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A two-stage preventive maintenance optimization model incorporating two-dimensional extended warranty

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ABSTRACT

In practice, customers can decide whether to buy an extended warranty or not, at the time of item sale or at the end of the basic warranty. In this paper, by taking into account the moments of customers purchasing two-dimensional extended warranty, the optimization of imperfect preventive maintenance for repairable items is investigated from the manufacturer's perspective. A two-dimensional preventive maintenance strategy is proposed, under which the item is preventively maintained according to a specified age interval or usage interval, whichever occurs first. It is highlighted that when the extended warranty is purchased upon the expiration of the basic warranty, the manufacturer faces a two-stage preventive maintenance optimization problem. Moreover, in the second stage, the possibility of reducing the servicing cost over the extended warranty period is explored by classifying customers on the basis of their usage rates and then providing them with customized preventive maintenance programs. Numerical examples show that offering customized preventive maintenance programs can reduce the manufacturer's warranty cost, while a larger saving in warranty cost comes from encouraging customers to buy the extended warranty at the time of item sale.

1. Introduction

1.1. Motivation

In today's competitive market, a variety of means have been used by manufacturers to capture more market share and customers satisfaction. Offering attractive warranty policies is a common incentive. A warranty policy is a statement in connection with the sale of an item on the scheme (e.g., free repair/replacement, refund, etc.) and extent (length of period) of compensation offered by the manufacturer in the event of item failure [1]. In fact, warranty has been widely used to serve multiple purposes in consumer and commercial transactions, and extensive attention has been paid to warranty management from various disciplines [2–5].

Based on the number of variables that are used to define the policy, warranty policies can be broadly divided into two categories, i.e., one-dimensional and two-dimensional [6]. One-dimensional warranties are defined on the basis of either age or usage; while two-dimensional warranties consider both age and usage and the potential interaction between them. In practice, two-dimensional warranties have been widely applied in automobile industry. For example, a new automobile is usually sold with a two-dimensional warranty, say free repair for 3 years or 60,000 km, whichever occurs first.

Moreover, during product marketing, manufacturers usually offer customers the option of buying an extended warranty (EW) that provides protection for an additional period after the basic warranty (BW) ceases. Considering the increasing complexity of many products in both of structures and functions, the maintenance of such products becomes more complicated and costly than ever before. Thus, EW is attracting more attention from both manufacturers and customers.

However, offering any types of warranty policies to customers will incur substantial cost to manufacturers resulting from the servicing of warranty claims. One effective way to reduce warranty servicing cost is to incorporate appropriate preventive maintenance (PM) programs into the warranty policy. Different from corrective maintenance actions which are intended to restore the failed item to an operating state, PM actions are planned actions either to reduce the probability of failures or to prolong the item's lifetime while it is still in the operating state [7]. From the cost-benefit point of view, it is worthwhile for a manufacturer to adopt a PM program only when the reduction of warranty servicing cost exceeds the additional cost incurred by the PM program [8].

In recent years, the demand growth for two-dimensional EW contracts raises a challenge for manufacturers, that is, how to implement appropriate PM actions for items covered by both two-dimensional BW and EW contracts so as to reduce the warranty

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cost. However, existing approaches and models are far from sufficient for assisting the manufacturers in making such decisions. This study intends to fill this gap.

1.2. Related literature

Warranty and PM are two topics that have been separately studied by many researchers. Integrated research on both of them has gained relatively limited attention [9]. This paper focuses on the optimization of imperfect PM strategies incorporating two-dimensional BW and EW policies. Below a brief literature review is conducted to better position the novelty and contributions of this paper.

Most existing studies on warranty are related to BW policies, while studies on EW policies are comparatively limited; see [10–17] for research on one-dimensional EW policies, and [18–21] for research on two-dimensional EW policies. Among them, some references addressed the optimization of PM strategies incorporating one-dimensional EW policies; see [13–15] for example. To our knowledge, there are only two journal papers [19,21] focusing on joint consideration of PM and two-dimensional EW policies. Shahanaghi et al. [19] developed a model to determine the optimal PM decisions during the two-dimensional EW period. They assumed that imperfect PM actions were conducted during only the EW period, and no PM was scheduled during the BW period. In a recent work, Wang et al. [21] attempted to link the two-dimensional BW and EW contracts in an integrated manner, and then to identify the optimal number of PM actions during both periods.

In addition, some other studies (see [8,22–26] for instance) on PM modeling and optimization are also related to our problem of interest, although they do not take EW into consideration. Among them, Kim et al. [8] developed a framework for cost analysis linking warranty and PM policy from a life-cycle perspective. In Ref. [8], the PM actions are performed at discrete time instants, and the effect of PM is characterized by the reduction of virtual age. A variation of the modeling framework in [8] is adopted in this study to serve the research purpose.

1.3. Contributions of this work

In practice, an EW contract is usually optional for interested customers, either at the time of item sale or when the BW expires [12,20,27]. Therefore, this paper outlines a framework for the optimization of PM strategies by considering the moments of customers purchasing the two-dimensional EW contract. It is highlighted that when the EW is purchased just before the BW expires, the PM optimization problem exhibits a *two-stage* nature. To our knowledge, this is the first attempt to investigate the two-stage imperfect PM optimization problem by taking into account the moments of customers purchasing the EW contract.

Moreover, conventional PM strategies are usually age-based or usage-based. For instance, under the age- [usage-] based strategy, PM actions are scheduled according to a specified age [usage] interval, irrespective of the item's usage intensity. By now, implementing PM actions based on a two-dimensional framework, by considering age and usage intervals simultaneously, has received little attention. A new strategy, called *two-dimensional PM strategy*, is considered in this work. Under this strategy, imperfect PM actions are conducted every K units of age or L units of usage, whichever occurs first. Recently, Wang and Su [28] numerically demonstrates that the two-dimensional PM strategy is superior, or at least identical, to the age-based and usage-based strategies in terms of warranty servicing cost.

Furthermore, most existing studies on PM assume that a unified PM program is specified for all customers, regardless of their usage intensities or operating conditions. As a matter of fact, different usage intensities would result in different rates of item deterioration. Thus, it is wiser for the manufacturer to design *customized* PM programs for different customer categories. Motivated by this consideration, the manufacturer can classify customers, at the end of the BW, based on their usage rates throughout the BW period and then provide them with customized PM programs over the EW period. An illustrative example is presented to demonstrate the advantage of the customized PM policy.

In short, the settings and contributions of this paper differ from existing literature (especially [19,21]) in the following aspects: (1) a two-dimensional PM strategy is proposed; (2) a two-stage imperfect PM optimization framework is developed and investigated; (3) for the two-stage problem, the possibility of reducing servicing cost over the EW period through offering customized PM programs is demonstrated.

The remaining of this paper is structured as follows. Section 2 introduces the modeling framework, including the item failure model, the PM model and the framework of two-stage PM optimization. Then, the two-stage PM optimization problem are presented and analyzed in Section 3. In Section 4, a numerical example is provided to illustrate the applicability of the proposed model. Finally, Section 5 concludes this paper.

2. Modeling framework

2.1. Assumptions

The following assumptions are made in developing the mathematical model.

(1) The item of interest is repairable and deteriorates with both age and usage, i.e., without maintenance intervention its failure rate will increase as age and/or usage increases.
(2) The usage rate varies randomly across the customer population but is constant over time for a specific customer.
(3) An EW is optional for interested customers at the time of item sale and/or when the BW terminates. It is also supposed that the EW is provided and serviced solely by the manufacturer.
(4) During both BW and EW periods, a two-dimensional PM strategy is offered and paid by the manufacturer.
(5) Since the manufacturer bears the PM costs, the customers have to take their items into designated service centers for scheduled PM services where their average usage rates throughout the BW period can be identified.
(6) All failures under warranty are rectified minimally, which means that the item's reliability after repair is the same as that just before failure.
(7) The time to repair a faulty item, as well as to perform a PM action, is sufficiently small compared to the mean time between failures, and thus is assumed to be negligible.

Remark 1. The assumption (2) above is fundamental for the marginal approach of two-dimensional failure modeling, which will be mentioned later. Rationality of this assumption is supported by automobile warranty data analysis. For example, Gupta et al. [29] claim that the correlation coefficient between age and mileage is usually over 0.7, while the same corresponding to the age and usage rate is usually very low and sometimes even less than 0.1. It is therefore reasonable to assume that the usage rate is primarily determined by the customer's usage pattern, and independent of the age. This assumption has been widely adopted in the literature; see [30–33] for example.

Remark 2. The assumption (5) is essential for developing customized PM programs. Previous literature usually bears the following consensus—the age is known for all sold items at any time (since the manufacturer generally knows the sale date), but the usage is only observed for those failed and claimed items [34,35]. However, in this study, due to the implementation of scheduled PM actions within the BW period, the maintenance crew can record the customers' usage details (such as item age, usage, failure modes) at each PM instant, and these data can then be used to identify the customers' average usage rates. Therefore, for those customers who purchase the EW just before the BW expires, the manufacturer is able to obtain their average usage rates throughout the BW period.

2.2. Modeling item failures

In order to model the failure process in terms of both age and usage, three main approaches have been developed in the literature, i.e., bivariate, composite scale and marginal (univariate) approaches; refer to Jack et al. [36] and Wu [37] for more discussions on these approaches.

In this study, the marginal approach is used to model the item failures in terms of age and usage. The repairable item is regarded as a black-box system, and the failure process is modeled at the system level. For an elapsed time t, the cumulative operating time and total usage of an item can be denoted by $T(t)$ and $U(t)$, respectively. For simplicity, we assume that the item's cumulative operating time equals its age, i.e., $T(t) = t$. Note that, this simplification can only be true when either no failure occurs in $[0, t)$ or all failures are repaired and the repair durations are negligible [30]. Based on the aforementioned assumption (2), the usage rate $R = U(t)/T(t)$ can be modeled as a random variable with distribution function $G(r) = P(R \le r)$, $r_{\min} \le r \le r_{\max}$, where $r_{\min}$ and $r_{\max}$ are the lower and upper limits of the usage rate, respectively. Besides, it is supposed that the distribution $G(r)$ is known for the manufacturer, either through past history of similar items or from a customer survey [32]. Conditional on $R = r$, the total usage $U(t)$ of an item at age t is given by $U(t) = rt$.

Item failures can thus be modeled by a point process with an intensity function that is dependent on both age and usage. Given $R = r$, the conditional intensity function can be denoted as $\lambda(t|r) = \varphi(T(t), U(t))$, where $\varphi(\bullet)$ is a non-decreasing function of both $T(t)$ and $U(t)$. Given that all failures are minimally repaired with negligible repair time and no PM is performed, failures over time occur according to a non-homogeneous Poisson process (NHPP) with conditional intensity function $\lambda(t|r)$. If appropriate data are available, model selection and parameter estimation for $\lambda(t|r)$ can be preformed; refer to Iskandar et al. [31] and Rigdon and Basu [38] for detailed approaches.

For the numerical example in this study, following Murthy and Wilson [39], the conditional intensity function is modeled as a polynomial function:

$$\begin{aligned}\lambda(t|r) &= \theta_0 + \theta_1 r + \theta_2 T(t) + \theta_3 U(t)\\ &= \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)t \end{aligned} \tag{1}$$

where θ_0, θ_1, θ_2, and θ_3 are positive constants.

As can be seen, the two-dimensional failure model is reduced to a one-dimensional model by using the marginal approach. It is worth mentioning that the polynomial intensity function is widely used in the literature on two-dimensional warranty; see [21,22,30,33] for reference.

2.3. Modeling the imperfect PM actions

The PM actions within fixed warranty period are normally performed according to a predetermined PM schedule, with which age reductions would occur between the actual age and virtual age. Here the modeling framework proposed by Kim et al. [8] is adopted to describe the age reduction effect of imperfect PM actions.

Suppose that a series of PM actions are scheduled at actual age $\tau_1, \tau_2, \ldots, \tau_j, \ldots$, with $\tau_0 = 0$. In [8], the jth PM action only compensates the damage accumulated during the time between the $(j-1)$th and the jth PM actions, which leads to a reduction of the item's virtual age. The virtual age immediately after performing the jth PM action is then given by [8]

$$v_j = v_{j-1} + \delta(m)(\tau_j - \tau_{j-1}) \tag{2}$$

where m is the level of PM effort, and $\delta(m)$, $m = 0, 1, \ldots, M$, is the age reduction factor of PM with level m.

Note that, larger value of m represents greater PM effort, and the age reduction factor $\delta(m)$ is a decreasing function of m with $\delta(0) = 1$ and $\delta(M) = 0$. If $m = M$, then the item is restored to as good as new (AGAN); If $m = 0$, then $v_j = \tau_j$, $j \ge 1$, which means that the item is restored to as bad as old (ABAO); More generally, $0 < m < M$ corresponds to imperfect PM, i.e., the item is partially restored. In this study, the functional relationship between $\delta(m)$ and m is characterized by an exponentially decreasing function, $\delta(m) = (1+m)e^{-m}$, as in Kim et al. [8].

Remark 3. Expression (2) shows the effect of PM on the item's virtual age, however, one may wonder how the PM actions impact the virtual usage. Since the two-dimensional failure modeling problem has been addressed as a one-dimensional problem by using the linear relationship $U(t) = rt$, this study assumes that the jth PM action has the same influence (as factor $\delta(m)$) on the virtual usage due to the reduction of virtual age. It is worth mentioning that PM actions may have different impacts on the age and usage, thus the virtual age and usage after each PM action should be simultaneously modeled. In this case, the bivariate failure modeling framework is required. However, this is out of the scope of this study. Interested readers can refer to [26,40] for useful references.

Expression (2) will be used in the next section to derive the expected number of failures when the two-dimensional PM strategy is applied.

2.4. Framework of two-stage PM optimization problem

Upon selling an item, e.g., an automobile, the manufacturer offers the customer a free two-dimensional BW $\Omega_B(W_B, U_B)$, where W_B and U_B are its age and usage limits, respectively. Under the BW policy, the manufacturer agrees to repair all failures free of charge up to age limit W_B or up to usage limit U_B from the point of item sale, whichever occurs first.

While a two-dimensional EW contract is usually optional, and interested customers can buy it with an additional premium. It offers the customer an opportunity to extend coverage by $\Omega_E(W_E, U_E)$ after the BW expiry for further protection, where W_E and U_E are its age and usage limits, respectively.

From the customers' perspective, there are at least three cases about purchasing the EW contract [12,20,27]:

Case 1 **:** purchase EW at the time of item sale;
Case 2 **:** purchase EW at the end of the BW;
Case 3 **:** do not purchase EW at any time.

Note that, in Case 3, the item is covered by only the BW, and this case is not our concern. The other two cases are described below.

*Case 1***:** When an EW is purchased at the sale of an item, the entire warranty region, which links the BW and EW together, is

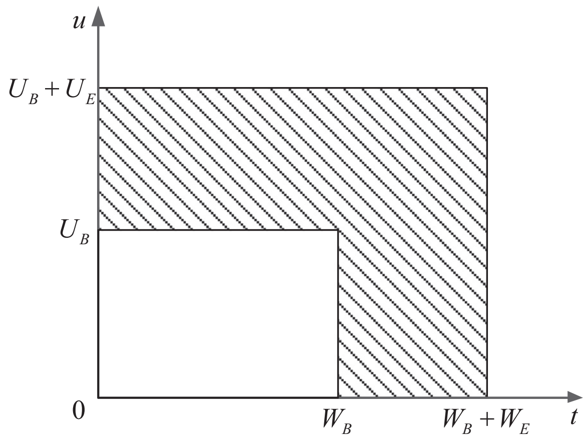

Fig. 1. Warranty regions in Case 1.

given by $[0, W_B+W_E] \times [0, U_B+U_E]$. The EW region is thus given by $[0, W_B+W_E] \times [0, U_B+U_E]$ minus $[0, W_B] \times [0, U_B]$, which is a rotated L-shaped region; see the shaded region in Fig. 1.

Case 2: When an EW is purchased just before the BW expires, the age and usage limits of the EW contract are now measured from the particular point at which the BW terminates. Therefore, as shown in Fig. 2, the EW region has a shift from the BW region [21,31]. With consideration of the item's usage rate r, two possible EW regions are $[W_B, W_B+W_E] \times [rW_B, rW_B+U_E]$ and $[U_B/r, U_B/r+W_E] \times [U_B, U_B+U_E]$; see the shaded regions in Fig. 2.

For more details on the two cases discussed above, please refer to Murthy and Jack [27].

It should be noted that when the EW is bought at different time points, the manufacturer faces different PM optimization problems, as illustrated in Fig. 3. The detailed discussions proceed as follows.

For customers who purchase the EW at the time of item sale (i.e., Case 1), the entire warranty region is equivalent to a larger BW region. In this case, the manufacturer can offer these customers an optimal PM program within the entire warranty region, i.e., K_0, L_0 and m_0.

For customers who do not buy the EW at the time of item sale, the manufacturer cannot judge whether they will need the EW at the end of the BW or not. Therefore, from the manufacturer's perspective, a practical strategy is to offer an optimal PM program within only the BW period, i.e., K_B, L_B and m_B, to these customers.

At the end of the BW, some customers would still have no need of the EW (i.e., Case 3), then no further PM actions are offered.

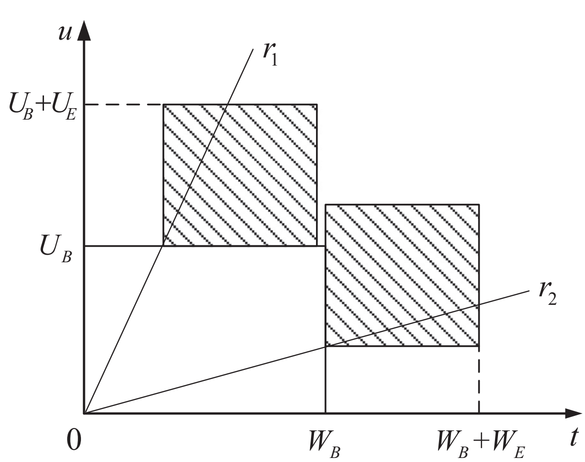

Fig. 2. Warranty regions in Case 2.

However, other customers may decide to buy the EW at this moment (i.e., Case 2). For this proportion of customers, the manufacturer can adopt the following two PM policies:

Policy 1. : The manufacturer can specify a unified PM program over the EW region, i.e., K_E, L_E and m_E, for all customers. Note that, this optimal PM program is not necessarily identical to that in the BW period.

Policy 2. : Alternatively, when the usage rates of customers who purchase the EW at the end of the BW can be estimated, these customers can be further classified into three usage categories: light, medium and heavy. Thus, the manufacturer can establish customized PM programs over the EW period for the three usage categories, i.e., K_{El}, L_{El} and m_{El} for light customers, K_{Ea}, L_{Ea} and m_{Ea} for medium customers, and K_{Eh}, L_{Eh} and m_{Eh} for heavy customers, respectively.

As can be seen, in Case 2, the manufacturer faces a *two-stage* PM optimization problem. The first stage aims to identify the optimal PM program within only the BW period, while the second stage focuses on determining the optimal PM program during the EW period based on the decision in the first stage. In other words, the PM optimization in Stage 2 depends on the PM program applied in Stage 1.

3. Model analysis

In order to reduce the number of failures within the warranty coverage, a two-dimensional PM program is implemented with the maintenance cost paid by the manufacturer. It is assumed that failures between two successive PM actions are minimally rectified with negligible repair time. Minimal repair can restore the failed item to an operational state and keep its reliability the same as that just before failure. This assumption has been employed in many studies related to product warranties, and it can ensure that the failure process between two successive PM actions follows a NHPP.

For the manufacturer, the total warranty servicing cost includes the PM cost and minimal repair cost. Let C_f and $C_p(m)$ denote the average cost for performing a minimal repair and a PM action with maintenance level m, respectively. $C_p(m)$ increases as m increases. Denote the manufacturer's total expected warranty cost per unit item within the BW period by $E[C_B(K_B, L_B, m_B)]$, where K_B and L_B are the age and usage intervals of PM actions, respectively, and m_B is the PM level applied within the BW period. Furthermore, the expected warranty cost per unit item within the EW period is denoted by $E[C_E(K_E, L_E, m_E \mid K_B, L_B, m_B)]$, where K_E and L_E are the age and usage intervals of PM actions, respectively, and m_E is the PM level applied within the EW period.

3.1. Stage 1

For an item with average usage rate r, its age at the BW expiry is given by

$$W_B^r = \min\{W_B, U_B/r\} = \begin{cases} W_B, \text{ if } r \le U_B/W_B \\ U_B/r, \text{ if } r > U_B/W_B \end{cases} \tag{3}$$

Similarly, under the two-dimensional PM strategy, the age-based PM interval during the BW period is given by $K_B^r = \min\{K_B, L_B/r\}$. Then, the number of PM actions performed over the BW period can be obtained as

$$n_B^r = \max\{j \mid jK_B^r < W_B^r, j \ge 0\} \tag{4}$$

In other words, for an item with average usage rate r, n_B^r PM

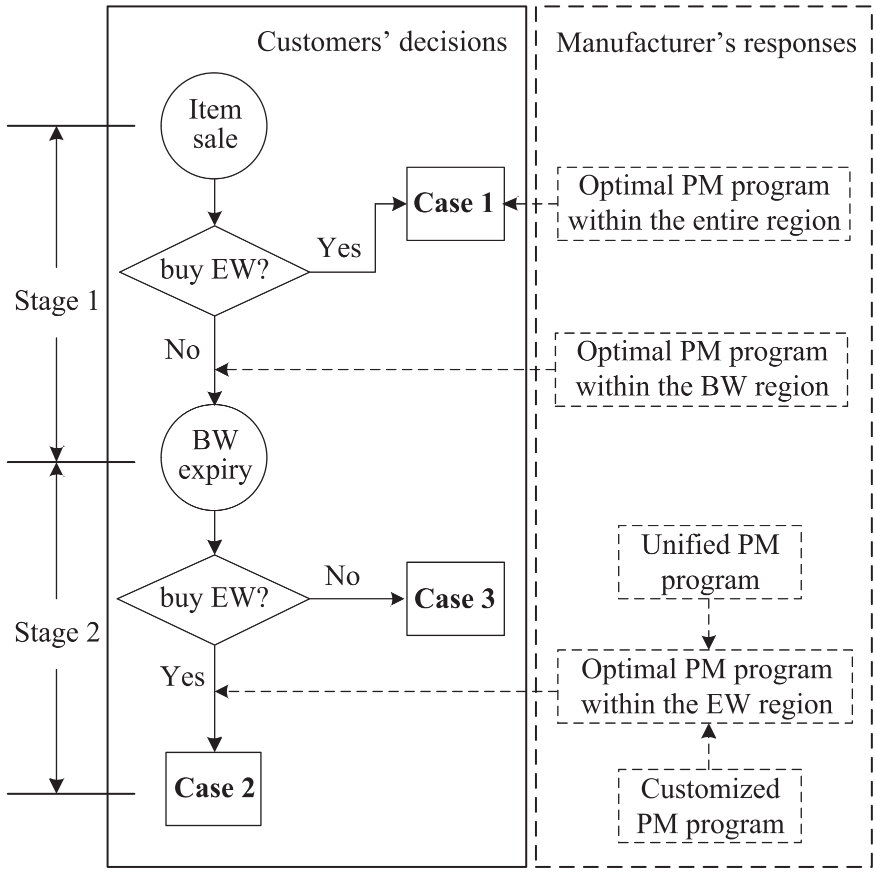

Fig. 3. The framework of PM optimization based on the moments of customers buying the EW.

actions are performed within the BW period, and after the n_B^r th PM action, the subsequent interval would end when the BW terminates, since the manufacturer would not perform an additional PM action. The same rule also applies to the case within the EW period, which will be analyzed in Section 3.2.

Assuming that the level of PM effort m_B is kept unchanged throughout the BW period, then based on Eq. (2), the virtual age immediately after the jth PM actions can be rewritten as $v_{B,j}^r = v_{B,j-1}^r + \delta(m_B)\left(\tau_{B,j}^r - \tau_{B,j-1}^r\right) = v_{B,j-1}^r + \delta(m_B)K_B^r$, which can be recursively derived as $v_{B,j}^r = j\delta(m_B)K_B^r$, $j = 1, \ldots, n_B^r$. Thus, for an item with average usage rate r, the expected number of failures over the BW period is given by

$$E\left[N_B^r(K_B, L_B, m_B)\right] = \sum_{j=0}^{n_B^r-1} \int_{j\delta(m_B)K_B^r}^{j\delta(m_B)K_B^r+K_B^r} \lambda(t|r)\mathrm{d}t + \int_{n_B^r\delta(m_B)K_B^r}^{n_B^r\delta(m_B)K_B^r+W_B^r-n_B^rK_B^r} \lambda(t|r)\mathrm{d}t \tag{5}$$

Combining the PM and minimal repair costs, we can obtain the total expected warranty cost over the BW period for an item with average usage rate r:

$$E\left[C_B^r(K_B, L_B, m_B)\right] = C_f E\left[N_B^r(K_B, L_B, m_B)\right] + n_B^r C_p(m_B) \tag{6}$$

Consequently, considering all usage rate scenarios, the total expected servicing cost within the BW period can be calculated by

$$E\left[C_B(K_B, L_B, m_B)\right] = \int_{r_{\min}}^{r_{\max}} E\left[C_B^r(K_B, L_B, m_B)\right]\mathrm{d}G(r) \tag{7}$$

In Stage 1, the manufacturer's optimization problem is to determine the optimal PM decisions over the BW period, i.e., K_B^*, L_B^* and m_B^*, so as to minimize Eq. (7). It can be described as

$$\begin{aligned} &\min\ E\left[C_B(K_B, L_B, m_B)\right] \\ &\quad s.t.\ K_B, L_B > 0 \\ &\quad 0 \le m_B \le M \end{aligned} \tag{8}$$

Remark 4. As discussed earlier, in Case 1 the entire warranty region can be treated as a larger BW region given by $[0, W_B+W_E] \times [0, U_B+U_E]$. Therefore, in this case the optimization of two-dimensional PM strategy follows directly from Stage 1 with W_B+W_E replacing W_B, and U_B+U_E replacing U_B.

3.2. Stage 2

At the end of the BW, some customers may decide to buy the EW for further protection. For these customers, the manufacturer will provide them with a new two-dimensional PM program free of charge along with the EW contract. In this study, two kinds of policies are considered: (1) offering a unified PM program to all customers who buy the EW, and (2) offering customized PM programs to different customers in accordance with their usage categories. In Section 3.2.1, the first policy is discussed, followed by the second policy in Section 3.2.2.

3.2.1. Policy 1: Unified PM program

Under this policy, the manufacturer chooses to offer a unified PM program along with the EW contract to all customers who buy the EW, irrespective of their usage rates.

For an item with average usage rate r, its actual age at the EW expiry is

$$W^r = \min\left\{W_B^r + W_E,\ W_B^r + U_E/r\right\} \tag{9}$$

where W_B^r is given by Eq. (3). The actual length of the EW period can thus be obtained as $W_E^r = W^r - W_B^r = \min\{W_E, U_E/r\}$.

According to the two-dimensional PM strategy, the age-based PM interval during the EW period is $K_E^r = \min\{K_E, L_E/r\}$. In this situation, the expected number of PM actions conducted within the EW period can be obtained as

$$n_E^r = \max\left\{j \middle| jK_E^r < W_E^r, j \ge 0\right\} \tag{10}$$

Assuming that the level of PM effort m_E is kept unchanged throughout the EW period, then for an item with average usage rate r, its virtual age right after the jth, $j = 1, 2, \ldots, n_E^r$, PM actions during the EW period is given by

$$\begin{aligned} v_{E,j}^r &= v_{E,j-1}^r + \delta(m_E)K_E^r \\ &= \ldots \\ &= v_{E,0}^r + j\delta(m_E)K_E^r \end{aligned} \tag{11}$$

In Eq. (11), $v_{E,0}^r$ is the item's virtual age at the beginning of the EW period, or equivalently at the end of the BW period, which is

$$v_{E,0}^r = W_B^r - \left(1 - \delta(m_B^*)\right)n_B^{r*}K_B^{r*} \tag{12}$$

where $K_B^{r*} = \min\{K_B^*, L_B^*/r\}$ and $n_B^{r*} = \max\left\{j \middle| jK_B^{r*} < W_B^r, j \ge 0\right\}$.

Then, for an item with average usage rate r, the expected number of failures over the EW period is given by

$$\begin{aligned} E\left[N_E^r\left(K_E, L_E, m_E|K_B^*, L_B^*, m_B^*\right)\right] = &\sum_{j=0}^{n_E^r-1} \int_{j\delta(m_E)K_E^r}^{j\delta(m_E)K_E^r+K_E^r} \lambda\Big(W_B^r \\ &- \left(1 - \delta(m_B^*)\right)n_B^{r*}K_B^{r*} + t|r\Big)\mathrm{d}t \\ &+ \int_{n_E^r\delta(m_E)K_E^r}^{n_E^r\delta(m_E)K_E^r+W_E^r-n_E^rK_E^r} \lambda\Big(W_B^r \\ &- \left(1 - \delta(m_B^*)\right)n_B^{r*}K_B^{r*} + t|r\Big)\mathrm{d}t \end{aligned} \tag{13}$$

For an item with average usage rate r, the total expected warranty cost over the EW period equals the PM cost plus the minimal repair cost, which is

$$\begin{aligned} &E\left[C_E^r\left(K_E, L_E, m_E|K_B^*, L_B^*, m_B^*\right)\right] \\ &\quad = C_f E\left[N_E^r\left(K_E, L_E, m_E|K_B^*, L_B^*, m_B^*\right)\right] + n_E^r C_p(m_E) \end{aligned} \tag{14}$$

Consequently, considering all the usage rate scenarios, the total expected servicing cost over the EW period is expressed as

$$\begin{aligned} &E\left[C_E\left(K_E, L_E, m_E|K_B^*, L_B^*, m_B^*\right)\right] \\ &\quad = \int_{r_{\min}}^{r_{\max}} E\left[C_E^r\left(K_E, L_E, m_E|K_B^*, L_B^*, m_B^*\right)\right]\mathrm{d}G(r) \end{aligned} \tag{15}$$

Remark 5. It's worth mentioning that for the sake of simplicity, this study assumes that the distribution of usage rates of those customers who buy the EW at the BW expiry is also $G(r)$, i.e., it is the same as that for all customers who bought the item. However, if this assumption is not valid, then $G(r)$ in Eq. (15) can be easily replaced by, say $G_1(r)$, which describes the new distribution of usage rates across the sub-population of customers who purchase the EW at the end of the BW [27].

In Stage 2, the optimization problem is to identify the optimal PM decisions over the EW period, i.e., K_E^*, L_E^* and m_E^*, so as to minimize Eq. (15), given the optimal decision in the first stage. It is described as follows.

$$\begin{aligned} &\min\ E\left[C_E\left(K_E, L_E, m_E|K_B^*, L_B^*, m_B^*\right)\right] \\ &\ s.\,t.\ K_E, L_E > 0 \\ &\quad\ 0 \le m_E \le M \end{aligned} \tag{16}$$

After the two-stage PM optimization, we can obtain the minimal expected warranty servicing cost over the entire warranty period, which is

$$E\left[C_t\right] = E\left[C_B\left(K_B^*, L_B^*, m_B^*\right)\right] + E\left[C_E\left(K_E^*, L_E^*, m_E^*|K_B^*, L_B^*, m_B^*\right)\right] \tag{17}$$

where $E\left[C_B\left(K_B^*, L_B^*, m_B^*\right)\right]$ is the minimal expected warranty cost over the BW period resulting from the first stage, and $E\left[C_E\left(K_E^*, L_E^*, m_E^*|K_B^*, L_B^*, m_B^*\right)\right]$ is the minimal expected warranty cost over the EW period resulting from the second stage.

3.2.2. Policy 2: Customized PM programs

As mentioned earlier, different usage intensities may result in different rates of item deterioration, thus it is wiser for the manufacturer to offer customized PM strategies to different customer categories. This argument has been supported by the studies of Wang et al. [21] and Wang and Su [28]. In this study, customers are categorized based on their usage rates. According to Murthy et al. [41], customers can be broadly divided into three groups, i.e., light, medium and heavy. This classification framework has been adopted by many subsequent studies, such as [21,28,30,31,33].

We assume that for those customers who buy the EW just before the BW ceases, the manufacturer can obtain their individual usage rates during the BW period; see the assumption (5) and Remark 2. Besides, based on the assumption (2), it is reasonable to further assume that for a specific customer, his/her usage pattern remains unchanged throughout the BW and EW periods. Therefore, obtaining the customers' usage rates during the BW period can help the manufacturer to understand their usage patterns during the EW period more accurately.

Now, we are ready to classify customers based on their usage rates throughout the BW period. One possible classification framework is that the last 25% of customers are considered as light users, the top 25% as heavy ones, and the middle 50% as medium ones. An alternative framework is that the last 1/3 of customers are considered as light ones, the top 1/3 as heavy ones, and the middle 1/3 as medium ones. Here, the former classification framework is applied. Denote the first and third quartiles of all usage rates by r_1 and r_2, respectively, i.e., $G(r_1) = 0.25$ and $G(r_2) = 0.75$. The three categories of customers are summarized as follows.

Light customers: customers with $r_{\min} \le r < r_1$, which are among the last 25%.
Medium customers: customers with $r_1 \le r \le r_2$, which are among the middle 50%.
Heavy customers: customers with $r_2 < r \le r_{\max}$, which are among the top 25%.

For example, when the usage rates obey uniform distribution, Fig. 4 illustrates the scheme of customer classification. Categorizing customers based on other distributions, such as Gamma, Lognormal, Weibull, etc., can be proceeded in a similar way.

On the basis of customer classification, customized PM programs for the three categories of customers can be established following the similar procedures in Section 3.2.1. To this end, Eq. (15) should be modified to

$$\begin{aligned} &E\left[C_{El}\left(K_{El}, L_{El}, m_{El}|K_B^*, L_B^*, m_B^*\right)\right] \\ &\quad = \int_{r_{\min}}^{r_1} E\left[C_{El}^r\left(K_{El}, L_{El}, m_{El}|K_B^*, L_B^*, m_B^*\right)\right]\mathrm{d}G(r) \end{aligned} \tag{18}$$

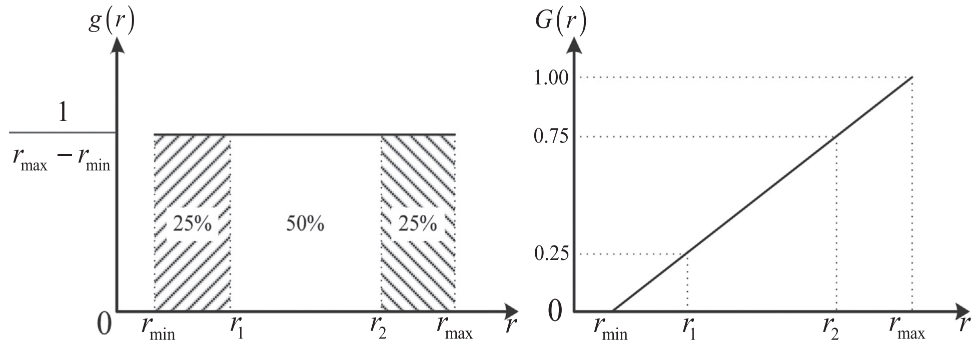

Fig. 4. Classifying customers when their usage rates obey uniform distribution.

Table 1
Usage rates for the three usage categories.

Usage category	Usage rates (10^4 km/year)
Light	$[r_{min}, r_1) = [0.5, 1.25)$
Medium	$[r_1, r_2] = [1.25, 2.75]$
Heavy	$(r_2, r_{max}] = (2.75, 3.5]$

Table 2
Numerical results when $W_E = 3$ years and $U_E = 3 \times 10^4$ km (K^* in months, L^* in 10^3 km).

	Case 1	Case 2-Unified PM		Case 2-Customized PM			
		BW	EW	BW	EW		
					Low	Medium	Heavy
K^*	11	8	8	8	10	10	7
L^*	15	10	10	10	8	10	15
m^*	4	3	3	3	3	3	3
		654.3	1208.1	654.3	427.2	563.3	180.7
Total cost ($)	1577.7	1862.4		1825.6			

Table 3
Numerical results when $W_E = 3$ years and $U_E = 6 \times 10^4$ km (K^* in months, L^* in 10^3 km).

	Case 1	Case 2-Unified PM		Case 2-Customized PM			
		BW	EW	BW	EW		
					Low	Medium	Heavy
K^*	11	8	10	8	10	10	7
L^*	15	10	15	10	10	15	15
m^*	4	3	4	3	3	4	3
		654.3	2008.9	654.3	449.5	1105.2	409.8
Total cost ($)	2227.4	2663.2		2618.8			

$$E\left[C_{Ea}\left(K_{Ea}, L_{Ea}, m_{Ea}|K_B^*, L_B^*, m_B^*\right)\right] = \int_{r_1}^{r_2} E\left[C_{Ea}^r\left(K_{Ea}, L_{Ea}, m_{Ea}|K_B^*, L_B^*, m_B^*\right)\right]\mathrm{d}G(r) \tag{19}$$

$$E\left[C_{Eh}\left(K_{Eh}, L_{Eh}, m_{Eh}|K_B^*, L_B^*, m_B^*\right)\right] = \int_{r_2}^{r_{max}} E\left[C_{Eh}^r\left(K_{Eh}, L_{Eh}, m_{Eh}|K_B^*, L_B^*, m_B^*\right)\right]\mathrm{d}G(r) \tag{20}$$

where the subscripts l, a, and h represent the case of light, medium and heavy types of customers, respectively.

Table 4
Numerical results when $W_E = 6$ years and $U_E = 3 \times 10^4$ km (K^* in months, L^* in 10^3 km).

	Case 1	Case 2-Unified PM		Case 2-Customized PM			
		BW	EW	BW	EW		
					Low	Medium	Heavy
K^*	11	8	8	8	11	10	7
L^*	15	10	10	10	8	10	15
m^*	4	3	3	3	3	3	3
		654.3	1335.2	654.3	554.0	563.3	180.7
Total cost ($)	1724.4	1989.5		1952.3			

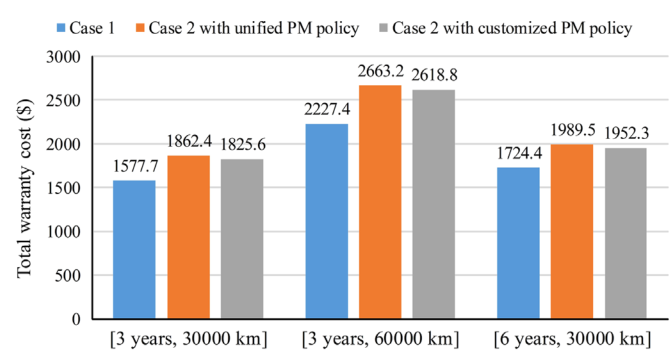

Fig. 5. Warranty cost comparisons.

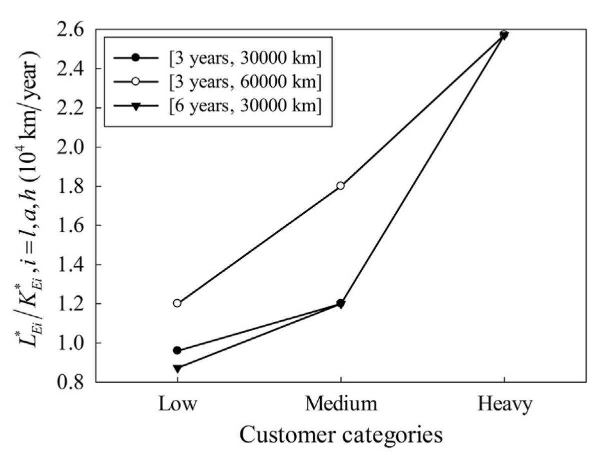

Fig. 6. Comparison of L_{Ei}^*/K_{Ei}^* under different customer categories.

Table 5
Comparison of single- and two-stage PM problem under unified PM policy (K^* in months, L^* in 10^3 km).

	[3 years, 3×10^4 km]				[3 years, 6×10^4 km]				[6 years, 3×10^4 km]			
	Single-stage		Two-stage		Single-stage		Two-stage		Single-stage		Two-stage	
	BW	EW	BW	EW	BW	EW	BW	EW	BW	EW	BW	EW
K^*	12	8	8	8	12	10	8	10	12	8	8	8
L^*	14	10	10	10	14	15	10	15	14	10	10	10
m^*	4	3	3	3	4	4	3	4	4	3	3	3
	743.0	867.9	654.3	1208.1	743.0	1477.6	654.3	2008.9	743.0	963.7	654.3	1335.2
Total cost ($)	1610.9		1862.4		2220.6		2663.2		1706.7		1989.5	

Table 6
Warranty servicing costs versus various minimal repair costs.

C_f ($)	[3 years, 3×10^4 km]			[3 years, 6×10^4 km]			[6 years, 3×10^4 km]		
	Case 1	Case 2-U	Case 2-C	Case 1	Case 2-U	Case 2-C	Case 1	Case 2-U	Case 2-C
100	801.8	1000.2	988.2	1152.8	1436.0	1428.5	885.0	1080.0	1069.9
200	1337.9	1647.6	1617.9	1887.7	2321.8	2292.2	1461.2	1770.7	1745.1
300	1807.1	2174.1	2133.6	2553.6	3105.4	3052.5	1976.5	2323.3	2278.2
400	2241.6	2797.5	2738.4	3171.3	3966.9	3901.1	2446.3	2977.8	2919.2
500	2648.5	3135.6	3055.1	3762.4	4446.6	4354.1	2892.7	3314.2	3244.6

Case 2-U: Case 2-Unified PM policy; Case 2-C: Case 2-Customized PM policy.

Note that, for the customized policy, the task in Stage 2 is to identify the optimal PM parameters over the EW period for the three customer categories, respectively, so as to minimize Eqs. (18)–(20), respectively, given the optimal decision in the first stage. Consequently, the total warranty servicing costs for all customers within the EW period are the sum of those for the light, medium and heavy usage categories, which is $\sum_{i=l,a,h} E\left[C_{Ei}\left(K_{Ei}^*, L_{Ei}^*, m_{Ei}^* | K_B^*, L_B^*, m_B^*\right)\right]$.

Combining the expected warranty servicing costs within both BW and EW periods, we can obtain the overall minimal expected warranty cost over the entire warranty period:

$$E\left[C_2\right] = E\left[C_B\left(K_B^*, L_B^*, m_B^*\right)\right] + \sum_{i=l,a,h} E\left[C_{Ei}\left(K_{Ei}^*, L_{Ei}^*, m_{Ei}^* | K_B^*, L_B^*, m_B^*\right)\right] \tag{21}$$

By comparing the warranty servicing cost in (21) with that in (17), the advantage of customized PM policy can be demonstrated.

4. Numerical examples

In this section, numerical examples are provided to demonstrate the applicability of the proposed two-stage PM optimization model. The item under study is a repairable automobile component sold with a two-dimensional warranty. The BW region is assumed to be rectangular with $W_B = 3$ years and $U_B = 3\times10^4$ km. In this study, three EW regions are considered, i.e., [3 years, 3×10^4 km], [3 years, 6×10^4 km], and [6 years, 3×10^4 km].

The initial conditional intensity function is $\lambda(t|r) = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)t$, with the parameter values given by $\theta_0 = 0.1$, $\theta_1 = 0.2$, $\theta_2 = 0.7$, $\theta_3 = 0.7$. The usage rate R is assumed to be uniformly distributed, with the density function given by $g(r) = 1/[r_{\max} - r_{\min}]$, for $r_{\min} \le r \le r_{\max}$. Here, we consider that $r_{\min} = 0.5\times10^4$ km/year and $r_{\max} = 3.5\times10^4$ km/year. Following Fig. 4, the first and third quartiles can be obtained as $r_1 = 1.25\times10^4$ km/year and $r_2 = 2.75\times10^4$ km/year, respectively. The customers are then classified as light, medium and heavy ones; see Table 1.

Moreover, we consider $\delta(m) = (1+m)e^{-m}$ for $m = 0, 1, \ldots, 5$, and the corresponding PM cost is $C_p(0) = \$0$, $C_p(1) = \$10$, $C_p(2) = \$30$, $C_p(3) = \$60$, $C_p(4) = \$100$, and $C_p(5) = \$160$, which is consistent with that of Kim et al. [8]. Assume that the average cost of a minimal repair is $C_f = \$250$.

Note that, the age and usage intervals of PM strategies are usually discrete in practice (e.g., $K = 1, 2, \ldots$ months; $L = 1, 2, \ldots \times 1000$ km), but most previous studies fail to recognize this fact by making an oversimplified assumption of continuity. In this work, the best solution is searched by enumerating possible candidates of K and L over a wide range. For the sake of efficient maintenance management, the search steps of K and L are 0.0833 year (i.e., one month) and 10^3 km, respectively.

The optimal PM parameters and corresponding warranty costs for the three EW regions—[3 years, 3×10^4 km], [3 years, 6×10^4 km], and [6 years, 3×10^4 km] are summarized in Tables 2–4, respectively. Here we take Table 2 as an example. In Case 1, within the entire warranty region, the item should be preventively maintained with level $m^* = 4$, every 11 months or 1.5×10^4 km, whichever occurs first. Therefore, a minimal warranty cost per unit item of $1577.7 is obtained. On the other hand, in Case 2, the manufacturer faces a two-stage PM optimization problem. The outcome of Stage 1 produces the optimal PM program within the BW period, i.e., $(K_B^*, L_B^*, m_B^*) = (8, 10, 3)$, and the corresponding warranty cost per unit item in this stage is $654.3. At Stage 2, the optimal unified PM program is $(K_B^*, L_B^*, m_B^*) = (8, 10, 3)$, and the corresponding warranty cost is $1208.1; while the optimal customized PM programs are (10, 8, 3), (10, 10, 3) and (7, 15, 3) for light, medium and heavy customers, respectively, and the corresponding warranty cost is $1171.2 (= 427.2+563.3+180.7).

From Tables 2–4, the following findings can be summarized:

(1) It can be observed that the warranty servicing cost in Case 2 is much higher than that in Case 1 (see Fig. 5). This can be interpreted by the fact that under the two-stage PM optimization problem, each stage focuses only on minimizing the warranty cost in the current stage, instead of considering the two stages as a whole.

(2) In Case 2, Tables 2–4 show that during the EW period, the customized PM policy leads to a lower warranty cost than the unified PM policy (see Fig. 5). Thus, the manufacturer should attempt to offer customized PM programs to different customer categories. This demonstrates the advantage of the customized PM policy.
(3) With regard to the customized PM policy over the EW period, the optimal PM parameters for different customer categories are quite different. It is interesting to observe that the ratio of PM usage interval to age interval, i.e., L_{Ei}^{*}/K_{Ei}^{*}, for $i = l, a, h$, increases as the usage intensity increases (see Fig. 6). This finding illustrates that the two-dimensional PM strategy can adjust the ratio of usage interval to age interval according to the customers' usage intensities.
(4) Comparing the results in Tables 2–4, it can be found that the rate of growth in the expected warranty cost is different over changing warranty age/usage limits. More specifically, the rate of growth in the warranty cost is steeper for increasing usage limit, especially when the usage intensity is high. For example, with regard to the unified PM policy, the EW region of $W_E =$ 3 years, $U_E = 3 \times 10^4$ km yields a servicing cost of $1208.1; if we extend W_E from 3 years to 6 years, and keep U_E unchanged, the warranty cost grows by 10.5%, to $1335.2; however, if we extend U_E from 3 years to 6 years, and keep W_E unchanged, the warranty cost grows by 66.3%, to $2008.9. This is because in our numerical example, the customers' usage rates are relatively high when compared with the ratio of warranty usage limit to age limit.

More importantly, in Case 2, the impact of improperly treating the two-stage PM optimization problem as a single-stage one is illustrated in Table 5. As can be seen, for the unified PM policy, warranty costs under the two-stage treatment are much higher than those under the single-stage treatment. This is because under the two-stage treatment, the PM applied within the BW period is insufficient (i.e., PM level is low), which cannot effectively reduce the item degradation. This in turn yields a much higher warranty cost within the EW period. In other words, the two-stage treatment is somewhat myopic. Similar observations can also be found for the customized PM policy, which are not presented here due to space consideration. Nevertheless, the authors emphasize that in practice the single-stage treatment may be inappropriate for Case 2.

Finally, the warranty servicing costs under various values of C_f are summarized in Table 6 in order to show the robustness of our findings. As expected, the aforementioned findings (1), (2), and (4) hold well in all the cases. This demonstrates the reproducibility of our findings.

5. Concluding remarks

In this paper, a two-dimensional PM strategy is investigated, under which the implementation of imperfect PM actions is dependent on both item age and usage. Based on this strategy, a framework of PM optimization for repairable items under two-dimensional warranty is developed, by considering the moments of customers purchasing the two-dimensional EW. The two-stage nature of the PM optimization problem, when the EW is purchased just before the BW terminates, is highlighted. The first stage is to identify the optimal PM program within only the BW period, while the second stage focuses on determining the optimal PM program within the EW period based on the outcome of the first stage. Moreover, in the second stage of the problem, this study proposes to classify customers based on their usage rates throughout the BW period, and then to provide customized PM programs during the EW period so as to reduce the warranty cost.

Numerical examples show that offering customized PM programs can reduce warranty cost for the manufacturer, while a larger saving in warranty cost comes from encouraging customers to buy the EW at the time of item sale. In view of this finding, two managerial suggestions can be obtained: (1) manufacturers should take some promotional actions, e.g., price discount, to encourage customers to purchase the EW when they acquire the items; and (2) for those customers who still purchase the EW at the end of the BW, the manufacturer should try to offer customized PM programs to them. Finally, it is noteworthy that improperly treating the two-stage PM optimization problem as a one-stage problem may be inappropriate in practice.

The customized PM programs in this paper differ from each other in terms of the PM level and the ratio of usage interval to age interval. Future research may consider more customized policies, e.g., introducing various pro-rata PM schemes to different customer categories. In addition, considering an upgrade action at the beginning of the EW as well as a two-dimensional PM program during the EW period as alternatives to improve the item reliability is also of interest.

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