition, we also examine other key metrics—namely, the optimal product prices, the product purchase probabilities, and the conditional warranty purchase probabilities—under the two consumer purchase patterns against γ_1 and γ_2. We can observe from Fig. 1(b) that there is a small zone in which the sequential purchase pattern results in a higher maximum expected profit yet a lower optimal product price than the simultaneous pattern. This is because the optimal product price under the sequential purchase pattern is negatively related to the expected warranty profit [see (22)], so that a higher maximum expected profit, deducting the expected warranty profit (which is positive), leads to a lower optimal product price. Moreover, Fig. 1(c) reveals that the pattern of product purchase probabilities is the same as that of the maximum expected profit. This is because the product purchase probability, at optimality, is exclusively dependent on the maximum expected profit under both purchase patterns [refer to (18) and (27)]. Furthermore, Fig. 1(d) shows a region (marked in light gray) in which the sequential purchase pattern leads to a higher maximum expected profit yet a lower conditional warranty purchase probability than the simultaneous pattern.

[8]FASB ASC 605 "Revenue Recognition" requires firms to recognize their extended warranty revenue on a straight-line basis over the contract period, yet to charge service cost performed under the contract to expense as incurred.

[9]For all model specifications considered in Jindal [6], the estimates of price sensitivity under the simultaneous pattern are constantly lower than those in its sequential counterpart (refer to Tables 5 and 6 therein).

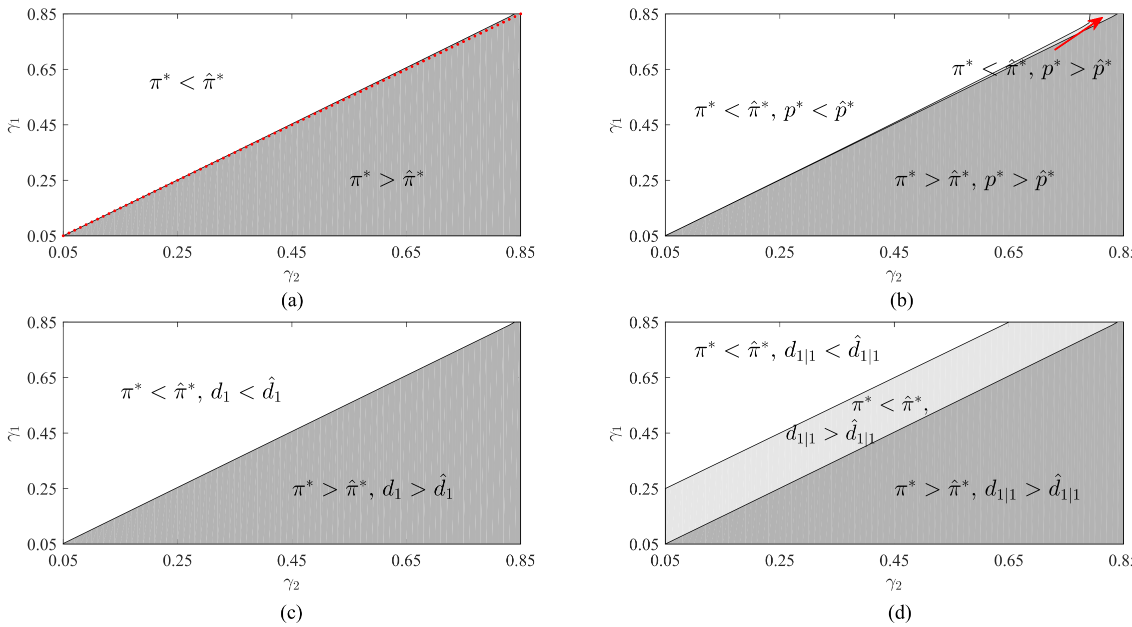

Fig. 1. Optimal product-warranty retail strategy with different γ_1 and γ_2. (Note: The red dotted line in panel (a) corresponds to $\gamma_1 = \gamma_2$.) (a) Maximum expected profit. (b) Optimal product price. (c) Product purchase probability. (d) Warranty purchase probability.

D. Comparative Statics

In this section, we further conduct comparative-statics analyses regarding several key model parameters. We focus on one parameter each time while keeping the others unchanged. This gives rise to managerial insights on how the change of model parameters impacts the firm's optimal retail decisions and associated metrics.

1) Comparative Statics of φ_0: We begin with parameter φ_0, which represents the product's intrinsic valuation. A larger φ_0 indicates that the product itself can bring more valuation to consumers, which, in turn, may affect consumer purchase behavior and the firm's retail decisions.

We have the following results regarding the comparative statics of φ_0.

Proposition 3: As φ_0 increases

i) π^* and $\hat{\pi}^*$ increase;

ii) p^* and $\hat{p}^*$ increase, whereas w_e^*, $\hat{w}_e^*$, p_e^*, and $\hat{p}_e^*$ remain constant.

iii) $d_1(p^*, w_e^*, p_e^*)$ and $\hat{d}_1(\hat{p}^*)$ increase, whereas $d_{1|1}(p^*, w_e^*, p_e^*)$ and $\hat{d}_{1|1}(\hat{w}_e^*, \hat{p}_e^*)$ remain constant.

Proposition 3 presents an intuitive finding about the firm's optimal retail decisions. That is, an increase in φ_0 only affects the optimal product price, whereas the optimal length and price of the warranty keep unaltered. This finding can be easily drawn by examining Propositions 1 and 2 in the sense that only p^* and $\hat{p}^*$ are affected—indirectly through π^* and $\hat{\pi}^*$, respectively—by φ_0; on the other hand, the optimal warranty lengths are determined solely by the product failure and repair cost information, whereas the optimal warranty prices depend only on warranty-related parameters. This finding is consistent across both consumer purchase patterns.

Moreover, a larger intrinsic valuation φ_0 would be helpful for increasing the product purchase probability, which, in turn, offers the firm more potential to raise product price and thus generate more profit. However, the conditional warranty purchase probabilities under both patterns are independent of φ_0, so that an increase in φ_0 has no impact on warranty-related decisions. This means that as φ_0 becomes larger, the growth in the expected total profit comes entirely from the main product itself. Fig. 2 illustrates the results in Proposition 3, in which we fix $\gamma_1 = \gamma_2 = 0.73$ [6] and keep the values of all the other parameters as in Section IV-C.

2) Comparative Statics of c_m: Next, we focus on the out-of-warranty repair cost paid by consumers, c_m. In particular, a larger c_m can lead to a larger warranty valuation φ_e. This is aligned with the reality, as a higher out-of-warranty repair cost would make the warranty more attractive (due to risk aversion); as a result, the warranty can bring more valuation to consumers.

The results regarding the comparative statics of c_m are summarized in the following proposition.

Proposition 4: As c_m increases

i) π^*, $\hat{\pi}_e^*$, and $\hat{\pi}^*$ increase;

ii) p^*, w_e^*, $\hat{w}_e^*$, p_e^*, and $\hat{p}_e^*$ increases, whereas $\hat{p}^*$ decreases.

iii) $d_1(p^*, w_e^*, p_e^*)$, $d_{1|1}(p^*, w_e^*, p_e^*)$, $\hat{d}_1(\hat{p}^*)$, and $\hat{d}_{1|1}(\hat{w}_e^*, \hat{p}_e^*)$ increase.

Proposition 4 shows that the out-of-warranty repair cost c_m can affect all retail decisions, albeit in different ways. Specifically, the impact of an increasing c_m on the optimal warranty length is direct (recall that $w_e^* = \hat{w}_e^* = f_e^{-1}(1 - c_r/c_m)$), while those on the optimal product and warranty prices are indirect through changing the valuation margin. As c_m increases, the optimal warranty lengths w_e^* and $\hat{w}_e^*$ become larger, which, in turn, results in a larger warranty valuation φ_e and cost c_e.

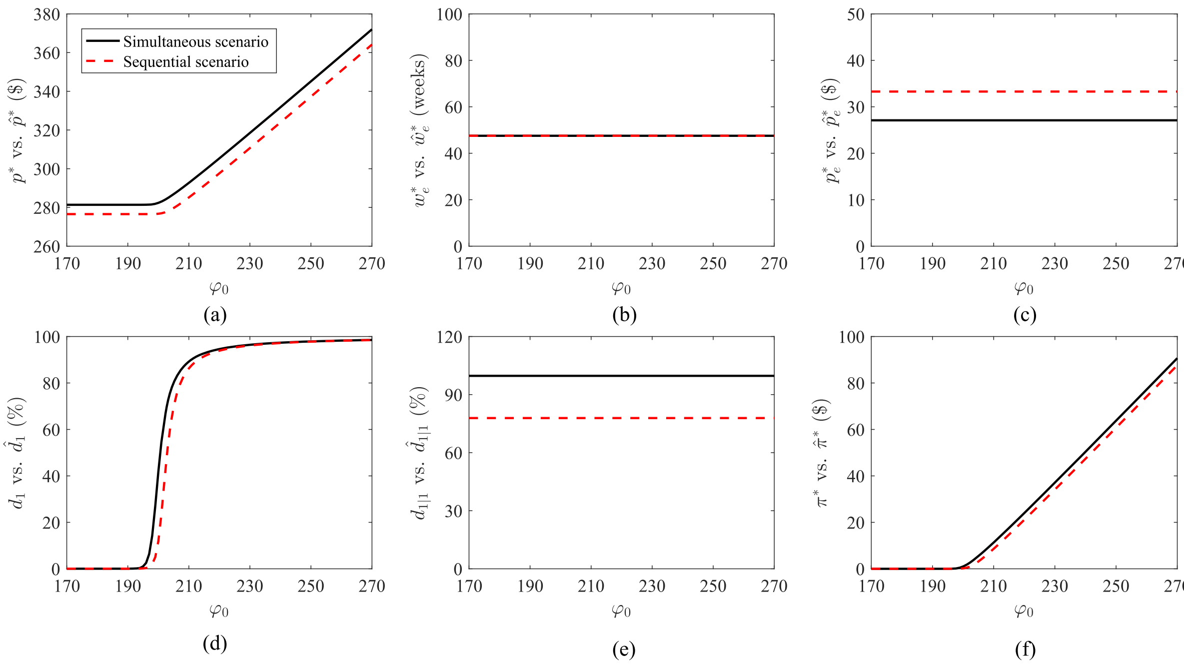

Fig. 2. Comparative statics of φ_0. (a) Optimal product price. (b) Optimal warranty length. (c) Optimal warranty price. (d) Product purchase probability. (e) Warranty purchase probability. (f) Maximum expected profit.

Interestingly, the valuation margin η turns out to be increasing in c_m as well, which further affects the optimal decisions on product and warranty prices.

Overall, Proposition 4 indicates that a larger c_m offers the firm more potential to extract profits from product and/or warranty sales. Under the simultaneous purchase pattern, an increase in c_m leads to a larger warranty length w_e^*, and thus a higher warranty cost c_e. As a result, the optimal warranty price p_e^* becomes higher, because the warranty is priced at its marginal cost (i.e., $p_e^* = c_e$); nevertheless, the warranty itself generates zero profit. In this scenario, however, a larger c_m would increase not only the conditional warranty purchase probability but also the product purchase probability. Thus, the firm can appropriately raise product price to gain more profit from the main product (i.e., π^* becomes larger).

However, an increasing c_m has a quite different impact on the optimal pricing decisions under the sequential purchase pattern. Recall from (6) and (8) that only the conditional warranty purchase probability $\hat{d}_{1|1}(w_e, p_e)$ is directly dependent on c_m (via φ_e). In this situation, as c_m grows, the firm could raise warranty price to extract more profit from warranty sales, with the conditional warranty purchase probability still becoming higher; meanwhile, the firm could lower product price to attract more customers to purchase the main product (though this reduces the product's markup, a larger consumer pool would be generated to enable warranty sales). As a consequence, the expected total profit $\hat{\pi}^*$ and the expected warranty profit $\hat{\pi}_e^*$ are both increasing in c_m.

To summarize, though the expected total profits grow with the unit out-of-warranty repair cost c_m under both simultaneous and sequential purchase patterns, the profit growth comes from product sales under the former pattern but from warranty sales under the latter. The results in Proposition 4 are illustrated in Fig. 3, in which we fix $\gamma_1 = \gamma_2 = 0.73$ [6] and keep the values of all the other parameters as in Section IV-C.

3) Comparative Statics of γ_1 and γ_2: Another key parameter in our model is the sensitivity to prices. In general, a larger γ_1 or γ_2 exerts a more serious negative impact of product/warranty prices on consumer utilities. However, it is difficult, if not impossible, to derive analytical results on the comparative statics of γ_1 or γ_2, except that w_e^* (and thus p_e^*) and $\hat{w}_e^*$ are independent of γ_1 and γ_2, respectively. We thus resort to numerical analysis. Fig. 4 shows the numerical results regarding comparative statics of γ_1 and γ_2, in which both of them increased from 0.5 to 0.9 and the values of all the other parameters are kept as in Section IV-C.

We can see that under both consumer purchase patterns, as the sensitivity to prices becomes larger, the optimal product prices become lower, the optimal warranty lengths and prices remain (almost) constant, the product purchase probabilities become smaller, the conditional warranty purchase probabilities become larger, and, finally, the expected total profits become smaller. This observation is reasonable in the sense that when the sensitivity to prices becomes larger, the optimal product price should be reduced to mitigate its negative impact on consumer utilities; even so, the product purchase probability and the resulting total profit still become significantly lower. In addition, we can observe that when the value of γ_1 or γ_2 becomes extremely large, the optimal product and warranty prices would remain constant at the lowest levels, and the expected total profits would be driven down to 0.

An interesting point to note is the pattern of optimal warranty prices and conditional warranty purchase probabilities. The optimal warranty price p_e^* remains constant, as it is equal to the marginal warranty servicing cost (i.e., $p_e^* = c_e$). According

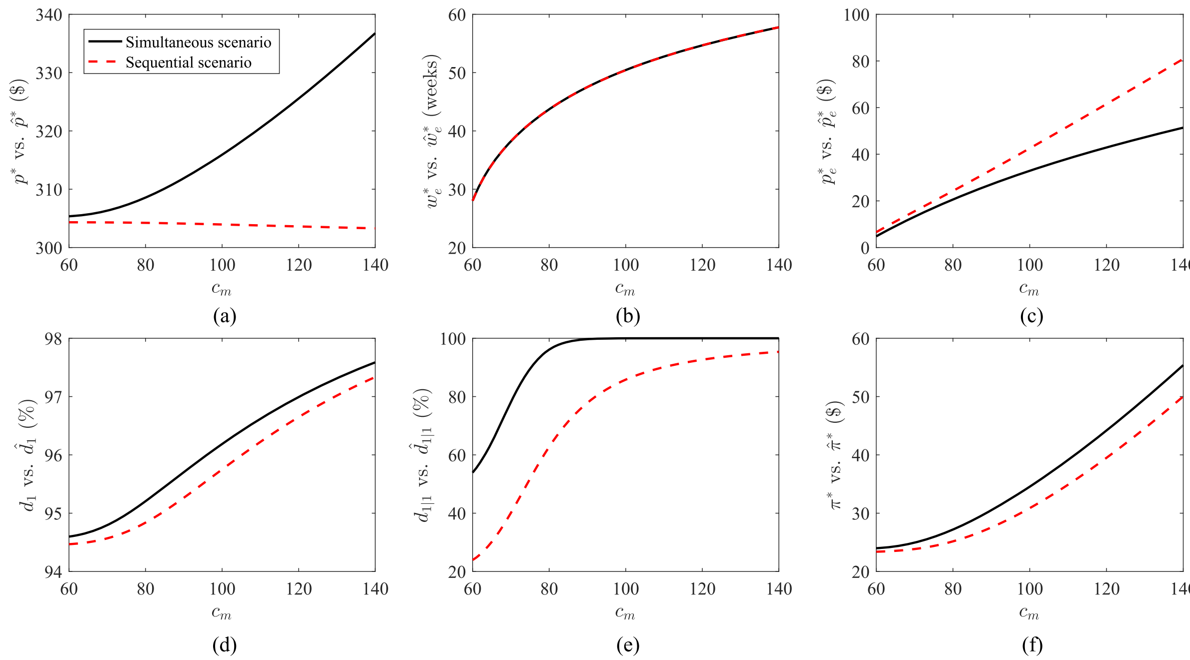

Fig. 3. Comparative statics of c_m. (a) Optimal product price. (b) Optimal warranty length. (c) Optimal warranty price. (d) Product purchase probability. (e) Warranty purchase probability. (f) Maximum expected profit.

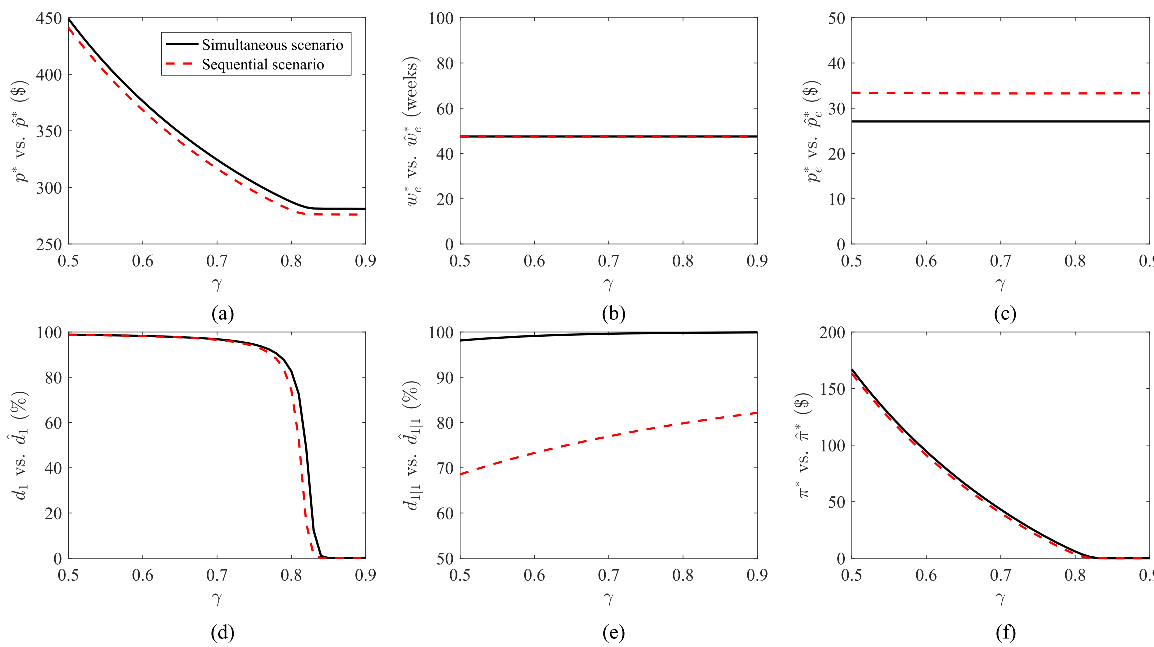

Fig. 4. Comparative statics of γ_1 and γ_2. (a) Optimal product price. (b) Optimal warranty length. (c) Optimal warranty price. (d) Product purchase probability. (e) Warranty purchase probability. (f) Maximum expected profit.

to (19), the warranty purchase probability $d_{1|1}(p^*, w_e^*, p_e^*)$ increases with γ_1. On the other hand, the pattern of $\hat{p}_e^*$ over γ_2 hinges on the relationship between the decrease in $1/\gamma_2$ and the increase in the expected warranty profit $\hat{\pi}_e^*$ (recall that $\hat{p}_e^* = c_e + \hat{\pi}_e^* + 1/\gamma_2$). In Fig. 4(c), $\hat{p}_e^*$ exhibits an insignificantly downward trend (however, we do observe a slightly upward trend under other parameter settings). In this scenario, an increasing $\hat{\pi}_e^*$ and an increasing γ_2, combined together, cause the conditional warranty purchase probability $\hat{d}_{1|1}(\hat{w}_e^*, \hat{p}_e^*)$ to rise as well [see (28)]. Nevertheless, the impact of price sensitivity on warranty-related metrics is much smaller than that on product-related metrics.

V. Extension: Failure Probability Distortion

In the base model, we implicitly assumed that consumers have full information on warranty valuation φ_e, which implies that they can accurately estimate product failure probability f_e (the out-of-warranty repair cost c_m is often known *a priori*). In

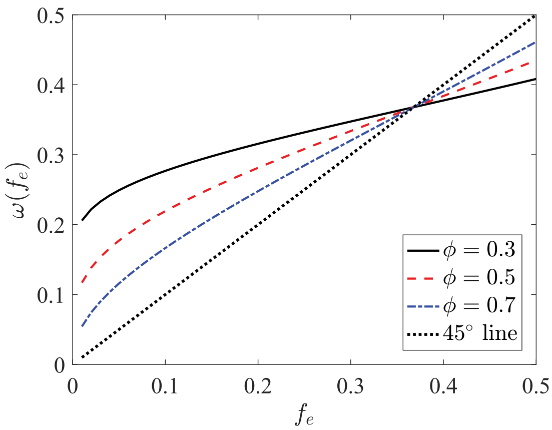

Fig. 5. Illustration of the probability distortion function $\omega(f_e)$.

reality, however, consumers often misperceive product failure probabilities, as precise reliability information is generally unavailable. This partially explains why consumers are willing to buy expensive warranties [7], [33]. In this section, we present a model extension by considering the consumers' probability distortion behavior. It can not only generalize the proposed models, but also checks the robustness of our findings and insights discussed previously.

A common way to characterize such behavior is to use a probability distortion function. Here, we adopt the well-known Prelec [47]'s functional form as follows:

$$\omega(f_e) = \exp(-(-\log(f_e))^{\phi}) \tag{29}$$

where $\phi \in (0, 1)$ is the distortion parameter, and a smaller value of ϕ implies a larger distortion. In particular, $\omega(f_e) \to f_e$ as $\phi \to 1$, corresponding to an ideal scenario in which consumers can accurately perceive the true failure probability. The form in (29) has been adopted by, for example, Abito and Salant [7] and Wang et al. [8] for warranty modeling purposes. A special property of this form is that when $f_e = e^{-1}$, we always have $\omega(f_e) = e^{-1}$. That is, the curve always passes through the point (e^{-1}, e^{-1}), regardless of the value of ϕ.

Another interesting property of the form in (29) is characterized in the following lemma.

Lemma 2: When $f_e \le e^{-1}$, $\omega(f_e) \ge f_e$; otherwise, $\omega(f_e) < f_e$.

Lemma 2 presents an interesting phenomenon: Consumers tend to overestimate (respectively, underestimate) actual product failure probability when it is small (respectively, large). This corresponds to an inverse S-shaped distortion function, which has been identified in the prior literature [6], [7]. Fig. 5 illustrates the probability distortion function under various values of ϕ.

In essence, the extended models with failure probability distortion are the same as those in Sections III and IV, except that the warranty valuation φ_e should be replaced by $\varphi'_e = c_m \cdot \omega(f_e)$. We use superscript $\dagger$ to represent the case in which failure probability distortion is considered. Then, we have the following results.

Proposition 5: When considering failure probability distortion, we have $w_e^\dagger = \hat{w}_e^\dagger = \arg\max_{w_e}\{\varphi'_e - c_e\}$. Moreover, the optimal product and warranty prices and the associated total profits have the same forms as those in Propositions 1 and 2, with φ'_e replacing φ_e.

TABLE II
RESULTS FOR THE NUMERICAL STUDY WITH PROBABILITY DISTORTION

Simultaneous purchase pattern		Sequential purchase pattern	
$\pi^\dagger$	32.69	$\hat{\pi}^\dagger$	29.30
$\pi_e^\dagger$	0	$\hat{\pi}_e^\dagger$	6.65
$p^\dagger$	314.06	$\hat{p}^\dagger$	304.02
$w_e^\dagger$	36.55	$\hat{w}_e^\dagger$	36.55
$p_e^\dagger$	11.45	$\hat{p}_e^\dagger$	19.47
$d_1^\dagger$	95.98%	$\hat{d}_1^\dagger$	95.53%
$d_{1\|1}^\dagger$	99.94%	$\hat{d}_{1\|1}^\dagger$	82.92%

Proposition 5 shows that the optimal warranty lengths under the simultaneous and sequential purchase patterns are still the same. However, unlike w_e^* and $\hat{w}_e^*$, now we are unable to derive closed-form expressions of $w_e^\dagger$ and $\hat{w}_e^\dagger$ due to the complicated form of $\omega(\cdot)$ in (29). As in Propositions 1 and 2, when solving for $\pi^\dagger$, $\hat{\pi}_e^\dagger$, and $\hat{\pi}^\dagger$, φ'_e and c_e should be the optimal values with $w_e^\dagger$ (or $\hat{w}_e^\dagger$) applied; that is, $\varphi'_e = c_m f_e(w_e^\dagger)$ and $c_e = -c_r \log(1 - f_e(w_e^\dagger))$. In short, incorporating failure probability distortion can indirectly affect the firm's product and warranty pricing decisions through influencing the optimal decision on warranty length.

We then compare the key metrics under the two consumer purchase patterns. For the special case of $\gamma_1 = \gamma_2$, we have the same results as in Corollary 1. We thus present the following results without proof.

Corollary 2: When $\gamma_1 = \gamma_2$, the following relationships hold: $\pi^\dagger > \hat{\pi}^\dagger$, $p^\dagger > \hat{p}^\dagger$, $w_e^\dagger = \hat{w}_e^\dagger$, $p_e^\dagger < \hat{p}_e^\dagger$, $d_1(p^\dagger, w_e^\dagger, p_e^\dagger) > \hat{d}_1(\hat{p}^\dagger)$, and $d_{1|1}(p^\dagger, w_e^\dagger, p_e^\dagger) > \hat{d}_{1|1}(\hat{w}_e^\dagger, \hat{p}_e^\dagger)$.

We revisit the numerical example in Section IV-C by setting the distortion factor to $\phi = 0.69$ [7] and keeping the values of the other parameters as before. Table II reports the optimal retail decisions under both consumer purchase patterns. We can see that the comparison results in Corollary 2 well hold. On the other hand, when $\gamma_1 \ne \gamma_2$, the optimal product-warranty retail strategy against different γ_1 and γ_2 is quite similar to that in Fig. 1, and thus omitted for brevity.

VI. Conclusion

In this article, we investigated a retail-strategy choice problem faced by a monopoly selling a durable product and an accompanying warranty directly to end consumers. In practice, some firms induce a sequential consumer purchase process regarding products and warranties in which consumers are presented with the warranty information only after they have bought the products, whereas others allow consumers to consider warranties together with products, resulting in a simultaneous purchase process. By developing logit-type product and/or warranty demand models and formulating the corresponding profit maximization problems, we studied the firm's optimal retail decisions as well as their interactions induced by different consumer purchase patterns. Comparative-statics analyses were further conducted to examine the impacts of parameter changes on the results.

Our research delivers two key managerial implications that can help practitioners to better manage their product and warranty retail operations.

1) First, we found that either retail strategy can result in a higher expected profit, depending on the relative magnitude of consumers' price sensitivities under the simultaneous and sequential purchase patterns (i.e., γ_1 and γ_2). As such, the firms could conduct a small-scale consumer survey or field experiment (similar to the one in Jindal [6]) to obtain a rough estimate of price sensitivities, so as to guide their retail-strategy choice for large-scale implementation. Having said that, a recent trend we observed is that many mainstream e-commerce platforms have started to display warranty information together with the product to facilitate a joint consideration. Warranty Week [4] also recommends firms to put warranty contents earlier on in the consumer purchasing cycle so as to increase their propensity to buy. These provide real-world support for the simultaneous product-warranty retail strategy.
2) Second, our research contributes another analytical finding to the mixed claims on the profitability of extended warranties. We found that an extended warranty, if presented together with the main product, should act as a *sales-promoting* tool, rather than a profit-generating tool. Given the long-standing criticism toward extended warranties (see, e.g., [35], [36], and [46]), it seems wise to offer low-price (even free) warranties so as to leverage their signaling role to attain competitive advantages. This might attract more consumers to purchase the product-warranty combination, and thus grants the firm a potential to charge a higher product price to gain more profit. Indeed, firms such as Planet Audio[10] and Miele[11] are even offering complimentary extended warranties to their consumers, often subject to some specific conditions.

We are interested in several extensions to the current study in the future. First, more influencing factors (e.g., consumer heterogeneity and reference effect) that are relevant to explaining warranty purchases could be involved in developing more general demand models. Second, only a monopolistic setting was considered in this work. Competition and decentralization in supply chains [24] could be examined in future research. Third, product quality/reliability was assumed to be exogenous in this work. It is interesting to incorporate quality/reliability decisions into the current problem, by considering a quality/reliability-dependent cost function [27]. Finally, it is of vital importance to empirically calibrate the demand models using real-world retail data, especially the relative magnitude of price sensitivities under the two purchase patterns, to better guide the practitioners in product and warranty retail operations.

[10][Online]. Available: https://planetaudio.com/free-extended-warranty (accessed on December 28, 2023).
[11][Online]. Available: https://www.mieleusa.com/c/free-5-year-warranty-professional-dishwashers-3569.htm (accessed on December 28, 2023).

TABLE III
FAILURE TIMES (IN WEEKS) OF THE 24 PRINTERS

Printer	1	2	3	4	5	6	7	8	9	10	11	12
	47	52	50	49	53	48	51	48	47	25	36	27 50
Printer	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
	42	52	47	49	33 46	36	41	45	39	42	43	16 35

APPENDIX

Printer Failure Dataset

A warranty claim data set of 32 printers is used in our numerical study in Section IV-C. The printers are from a printer brand in Southeast Asia, and we mask the company name for confidentiality. All of the printers were sold during the first week of August 2015 (week 1), and 24 of them had warranty claim records during our study period from August 2015 to September 2016. Table III reports the warranty claim times (failure times) of these printers. Under the power-law failure intensity as in (10) and the minimal repair assumption, we obtain the estimates of α and β by numerically maximizing the following log-likelihood function (see, e.g., [48]):

$$\ell(\alpha, \beta) = \sum_{i=1}^{N} \left[\sum_{j=1}^{m_i} \log \left(\frac{\beta}{\alpha} \left(\frac{\tau_{ij}}{\alpha} \right)^{\beta-1} \right) - \left(\frac{\overline{\tau}_i}{\alpha} \right)^{\beta} \right]$$

where N is the total number of printers, m_i is the number of warranty claims of printer i, $\overline{\tau}_i$ is the length of the study period of printer i, and τ_{ij} is the jth failure time of printer i. For the case study, we have $N = 32$ and $\overline{\tau}_i = 56$ for all $i = 1, 2, \ldots, 32$; three printers have two claims over our study period (namely, $m_{12} = m_{17} = m_{24} = 2$) and the others have only one claim.

Data analysis shows that printer failures occur according to a power law process with intensity $\lambda(t) = \beta/\alpha(t/\alpha)^{\beta-1}$, with $\alpha = 58.98$ weeks and $\beta = 3.28$. Moreover, the average cost of minimal repair is estimated to be $c_r = \$55$; however, consumers have to pay $c_m = \$90$, on average, if a repair is not covered by a warranty. As a result, the marginal warranty cost is $c_e = 55 \cdot (w_e/58.98)^{3.28}$ and the consumer valuation for the warranty is $\varphi_e = 90 \cdot [1 - \exp(-(w_e/58.98)^{3.28})]$. In addition, we assume that the printer's intrinsic valuation is $\varphi_0 = \$225$, its marginal production cost is $c_0 = \$280$, and the sensitivities to prices are $\gamma_1 = \gamma_2 = 0.73$ (according to [6]).

Model With Nonzero Base Warranty

We present here an alternative model in which the base-warranty length w_b is assumed to be *fixed* yet *nonzero*. We shall show that the findings in our main text are still valid.

In this setting, the product's intrinsic valuation φ_0 can be decomposed into two parts as follows:

$$\varphi_0 = \varphi_0' + \varphi_b = \varphi_0' + c_m f_b \tag{30}$$

where φ_0' is the product's intrinsic valuation without base warranty, φ_b is the base warranty's valuation, and f_b is the product's failure probability over the base warranty period $[0, w_b]$, which is given by $f_b = f_b(w_b) = 1 - \exp(-\int_0^{w_b} \lambda(t)\mathrm{d}t)$.

Similarly, the marginal cost c_0 should be decomposed into

$$c_0 = c_0' + c_b$$

where c_0' is the marginal production cost and c_b is the expected cost of honoring the base warranty, which is given by $c_b = c_b(w_b) = c_r \int_0^{w_b} \lambda(t)\mathrm{d}t$, under the assumption that minimal repair is conducted upon any failure over $[0, w_b]$.

In addition, the product's failure probability over the extended warranty period $[w_b, w_b + w_e]$ and the expected cost of servicing the extended warranty should be revised to $f_e = 1 - \exp(-\int_{w_b}^{w_b+w_e} \lambda(t)\mathrm{d}t)$ and $c_e = c_r \int_{w_b}^{w_b+w_e} \lambda(t)\mathrm{d}t$, respectively.

Then, the expected profits under the simultaneous and sequential purchase patterns are still given by (12) and (20), respectively, with φ_0, c_0, f_e, and c_e being revised properly as above. It is easy to verify that the optimal retail decisions and maximum profits in this setting are the same as those in Propositions 1 and 2 for both consumer purchase patterns. Hence, incorporating a fixed, nonzero base warranty into the model would not change our findings.

We note that a slight difference induced by the nonzero base warranty lies in the optimal extended warranty lengths w_e^* and $\hat{w}_e^*$, which can be determined by solving $f_e(w_e) = 1 - c_r/c_m$. Though the solution form is identical to that in Propositions 1 and 2, the expression of $f_e(w_e)$ has indeed been changed. In particular, if the product has a power-law failure intensity as in (10), then the optimal extended warranty lengths would be $w_e^* = \hat{w}_e^* = \alpha[\log(c_m/c_r) + (w_b/\alpha)^\beta]^{1/\beta} - w_b$.

Technical Proofs: Before providing detailed proofs, we first present the following lemma that will be used in the subsequent proofs.

Lemma 3: i) Let π_1° and π_2° be the unique solutions to $A_1 \exp(B - \gamma\pi) = \pi$ and $A_2 \exp(B - \gamma\pi) = \pi$, respectively, where A_1, A_2, B, and γ are positive constants. If $A_1 > A_2$, then $\pi_1^\circ > \pi_2^\circ$. ii) Let $\pi_1^\dagger$ and $\pi_2^\dagger$ be the unique solutions to $A \exp(B_1 - \gamma\pi) = \pi$ and $A \exp(B_2 - \gamma\pi) = \pi$, respectively, where A, B_1, B_2, and γ are positive constants. If $B_1 > B_2$, then $\pi_1^\dagger > \pi_2^\dagger$.

Proof.: The results can be proved by contradiction in a similar way, thus we only provide the proof for part i). Suppose $\pi_1^\circ < \pi_2^\circ$. Because $\gamma > 0$, we have $A_1 \exp(B - \gamma\pi_1^\circ) > A_2 \exp(B - \gamma\pi_2^\circ)$, and thus $\pi_1^\circ > \pi_2^\circ$. This contradicts our initial hypothesis. Hence, if $A_1 > A_2$, we must have $\pi_1^\circ > \pi_2^\circ$.

Proof of Lemma 1: i) It is intuitive that $d_0(p, w_e, p_e)$, the probability for the outside option, is increasing in p_e. Because $d_1(p, w_e, p_e) = 1 - d_0(p, w_e, p_e)$, we know that $d_1(p, w_e, p_e)$ is decreasing in p_e. Moreover, the conditional warranty purchase probability can be rewritten as follows:

$$d_{1|1}(p, w_e, p_e) = \frac{1}{1 + 1/\exp(\gamma_1\varphi_e - \gamma_1 p_e)}$$

which is clearly decreasing in p_e.

ii) Notice that $\varphi_e = c_m f_e$ is increasing in w_e. Then, it is straightforward that both $d_1(p, w_e, p_e)$ and $d_{1|1}(p, w_e, p_e)$ are increasing in w_e.

Proof of Proposition 1: Let $\Pi(p, w_e, p_e) = \pi$. Rearranging terms and after some algebraic manipulations, we have the following:

$$\begin{aligned}&(p - c_0 + p_e - c_e - \pi)\exp(\varphi_0 - \gamma_1 p + \gamma_1\varphi_e - \gamma_1 p_e)\\ &\quad = \pi - (p - c_0 - \pi)\exp(\varphi_0 - \gamma_1 p).\end{aligned}$$

It is clear that the left-hand side (LHS) is strictly decreasing in π, whereas the right-hand side (RHS) is strictly increasing in π. To solve the joint optimization problem in (13), it is helpful to think the firm can focus first on the problem $\max_{p_e} \Pi(p, w_e, p_e)$, which is equivalent to

$$\pi - (p - c_0 - \pi)\exp(\varphi_0 - \gamma_1 p) = \max_{p_e} h_1(p_e)$$

where $h_1(p_e) := (p - c_0 + p_e - c_e - \pi)\exp(\varphi_0 - \gamma_1 p + \gamma_1 \varphi_e - \gamma_1 p_e)$.

Taking the first derivative of $h_1(p_e)$ with respect to p_e gives the following:

$$\begin{aligned}\frac{\partial h_1(p_e)}{\partial p_e} = &\exp(\varphi_0 - \gamma_1 p + \gamma_1\varphi_e - \gamma_1 p_e)\\ &\times (1 - \gamma_1(p - c_0 + p_e - c_e - \pi)).\end{aligned}$$

We can see that $h_1(p_e)$ is increasing in p_e for $p_e \le c_0 - p + c_e + \pi + 1/\gamma_1$, and becomes decreasing thereafter. Hence, it reaches its maximum at $p_e^*(p) = c_0 - p + c_e + \pi + 1/\gamma_1$, where π is the unique solution to $\pi - (p - c_0 - \pi)\exp(\varphi_0 - \gamma_1 p) = \exp(\varphi_0 + \gamma_1\varphi_e - \gamma_1(c_0 + c_e + \pi) - 1)/\gamma_1$.

The optimal warranty length can then be obtained by solving

$$\begin{aligned}&\pi - (p - c_0 - \pi)\exp(\varphi_0 - \gamma_1 p)\\ &\quad = \max_{w_e} \frac{1}{\gamma_1}\exp(\varphi_0 + \gamma_1\varphi_e - \gamma_1(c_0 + c_e + \pi) - 1). \quad (31)\end{aligned}$$

Notice that on the RHS of (31), only φ_e and c_e are related to w_e. Thus, the optimal value of w_e can be determined by solving $\max_{w_e} \eta = \varphi_e - c_e$. Recall that $\varphi_e = c_m f_e$ and $c_e = -c_r \log(1 - f_e)$. Taking the first and second derivatives of η with respect to f_e yields $\partial\eta/\partial f_e = c_m - c_r/(1 - f_e)$ and $\partial^2\eta/\partial f_e^2 = -c_r/(1 - f_e)^2 < 0$. As f_e is monotonically increasing in w_e, the optimal $w_e^*(p)$ is the unique solution to $f_e = 1 - c_r/c_m$ or, equivalently, $w_e^*(p) = f_e^{-1}(1 - c_r/c_m)$. In fact, $w_e^*(p)$ is independent of p.

We further determine the optimal product price such that $\Pi(p, w_e^*(p), p_e^*(p))$ is maximized. This problem is immediately equivalent to finding a unique solution to

$$\pi - \frac{1}{\gamma_1}\exp(\varphi_0 + \gamma_1\varphi_e - \gamma_1(c_0 + c_e + \pi) - 1) = \max_p h_2(p)$$

where $h_2(p) := (p - c_0 - \pi)\exp(\varphi_0 - \gamma_1 p)$. It is easily known that $h_2(p)$ is unimodal in p, and attains its maximum at $p^* = c_0 + \pi^* + 1/\gamma_1$, where π^* is the unique solution to (17). By substituting p^* into $p_e^*(p)$, the optimal warranty price is $p_e^* = c_e$, which is exactly equal to the marginal warranty cost.

Moreover, the product purchase probability, at optimality, is as follows:

$$\begin{aligned} d_1(p^*, w_e^*, p_e^*) &= d_{01}(p^*, w_e^*, p_e^*) + d_{11}(p^*, w_e^*, p_e^*) \\ &= \frac{\gamma_1 \pi^*}{1+\gamma_1 \pi^*} \end{aligned}$$

and the conditional warranty purchase probability, at optimality, can be derived as follows:

$$\begin{aligned} d_{1|1}(p^*, w_e^*, p_e^*) &= \frac{d_{11}(p^*, w_e^*, p_e^*)}{d_{01}(p^*, w_e^*, p_e^*) + d_{11}(p^*, w_e^*, p_e^*)} \\ &= \frac{\exp(\gamma_1 \varphi_e - \gamma_1 p_e^*)}{1+\exp(\gamma_1 \varphi_e - \gamma_1 p_e^*)} \\ &= \frac{\exp(\gamma_1 \varphi_e - \gamma_1 c_e)}{1+\exp(\gamma_1 \varphi_e - \gamma_1 c_e)}. \end{aligned}$$

The first result is obtained by combining (17) with (3) and (4).

Proof of Proposition 2: Following the discussion in Section IV-B, we first determine the optimal warranty length $\hat{w}_e^*$ and warranty price $\hat{p}_e^*$ to maximize $\hat{\Pi}_e(w_e, p_e) = \hat{d}_{1|1}(w_e, p_e)$ $(p_e - c_e)$. Let $\hat{\Pi}_e(w_e, p_e) = \hat{\pi}_e$. Rearranging terms and using some algebra yield

$$(p_e - c_e - \hat{\pi}_e)\exp(\gamma_2 \varphi_e - \gamma_2 p_e) = \hat{\pi}_e.$$

Analogous to the proof of Proposition 1, we first consider the problem $\max_{p_e} \hat{\Pi}_e(w_e, p_e)$, which is equivalent to

$$\hat{\pi}_e = \max_{p_e} g_1(p_e)$$

where $g_1(p_e) := (p_e - c_e - \hat{\pi}_e)\exp(\gamma_2\varphi_e - \gamma_2 p_e)$. Clearly, $g_1(p_e)$ is unimodal in p_e, and attains its maximum at $\hat{p}_e^* = c_e + \hat{\pi}_e^* + 1/\gamma_2$, where $\hat{\pi}_e^*$ is the unique solution to $\hat{\pi}_e = \exp(\gamma_2\varphi_e - \gamma_2(c_e + \hat{\pi}_e) - 1)/\gamma_2$. Then, the optimal warranty length can be obtained by solving

$$\hat{\pi}_e = \max_{w_e} \frac{1}{\gamma_2}\exp(\gamma_2\varphi_e - \gamma_2(c_e + \hat{\pi}_e) - 1).$$

It is straightforward that $\hat{w}_e^*$ can be derived as $\hat{w}_e^* = \arg\max_{w_e} \eta = f_e^{-1}(1 - c_r/c_m)$.

Next, we determine an optimal product price to maximize $\hat{\Pi}(p, \hat{w}_e^*, \hat{p}_e^*) = \hat{d}_1(p)(p - c_0 + \hat{\pi}_e^*)$. Let $\hat{\Pi}(p, \hat{w}_e^*, \hat{p}_e^*) = \hat{\pi}$, then we have

$$(p - c_0 + \hat{\pi}_e^* - \hat{\pi})\exp(\varphi_0 - \gamma_2 p) = \hat{\pi}.$$

The problem $\max_p \hat{\Pi}(p, \hat{w}_e^*, \hat{p}_e^*)$ is immediately equivalent to

$$\hat{\pi} = \max_p g_2(p)$$

where $g_2(p) := (p - c_0 + \hat{\pi}_e^* - \hat{\pi})\exp(\gamma_2\varphi_e - \gamma_2 p)$. We know that $g_2(p)$ is unimodal in p and attains its maximum at $\hat{p}^* = c_0 - \hat{\pi}_e^* + \hat{\pi}^* + 1/\gamma_2$, where $\hat{\pi}^*$ is the unique solution to (26).

Therefore, the product purchase probability and the conditional warranty purchase probability, at optimality, are as follows:

$$\hat{d}_1(\hat{p}^*) = \frac{\exp(\varphi_0 - \gamma_2\hat{p}^*)}{1+\exp(\varphi_0 - \gamma_2\hat{p}^*)} = \frac{\gamma_2\hat{\pi}^*}{1+\gamma_2\hat{\pi}^*}$$

and

$$\hat{d}_{1|1}(\hat{w}_e^*, \hat{p}_e^*) = \frac{\exp(\gamma_2\varphi_e - \gamma_2\hat{p}_e^*)}{1+\exp(\gamma_2\varphi_e - \gamma_2\hat{p}_e^*)} = \frac{\gamma_2\hat{\pi}_e^*}{1+\gamma_2\hat{\pi}_e^*},$$

respectively. The results follow directly from (25) and (26), respectively.

Proof of Corollary 1: Because $\gamma_1 = \gamma_2$, we simply use γ without confusion. Recall that π^* and $\hat{\pi}^*$ are the unique solutions to (17) and (26), respectively, and notice that (26) can be rewritten as follows:

$$\hat{\pi}^* = \frac{1}{\gamma}\exp(\varphi_0 - \gamma(c_0 + \hat{\pi}^*) - 1)\cdot\exp(\gamma\hat{\pi}_e^*).$$

Based on Lemma 3(i), we claim that if $1 + \exp(\gamma\varphi_e - \gamma c_e) > \exp(\gamma\hat{\pi}_e^*)$, then $\pi^* > \hat{\pi}^*$ holds. Rearranging terms for (25) yields $\exp(\gamma\varphi_e - \gamma c_e) = \gamma\hat{\pi}_e^* e^{\gamma\hat{\pi}_e^*+1}$. Hence, the condition $1 + \exp(\gamma\varphi_e - \gamma c_e) > \exp(\gamma\hat{\pi}_e^*)$ is equivalent to $\gamma\hat{\pi}_e^* e^{\gamma\hat{\pi}_e^*+1} - e^{\gamma\hat{\pi}_e^*} + 1 > 0$. Let $\xi(x) = e^x(e\cdot x - 1) + 1$, and $x \in [0, \infty)$. It is easy to verify that $\xi(x)$ is increasing in x, from $\xi(x = 0) = 0$ to ∞. From (25), we know that $\gamma\hat{\pi}_e^* > 0$. Therefore, $\xi(\gamma\hat{\pi}_e^*) = \gamma\hat{\pi}_e^* e^{\gamma\hat{\pi}_e^*+1} - e^{\gamma\hat{\pi}_e^*} + 1 > 0$ always holds and thus $\pi^* > \hat{\pi}^*$.

Based on Propositions 1 and 2, it is obvious to have $\hat{p}_e^* > p_e^*$. Because $\pi^* > \hat{\pi}^*$, we have $\pi^* > \hat{\pi}^* - \hat{\pi}_e^*$. Then, $p^* > \hat{p}^*$ follows directly. Propositions 1 and 2 also tell us that $d_1(p^*, w_e^*, p_e^*) = \gamma\pi^{*}/(1+\gamma\pi^*)$ and $\hat{d}_1(\hat{p}^*) = \gamma\hat{\pi}^*/(1+\gamma\hat{\pi}^*)$. It is intuitive that $d_1(p^*, w_e^*, p_e^*)$ and $\hat{d}_1(\hat{p}^*)$ are increasing functions of π^* and $\hat{\pi}^*$, respectively. As $\pi^* > \hat{\pi}^*$, we have $d_1(p^*, w_e^*, p_e^*) > \hat{d}_1(\hat{p}^*)$. Finally, according to Lemma 1, $d_{1|1}(p, w_e, p_e)$ (and also $\hat{d}_{1|1}(w_e, p_e)$) are decreasing in p_e. As $p_e^* < \hat{p}_e^*$, we have $d_{1|1}(p^*, w_e^*, p_e^*) > \hat{d}_{1|1}(\hat{w}_e^*, \hat{p}_e^*)$.

Proof of Proposition 3: i) From (17) and (26), we know that π^* and $\hat{\pi}^*$ are increasing in φ_0 [according to Lemma 3(ii)]; whereas (25) implies that $\hat{\pi}_e^*$ is independent of φ_0.

ii) Because $w_e^* = \hat{w}_e^* = f_e^{-1}(1 - c_r/c_m)$ are independent of φ_0, we know that w_e^* and $\hat{w}_e^*$ remain constant as φ_0 increases. Other results in part ii) follow directly from part i).

iii) It is easy to show that as φ_0 increases, $d_1(p^*, w_e^*, p_e^*)$ and $\hat{d}_1(\hat{p}^*)$ increase, whereas $d_{1|1}(p^*, w_e^*, p_e^*)$ and $\hat{d}_{1|1}(\hat{w}_e^*, \hat{p}_e^*)$ remain constant.

Proof of Proposition 4: i) We first examine the valuation margin η. Recall that under both consumer purchase patterns, $\varphi_e = c_m - c_r$ and $c_e = -c_r\log(c_r/c_m)$. Then, we have $\partial\eta/\partial c_m = 1 - c_r/c_m > 0$ and thus η is increasing in c_m for any $c_m > c_r$.

It is obvious that the RHS of (17) is increasing in η. As a result, when η becomes larger, π^* must become larger to make the equality valid [see Lemma 3(i)]. Likewise, (25) implies that $\hat{\pi}_e^*$ is also increasing in η. From (26), we know that $\hat{\pi}^*$ increases as $\hat{\pi}_e^*$ increases; based on the chain rule, $\hat{\pi}^*$ is also increasing in η. Applying the chain rule again yields the result that π^*, $\hat{\pi}^*$, and $\hat{\pi}_e^*$ are all increasing functions of c_m.

ii) According to the fact that $f_e(w_e)$ is a monotonically increasing function of w_e, we can easily know that w_e^* and $\hat{w}_e^*$ are increasing in c_m. As $p_e^* = c_e = -c_r\log(c_r/c_m)$, we know that p_e^* is also increasing in c_m. In addition, the increasing pattern of p^* and $\hat{p}_e^*$ with respect to c_m follows directly from the fact that π^* and $\hat{\pi}_e^*$ (and c_e) are increasing in c_m.

For $\hat{p}^*$, we need to examine the term $\hat{\pi}^* - \hat{\pi}_e^*$. Notice that (26) is equivalent to

$$\hat{\pi}^* - \hat{\pi}_e^* = \frac{1}{\gamma_2}\exp(\varphi_0 - \gamma_2(c_0 + \hat{\pi}^* - \hat{\pi}_e^*) - 1) - \hat{\pi}_e^*. \quad (32)$$

Therefore, $\hat{\pi}^* - \hat{\pi}_e^*$, as a whole, is decreasing in $\hat{\pi}_e^*$, and thus $\hat{p}^*$ is decreasing in c_m (by the chain rule).

iii) Based on part i), it is easy to show that $d_1(p^*, w_e^*, p_e^*)$, $d_{1|1}(p^*, w_e^*, p_e^*)$, $\hat{d}_1(\hat{p}^*)$, and $\hat{d}_{1|1}(\hat{w}_e^*, \hat{p}_e^*)$ are all increasing in c_m.

Proof of Lemma 2: We first examine the functional form x^ϕ, for $x \in [0, \infty)$ and $\phi \in (0, 1)$. It is clear that x^ϕ is concave increasing in x, and $x^\phi > x$ for $x \in [0, 1)$; $x^\phi \leq x$, otherwise.

Now, let $x = -\log(f_e)$. Because the range of failure probability is $f_e \in [0, 1]$, we have $-\log(f_e) \in [0, \infty)$. Therefore, when $-\log(f_e) < 1$ (i.e., $f_e > e^{-1}$), we have $(-\log(f_e))^\phi > -\log(f_e)$ and thus $\omega(f_e) = \exp(-(-\log(f_e))^\phi) < f_e$; otherwise, we have $\omega(f_e) \geq f_e$.

Proof of Proposition 5: The proof is quite similar to those of Propositions 1 and 2, and thus omitted for simplicity.

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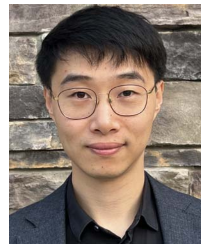

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