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Design and pricing of usage-driven customized two-dimensional extended warranty menus

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ABSTRACT
In this article, we study the design and pricing of customized two-dimensional extended warranty menus from the provider's perspective, where each consumer is presented with a tailored warranty menu based on her usage rate. We propose to design such usage-driven customized menus in two steps. First, for a fixed usage rate, we design the corresponding menu by seeking optimal prices for given warranty limits. Then, the menu for any other usage rate is designed by fixing the optimal prices and determining the associated warranty limits. We prove that under the multinomial logit choice framework, if the product failure probabilities for respective options in two customized menus are the same, then the menus will have identical expected profits and attach rates. A case study on commercial vehicles is presented to demonstrate the customization methodology. We find that customized menus can generate a higher profit and attach rate than a uniform menu without customization, indicating the benefit of customization. In addition, we extend the original model to incorporate ancillary preventive maintenance programs. Our analysis reveals that the optimal preventive maintenance policy for each option is independent of those for other options; moreover, bundling preventive maintenance programs can increase profit and attach rate.

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Customization; heterogeneous usage rates; multinomial logit choice; preventive maintenance; two-dimensional extended warranties

1. Introduction

In recent years, the increasing prominence of e-commerce and online platforms, as well as the rapid advance in monitoring and tracking technologies, offers firms an unprecedented capability to understand and predict consumer features or preferences on a highly detailed level. Benefiting by this, the practice of *customization* or *personalization*—that is, tailoring a product/service offering to each consumer segment or even each individual consumer according to observed or predicted information on this segment or individual—has become prevailing in, for example, airline, e-commerce, and online advertising industries (Klein *et al.*, 2020). In this article, we apply the customization concept to a new problem concerning product after-sales service contracts: specifically, warranty menu design and pricing.

A warranty is essentially a type of after-sales service contract that protects consumers from premature product failures, by specifying the terms and situations in which specific compensations (e.g., free repair, replacement, or refund) will be provided if the product fails to function as described or intended (Murthy and Djamaludin, 2002; Wang *et al.*, 2022b). In this work, we focus specifically on *two-dimensional warranties* that are characterized by a region in a two-dimensional plane, typically with one axis representing time or age and the other one usage (Xie *et al.*, 2017; Peng *et al.*, 2021). In practice, many capital-intensive products, such as vehicles and heavy machinary, are sold with two-dimensional warranties. For instance, each new vehicle sold by Ford Motor Company is covered by a 3-year/36,000-mile warranty, on a whichever-occurs-first basis. The prime motivation of offering two-dimensional warranties is the heterogeneity in consumer usage (Wang and Xie, 2018). By setting an appropriate usage limit in addition to an age limit, a two-dimensional warranty can help manufacturers hedge against excessive failures from heavy-usage consumers.

The Ford's warranty policy mentioned above is essentially a manufacturer's base warranty that is an integrated element of product sales, whose cost has been factored into the product price. Recent years witness a growing popularity of *extended warranties* that entitle consumers to enjoy an additional protection after expiration of the base warranty (Murthy and Jack, 2014). Although an extended warranty has to be bought with an extra premium, it provides consumers with "peace of mind" (Huysentruyt and Read, 2010). This benefit is particularly important for capital-intenive products whose repairs are highly expensive. Currently, more and more manufacturers, retailers, or even third-party insurers (hereafter referred to as providers) are selling extended warranties, largely due to its high profitability. For example, in the automobile industry, approximately 40% of vehicle transactions include the purchase of extended warranties; moreover, the profit margins on extended warranties are over 20% and dwarf those on the vehicles alone (Lee and Venkataraman, 2022).

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The main focus of this work is on two-dimensional extended warranties, which are predominately provided for capital-intensive products. In the current aftermarket, a common practice adopted by most providers is to offer *one-size-fits-all* two-dimensional extended warranties (Wang and Xie, 2018). That is, all consumers are presented with the same contract (with a fixed coverage and price), and they can only decide on whether to buy it or not. However, it is evident that consumers are highly heterogeneous in, for example, usage pattern, income, and risk attitude (Padmanabhan, 1995; Huysentruyt and Read, 2010; Abito and Salant, 2019). For example, the owners of two cars might drive, on average, different distances per day or have distinct degrees of tolerance for car breakdowns. In these situations, one size cannot fit all, and offering a warranty menu can largely satisfy diverse consumer desires. A real-world example is the flexible (base) warranty menu offered by the Kuhn group for its building machines, which includes options from 1500 hours/12 months to 6000 hours/48 months (Kuhn, 2021).

Although the menu offered by Kuhn can satisfy consumers' desire for different warranty lengths, it is incapable of coping with the heterogeneity in consumer usage. More specifically, such heterogeneity makes the warranty options only favorable for consumers whose usage rates are around 1500 hours/year, yet economically unattractive to lower- or higher-usage consumers. This results in selection bias towards moderate-usage consumers, which might affect the warranty attach rate (i.e., the proportion of product buyers who also purchase an extended warranty) and the provider's profitability. A promising way of addressing this issue is through usage-driven customization—that is, offering each consumer a tailored warranty menu according to her specific usage rate. Successful customization in this context hinges on the provider's capability of accessing each consumer's usage information. In practice, it is not difficult to obtain this information for capital-intensive products through, for example, usage or maintenance records during the base warranty period (Su and Wang, 2016; Wang and Ye, 2021). However, optimal design of such customized two-dimensional extended warranty menus remains an open problem.

A complicating factor in the warranty-menu design and pricing problem is the presence of *substitution* behavior; that is, different options serve as substitutes so that the demand of a specific option depends not only on its own features, but also on those of the other options (Wang *et al.*, 2020). In the literature, numerous consumer choice models have been proposed to characterize how consumers make purchase decisions in this situation (see Strauss *et al.* (2018), for an overview). In this research, the *MultiNomial Logit* (MNL) choice model in McFadden (1974)—derived from the random utility maximization theory—is adopted to describe consumer choices among the warranty options available in each customized menu.

In this article, we aim to extend the extended-warranty-menu design and pricing research in Wang *et al.* (2020) from a one-dimensional context to a two-dimensional one with customization. The objective is to design a two-dimensional extended warranty menu for each consumer based on her specific usage rate to attain a degree of fairness. From the provider's perspective, the problem is to determine an age limit, usage limit, and price for each warranty option in each customized menu, so as to maximize the expected total profit. We prove that under the MNL model, if we can design two customized menus (for two distinct usage rates) such that the product failure probabilities for their respective options are identical, then the menus will result in the same expected profits. On this basis, we propose to solve this problem in two steps: First, focusing on a fixed usage rate, we design the corresponding warranty menu by determining optimal warranty prices for given age and usage limits. Second, the warranty menu for any other usage rate is then designed by fixing the optimal prices as in the first step and deriving the age and usage limits accordingly. In particular, we find that in the first step the optimal pricing policy is a cost-plus-margin policy with identical margins for all options; in the second step, the derived age and usage limits might not guarantee consumer self-selection. A case study on commercial vehicles is presented to showcase the implementation of the proposed customization methodology.

Moreover, we formulate a uniform two-dimensional extended warranty menu in which all consumers are presented with the same options without customization (similar to the Kuhn's case mentioned previously). Numerical comparison shows that customized menus can generate a higher expected profit and attach rate than its uniform counterpart. Furthermore, we extend the original model by incorporating ancillary Preventive Maintenance (PM) programs. We find that the provider should specify the optimal number of PM services for each option independently of those for other options, despite the substitution characteristic of the options. In particular, the PM optimization problem of interest turns out to be a utility-maximizing problem, rather than cost-minimizing ones as in most of the existing maintenance literature. Our analysis further shows that bundling ancillary PM programs with warranty options can help increase the expected profit and attach rate.

The remainder of this article is structured as follows. Section 2 reviews the relevant literature. Sections 3 and 4 formulate the model elements and then study the design and pricing of customized two-dimensional extended warranty menus, respectively. Section 5 further incorporates ancillary PM programs into the customized menu design. Finally, Section 6 concludes this article and presents some potential topics for future research. Additional numerical results and all technical proofs can be found in the online supplement.

2. Literature review

Extended warranties have received extensive investigations in the literature; interested readers are referred to Murthy and Jack (2014) for a comprehensive overview. The first stream of research related to this work focuses on extended warranties with certain flexibility features. Lam and Lam

(2001) study an extended warranty model that, at the end of a base warranty, offers consumers the option to either buy an extended warranty or repair the product. Jack and Murthy (2007) propose a flexible extended warranty policy that allows consumers to choose its starting point and length of protection. Hartman and Laksana (2009) analyze a number of extended warranty contracts that allow deferrals and renewals. In particular, the design and pricing of a warranty menu are considered as well, in order to address consumer heterogeneity in risk attitude. Gallego *et al.* (2014) study a flexible-duration extended warranty with month-by-month commitments under consumers' reliability learning. They find that the consumers' optimal renewal policy has a threshold structure under some mild conditions. Closest in spirit to our work is the paper by Wang and Ye (2021) that designs customized two-dimensional extended warranty contracts by considering consumer heterogeneities in usage rate and failure history. However, our work differs from theirs in two aspects: First, our aim is to design a customized warranty menu, instead of a single customized contract, for each consumer according to her usage rate. Second, the consumer choice model in our work is derived from the random utility maximization theory, whereas the consumer purchase model in their work is derived from an expected utility-based formulation of heterogeneous external stress factors.

Another stream of related research deals with flexible two-dimensional (base) warranties (see, e.g., Chun and Tang, 1999; Manna *et al.*, 2006; Shahanaghi *et al.*, 2013; Ye and Murthy, 2016). The goal is to derive a series of combinations of age and usage limits that generate the same warranty cost or the same failure probability over the warranty period, without considering consumer choices. When the minimal repair policy is adopted to rectify each failure under warranty, the iso-cost warranty region is equivalent to the iso-PoF (Probability of Failure) region. In particular, Ye and Murthy (2016) divide consumers into n different groups in terms of usage rates, and then design a specific iso-PoF region for each group. Interestingly, they show that when the marginal effect of usage rate on product failures is not significant, the warranty menu can guarantee consumer self-selection; that is, consumers are able to self-select into the right region designated for them. To our knowledge, Shahanaghi *et al.* (2013) make the only effort to design flexible two-dimensional extended warranty contracts, with optimized PM strategies incorporated. Different from this stream of research, the objective of our work is to design profit-maximizing customized two-dimensional extended warranty menus by characterizing consumer choices.

Moreover, studies on flexible PM strategies under two-dimensional warranties also come close to our work. Huang *et al.* (2017) propose to classify consumers into three groups based on their usage rates within the base warranty period (i.e., high, moderate, and low) and to offer a tailored PM strategy during the extended warranty period to each group, so as to reduce the warranty cost. Su and Wang (2016) address the optimization of PM in both two-dimensional base and extended warranty periods, where in the latter period, similar to Huang *et al.* (2017), tailored PM policies are applied to different consumer groups. Wang *et al.* (2017) consider that in the extended warranty period, consumers have the option to either accept or reject PM at each time. Dai *et al.* (2020) study a problem similar to that in Su and Wang (2016), while only consider the base warranty period. In the present work, we incorporate a utility-maximizing PM problem into the framework of customized two-dimensional extended warranty menu design and pricing.

This article also builds upon recent advances in product-line pricing based on MNL and its variants. Li and Huh (2011) study a product-line pricing problem under the nested logit model (and MNL as a special case) and examine concavity of the profit function with respect to market share. Wang (2012) considers an MNL-based assortment and price optimization problem subject to a capacity constraint. In particular, he simplifies this problem to the determination of a unique fixed point of a single-variable function. This simplification technique is also adopted in our work. Wang *et al.* (2022a) develop an integrated framework to study joint optimization of product price, quality, and service duration under the MNL and nested logit models. Moreover, Li *et al.* (2019) investigate a product-line pricing problem with demand given by a discrete Mixed MNL (MMNL) model. To the best of our knowledge, Wang *et al.* (2020) make the first attempt to study MNL-based (one-dimensional) extended-warranty-menu design and pricing. Our work extends the research in Wang *et al.* (2020) to a two-dimensional context. The customization concept is applied to deal with the heterogeneity in consumer usage rate. Specifically, each consumer is presented with a tailored extended warranty menu based on her usage rate, while the menus for consumers with distinct usage rates are different. A final point to note is that the proposed model differs from the *nested logit* model (see, e.g., Li and Huh, 2011; Wang *et al.*, 2022a) in which a consumer first chooses a group of products among several possible groups (called "nests") and then limits her subsequent selection within the chosen group. Instead, the first choice step in our work is governed by usage-driven customization.

3. The model

Consider that a new capital-intensive product is sold with a two-dimensional base warranty policy, which is characterized by a rectangular region $\Omega(W_b, L_b) = \{(t, u) : t \in (0, W_b) \text{ and } u \in (0, L_b)\}$, as shown in Figure 1. Under this policy, any eligible failures over the warranty period will be rectified by the manufacturer at no cost to the consumer. The base warranty terminates when either the product age reaches its limit W_b or the cumulative usage exceeds its limit L_b, whichever occurs earlier.

3.1. Consumer heterogeneity and customized menus

The consumer population is heterogeneous in terms of usage rate. An assumption adopted here is that each

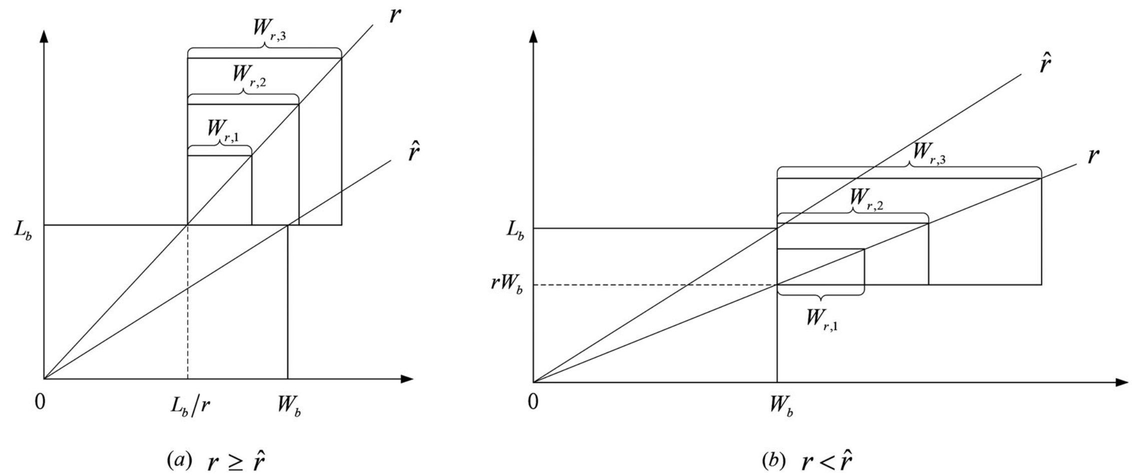

Figure 1. Illustration of usage-driven customized two-dimensional extended warranty menus.

consumer has a constant usage rate during the warranty period (both base and extended), whereas the usage rate varies randomly across the population. Existing literature on warranty data analysis has revealed that the cumulative usage of a car is approximately linear in its age (see, e.g., Lawless *et al.*, 1995; Lawless *et al.*, 2009), so that the assumption of constant usage rate is practically reasonable. In this manner, usage rate R can be modeled as a random variable with Cumulative Distribution Function (CDF) $G(r)$, $r_{\min} \leq r \leq r_{\max}$.

Upon expiration of the base warranty, the average usage rate of an individual consumer over the base warranty period can be accurately estimated, for example, from the odometer installed in a car. Hence, we impose the following assumption.

Assumption 1. The usage rate of each consumer is available to both the consumer herself and the provider upon signing an extended warranty contract.

This assumption is reasonable in real applications, as most providers would perform certain inspections on product health status as well as checks regarding product maintenance history before signing an extended warranty contract. Such inspections and checks can reveal the real usage pattern of each consumer in a fairly accurate fashion.

The provider then offers each consumer a tailored extended warranty menu based on her specific usage rate, from which she can buy any option (or buy nothing) that best fits her protection needs after the base warranty coverage (see Figure 2). Suppose that n options, denoted by $\mathcal{N} = \{1, 2, ..., n\}$, are available in the menu recommended to consumers with usage rate r, and each option has a distinct age limit $W_{r,i}$, usage limit $L_{r,i}$, and warranty price P_i, $i \in \mathcal{N}$. Then, the menu can be represented by $\mathcal{M}_r = \{\boldsymbol{W}_r, \boldsymbol{L}_r, \boldsymbol{P}\}$, where $\boldsymbol{W}_r := \{W_{r,i}\}_{i\in\mathcal{N}}$, $\boldsymbol{L}_r := \{L_{r,i}\}_{i\in\mathcal{N}}$, and $\boldsymbol{P} := \{P_i\}_{i\in\mathcal{N}}$. For convenience, the options are ordered such that $W_{r,1} < W_{r,2} < \cdots < W_{r,n}$. In essence, menu $\mathcal{M}_r$ is *customized* to the consumers' usage rate r (the usage limits are specified as $\boldsymbol{L}_r = r\boldsymbol{W}_r$), so that the actual age limit the consumers will enjoy is exactly equal to $W_{r,i}$, $\forall i \in \mathcal{N}$ (see Figure 1). Moreover, it is worth pointing out that the warranty price of each option i in $\mathcal{M}_r$ is independent of r. In other words,

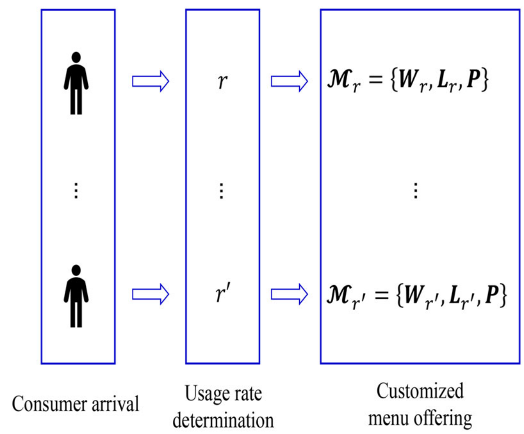

Figure 2. Illustration of usage-driven customization.

the price P_i of option i is the same across all $\mathcal{M}_r$'s ($r \in [r_{\min}, r_{\max}]$), and only warranty limits $W_{r,i}$ and $L_{r,i}$ vary with usage rate r.

3.2. Warranty cost model

The product is designed for operating under some nominal usage rate r_0, conditional on which the distribution for the time-to-first-failure, T_0, is $F_0(t)$ and the associated failure rate is $\lambda_0(t)$. When a consumer's actual usage rate r differs from the nominal value, the product failure time would be affected. In this work, we adopt the accelerated failure time model (Lawless *et al.*, 1995; Jack *et al.*, 2009) to describe the effect of usage rate on the time to first failure. Let T_r represent the time to first failure under actual usage rate r. Conditional on $R = r$, T_r is linked to r through $T_r/T_0 = (r_0/r)^\gamma$, where $\gamma > 0$ is the acceleration factor. It implies that a heavy usage increases the deterioration rate and this, in turn, shortens the time to failure. This way, the CDF $F(t|r)$ and failure rate $\lambda(t|r)$ associated with T_r can be easily linked to $F_0(t)$ and $\lambda_0(t)$, respectively, through $F(t|r) = F_0(t(r/r_0)^\gamma)$ and $\lambda(t|r) = (r/r_0)^\gamma \lambda_0(t(r/r_0)^\gamma)$.

We confine product rectification to minimal repair with negligible duration, as is commonly done in the warranty literature (see, e.g., Shafiee *et al.*, 2013; Huang *et al.*, 2017; Wang *et al.*, 2020). Under this assumption, product failure intensity after repair is nearly the same as that immediately prior to the failure. As a result, product failures will occur according to a Non-Homogeneous Poisson Process (NHPP) with an intensity identical to the failure rate. The expected warranty servicing cost of option i in $\mathcal{M}_r$ is thus given by

$$\begin{aligned} \mathcal{C}(W_{r,i};r) &= c_f \int_{W_b(r)}^{W_b(r)+W_{r,i}} \lambda(t|r)\mathrm{d}t \\ &= c_f\Big[\Lambda(W_b(r)+W_{r,i}|r) - \Lambda(W_b(r)|r)\Big], \end{aligned} \tag{1}$$

where $\Lambda(t|r) = \int_0^t \lambda(x|r)\mathrm{d}x = \Lambda_0(t(r/r_0)^\gamma)$ is the conditional cumulative failure intensity, $W_b(r) = \min\{W_b, L_b/r\}$ is the actual expiry time of the base warranty, and c_f is the average repair cost borne by the provider.

Assumption 2. The warranty cost $\mathcal{C}(W_{r,i};r)$ is a convex increasing function of $W_{r,i}$.

Assumption 2 implies that $\lambda(t|r)$ is an increasing function of t; that is, we only consider aging products whose failure intensities are growing over time. This is appropriate for most capital-intensive products that are sold with two-dimensional warranties.

3.3. Consumer choice model

As there are multiple options in a warranty menu, it is essential—for warranty-menu design and pricing—to model consumer choice behavior among the available options. For this purpose, we adopt the well known random utility maximization framework, in which consumers are assumed to act as utility maximizers.

When making a purchase decision regarding a specific option, the consumer first perceives the repair expenditure, if that option is not bought, and then compares it with the price to be paid for buying that option. For a consumer with usage rate r, her random utility from buying option $i \in \mathcal{N}$ in $\mathcal{M}_r$ is defined as

$$\mathcal{U}_{r,i} = \mathcal{V}(W_{r,i};r) - P_i + \varepsilon_{r,i}, \tag{2}$$

where $\mathcal{V}(W_{r,i};r)$ represents the average consumer-perceived valuation (measured in money units) of option i in menu $\mathcal{M}_r$, P_i is the associated warranty price, and $\varepsilon_{r,i}$ captures the randomness in the valuation function that is affected by unobservable characteristics.

The consumer-perceived valuation $\mathcal{V}(W_{r,i};r)$—also known as the *reservation price*—represents the maximal price of option i that is acceptable to a consumer with usage rate r. Following Wang *et al.* (2020), for each option $i \in \mathcal{N}$ in $\mathcal{M}_r$, $\mathcal{V}(W_{r,i};r)$ can be modeled as

$$\mathcal{V}(W_{r,i};r) = c_m \cdot \delta(\mathcal{F}_{r,i}), \tag{3}$$

where c_m is the unit out-of-warranty repair cost paid by consumers (generally, c_m is much larger than c_f, and known *a priori* for consumers) and $\mathcal{F}_{r,i}$ is the product failure probability over the protection period of option i in menu $\mathcal{M}_r$, which can be expressed as

$$\begin{aligned} \mathcal{F}_{r,i} &= \frac{F(W_b(r)+W_{r,i}|r) - F(W_b(r)|r)}{1-F(W_b(r)|r)} \\ &= 1 - \exp\left(-\int_{W_b(r)}^{W_b(r)+W_{r,i}} \lambda(t|r)\mathrm{d}t\right). \end{aligned} \tag{4}$$

Notice that $\mathcal{F}_{r,i}$ is increasing in $W_{r,i}$ and, according to Equation (1), $\mathcal{C}(W_{r,i};r) = -c_f \log(1-\mathcal{F}_{r,i})$.

Essentially, for a consumer with usage rate r, the valuation of option i in $\mathcal{M}_r$ resolves only when the warranted item fails within the protection region $[W_b(r), W_b(r)+W_{r,i}] \times [rW_b(r), rW_b(r)+rW_{r,i}]$ and the associated repair expense is borne by the provider (Wang *et al.*, 2020). Specifically, if an item fails in the protection period of option i with probability $\mathcal{F}_{r,i}$, then the consumer valuation would be c_m; otherwise, the consumer valuation would be zero. Therefore, if consumers have complete information on the failure probability $\mathcal{F}_{r,i}$, then the average consumer valuation of option i in $\mathcal{M}_r$ would be $c_m\mathcal{F}_{r,i}$.

In practice, however, the exact information on $\mathcal{F}_{r,i}$ is not easy to obtain from the consumer's perspective, as it is closely related to product reliability information that is mainly available to the provider. Nevertheless, consumers are able to perceive the failure probability based on their usage experiences during the base warranty period or based on public reliability information (made available by, for example, consumer magazines or websites). Existing empirical studies have revealed that consumers tend to distort actual failure probability $\mathcal{F}_{r,i}$ (see, e.g., Huysentruyt and Read, 2010; Abito and Salant, 2019). The functional form $\delta(\cdot)$ in Equation (3) is employed to characterize such probability distortion behavior. We impose the following mild assumption on $\delta(\cdot)$.

Assumption 3. The probability distortion function $\delta(\cdot)$ is a monotonically increasing function.

Assumption 3 is practically reasonable, because consumers would tend to preceive a high failure probability if the actual probability is indeed high. Commonly adopted forms of $\delta(\cdot)$ that satisfy Assumption 3 include $\delta(\mathcal{F}_{r,i}) = \exp(-(-\log(\mathcal{F}_{r,i}))^\lambda)$ in Abito and Salant (2019) and $\delta(\mathcal{F}_{r,i}) = (\mathcal{F}_{r,i})^\lambda/((\mathcal{F}_{r,i})^\lambda + (1-\mathcal{F}_{r,i})^\lambda)^{1/\lambda}$ in Jindal (2015), where $\lambda \in (0,1)$ is the model parameter and a larger value of λ reflects a smaller distortion. Based on the two forms above, consumers tend to overestimate actual failure probability when it is small, and underestimate this probability when it is large (Wang *et al.*, 2020). Notice that when $\lambda \to 1$, we have $\delta(\mathcal{F}_{r,i}) \to \mathcal{F}_{r,i}$ for both forms, corresponding to an ideal situation that consumers are able to estimate the true failure probability precisely.

We describe consumer choice behavior via the well known MNL model (McFadden, 1974). An underlying assumption here is that $\varepsilon_{r,i}$'s are independent and identically distributed Gumbel random variables; that is, $\Pr(\varepsilon_{r,i} \le x) = \exp(-\exp(-(x/\mu+\chi)))$, where $\chi \approx 0.5772$ is Euler's constant and $\mu > 0$ is a scale parameter. Let $\mathcal{U}_{r,0} \equiv \varepsilon_{r,0}$ be the utility of the outside option, corresponding to the case that consumers with usage rate r do not buy any option from $\mathcal{M}_r$. Rational consumers (with usage rate r) would select an option from $\mathcal{M}_r$ that has the highest utility after $\varepsilon_{r,i}$'s are fully resolved. Under the random utility maximization framework, it is a standard result (see Ben-Akiva and Lerman, 1985) that for a given $\mathcal{M}_r$, consumers with usage rate r will choose option $i \in \mathcal{N}$ with probability

$$d_{r,i}(\mathcal{M}_r) = \Pr\left(\mathcal{U}_{r,i} = \max_{j \in \mathcal{N} \cup \{0\}} \mathcal{U}_{r,j}\right) = \frac{\exp\left(\dfrac{\mathcal{V}(W_{r,i}; r) - P_i}{\mu}\right)}{1 + \sum_{j \in \mathcal{N}} \exp\left(\dfrac{\mathcal{V}(W_{r,j}; r) - P_j}{\mu}\right)}, \tag{5}$$

and turn to the outside option with probability $d_{r,0}(\mathcal{M}_r) = 1 - \sum_{i \in \mathcal{N}} d_{r,i}(\mathcal{M}_r)$.

3.4. Expected total profit

An extended warranty is contingent on the consumers purchasing a product. The potential consumer population for extended warranty menus thus consists of those who have bought the main product (Wang *et al.*, 2020). As a result, the potential market size can be considered exogenous for this problem, and the choice probability $d_{r,i}(\mathcal{M}_r)$ also characterizes the expected proportion of market share for option $i \in \mathcal{N}$ in $\mathcal{M}_r$. Because the number of product owners with usage rate being r is assumed to be fixed, we can focus simply on the expected warranty profit per unit sold:

$$\Omega_r(\mathcal{M}_r) = \sum_{i \in \mathcal{N}} (P_i - \mathcal{C}(W_{r,i}; r)) d_{r,i}(\mathcal{M}_r), \tag{6}$$

where $P_i - \mathcal{C}(W_{r,i}; r)$ is the expected profit margin (markup) of selling option i in $\mathcal{M}_r$.

If we can design a customized menu $\mathcal{M}_r$ for each usage rate $r \in [r_{\min}, r_{\max}]$, then the expected total profit can be expressed as

$$\Omega = \int_{r_{\min}}^{r_{\max}} \Omega_r(\mathcal{M}_r) \mathrm{d}G(r), \tag{7}$$

where $G(r)$ is the CDF of consumer usage rates. Our analysis below (Lemma 1) shows that the information on $G(r)$ is essentially unnecessary for this problem.

In addition to the profit, another metric of interest is the attach rate—that is, the proportion of product owners who also buy an extended warranty. The attach rate of $\mathcal{M}_r$ is $\mathcal{D}_r(\mathcal{M}_r) = \sum_{i \in \mathcal{N}} d_{r,i}(\mathcal{M}_r)$, and the overall attach rate is given by $\mathcal{D} = \int_{r_{\min}}^{r_{\max}} \mathcal{D}_r(\mathcal{M}_r) \mathrm{d}G(r)$.

4. Design and pricing of customized warranty menus

In this section, we deal with the design and pricing of customized two-dimensional extended warranty menus for consumers with heterogeneous usage rates. As mentioned earlier, the existing studies on flexible two-dimensional warranties focus predominately on determining multiple combinations of warranty limits under a given warranty cost or product failure probablity. This can give a sense of fairness to consumers with distinct usage rates. In this work, we combine this idea with the MNL choice model to design usage-driven, profit-maximizing customized menus of two-dimensional extended warranties.

We first present the following result that can facilitate the design of such menus.

Lemma 1. *If there are two menus $\mathcal{M}_r$ and $\mathcal{M}_{r'}$ satisfying $\mathcal{F}_{r,i} = \mathcal{F}_{r',i}, \forall i \in \mathcal{N}$, then we have $\Omega_r(\mathcal{M}_r) = \Omega_{r'}(\mathcal{M}_{r'})$ and $\mathcal{D}_r(\mathcal{M}_r) = \mathcal{D}_{r'}(\mathcal{M}_{r'})$.*

Lemma 1 implies that if we can design two customized menus $\mathcal{M}_r$ and $\mathcal{M}_{r'}$ such that their respective options i lead to the same product failure probability over the extended warranty period (i.e., $\mathcal{F}_{r,i} = \mathcal{F}_{r',i}, \forall i \in \mathcal{N}$), then their expected profits and attach rates are identical, respectively. A follow-up result of Lemma 1 is that $\Omega = \int_{r_{\min}}^{r_{\max}} \Omega_r(\mathcal{M}_r) \mathrm{d}G(r) = \Omega_{\hat{r}}(\mathcal{M}_{\hat{r}})$ and $\mathcal{D} = \int_{r_{\min}}^{r_{\max}} \mathcal{D}_r(\mathcal{M}_r) \mathrm{d}G(r) = \mathcal{D}_{\hat{r}}(\mathcal{M}_{\hat{r}})$. That is, the expected total profit (resp. overall attach rate) is equal to the expected profit (resp. attach rate) of each customized menu $\mathcal{M}_r$ (here we use $\mathcal{M}_{\hat{r}}$, $\hat{r} = L_b / W_b$). This implies that the exact information on the usage rate distribution $G(r)$ is essentially unnecessary for this problem.

In essence, Lemma 1 reveals the equivalence of iso-PoF options and iso-profit options in the MNL choice framework. This result is essential for the design and pricing of customized extended warranty menus in terms of coping with consumer heterogeneity in usage rate. Based on the result above, the customized menus can be designed in two steps:

1. We first design $\mathcal{M}_{\hat{r}}$ by, for given warranty limits $\boldsymbol{W}_{\hat{r}}$ and $\boldsymbol{L}_{\hat{r}}$ $(= \hat{r} \boldsymbol{W}_{\hat{r}})$, seeking an optimal price P_i^* for each option $i \in \mathcal{N}$.
2. We then design $\mathcal{M}_r$ $(r \neq \hat{r})$ by, for given $\boldsymbol{P}^*$, determining warranty limits $\boldsymbol{W}_r$ and $\boldsymbol{L}_r$ $(= r \boldsymbol{W}_r)$ that satisfy $\mathcal{F}_{r,i} = \mathcal{F}_{\hat{r},i}, \forall i \in \mathcal{N}$.

By doing so, we can design a series of customized warranty menus for consumers with any usage rates. Each of these menus leads to the same expected profit and attach rate.

4.1. Design and pricing of $\mathcal{M}_{\hat{r}}$

We first design the menu $\mathcal{M}_{\hat{r}}$ for consumers with usage rate $\hat{r}$, in which the warranty limits are pre-specified as $\boldsymbol{W}_{\hat{r}}$ and $\boldsymbol{L}_{\hat{r}}$ $(= \hat{r} \boldsymbol{W}_{\hat{r}})$. There are two points noteworthy for this problem: First, the usage rate $\hat{r} = L_b / W_b$ is chosen for convenience, although we can base the initial design on any value of usage rate (e.g., the mean usage rate $\bar{r}$). Second, the age limits $\boldsymbol{W}_{\hat{r}}$ are specified as positive integers (say, 6 months, 12 months, and 18 months), so as to keep consistent with the current warranty practice.

In this context, the provider first pre-specifies a list of warranty limits and then decides on an optimal price associated with each warranty limit, so as to maximize the expected profit. Mathematically, the provider's problem can be formulated as

$$\max_{\boldsymbol{P}} \Omega_{\hat{r}}(\mathcal{M}_{\hat{r}}). \tag{8}$$

It is not easy to solve this multi-variable pricing problem using standard optimization techniques, because it is not well behaved (Wang, 2012; Wang *et al.*, 2020). Interestingly, the following theorem shows that it can be converted into a simple root-finding problem.

Theorem 1. *Let $\omega_{\hat{r}}^* = \max_{\boldsymbol{P}} \Omega_{\hat{r}}(\mathcal{M}_{\hat{r}})$. For given warranty limits $\boldsymbol{W}_{\hat{r}}$ and $\boldsymbol{L}_{\hat{r}}$ of $\mathcal{M}_{\hat{r}}$:*

(i) *The optimal price of each option i is given by $P_i^* = \mathcal{C}(W_{\hat{r},i}; \hat{r}) + \omega_{\hat{r}}^* + \mu$, where $\omega_{\hat{r}}^*$ is the unique solution to*

$$\mu \sum_{i \in \mathcal{N}} \exp\left(\frac{\mathcal{V}(W_{\hat{r},i}; \hat{r}) - \mathcal{C}(W_{\hat{r},i}; \hat{r}) - \omega_{\hat{r}} - \mu}{\mu}\right) = \omega_{\hat{r}}. \tag{9}$$

(ii) *The resultant attach rate of $\mathcal{M}_{\hat{r}}$ is given by*

$$\mathcal{D}_{\hat{r}}^*(\mathcal{M}_{\hat{r}}) = \sum_{i \in \mathcal{N}} d_{\hat{r},i}^*(\mathcal{M}_{\hat{r}}) = \frac{\omega_{\hat{r}}^*}{\mu + \omega_{\hat{r}}^*}. \tag{10}$$

One can see that the multi-variable pricing problem in (8) has been transformed into the problem of determining a unique intersection between a strictly decreasing function and a 45° line (see Equation (9)). By doing so, the problem becomes much easier to solve. More importantly, Theorem 1 indicates that the optimal pricing policy for given $\boldsymbol{W}_{\hat{r}}$ and $\boldsymbol{L}_{\hat{r}}$ is a *cost-plus-margin* policy; the optimal warranty prices for given $\boldsymbol{W}_{\hat{r}}$ and $\boldsymbol{L}_{\hat{r}}$ should be specified in a way that the margins are the same for all options (given by $\omega_{\hat{r}}^* + \mu$, $\forall i \in \mathcal{N}$). This means that although the price of a longer-duration warranty option is higher than that of a shorter-duration one, their unit profits are essentially the same; however, their contributions to the total profit are indeed different, given their distinct choice probabilities. The *equal-margin pricing* policy is consistent with the existing literature (see, e.g., Li and Huh, 2011; Wang *et al.*, 2020; Wang *et al.*, 2022a). In addition, the attach rate of $\mathcal{M}_{\hat{r}}$, at optimality, is found to be an increasing function of the expected profit $\omega_{\hat{r}}^*$. This implies that, at optimality, a larger attach rate corresponds to a higher expected profit and vice versa.

4.2. Design and pricing of $\mathcal{M}_r$ $(r \neq \hat{r})$

Based on the optimally designed $\mathcal{M}_{\hat{r}}$ obtained above, we then design $\mathcal{M}_r$ for any $r \neq \hat{r}$. Let $\boldsymbol{P}^* = \{P_i^*\}_{i \in \mathcal{N}}$, where P_i^* is the optimal warranty price as in Theorem 1. For designing $\mathcal{M}_r$, the provider should specify the limits $\boldsymbol{W}_r$ and $\boldsymbol{L}_r$ $(= r\boldsymbol{W}_r)$, for given $\boldsymbol{P}^*$, such that the product failure probability over the extended warranty period satistfies $\mathcal{F}_{r,i} = \mathcal{F}_{\hat{r},i}$, $\forall i \in \mathcal{N}$. Thinking about this problem in another way, the provider first specifies the limits $\boldsymbol{W}_r$ and $\boldsymbol{L}_r$ (through $\mathcal{F}_{r,i} = \mathcal{F}_{\hat{r},i}$, $\forall i \in \mathcal{N}$) and then determines the optimal prices

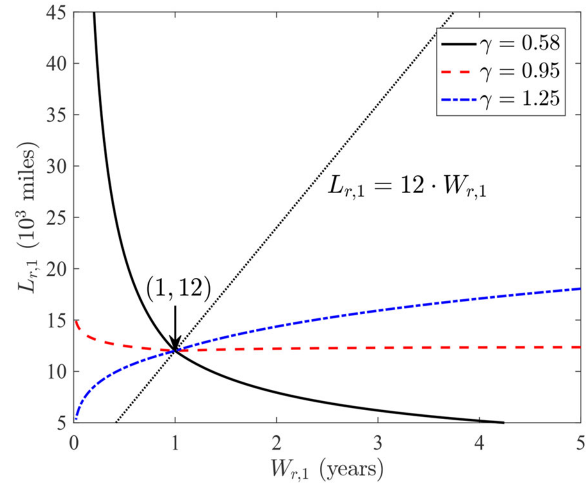

Figure 3. Iso-PoF curves for different values of γ.

by solving $\max_{\boldsymbol{P}} \Omega_r(\mathcal{M}_r)$. Based on the cost-plus-margin pricing policy in Theorem 1, the optimal prices and the resulting profit for $\mathcal{M}_r$ should be the same as those for $\mathcal{M}_{\hat{r}}$.

The following proposition presents the age limit $W_{r,i}$ for each option i in $\mathcal{M}_r$:

Proposition 1. *The age limit for option i in any menu $\mathcal{M}_r$ $(r \neq \hat{r})$ is given by*

$$W_{r,i} = (r_0/r)^{\gamma} \Lambda_0^{-1}\left[\mathcal{A} + \Lambda_0((r/r_0)^{\gamma} W_b(r))\right] - W_b(r), \tag{11}$$

where $\mathcal{A} = \Lambda_0[(\hat{r}/r_0)^{\gamma}(W_b(\hat{r}) + W_{\hat{r},i})] - \Lambda_0[(\hat{r}/r_0)^{\gamma} W_b(\hat{r})]$ *and $\Lambda_0^{-1}(\cdot)$ is the inverse function of $\Lambda_0(\cdot)$.*

By calculating $(W_{r,i}, rW_{r,i})$ for each usage rate r, we can readily obtain the iso-PoF curve for the ith option. More importantly, based on Lemma 1, an iso-PoF curve is equivalent to an iso-profit curve under the proposed design methodology.

Corollary 1. *If time-to-first-failure, T_0, is Weibull distributed, that is, $\lambda_0(t) = (\beta/\alpha)(t/\alpha)^{\beta-1}$, then the age limit $W_{r,i}$ can be derived as*

$$W_{r,i} = \left[(r_0/r)^{\gamma\beta} \alpha^{\beta} \mathcal{A} + (W_b(r))^{\beta}\right]^{\frac{1}{\beta}} - W_b(r), \tag{12}$$

where

$$\mathcal{A} = \left(\frac{\hat{r}}{r_0}\right)^{\gamma\beta}\left[\left(\frac{W_b(\hat{r}) + W_{\hat{r},i}}{\alpha}\right)^{\beta} - \left(\frac{W_b(\hat{r})}{\alpha}\right)^{\beta}\right].$$

It is worth discussing that Ye and Murthy (2016) find that for a customized two-dimensional base warranty menu, when the acceleration factor $\gamma > 1$, consumers have an incentive to report usage rates lower than their true usage rates; otherwise, consumers will report their true usage rates (see Theorem 4 therein). That is to say, only when $\gamma < 1$, can *self-selection* be guaranteed for a customized two-dimensional base warranty menu. This is because when $\gamma > 1$, reporting a lower usage rate will help consumers receive a larger age limit and a larger usage limit at the same time. However, it is difficult, if not impossible, to derive a closed-form condition under which self-selection can be guaranteed in the two-dimensional extended warranty setting, as the starting point of an extended warranty, $W_b(r)$, is dependent on usage rate r. Using the parameter setting in Section 4.4 and considering $W_{\hat{r},1} = 1$ year, Figure 3 shows the iso-PoF curves for option $i = 1$ when $\gamma = 0.58$, 0.95, and 1.25, respectively. The dotted straight line is $L_{r,1} = \hat{r} W_{r,1}$ (with $\hat{r} = 12$), and the curves intersect at point (1, 12).

One can observe that among the three values of γ, self-selection can only be attained when $\gamma = 0.58$. In this case, if there are two usage-rate values r and r' such that $W_{r,1} < W_{r',1}$, then we must have $L_{r,1} > L_{r',1}$. This ensures consumer self-selection, as no option can dominate others in terms of both age and usage limits. For the case of $\gamma = 1.25$, we can always find two points on the associated curve satisfying $W_{r,1} < W_{r',1}$ and $L_{r,1} < L_{r',1}$, simultaneously. In this situation, consumers will not self-select into the customized menus tailored to their usage rates. Interestingly, when $\gamma = 0.95$—which satisfies the condition of $\gamma < 1$ in Ye and Murthy (2016), self-selection can be guaranteed for $r < \hat{r}$, but not for $r > \hat{r}$ (The curve has a slightly increasing trend in this region so that we have $W_{r,1} < W_{r',1}$ and $L_{r,1} < L_{r',1}$ for any $\hat{r} < r' < r$). This implies that the self-selecting condition in the extended warranty context is *stricter* than that in the base warranty context.

We emphasize that the usage-driven customization framework developed in our work does not rely on consumer self-selection, because the provider will have enough information on each consumer's usage rate before signing an extended warranty contract (Assumption 1). Nevertheless, in view of the significant role of the acceleration factor γ, it is important to implement robust product design to reduce the value of γ (Ye and Murthy, 2016), so that the product can be robust to usage rate and self-selection can thus be guaranteed.

4.3. Uniform warranty menu without customization

In this subsection, we present a uniform two-dimensional extended warranty menu for comparison purposes, where all consumers are offered the same menu without customization with respect to their usage rates. Let $\mathcal{M}^\dagger$ denote the uniform menu, which can be represented as $\mathcal{M}^\dagger = \{\boldsymbol{W}^\dagger, \boldsymbol{L}^\dagger, \boldsymbol{P}^\dagger\} = \{W_i^\dagger, L_i^\dagger, P_i^\dagger\}_{i\in\mathcal{N}}$, where $L_i^\dagger = \bar{r} W_i^\dagger$ and $\bar{r}$ is the average usage rate.

Given the uniform menu $\mathcal{M}^\dagger$, a consumer with usage rate r will buy option $i \in \mathcal{N}$ with probability

$$d_{r,i}^\dagger(\mathcal{M}^\dagger) = \frac{\exp\left(\frac{\mathcal{V}(W_i^\dagger(r);r) - P_i^\dagger}{\mu}\right)}{1 + \sum_{j\in\mathcal{N}} \exp\left(\frac{\mathcal{V}(W_j^\dagger(r);r) - P_j^\dagger}{\mu}\right)},$$

where $W_i^\dagger(r) = \min\{W_i^\dagger, \bar{r} W_i^\dagger / r\}$ is the actual warranty expiry time of option i, and the valuation function $\mathcal{V}(W_i^\dagger(r); r)$ is given by Equation (3), with $W_i^\dagger(r)$ replacing $W_{r,i}$.

Therefore, the overall attach rate of $\mathcal{M}^\dagger$ is given by $\mathcal{D}^\dagger(\mathcal{M}^\dagger) = \int_{r_{\min}}^{r_{\max}} \sum_{i\in\mathcal{N}} d_{r,i}^\dagger(\mathcal{M}^\dagger) \mathrm{d}G(r)$, and the corresponding expected total profit is given by

$$\Omega^\dagger(\mathcal{M}^\dagger) = \int_{r_{\min}}^{r_{\max}} \left[\sum_{i\in\mathcal{N}} (P_i^\dagger - \mathcal{C}(W_i^\dagger(r); r)) d_{r,i}^\dagger(\mathcal{M}^\dagger)\right] \mathrm{d}G(r),$$

where $\mathcal{C}(W_i^\dagger(r); r)$ is given by Equation (1), with $W_i^\dagger(r)$ replacing $W_{r,i}$ as well.

Provided that warranty limits $\boldsymbol{W}^\dagger$ and $\boldsymbol{L}^\dagger$ are pre-specified (we let $\boldsymbol{W}^\dagger = \boldsymbol{W}_{\hat{r}}$ for convenience), the provider's problem is to determine the optimal prices to maximize the expected total profit. Mathematically, this problem can be formulated as

$$\max_{\boldsymbol{P}^\dagger} \Omega^\dagger(\mathcal{M}^\dagger). \tag{13}$$

The problem in Equation (13) is essentially a product-line pricing problem with demand given by a continuous MMNL choice model, in which the MMNL model captures consumer heterogeneity in usage rate. It should be noted that in traditional MMNL-based product-line pricing problems, only the choice probabilities depend on a specific random variable, whereas in our problem both the choice probabilities and the warranty costs are dependent on the random usage rate. According to Li *et al.* (2019) and Wang *et al.* (2020), the MMNL-based product-line pricing problem is highly ill-behaved. Because this problem is not the main focus of our work, we simply adopt heuristics methods to solve it. Our numerical results (see Appendix A in the online supplement) show that customized menus can lead to a higher expected profit and attach rate than a uniform menu, demonstrating the value of customization.

4.4. Application

In this subsection, we adopt the commercial vehicle example presented in Lawless *et al.* (1995) and Lawless *et al.* (2009) to showcase the proposed customization methodology. The vehicle of interest was sold with a 3-year/36,000-mile base warranty, that is, $W_b = 3$, $L_b = 36$, and thus $\hat{r} = 12$. The units for age and usage are years and 10^3 miles, respectively. The vehicle was designed for operating under a nominal usage rate $r_0 = 1$. According to a consumer survey, usage rates follow a log-normal distribution, that is, $\log(R) \sim \mathrm{N}(\varphi, \sigma^2)$, with $\varphi = 2.37$ and $\sigma = 0.58$; the average usage rate is thus $\bar{r} = \exp(\varphi + \sigma^2/2) = 12.67$. Moreover, warranty claims analysis shows that the time to first failure is Weibull distributed, that is, $\lambda_0(t) = (\beta/\alpha)(t/\alpha)^{\beta-1}$, with $\alpha = 60.45$ and $\beta = 1.10$; while the acceleration factor is estimated to be $\gamma = 0.58$. We assume that the average repair cost borne by the provider is $c_f = 100$; however, if a consumer does not buy any extended warranty, then she has to pay, on average, $c_m = 180$ for an out-of-warranty repair. Furthermore, the consumer's probability distortion function takes the form $\delta(\mathcal{F}) = \exp(-(-\log(\mathcal{F}))^\lambda)$, with $\lambda = 0.69$. In addition, the scale parameter of MNL is set to $\mu = 5$.

Suppose that the provider would like to design three options for each customized extended warranty menu. We first focus on the design and pricing of menu $\mathcal{M}_{\hat{r}}$, with given age limits $W_{\hat{r},1} = 1$, $W_{\hat{r},2} = 2$, and $W_{\hat{r},3} = 3$. As $\hat{r} = 12$, the corresponding usage limits are set to $L_{\hat{r},1} = 12$, $L_{\hat{r},2} = 24$, and $L_{\hat{r},3} = 36$, respectively. Table 1 presents the optimally designed $\mathcal{M}_{\hat{r}}$ and related metrics. Based on the cost-plus-margin pricing policy in Theorem 1, the

Table 1. Optimally designed warranty menu $\mathcal{M}\hat{r}$ and related metrics.

$W_{\hat{r},i}$	$L_{\hat{r},i}$	$\mathcal{F}_{\hat{r},i}$	$\mathcal{V}(W_{\hat{r},i};\hat{r})$	$\mathcal{C}(W_{\hat{r},i};\hat{r})$	P_i^*	$d_{\hat{r},i}^*$ (%)
1	12	0.065	24.25	6.68	27.22	13.44
2	24	0.127	34.55	13.53	34.07	26.79
3	36	0.185	42.94	20.51	41.06	35.43

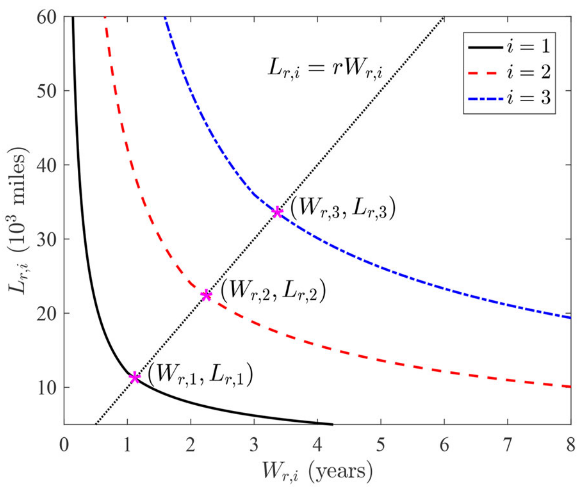

Figure 4. Iso-PoF curves for customized warranty options.

optimal prices for the three options are given by \$27.22, \$34.07, and \$41.06, respectively. The resulting expected per-unit profit is $\omega_{\hat{r}}^* = 15.54$ and the corresponding attach rate is $\mathcal{D}_{\hat{r}}^* = 75.66\%$. One can observe that the choice probabilities of options 2 and 3 are remarkably higher than that of option 1. This is because the average consumer utilities $\mathcal{V}(W_{\hat{r},i};\hat{r}) - P_i^*$ for $i = 2$, 3 are positive, leading to notably large choice probabilities. However, despite that the average consumer utility for option 1 is negative, the associated choice probability is still larger than zero. This is due to the presence of random utility term $\varepsilon_{r,i}$ in Equation (2) and the exponent terms $\exp(\cdot)$ in MNL that always produce positive choice probabilities.

Based on the optimally designed $\mathcal{M}_{\hat{r}}$, we further design menu $\mathcal{M}_r$ for any $r \neq \hat{r}$. For this purpose, we fix the prices at $P_1^* = \$27.22$, $P_2^* = \$34.07$, and $P_3^* = \$41.06$, and derive age limits $W_{r,i}$'s (the associated usage limits are immediately $L_{r,i} = rW_{r,i}$) that satisfy $\mathcal{F}_{r,i} = \mathcal{F}_{\hat{r},i}$, $i = 1$, 2, 3. According to Corollary 1, Figure 4 illustrates the iso-PoF curves for the three options. If we draw a straight line $L_{r,i} = rW_{r,i}$, then the intersecting points of this line with the three iso-PoF curves determine the age and usage limits for $\mathcal{M}_r$. One can see that on each curve i, if there are two usage-rate values r and r' such that $W_{r,i} < W_{r',i}$, then we must have $L_{r,i} > L_{r',i}$. This property guarantees consumer self-selection, which is not consistently attained in the extended warranty context, as discussed in Section 4.2. Some additional numerical results can be found in Appendix A in the online supplement.

5. Incorporating ancillary PM programs

In real applications, many capital-intensive products are sold with PM warranties, under which regular warranty-covered PM activities are executed to keep the products operating in a reliable and cost-effective manner (Wang and Xie, 2018). For instance, commercial vehicles protected by warranties (base and/or extended) are usually subject to regular PM activities. A real example is the so-called 6615 program launched by Nissan in Taiwan, which offers registered members a 6-year/150,000-kilometer extended warranty on six key systems, subject to PM every 6 months or 10,000 kilometers by designated maintenance centers (Huang *et al.*, 2017). Since many capital-intensive products are sold with two-dimensional warranties, it is of vital importance to integrate appropriate PM policies when designing customized two-dimensional extended warranty menus. In this section, we extend the original model in Sections 3 and 4 by incorporating ancillary PM programs.

It should be noted that the proposed PM model is different from that in Wang *et al.* (2020). Specifically, Wang *et al.* (2020) concentrates on a PM bundling problem in which the provider only decides on whether to bundle a given PM program (with prespecified PM interval and degree) with each option or not, whereas our model described below focuses on a PM planning problem in which the provider should decide on the optimal number of PM services attached to each option with the aim of profit maximization.

5.1. Model formulation

Suppose now that a customized two-dimensional extended warranty menu $\widetilde{\mathcal{M}}_r = \{\boldsymbol{W}_r, \boldsymbol{L}_r, \boldsymbol{K}, \boldsymbol{P}\}$ is recommended to consumers with usage rate r, where $\boldsymbol{K} = \{k_i\}_{i\in\mathcal{N}}$ and k_i is the number of PM services bundled with option i. It should be noted that the ancillary PM program is not separately priced, rather its price is factored into the warranty price $\boldsymbol{P}$. Here, we consider a periodic PM strategy for ease of implementation, and the maintenance interval for option i in $\widetilde{\mathcal{M}}_r$ is $\Delta_{r,i} = W_{r,i}/(k_i + 1)$. This implies that there is no PM activity at the end of the extended warranty period. Note that, analogous to the prices $\boldsymbol{P}$, consumers will always receive k_i PM services if the respective options i in $\widetilde{\mathcal{M}}_r$, $r \in [r_{\min}, r_{\max}]$, are bought. This ensures a large degree of fairness for consumers with distinct usage rates.

In this work, we employ the widely adopted age reduction model (see, e.g., Shafiee *et al.*, 2013; Shahanaghi *et al.*, 2013; Wang *et al.*, 2020) to characterize the effect of imperfect PM on product reliability. This model assumes that a PM action can result in a certain amount of age reduction for the product, from its current age to a younger (virtual) age. Considering the ith warranty option in $\widetilde{\mathcal{M}}_r$, let $\tau_{i,j}^r = W_b(r) + j\Delta_{r,i}$, $j = 1, 2, \ldots, k_i$, represent the product's actual age upon the jth PM service, with $\tau_{i,0}^r = W_b(r)$. The product's virtual age right after the jth PM service is expressed as $a_{i,j}^r = a_{i,j-1}^r + (1-\phi)(\tau_{i,j}^r - \tau_{i,j-1}^r)$, where $a_{i,0}^r = W_b(r)$ and ϕ is the age reduction factor. A larger value of ϕ reflects a larger maintenance effect: $\phi = 0$ implies that no PM activity is performed (i.e., $a_{i,j}^r = \tau_{i,j}^r$); $\phi = 1$ indicates that the PM effect is perfect (i.e., $a^r_{i,j} = W_b(r)$); while $\phi \in (0,1)$ corresponds to an imperfect PM effect.

It is straightforward that $a^r_{i,j}$ can be recursively derived as $a^r_{i,j} = W_b(r) + j(1-\phi)\Delta_{r,i}$, $j = 1,2,\ldots,k_i$. By NHPP, the expected repair cost of option i in $\widetilde{\mathcal{M}}_r$ can be expressed as

$$\mathcal{C}'(W_{r,i}, k_i; r) = c_f \sum_{j=0}^{k_i} \int_{W_b(r)+j(1-\phi)\Delta_{r,i}}^{W_b(r)+j(1-\phi)\Delta_{r,i}+\Delta_{r,i}} \lambda(t|r)\mathrm{d}t. \tag{14}$$

Note that $\mathcal{C}'(W_{r,i}, k_i; r)$ should be smaller than or at most equal to $\mathcal{C}(W_{r,i}; r)$ in Equation (1), due to the age reduction effect of the PM implemented.

Assume that the cost of each PM service is a constant, denoted by c_{pm}. Then, the total servicing cost of option i in $\widetilde{\mathcal{M}}_r$ becomes $\widetilde{\mathcal{C}}(W_{r,i}, k_i; r) = \mathcal{C}'(W_{r,i}, k_i; r) + k_i c_{pm}$. Further assume that each PM service brings an additional utility u^{pm} to the consumer. Then, the random consumer utility of option i in $\widetilde{\mathcal{M}}_r$ is given by $\widetilde{\mathcal{U}}_{r,i} = \widetilde{\mathcal{V}}(W_{r,i}, k_i; r) + k_i u^{pm} - P_i + \varepsilon_{r,i}$, where $\widetilde{\mathcal{V}}(W_{r,i}, k_i; r) = c_m \delta(\widetilde{\mathcal{F}}_{r,i})$ and

$$\widetilde{\mathcal{F}}_{r,i} = 1 - \exp\left(-\sum_{j=0}^{k_i} \int_{W_b(r)+j(1-\phi)\Delta_{r,i}}^{W_b(r)+j(1-\phi)\Delta_{r,i}+\Delta_{r,i}} \lambda(t|r)\mathrm{d}t\right). \tag{15}$$

According to the MNL choice model, consumers with usage rate r will choose option $i \in \mathcal{N}$ in $\widetilde{\mathcal{M}}_r$ with probability

$$\widetilde{d}_{r,i}(\widetilde{\mathcal{M}}_r) = \frac{\exp\left(\frac{\widetilde{\mathcal{V}}(W_{r,i}, k_i; r) + k_i u^{pm} - P_i}{\mu}\right)}{1 + \sum_{j\in\mathcal{N}} \exp\left(\frac{\widetilde{\mathcal{V}}(W_{r,j}, k_j; r) + k_j u^{pm} - P_j}{\mu}\right)}.$$

Thus, the attach rate of $\widetilde{\mathcal{M}}_r$ is $\widetilde{\mathcal{D}}_r(\widetilde{\mathcal{M}}_r) = \sum_{i\in\mathcal{N}} \widetilde{d}_{r,i}(\widetilde{\mathcal{M}}_r)$, and the expected profit is $\widetilde{\Omega}_r(\widetilde{\mathcal{M}}_r) = \sum_{i\in\mathcal{N}}(P_i - \widetilde{\mathcal{C}}(W_{r,i}, k_i; r))\widetilde{d}_{r,i}(\widetilde{\mathcal{M}}_r)$. Furthermore, the overall attach rate is $\widetilde{\mathcal{D}} = \int_{r_{\min}}^{r_{\max}} \widetilde{\mathcal{D}}_r(\widetilde{\mathcal{M}}_r)\mathrm{d}G(r)$, and the expected total profit from selling all customized menus is given by $\widetilde{\Omega} = \int_{r_{\min}}^{r_{\max}} \widetilde{\Omega}_r(\widetilde{\mathcal{M}}_r)\mathrm{d}G(r)$.

5.2. Customized menu design and pricing

Now, we investigate the design and pricing of customized two-dimensional extended warranty menus bundled with ancillary PM programs. Similar to Lemma 1, we have the following result that can facilitate customized menu design.

Lemma 2. *If there are two menus $\widetilde{\mathcal{M}}_r$ and $\widetilde{\mathcal{M}}_{r'}$ satisfying $\widetilde{\mathcal{F}}_{r,i} = \widetilde{\mathcal{F}}_{r',i}$, $\forall i \in \mathcal{N}$, then we have $\widetilde{\Omega}_r(\widetilde{\mathcal{M}}_r) = \widetilde{\Omega}_{r'}(\widetilde{\mathcal{M}}_{r'})$ and $\widetilde{\mathcal{D}}_r(\widetilde{\mathcal{M}}_r) = \widetilde{\mathcal{D}}_{r'}(\widetilde{\mathcal{M}}_{r'})$.*

Lemma 2 directly leads to $\widetilde{\Omega} = \int_{r_{\min}}^{r_{\max}} \widetilde{\Omega}_r(\widetilde{\mathcal{M}}_r)\mathrm{d}G(r) = \widetilde{\Omega}_{\hat{r}}(\widetilde{\mathcal{M}}_{\hat{r}})$ and $\widetilde{\mathcal{D}} = \int_{r_{\min}}^{r_{\max}} \widetilde{\mathcal{D}}_r(\widetilde{\mathcal{M}}_r)\mathrm{d}G(r) = \widetilde{\mathcal{D}}_{\hat{r}}(\widetilde{\mathcal{M}}_{\hat{r}})$. That is, the expected total profit and overall attach rate are equal to those of each customized menu $\widetilde{\mathcal{M}}_r$, respectively. More importantly, the equivalence of iso-PoF options and iso-profit options holds as well, even when ancillary PM programs are bundled with the extended warranty options.

Based on the result in Lemma 2, the customized extended warranty menus with ancillary PM programs can be designed in the following two steps:

1. We first design $\widetilde{\mathcal{M}}_{\hat{r}}$ by, for given $\boldsymbol{W}_{\hat{r}}$ and $\boldsymbol{L}_{\hat{r}}$, seeking an optimal number of PM services, k_i^*, and an optimal price, P_i^*, for each option $i \in \mathcal{N}$.
2. We then design $\widetilde{\mathcal{M}}_r$ $(r \neq \hat{r})$ by, for given $\boldsymbol{P}^*$ and $\boldsymbol{K}^*$, obtaining the associated warranty limits $\boldsymbol{W}_r$ and $\boldsymbol{L}_r$ that satisfy $\widetilde{\mathcal{F}}_{r,i} = \widetilde{\mathcal{F}}_{\hat{r},i}$, $\forall i \in \mathcal{N}$.

The optimization problem in the first step can be formulated as $\max_{\boldsymbol{P},\boldsymbol{K}} \widetilde{\Omega}_{\hat{r}}(\widetilde{\mathcal{M}}_{\hat{r}})$. The optimal solution is presented in the following theorem:

Theorem 2. *Let $\widetilde{\omega}^*_{\hat{r}} = \max_{\boldsymbol{P},\boldsymbol{K}} \widetilde{\Omega}_{\hat{r}}(\widetilde{\mathcal{M}}_{\hat{r}})$. For given warranty limits $\boldsymbol{W}_{\hat{r}}$ and $\boldsymbol{L}_{\hat{r}}$ of $\widetilde{\mathcal{M}}_{\hat{r}}$:*

(i) *The optimal number of PM services attached to option $i \in \mathcal{N}$ should be $k_i^* = \mathrm{argmax}_{k_i}\{\zeta(k_i)\}$, where $\zeta(k_i) = \widetilde{\mathcal{V}}(W_{\hat{r},i}, k_i; \hat{r}) + k_i u^{pm} - \widetilde{\mathcal{C}}(W_{\hat{r},i}, k_i; \hat{r})$.*

(ii) *The optimal price of each option i is given by $P_i^* = \widetilde{\mathcal{C}}(W_{\hat{r},i}, k_i^*; \hat{r}) + \widetilde{\omega}^*_{\hat{r}} + \mu$, where $\widetilde{\omega}^*_{\hat{r}}$ is the unique solution to*

$$\mu \sum_{i\in\mathcal{N}} \exp\left(\frac{\zeta(k_i^*) - \widetilde{\omega}_{\hat{r}} - \mu}{\mu}\right) = \widetilde{\omega}_{\hat{r}}. \tag{16}$$

(iii) *The resultant attach rate of $\widetilde{\mathcal{M}}_{\hat{r}}$ is thus given by*

$$\widetilde{\mathcal{D}}^*_{\hat{r}}(\widetilde{\mathcal{M}}_{\hat{r}}) = \sum_{i\in\mathcal{N}} \widetilde{d}^*_{\hat{r},i}(\widetilde{\mathcal{M}}_{\hat{r}}) = \frac{\widetilde{\omega}^*_{\hat{r}}}{\mu + \widetilde{\omega}^*_{\hat{r}}}. \tag{17}$$

Theorem 2 reveals an interesting result: For the joint price and PM optimization problem, the provider should specify the optimal number of PM services for each option independently of the PM or pricing decisions of other options, despite that all options are substitutable. In particular, here we are focusing on a utility-maximizing PM optimization problem—rather than cost-minimizing ones as in most of the existing literature (see, e.g., Shafiee *et al.*, 2013; Wang *et al.*, 2020; Peng *et al*, 2021)—where the objective function $\zeta(k_i)$ can be rewritten as $\zeta(k_i) = c_m\delta(\widetilde{\mathcal{F}}_{r,i}) + k_i(u^{pm} - c_{pm}) + c_f \log(1 - \widetilde{\mathcal{F}}_{r,i})$. Because the optimal number of PM services should be an integer in nature, k_i^* can be easily obtained by searching in its feasible domain $\{0, 1, \ldots, k_i^{\max}\}$. By defining the minimum acceptable PM interval under usage rate $\hat{r}$ as $\Delta^{\min}_{\hat{r}}$, the maximum allowable number of PM services for option i can be specified as $k_i^{\max} = W_{\hat{r},i}/\Delta^{\min}_{\hat{r}} - 1$. Moreover, the optimal pricing policy is still a cost-plus-margin policy, with the same profit margins for all options. However, the optimal prices for options with ancillary PM programs might be lower or higher than those without PM, depending on the relative magnitude of $\widetilde{\mathcal{C}}(W_{\hat{r},i}, k_i^*; \hat{r})$ and $\mathcal{C}(W_{\hat{r},i}; \hat{r})$. Furthermore, the attach rate of

Table 2. Optimally designed warranty menu $\widetilde{\mathcal{M}}\hat{r}$ and related metrics.

$W_{\hat{r},i}$	$L_{\hat{r},i}$	k_i^*	$\widetilde{\mathcal{F}}_{\hat{r},i}$	$\widetilde{\mathcal{V}}(W_{\hat{r},i},k_i^*;\hat{r})$	$k_i^* u^{pm}$	$\widetilde{\mathcal{C}}(W_{\hat{r},i},k_i^*;\hat{r})$	P_i^*	$\widetilde{d}_{\hat{r},i}^*$ (%)
1	12	3	0.064	24.18	16.5	21.64	45.47	8.04
2	24	7	0.125	34.35	28.5	48.38	72.21	23.83
3	36	11	0.183	42.59	60.5	75.20	99.04	47.15

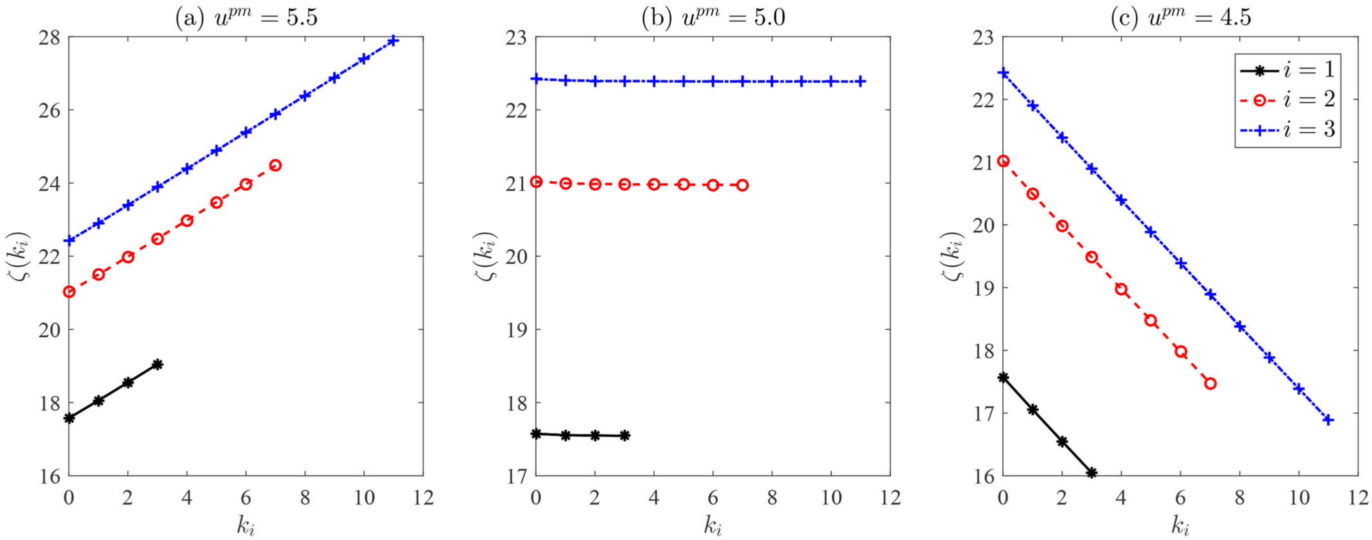

Figure 5. Value of $\zeta(k_i)$, $i = 1, 2, 3$, under different values of u^{pm}.

$\widetilde{\mathcal{M}}_{\hat{r}}$, at optimality, is still an increasing function of the associated expected total profit $\widetilde{\omega}_{\hat{r}}^*$.

By comparing the results in Theorems 1 and 2, we have the following corollary:

Corollary 2. *$\widetilde{\omega}_{\hat{r}}^* \geq \omega_{\hat{r}}^*$ and $\widetilde{\mathcal{D}}_{\hat{r}}^*(\widetilde{\mathcal{M}}_{\hat{r}}) \geq \mathcal{D}_{\hat{r}}^*(\mathcal{M}_{\hat{r}})$.*

Corollary 2 informs us that the expected profit and attach rate of menu $\widetilde{\mathcal{M}}_{\hat{r}}$ bundled with ancillary PM programs, at optimality, are higher than, or at least equal to, those of $\mathcal{M}_{\hat{r}}$ without PM (in Section 4.1), respectively. In other words, among the product owners, more of them would like to buy an extended warranty option if an ancillary PM program is attached. As a consequence, the expected profit becomes higher. This stems from the net benefits brought by the ancillary PM programs, and it shows the necessity of bundling optimal PM programs with the warranty options. In addition, a direct follow-up result of Corollary 2 is that $\widetilde{\Omega}^* \geq \Omega^*$ and $\widetilde{\mathcal{D}}^* \geq \mathcal{D}^*$.

In the second step of customized menu design, the provider should specify $\widetilde{\mathcal{M}}_r$ for any $r \neq \hat{r}$. Specifically, the provider's problem is to determine warranty limits $\boldsymbol{W}_r$ and $\boldsymbol{L}_r$ $(= r\boldsymbol{W}_r)$ for given $\boldsymbol{P}^*$ and $\boldsymbol{K}^*$ (as in Theorem 2), such that the product failure probability over the extended warranty period satisfies $\widetilde{\mathcal{F}}_{r,i} = \widetilde{\mathcal{F}}_{\hat{r},i}$, $\forall i \in \mathcal{N}$. Based on the expression of $\widetilde{\mathcal{F}}_{r,i}$ in Equation (15), solving $\widetilde{\mathcal{F}}_{r,i} = \widetilde{\mathcal{F}}_{\hat{r},i}$ for $W_{r,i}$ yields

$$\sum_{j=0}^{k_i}\int_{W_b(r)+j(1-\phi)\frac{W_{r,i}}{(k_i+1)}}^{W_b(r)+j(1-\phi)\frac{W_{r,i}}{(k_i+1)}+\frac{W_{r,i}}{(k_i+1)}}\lambda(t|r)\mathrm{d}t = \sum_{j=0}^{k_i}\int_{W_b(\hat{r})+j(1-\phi)\frac{W_{\hat{r},i}}{(k_i+1)}}^{W_b(\hat{r})+j(1-\phi)\frac{W_{\hat{r},i}}{(k_i+1)}+\frac{W_{\hat{r},i}}{(k_i+1)}}\lambda(t|\hat{r})\mathrm{d}t. \tag{18}$$

One can see that it is difficult, if not impossible, to obtain a closed-form expression of age limit $W_{r,i}$, due to the mathematical complexity. Fortunately, $W_{r,i}$ can be efficiently obtained by numerical search methods, such as the bisection and golden-section search methods.

5.3. Application: Revisited

We revisit the commercial vehicle example in Section 4.4. In addition to the parameter setting above, we assume that the age reduction factor is $\phi = 0.5$, the average cost of each PM service is $c_{pm} = \$5$, the additional utility due to each PM service is $u^{pm} = \$5.5$, and the consumers' minimal acceptable PM interval is $\Delta_{\hat{r}}^{\min} = 3$ months. As in Section 4.4, the customized two-dimensional extended warranty menu $\widetilde{\mathcal{M}}_{\hat{r}}$ consists of three options with $W_{\hat{r},1} = 1$ year, $W_{\hat{r},2} = 2$ years, and $W_{\hat{r},3} = 3$ years. According to the minimal acceptable PM interval, the maximum allowable numbers of PM services for the three options are $k_1^{\max} = 3$, $k_2^{\max} = 7$, and $k_3^{\max} = 11$, respectively.

Because the design of $\widetilde{\mathcal{M}}_r$, $r \neq \hat{r}$, is quite similar to that in Section 4.4, here we focus only on the design and pricing of $\widetilde{\mathcal{M}}_{\hat{r}}$. Table 2 summarizes the optimally designed menu $\widetilde{\mathcal{M}}_{\hat{r}}$ and related metrics. At optimality, the expected profit is $\widetilde{\omega}_{\hat{r}}^* = \$18.84$ and the associated attach rate is $\widetilde{\mathcal{D}}_{\hat{r}}^* = 79.02\%$, which are higher than those without PM, respectively. This indicates that bundling ancillary PM programs with extended warranty menus can indeed help providers attract more consumers and extract more profits. This observation is consistent with the results in Corollary 2. One can also observe that the optimal warranty prices with ancillary PM services bundled are higher than those without PM. In addition, when compared with the case without PM (see Table

1), the choice probabilities of options 1 and 2, at optimality, become smaller, whereas that of option 3 becomes much larger. This implies that attaching ancillary PM programs offers more benefit to longer-duration options.

An interesting phenomenon to note is that the optimal number of PM services bundled with each option i is exactly equal to the corresponding maximum allowable number $k_i^{\max}$. This originates from the increasing pattern of $\zeta(k_i)$, $i = 1,\ 2,\ 3$, as shown in Figure 5(a). Recall that $\zeta(k_i) = \widetilde{\mathcal{V}}(W_{\hat{r},i}, k_i; \hat{r}) - \mathcal{C}'(W_{\hat{r},i}, k_i; \hat{r}) + k_i(u^{pm} - c_{pm})$. In this figure, one can see that $\zeta(k_i)$'s are increasing in k_i in a nearly linear manner. This is bacause $\widetilde{\mathcal{V}}(W_{\hat{r},i}, k_i; \hat{r}) - \mathcal{C}'(W_{\hat{r},i}, k_i; \hat{r})$ is almost constant over k_i (with a slightly convex decreasing trend), so that the pattern of $\zeta(k_i)$ is determined by the linearity of $k_i(u^{pm} - c_{pm})$.

In addition, we examine the pattern of $\zeta(k_i)$ under another two values of u^{pm}: $u^{pm} = 5.0$ and $u^{pm} = 4.5$. In our parameter setting, when $u^{pm} = c_{pm} = 5.0$, $\zeta(k_i)$ reduces to $\widetilde{\mathcal{V}}(W_{\hat{r},i}, k_i; \hat{r}) - \mathcal{C}'(W_{\hat{r},i}, k_i; \hat{r})$, which is almost constant but has a slightly decreasing trend (see Figure 5(b)), as discussed above. In this case, the optimal number of PM services for each option i is given by $k_i^* = 0$. When $u^{pm} = 4.5 < c_{pm}$, $\zeta(k_i)$'s exhibit a sharp downward tendency; see Figure 5(c). This is largely due to the decreasing linearity of $k_i(u^{pm} - c_{pm})$. In this case, the optimal number of PM services for each option i is $k_i^* = 0$ as well. In short, for the three cases in Figure 5, we are maximizing an objective function that is either decreasing or increasing in the decision variable k_i, so that the optimal k_i^* should be either zero or $k_i^{\max}$.

Following the discussions above, an interesting question arises, that is, *under which condition the optimal number of PM services is between zero and* $k_i^{\max}$? Our numerical investigations show that there are indeed some cases in which the optimal number of PM services is between zero and $k_i^{\max}$. Interested readers are referred to Appendix B in the online supplement for more details.

6. Concluding remarks

In this article, we studied the design and pricing of customized two-dimensional extended warranty menus from the provider's standpoint. For this purpose, we bridged two distinct disciplines—reliability/maintenance and economics. Specifically, the NHPP failure process and imperfect PM modeling from the reliability/maintenance discipline were adopted to formulate product failure probability as well as warranty cost; the well-known MNL choice model from the economics discipline was employed to characterize consumer choice behavior. Consumer heterogeneity in usage rate was addressed by deriving a series of combinations of warranty limits under a given failure probablity (i.e., iso-PoF curves) so as to offer a sense of fairness to consumers. We proved that under the MNL choice model, the iso-PoF curves are equivalent to iso-profit curves, which can help providers to design usage-driven, profit-maximizing customized extended warranty menus.

Our case study showed that customized warranty menus can result in a higher profit and attach rate than a uniform warranty menu, demonstrating the benefit of usage-driven customization; moreover, by bundling optimally designed ancillary PM programs with the warranty options, the profit and attach rate could be even higher. In addition to flexibility and profitability, another advantage of the proposed customization methodology is the robustness to consumers' usage-rate changes after buying an extended warranty (see also Wang and Ye, 2021). That is, no matter how a consumer changes her usage rate after purchasing an extended warranty, the provider's profit would not decrease. This is due to the fact that given an extended warranty region $(W_{r,i}, L_{r,i})$, the largest warranty cost is realized when the consumer's usage rate is exactly equal to $L_{r,i}/W_{r,i}$.

This work can be extended in several aspects. First, we dealt with consumer heterogeneity in usage rate only, implicitly assuming that consumers are homogeneous in other ways. It would be of interest to consider consumer heterogeneity in risk attitude or perception of failure probability. Second, in this work the warranty coverage was implicitly assumed to be homogeneous for all menus. In reality, providers might offer menus with distinct coverages. For instance, Ford Protect Extended Service Plans include PremiumCARE, ExtraCARE, BaseCARE, and PowertrainCARE, which cover 1000+, 113, 84, and 29 components, respectively. Extending the proposed customization methodology to cope with heterogeneous coverages is an interesting problem. Finally, seasonality is a common issue in warranty claims. It is valuable to incorporate dynamic covariates into the NHPP failure model to characterize potential seasonal patterns in warranty claims (see, e.g., Shan *et al.*, 2020) so as to improve warranty cost estimation.

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Supplemental Online Materials to "Design and pricing of usage-driven customized two-dimensional extended warranty menus" by Xiao-Lin Wang

A Additional numerical results in Section 4.4

As with Wang et al. (2020), we examine an incremental problem in which a series of warranty menus are nested with an increasing number of options, that is, $\{(W_{\hat{r},1} = 1, L_{\hat{r},1} = \hat{r}, P_1)\}$, $\{(W_{\hat{r},1} = 1, L_{\hat{r},1} = \hat{r}, P_1), (W_{\hat{r},2} = 2, L_{\hat{r},1} = 2\hat{r}, P_2)\}$, $\cdots$. Figure 6 plots the expected profit versus the number of options under different values of λ. One can observe that the expected profit grows with the number of available options, until reaching a specific limit. This is because long-duration warranty options would lead to negative average consumer utilities, thus contribute little to the profit. Moreover, the smaller the value of λ, the faster the profit curve reaches its limiting value. This is due to the fact that when λ becomes smaller, the concavity of $\mathcal{V}(W_{\hat{r},i}; \hat{r})$ is severer, and thus the average consumer utility would become negative even for small $W_{\hat{r},i}$. Another observation noteworthy is that the expected profit is larger when λ becomes smaller. This originates from the fact that a smaller λ reflects a larger probability distortion, implying that consumers are less capable of estimating the true failure probability accurately. Therefore, the provider's expected profit will become larger due to the asymmetric advantage in product reliability information. The aforementioned findings are essentially consistent with those in Wang et al. (2020).

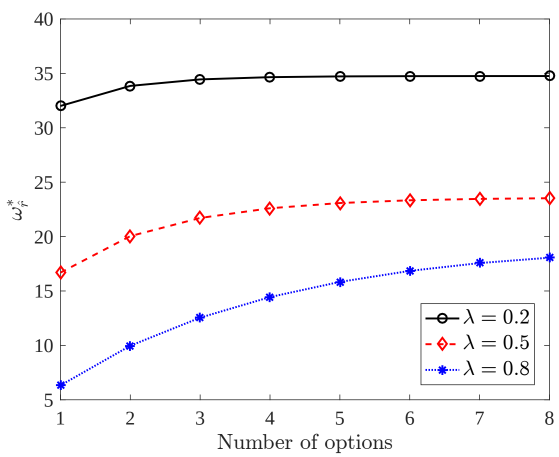

Figure 6: Expected profit versus the number of options under different values of λ.

We further examine the impact of acceleration factor γ on the incremental problem (see Figure 7). As in Figure 6, one can still observe that the expected profit increases with the number of available options and then reaches a specific limit, regardless of the value of γ; the larger the value of γ, the faster the profit curve approaches its limit. More interestingly, we find that when the number of options available in the menu is small (say, $n \leq 2$), a larger value of γ leads to a higher profit; however, when the number of available options is large (say, $n \geq 6$), the situation is reversed, that is, a smaller value of γ results in a higher profit. Recall that the value of γ can affect both the valuation $\mathcal{V}(W_{\hat{r},i}; \hat{r})$ and the warranty cost $\mathcal{C}(W_{\hat{r},i}; \hat{r})$. In general, a larger value of γ amplifies the concavity of $\mathcal{V}(W_{\hat{r},i}; \hat{r})$ and the

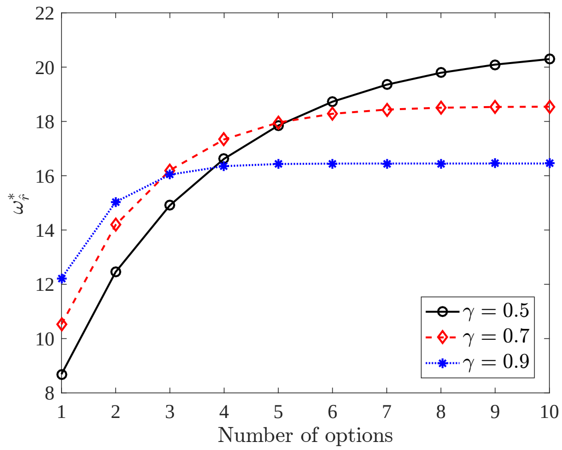

Figure 7: Expected profit versus the number of options under different values of γ.

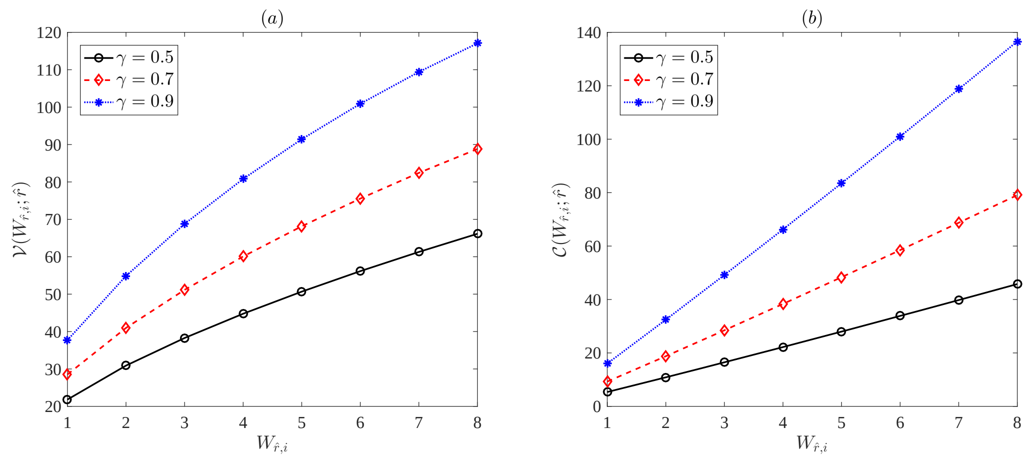

Figure 8: Valuation $\mathcal{V}(W_{\hat{r},i};\hat{r})$ and warranty cost $\mathcal{C}(W_{\hat{r},i};\hat{r})$ under different values of γ.

convexity of $\mathcal{C}(W_{\hat{r},i};\hat{r})$, simultaneously (see Figure 8). This is why the influence of γ on the profit is more significant than that of λ (which can only affect the valuation).

Finally, we evaluate the benefit of usage-driven customization by comparing the performance of customized and uniform warranty menus in a nested fashion as well. Table 3 summarizes the optimal prices and associated expected profit and attach rate when the number of options, n, in each menu ranges from 1 to 5. As can be seen, the optimal prices, expected profit, and attach rate for the uniform menu are always smaller than those for customized menus, respectively, irrespective of the number of options. The profit growths of offering customized menus in comparison with offering a uniform menu are 11.49%, 10.74%, 10.53%, 10.48%, and 10.50%, respectively, when the number of options ranges from 1 to 5. The figures of attach-rate growth are 5.08%, 4.15%, 3.90%, 3.81%, and 3.80%, respectively. This implies that, by customization, the provider is able to attract more consumers and extract more profit from them by specifying higher prices. This demonstates the benefit of customizing two-dimensional extended warranty menu offerings.

Table 3: Performance comparison of customized and uniform menus.

	$n=1$		$n=2$		$n=3$		$n=4$		$n=5$	
	$P_i^\dagger$	P_i^*	$P_i^\dagger$	P_i^*	$P_i^\dagger$	P_i^*	$P_i^\dagger$	P_i^*	$P_i^\dagger$	P_i^*
$i=1$	19.09	21.09	22.40	24.88	24.39	27.22	25.66	28.76	26.49	29.78
$i=2$			28.50	31.73	30.45	34.07	31.68	35.61	32.49	36.63
$i=3$					36.79	41.06	37.99	42.59	38.76	43.61
$i=4$							44.53	49.70	45.26	50.72
$i=5$									51.96	57.93
Ω	8.44	9.41	11.92	13.2	14.06	15.54	15.46	17.08	16.38	18.10
$\mathcal{D}$ (%)	62.15	65.31	69.64	72.53	72.82	75.66	74.52	77.36	75.48	78.35

B Additional numerical results in Section 5.3

We now attempt to answer the question raised at the end of Section 5.3, that is, *under which condition the optimal number of PM services is between zero and $k_i^{\max}$?* To answer this question, Figure 9 shows the optimal number of PM services versus u^{pm} under different combinations of γ ($=0.5, 0.7, 0.9$) and λ ($=0.2, 0.5, 0.8$). From this figure, one can draw the following observations:

- In general, it is not worthwhile to offer any ancillary PM services when u^{pm} is small, whereas it is beneficial to provide ancillary PM services at its maximum allowable number when u^{pm} is large. This is intuitive, following the discussions regarding Figure 5, in the sense that a larger (resp. smaller) value of u^{pm} can lead to an increasing (resp. decreasing) $\zeta(k_i)$, which in turn generates an optimal solution $k_i^* = k_i^{\max}$ (resp. $k_i^* = 0$). It is noteworthy that though the transition between the two scenarios above occurs in a small range of the value of u^{pm}, there are indeed some cases in which the value of k_i^* is between zero and $k_i^{\max}$.

- Moreover, the shift of k_i^* from zero towards $k_i^{\max}$ occurs the earliest for the third option (even when $u^{pm} < 5 = c_{pm}$), followed by the second and then the first ones. Recall that the lengths of the three options are $W_{\hat{r},1} = 1$ year, $W_{\hat{r},2} = 2$ years, and $W_{\hat{r},3} = 3$ years, respectively. The longer the extended warranty period, the more ancillary PM services are needed to reduce product failure probability and thus lower the warranty cost.

- Furthermore, in terms of acceleration factor γ, one can observe that the shift of k_i^* from zero towards $k_i^{\max}$ occurs earlier when γ is larger. This can be explained by the fact that a larger value of γ amplifies the impact of usage rate on product reliability, so that more PM actions are needed to reduce product failure probability.

- Finally, the shift of k_i^* occurs earlier when the probability distortion parameter λ is smaller. Recall that by probability distortion, the actual failure probability tends to be overestimated when it is small (i.e., when $\widetilde{\mathcal{F}}_{\hat{r},i} < e^{-1}$ for the form of $\delta(\cdot)$ adopted in this study), while underestimated when it is large. According to Tables 1 and 2, the actual failure probabilities are smaller than e^{-1}, even without any PM, thus they should be overestimated. Because a smaller value of λ reflects a larger distortion, we know that when λ is smaller, a specific failure probability would lead to a higher

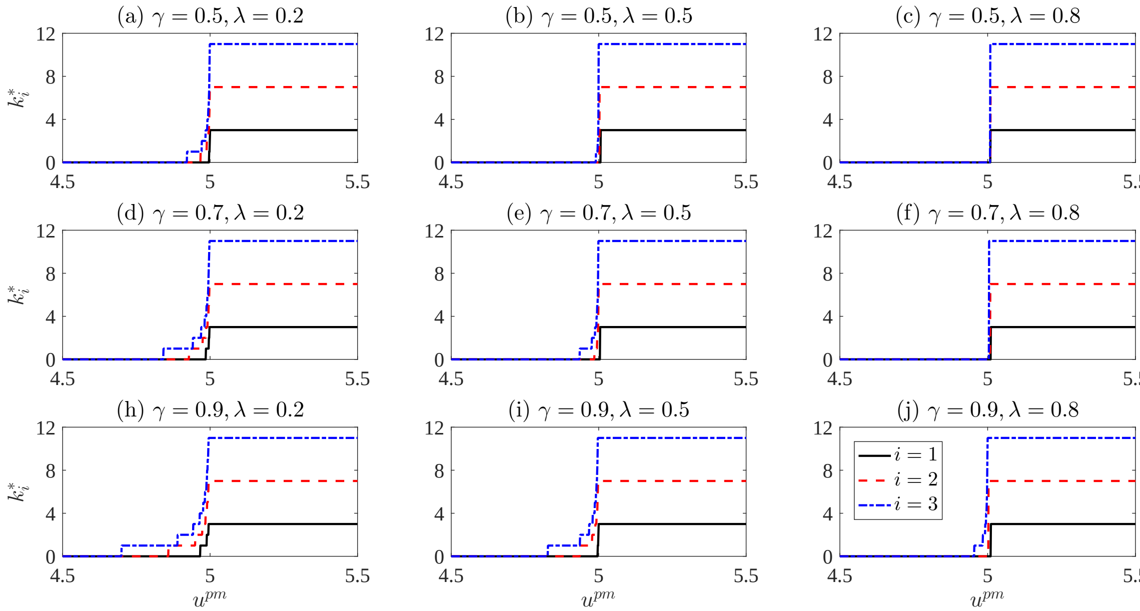

Figure 9: Optimal number of PM services, k_i^*, $i = 1, 2, 3$, under different combinations of γ and λ.

consumer-perceived valuation but a constant warranty cost. It is thus beneficial to perform more PM services to reduce the failure probability and thus the warranty cost, with the consumer-perceived valuation being not reduced too much. Overall, offering ancillary PM services tends to be beneficial for smaller λ, even when $u^{pm} < c_{pm}$.

Now, we are in a position to answer the question above. Through numerical investigations, we find that there are indeed some cases in which the optimal number of PM services is between zero and $k_i^{\max}$. Such cases occur when the additional utility u^{pm} brought by PM is slightly smaller than the PM cost c_{pm}, the warranty period $W_{\hat{r},i}$ is longer, the acceleration factor γ is larger, and/or the probability distortion parameter λ is smaller.

C Technical proofs

Proof of Lemma 1. First of all, it is clear that if $\mathcal{F}_{r,i} = \mathcal{F}_{r',i}$, then $\delta(\mathcal{F}_{r,i}) = \delta(\mathcal{F}_{r',i})$, $\forall i \in \mathcal{N}$, because the probability distortion function $\delta(\cdot)$ is a monotonically increasing function (Assumption 3). This further leads to $\mathcal{V}(W_{r,i}; r) = \mathcal{V}(W_{r',i}; r')$ and thus $d_{r,i}(\mathcal{M}_r) = d_{r',i}(\mathcal{M}_{r'})$, $\forall i \in \mathcal{N}$. On the other hand, from Equations (1) and (4) we know that $\mathcal{C}(W_{r,i}; r) = -c_f \log(1 - \mathcal{F}_{r,i})$. Hence, the condition $\mathcal{F}_{r,i} = \mathcal{F}_{r',i}$ can also lead to $\mathcal{C}(W_{r,i}; r) = \mathcal{C}(W_{r',i}; r')$, $\forall i \in \mathcal{N}$. As a result, it is easy to know, from Equation (6), that $\Omega_r(\mathcal{M}_r) = \Omega_{r'}(\mathcal{M}_{r'})$. Moreover, $\mathcal{D}_r(\mathcal{M}_r) = \mathcal{D}_{r'}(\mathcal{M}_{r'})$ can be easily verified, as $d_{r,i}(\mathcal{M}_r) = d_{r',i}(\mathcal{M}_{r'})$, $\forall i \in \mathcal{N}$. $\square$

Proof of Theorem 1. Let $\Omega_{\hat{r}}(\mathcal{M}_{\hat{r}}) = \omega_{\hat{r}}$ for any prices $\boldsymbol{P}$. Rearranging terms and using some algebra yield

$$\sum_{i \in \mathcal{N}} \left(P_i - \mathcal{C}(W_{\hat{r},i}; \hat{r}) - \omega_{\hat{r}} \right) \exp\left(\frac{\mathcal{V}(W_{\hat{r},i}; \hat{r}) - P_i}{\mu} \right) = \omega_{\hat{r}}.$$

One can see that the left-hand side is a strictly decreasing function of $\omega_{\hat{r}}$ for any given $\boldsymbol{P}$, while the right-hand side is a 45-degree straight line. Thus, the optimization problem in (8) has been transformed to the problem of determining a unique intersection between a strictly decreasing function and a 45-degree line, that is,

$$\omega_{\hat{r}} = \Theta(\omega_{\hat{r}}) := \sum_{i \in \mathcal{N}} \theta_i(\omega_{\hat{r}}), \tag{19}$$

where $\theta_i(\omega_{\hat{r}}) = \max_{P_i} \theta_i(\omega_{\hat{r}}, P_i)$ and

$$\theta_i(\omega_{\hat{r}}, P_i) = \Big(P_i - \mathcal{C}(W_{\hat{r},i}; \hat{r}) - \omega_{\hat{r}}\Big) \exp\left(\frac{\mathcal{V}(W_{\hat{r},i}; \hat{r}) - P_i}{\mu}\right).$$

Taking the first order derivative of $\theta_i(\omega_{\hat{r}}, P_i)$ with respect to P_i yields

$$\frac{\partial \theta_i(\omega_{\hat{r}}, P_i)}{\partial P_i} = \frac{1}{\mu} \exp\left(\frac{\mathcal{V}(W_{\hat{r},i}; \hat{r}) - P_i}{\mu}\right) \Big[\mu - \big(P_i - \mathcal{C}(W_{\hat{r},i}; \hat{r}) - \omega_{\hat{r}}\big)\Big].$$

One can easily know, given $\omega_{\hat{r}}$, that $\theta_i(\omega_{\hat{r}}, P_i)$ is increasing in P_i for $P_i \leq \mathcal{C}(W_{\hat{r},i}; \hat{r}) + \omega_{\hat{r}} + \mu$ and is decreasing thereafter. Therefore, $\theta_i(\omega_{\hat{r}}, P_i)$ achieves its maximum at $P_i = \mathcal{C}(W_{\hat{r},i}; \hat{r}) + \omega_{\hat{r}} + \mu$, and $\theta_i(\omega_{\hat{r}})$ thus becomes

$$\theta_i(\omega_{\hat{r}}) = \mu \exp\left(\frac{\mathcal{V}(W_{\hat{r},i}; \hat{r}) - \mathcal{C}(W_{\hat{r},i}; \hat{r}) - \omega_{\hat{r}} - \mu}{\mu}\right) > 0.$$

It is thus clear that there exists a unique $\omega^*_{\hat{r}}$ such that Equation (19) (or equivalently, Equation (9)) is satisfied. For a given option of age limit $W_{\hat{r},i}$ and usage limit $\hat{r}W_{\hat{r},i}$, its optimal price should be set to $P^*_i = \mathcal{C}(W_{\hat{r},i}; \hat{r}) + \omega^*_{\hat{r}} + \mu$.

When the optimal pricing policy above is implemented, the resulting attach rate of $\mathcal{M}_{\hat{r}}$ becomes

$$\mathcal{D}^*_{\hat{r}}(\mathcal{M}_{\hat{r}}) = \sum_{i \in \mathcal{N}} d^*_{\hat{r},i}(\mathcal{M}_{\hat{r}}) = \frac{\sum_{i \in \mathcal{N}} \exp\left(\frac{\mathcal{V}(W_{\hat{r},i}; \hat{r}) - P^*_i}{\mu}\right)}{1 + \sum_{i \in \mathcal{N}} \exp\left(\frac{\mathcal{V}(W_{\hat{r},i}; \hat{r}) - P^*_i}{\mu}\right)} = \frac{\omega^*_{\hat{r}}/\mu}{1 + \omega^*_{\hat{r}}/\mu} = \frac{\omega^*_{\hat{r}}}{\mu + \omega^*_{\hat{r}}}.$$

The third equality follows directly from Equation (9). This completes the proof. $\square$

Proof of Proposition 1. According to Lemma 1, the warranty limits of any menu $\mathcal{M}_r$ ($r \neq \hat{r}$) can be determined through the relationship $\mathcal{F}_{r,i} = \mathcal{F}_{\hat{r},i}$, $\forall i \in \mathcal{N}$. Based on the expression of $\mathcal{F}_{r,i}$ in Equation (4), solving $\mathcal{F}_{r,i} = \mathcal{F}_{\hat{r},i}$ for $W_{r,i}$ yields Equation (11). $\square$

Proof of Lemma 2. The proof is quite similar to that of Lemma 1 and thus omitted. $\square$

Proof of Theorem 2. To solve the problem $\max_{\boldsymbol{P},\boldsymbol{K}} \widetilde{\Omega}_{\hat{r}}(\widetilde{\mathcal{M}}_{\hat{r}})$, it is helpful to think that the scenario is equivalent to the provider first determining the numbers of PM services and then immediately specifying prices.

Let $\widetilde{\Omega}_{\hat{r}}(\widetilde{\mathcal{M}}_{\hat{r}}) = \widetilde{\omega}_{\hat{r}}$ for any $\boldsymbol{P}$ and $\boldsymbol{K}$. Following the same procedures as in the proof of Theorem 1, we know that the optimal price of option i in $\widetilde{\mathcal{M}}_{\hat{r}}$ should be set to $P_i = \widetilde{\mathcal{C}}(W_{\hat{r},i}, k_i; \hat{r}) + \widetilde{\omega}_{\hat{r}} + \mu$, where $\widetilde{\omega}_{\hat{r}}$ is the unique solution to

$$\max_{\boldsymbol{K}} \sum_{i \in \mathcal{N}} \exp\left(\frac{\widetilde{\mathcal{V}}(W_{\hat{r},i}, k_i; \hat{r}) + k_i u^{pm} - \widetilde{\mathcal{C}}(W_{\hat{r},i}, k_i; \hat{r}) - \widetilde{\omega}_{\hat{r}} - \mu}{\mu}\right) = \frac{\widetilde{\omega}_{\hat{r}}}{\mu}.$$

Notice that the left-hand side can be completely separated for each option i, implying that the multi-dimensional PM optimization problem can be decoupled into multiple single-variable optimization problems for each k_i. Therefore, the optimal number of PM services attached to option $i \in \mathcal{N}$ is given by $k_i^* = \arg\max_{k_i}\{\zeta(k_i)\}$, where $\zeta(k_i) = \widetilde{\mathcal{V}}(W_{\hat{r},i}, k_i; \hat{r}) + k_i u^{pm} - \widetilde{\mathcal{C}}(W_{\hat{r},i}, k_i; \hat{r})$.

Furthermore, when the optimal prices $P_i^* = \widetilde{\mathcal{C}}(W_{\hat{r},i}, k_i^*; \hat{r}) + \widetilde{\omega}_{\hat{r}}^* + \mu$, $\forall i \in \mathcal{N}$, are implemented, the resulting attach rate of $\mathcal{M}_{\hat{r}}$ becomes

$$\widetilde{\mathcal{D}}_{\hat{r}}^*(\widetilde{\mathcal{M}}_{\hat{r}}) = \sum_{i\in\mathcal{N}} \widetilde{d}_{\hat{r},i}^*(\widetilde{\mathcal{M}}_{\hat{r}}) = \frac{\sum_{i\in\mathcal{N}} \exp\left(\frac{\widetilde{\mathcal{V}}(W_{\hat{r},i};\hat{r}) + k_i^* u^{pm} - P_i^*}{\mu}\right)}{1 + \sum_{i\in\mathcal{N}} \exp\left(\frac{\widetilde{\mathcal{V}}(W_{\hat{r},i};\hat{r}) + k_i^* u^{pm} - P_i^*}{\mu}\right)} = \frac{\widetilde{\omega}_{\hat{r}}^*}{\mu + \widetilde{\omega}_{\hat{r}}^*}.$$

The third equality follows directly from Equation (16). This completes the proof. □

Proof of Corollary 2. Recall that $\zeta(k_i) = \widetilde{\mathcal{V}}(W_{\hat{r},i}, k_i; \hat{r}) + k_i u^{pm} - \widetilde{\mathcal{C}}(W_{\hat{r},i}, k_i; \hat{r})$. Then, we have $\zeta(0) = \mathcal{V}(W_{\hat{r},i}; \hat{r}) - \mathcal{C}(W_{\hat{r},i}; \hat{r})$. Notice that the feasible domain of k_i is $\{0, 1, \ldots, k_i^{\max}\}$, and the optimal number of PM services, k_i^*, is determined by solving $\max_{k_i} \zeta(k_i)$. Then, it is clear that $\zeta(k_i^*) \geq \zeta(0)$. By comparing Equations (9) and (16), it is easy to verify, via contradiction, that $\widetilde{\omega}_{\hat{r}}^* \geq \omega_{\hat{r}}^*$.

In addition, according to Equations (10) and (17), we have

$$\widetilde{\mathcal{D}}_{\hat{r}}^*(\widetilde{\mathcal{M}}_{\hat{r}}) = \frac{\widetilde{\omega}_{\hat{r}}^*}{\mu + \widetilde{\omega}_{\hat{r}}^*} = \frac{1}{1 + \mu/\widetilde{\omega}_{\hat{r}}^*} \geq \frac{1}{1 + \mu/\omega_{\hat{r}}^*} = \mathcal{D}_{\hat{r}}^*(\mathcal{M}_{\hat{r}}).$$

This completes the proof. □