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Optimal Upgrade Policy for Used Products Sold with Two-dimensional Warranty

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In this paper, a pre-sale upgrade model is developed for repairable used products sold with two-dimensional warranty policies. Two kinds of two-dimensional warranty regions are considered, that is, rectangular and L-shaped regions. The marginal approach is used to describe the effect of age and usage on product reliability by treating usage as a random function of age. The optimal upgrade policy is determined to minimize the dealer's total expected servicing cost which is the sum of upgrade cost and warranty servicing cost. Analytical properties of the optimal solutions are also investigated. Furthermore, a five-step approach is presented to design flexible warranty contracts with optimized upgrade policies for used products, which allow customers to choose the most suitable warranty plans among the available options offered by the dealer. Numerical example and sensitivity analysis are provided to demonstrate the proposed models. This study provides the dealers of used products with useful guidelines on designing flexible and optimized warranty contracts. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: used products; two-dimensional warranty; upgrade; flexible warranty

1. Introduction

Warranty is a contractual obligation incurred by a manufacturer (vendor or seller) in connection with the sale of a product. In broad terms, the purpose of warranty is to establish liability in the event of a premature failure of an item or the inability of the item to perform its intended function.[1] Nowadays, product warranty has become an integral part of most manufacturers' and dealers' marketing strategies, and many types of warranty policies are offered to promote the products.[2–6]

Based on the number of variables that are used to define the policy, warranty can be broadly categorized into two types, that is, one-dimensional (1D) and two-dimensional (2D) policies.[1] A 1D policy is characterized by an interval defined in terms of a single variable, for example, only time or only usage. In contrast, a 2D policy is represented by a region in a 2D plane, generally with one dimension denoting time and the other one usage. In practice, many products deteriorate with both time and usage, for example, automobiles and printers, and so on. Thus, 2D warranty policies are more suitable for such products. For instance, a new automobile is usually covered by a 2D warranty, say for 3 years or 100,000 km, whichever comes first.

The successful application of 2D warranty policies, especially in the automobile industry, booms the academic research. In the last decades, considerable research has been conducted on 2D warranty policies for new products. Su and Shen[7] presented cost and profit models for two types of extended warranty policies with different corrective repair options. Considering two types of failures, namely, normal and defect failures, Ye *et al.*[8] proposed a new burn-in modeling approach for repairable products sold with 2D warranty. Huang *et al.*[9] developed a 2D warranty model with reliability-based preventive maintenance (PM) to maximize the manufacturer's profit. Moreover, Wu[10] proposed a new method of constructing asymmetric copulas, which can effectively capture the tail dependence that exists between the pair of age and usage, to fit the real 2D reliability data. When the future age of the system is easily predicted whereas future usage values is typically unknown, Lu and Anderson-Cook[11] developed a methodology for predicting both individual and population reliability. Tong *et al.*[12] presented a designing and pricing model of 2D extended warranty contracts with consideration of various maintenance strategies. Recently, Wang *et al.*[13] studied the optimal PM strategy for products during both 2D basic and extended warranty regions from the manufacturer's perspective.

Comparing with the research on 2D warranties for new products, 2D warranty policies for used/second-hand products receive much less attention. To the best of our knowledge, there are three reported papers on 2D warranty policies for used products. Chattopadhyay and Yun[14] proposed a mathematical model for estimating warranty cost and determining the best decision on price and 2D warranty terms for second-hand products. Shafiee *et al.*[15] developed statistical models to estimate

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the dealer's expected warranty cost for second-hand products sold with 2D warranty policies. Moreover, two stochastic models were proposed in Sarada and Mubashirunnissa[16] to calculate the expected warranty cost for second-hand products under various 2D warranty policies.

Used products often incur higher warranty servicing cost for the dealer, because they usually have more breakdowns than new ones. In such case, reliability improvement strategies for used products, such as overhaul, upgrade, and remanufacturing, are effective ways to improve product reliability and reduce warranty servicing cost. However, existing literature on this topic focuses only on 1D warranty policies. For example, Chattopadhyay and Murthy[17] developed two models to assess the reliability improvement strategies for used items sold with free repair/replacement warranty policies. Lo and Yu[18] proposed a mathematical model to optimize the upgrade policy and warranty length for used products. In addition, Yazdian *et al.*[19] investigated joint optimization of remanufacturing degree, pricing, and warranty decision-making for end-of-life products to maximize the remanufacturer's profit. To maximize the dealer's profit, Su and Wang[20] developed a mathematical model to determine the optimal upgrade level and PM policy for used products. Kim *et al.*[21] optimized a periodic inspection/upgrade policy for second-hand products under warranty in order to minimize the dealer's total expected cost. Moreover, some other related work has been carried out by Shafiee and his co-authors. The reader can refer to Shafiee and Chukova[22] and the references therein on their latest development in this area.

Up to now, modeling and optimizing upgrade strategies for used products under 2D warranty policies, nevertheless, have received little attention. This study aims to fulfill this gap. In the present paper, a model of upgrade action is proposed for used products sold with 2D free repair warranty policies. Meanwhile, two kinds of warranty regions are considered, that is, rectangular and L-shaped regions. The objective of this study is to determine the optimal upgrade policies to minimize the dealer's total expected servicing costs, and analytical properties of the optimal solutions are also investigated. The proposed models are then applied to determine flexible and optimized warranty contracts for used products, which can offer various warranty options to customers with different usage rates.

It is worth noting that the problem of interest has strong engineering practice background. In recent years, the market of used/second-hand items has been on the rise. For example, the ratio of used-to-new car sale in the USA has increased from 2.4 to 3.2 during 2000–2010. In 2010, approximately 11.58 million new cars were sold in the USA, compared with the sale of 36.88 million used cars from both dealers and end users.[23] In practice, a used car is usually sold with a new 2D warranty. For instance, in the USA, every Ford certified pre-owned used car comes with a comprehensive 12 months/12000 miles warranty (www.ford.com). With economy and safety considerations, for a used car, the dealer often carry out some pre-sale upgrade and inspection actions on certain sub-systems to restore the car to a better state. Modeling and optimizing upgrade policy for used products under 2D warranty policies are the main concern of this paper.

The reminder of this paper is structured as follows. Section 2 develops an optimization model for determining the optimal upgrade policies under rectangular and L-shaped 2D warranty contracts. In Section 3, the proposed approach is used to design flexible and optimized warranty contracts. Numerical example and sensitivity analysis are presented in Section 4. The last section concludes this work and suggests the future work.

2. Model development

2.1. Modeling product failures

In the literature, the approaches to modeling 2D failure process can be divided into three main categories, that is, marginal (univariate), composite scale, and bivariate approaches, see Jack *et al.*[24] and Wu[10,25] for more discussions on these approaches.

In this study, the marginal approach is used to describe the product failure process in the context of 2D warranty. Here, we consider repairable products, which deteriorate with both time and usage. For an elapsed time t, the cumulative operating time and total usage of a product can be denoted by $T(t)$ and $U(t)$, respectively. These two time-dependent measures would be the metrics used in the context of 2D warranty modeling.[9] For simplicity, it is assumed that the age of a product is equivalent to its cumulative operating time, that is, $T(t) = t$. Note that this simplification can only be true when either no failure occurs in $[0, t)$ or all failures are repaired and the repair times are negligible.[26] Let R be the usage rate (ratio of usage to time), that is, $R = U(t)/T(t)$, and assume that the usage rate varies from customer to customer but is constant for a given customer. In this way, R can be modeled as a random variable with a distribution function $G(r) = P\{R \leq r\}$, $0 \leq r < \infty$. This indicates a linear relationship between the cumulative operating time and total usage. In our simplified case, the usage rate becomes $R = U(t)/t$.

Thus, item failures can be modeled by a point process with an intensity function that is dependent on both age and usage. Given $R = r$, the conditional intensity function can be denoted as $\lambda(t|r) = \varphi(T(t), U(t))$, where $\varphi(\cdot)$ is a non-decreasing function of both $T(t)$ and $U(t)$. Suppose that all failures are minimally repaired and the repair times are negligible, then failures over time occur according to a non-homogeneous Poisson process (NHPP) with the conditional failure intensity function $\lambda(t|r)$. Following Iskandar and Murthy,[26] the conditional intensity function of the NHPP is modeled by

$$\lambda(t|r) = \theta_0 + \theta_1 r + \theta_2 T(t) + \theta_3 U(t) \tag{1}$$

where θ_0, θ_1, θ_2, and θ_3 are positive constants, which can be estimated through the historical records of warranty claims and maintenance activities kept by the dealer.

Note that the intensity function given by (1) is widely used in the literature on 2D warranty, see Iskandar and Murthy,[26] Huang *et al.*[9] and Wang *et al.*[13] for example. In our simplified case $T(t)=t$ and $U(t)=r{\cdot}t$ given $R=r$, and (1) can thus be simplified to

$$\lambda(t|r) = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)t \tag{2}$$

It can be seen that the 2D failure modeling problem is reduced to a 1D one by using the marginal method.

2.2. Modeling upgrade effect

Suppose that a used item with past age A and usage B is bought by the dealer from its first user, and it has been maintained or repaired minimally with negligible repair times during its past life until the age of A and usage of B. Afterwards, the used item is subjected to an instantaneous upgrade action prior to its re-sale. We model the upgrade action by using the black-box approach and ignoring its detailed procedures. Here, the effect of upgrade action on product reliability is characterized by the reduction of conditional intensity function of the failure process.[27]

In this study, we assume that the used item's conditional failure intensity after upgrade is equal to the residual failure intensity immediately after upgrade plus the failure intensity at renewed age of t. Therefore, given an item has survived for A time units and B usage units, after upgrade action its conditional failure intensity at time t is given by

$$\lambda_p(t|r) = (1-p)[\lambda(A|r_1) - \lambda(0|r_1)] + \lambda(t|r), t\geq 0 \tag{3}$$

where $r_1=B/A$ is the used item's average usage rate during its past life; $p\in[0, 1]$ is the upgrade degree, which is used to describe the effect of upgrade action; while $\lambda(0|r_1)$ is the initial failure intensity under usage rate r_1, which is non-zero.

Note that $p=0$ corresponds to no reliability improvement; $p=1$ indicates that the item is restored to as good as new; while $p\in(0, 1)$ implies that the failure intensity is partially reduced. After upgrade, the used item's failure intensity at time t is influenced by the upgrade degree p, but its degradation pattern remains unchanged.

Suppose that the cost of an upgrade action equals the setup cost plus the term modeled by an increasing power function of factor p. Thus, the upgrade cost $C_u(p)$ is modeled by

$$C_u(p) = C_s + C_v p^{\varphi} A^{\xi} B^{\zeta} \tag{4}$$

where C_s is the setup cost; C_v, φ, ξ and ζ are positive real parameters. The proposed cost model is a 2D extension of the cost model introduced by Chattopadhyay and Murthy.[17] It implies that the upgrade cost increases with the increase of A, B, and/or p.

2.3. Two-dimensional warranty policies for used products

Different regions define different 2D warranty policies. Blischke and Murthy[1] introduce several possible shapes of 2D warranty policies. In this paper, rectangular and L-shaped regions are considered because of their popularity in the literature and their simplicity in comprehension by customers. Considering a used product with past age A and past usage B, denote the time limit and usage limit of the 2D warranty policies by W and U, respectively. Then, the two kinds of warranty regions can be defined as follows.

Rectangular warranty region: This policy is defined by a rectangular region: $\Omega_R(W,U)=\{(t,u): t\in(0,W) \text{ and } u\in(0,U)\}$, as shown in Figure 1(a). Under this policy, the dealer agrees to repair all failures free of charge up to time limit W or up to usage limit U from the point of the new sale, whichever occurs first. In other words, the customer is guaranteed a maximum coverage for W units of time and for U units of usage after sale.[1,28]

L-shaped warranty region: This policy is characterized by a L-shaped region: $\Omega_L(W,U)=\{(t,u): t\in(0,W) \text{ or } u\in(0,U)\}$, as depicted in Figure 1(b). Under this policy, the dealer agrees to provide corrective repairs free of charge up to a minimum time W and up to a minimum total usage U from the point of the new sale. In other words, this policy assures the customer a minimum coverage for W units of time and U units of usage after sale.[1,28]

Murthy *et al.*[28] argue that the rectangular policy tends to favor customers with medium usage intensity, whereas the L-shaped policy tends to favor customers with both light and heavy usage intensities. This argument will be supported by our numerical results in Section 4.1.1. The qualitative explanations are as follows.

Under the rectangular policy, for a customer with high usage rate ($r>\eta=U/W$), the warranty will expire before the time limit W because of the usage exceeding its limit U; while for a customer with low usage rate ($r<\eta$), the warranty would terminate at W with its total usage at expiry below U. That is to say, for customers with medium usage intensity, their actual warranty regions are generally larger than those for customers with either heavy or light usage intensity. On the other hand, under the L-shaped policy, for customers with high usage intensity, the warranty ceases mainly because of time limit W with the total usage at expiry beyond limit U; while for the low usage intensity, the warranty terminates mainly because of the usage limit U with the time at expiry beyond limit W. This implies that the case of L-shaped policy is opposite to that of rectangular policy.

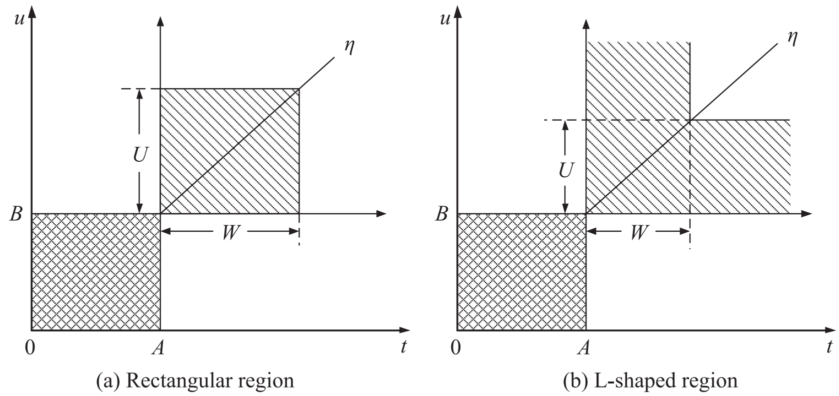

Figure 1. Two-dimensional warranty regions for used products

2.4. Modeling expected warranty cost

In this section, the expected warranty costs for used products under rectangular and L-shaped 2D warranty policies are derived from the dealer's perspective. Assume that all product failures under warranty are rectified minimally at no cost to the customers then the failure process during the warranty period is also a NHPP. Besides, assume that product failures can be detected immediately and then result in warranty claims, and all warranty claims are valid.

Under the rectangular policy, two cases should be considered: $r \le \eta$ and $r > \eta$. When $r \le \eta$, the warranty expires at time W and an estimate of the total usage is $r{\cdot}W$; when $r > \eta$, the warranty expires at age U/r when the usage limit U is reached. Hence, given $R = r$, the conditional expected number of failures over the rectangular warranty region is given by

$$\begin{aligned} E[N_R(W,U,p)|r] &= \begin{cases} \int_0^W \lambda_p(t|r)\mathrm{d}t, & \text{if } r \le \eta \\ \int_0^{U/r} \lambda_p(t|r)\mathrm{d}t, & \text{if } r > \eta \end{cases} \\ &= \begin{cases} (1-p)[\lambda(A|r_1) - \lambda(0|r_1)]W + \Lambda(W|r), & \text{if } r \le \eta \\ (1-p)[\lambda(A|r_1) - \lambda(0|r_1)]\dfrac{U}{r} + \Lambda\left(\dfrac{U}{r}|r\right), & \text{if } r > \eta \end{cases} \end{aligned} \tag{5}$$

where $\Lambda(t|r) = \int_0^t \lambda(x|r)\mathrm{d}x$.

Removing the condition $R = r$, the total expected number of failures over the rectangular warranty region is obtained as

$$\begin{aligned} E[N_R(W,U,p)] = &\int_0^\eta \{(1-p)[\lambda(A|r_1) - \lambda(0|r_1)]W + \Lambda(W|r)\}\mathrm{d}G(r) \\ &+ \int_\eta^\infty \left\{(1-p)[\lambda(A|r_1) - \lambda(0|r_1)]\frac{U}{r} + \Lambda\left(\frac{U}{r}|r\right)\right\}\mathrm{d}G(r) \end{aligned} \tag{6}$$

Following similar procedures, the total expected number of failures over the L-shaped warranty region can be calculated by

$$\begin{aligned} E[N_L(W,U,p)] = &\int_0^\eta \left\{(1-p)[\lambda(A|r_1) - \lambda(0|r_1)]\frac{U}{r} + \Lambda\left(\frac{U}{r}|r\right)\right\}\mathrm{d}G(r) \\ &+ \int_\eta^\infty \{(1-p)[\lambda(A|r_1) - \lambda(0|r_1)]W + \Lambda(W|r)\}\mathrm{d}G(r) \end{aligned} \tag{7}$$

Suppose that each failure during the warranty period incurs an average minimal repair cost C_r. Then, under 2D free repair warranty policies, the expected warranty servicing cost for a used product with time limit W, usage limit U and upgrade degree p is given by

$$E[C_{wi}(W,U,p)] = C_r E[N_i(W,U,p)], i = R, L \tag{8}$$

where $E[N_R(W, U, p)]$ and $E[N_L(W, U, p)]$ are given by (6) and (7), respectively.

2.5. Optimization model

The total expected servicing cost of a used product includes the investment in the upgrade program and the expected warranty servicing cost. It can be obtained by

$$C_i(W,U,p) = C_u(p) + E[C_{wi}(W,U,p)], i = R, L \tag{9}$$

where $C_u(p)$ is given by (4) and $E[C_{wi}(W, U, p)]$ by (8).

The objective of this paper is to determine the optimal upgrade degree (p^*) to minimize the dealer's total expected servicing cost. Hence, the optimization problem is

$$\min_p C_i(W,U,p), i = R, L \ \text{ subject to } 0 \le p \le 1 \tag{10}$$

First of all, we consider the optimization problem in the case of rectangular warranty policy. To investigate the optimal upgrade policy, we take the first derivative of $C_R(W, U, p)$ with respect to p, which is

$$\frac{\partial C_R(W,U,p)}{\partial p} = \varphi C_v p^{\varphi-1} A^{\xi} B^{\zeta} - C_r[\lambda(A|r_1) - \lambda(0|r_1)]\left[\int_0^{\eta} W \mathrm{d}G(r) + \int_{\eta}^{\infty} (U/r)\mathrm{d}G(r)\right] \tag{11}$$

Setting (11) equal to zero, we have

$$p_R = \left\{ C_r[\lambda(A|r_1) - \lambda(0|r_1)] \frac{\int_0^{\eta} W \mathrm{d}G(r) + \int_{\eta}^{\infty} (U/r)\mathrm{d}G(r)}{\varphi C_v A^{\xi} B^{\zeta}} \right\}^{\frac{1}{\varphi-1}} \tag{12}$$

According to the boundary condition $0 \le p \le 1$, there are two extreme points of $\partial C_R(W,U, p)/\partial p$. Substituting $p = 0$ and $p = 1$ into (11), respectively, we have

$$\frac{\partial C_R(W,U,p)}{\partial p}\Big|_{p=0} = \lim_{p \to 0} \varphi C_v p^{\varphi-1} A^{\xi} B^{\zeta} - C_r[\lambda(A|r_1) - \lambda(0|r_1)]\left[\int_0^{\eta} W \mathrm{d}G(r) + \int_{\eta}^{\infty} (U/r)\mathrm{d}G(r)\right]$$

and

$$\frac{\partial C_R(W,U,p)}{\partial p}\Big|_{p=1} = \varphi C_v A^{\xi} B^{\zeta} - C_r[\lambda(A|r_1) - \lambda(0|r_1)]\left[\int_0^{\eta} W \mathrm{d}G(r) + \int_{\eta}^{\infty} (U/r)\mathrm{d}G(r)\right]$$

Further taking the second derivative of $C_R(W, U, p)$ with respect to p, we have

$$\frac{\partial^2 C_R(W,U,p)}{\partial p^2} = \varphi(\varphi - 1) C_v p^{\varphi-2} A^{\xi} B^{\zeta} \tag{13}$$

When $\varphi \le 1$, $\partial^2 C_R(W, U, p)/\partial p^2$ is negative or zero for all p, which implies that $C_R(W, U, p)$ is a concave function. According to the boundary condition $0 \le p \le 1$, we know that the minimum total servicing cost is achieved at one of the two boundaries, that is, either at $p = 0$ or at $p = 1$, depending on which one leading to a lower total cost. Hereafter, we only consider the case of $\varphi > 1$.

The following theorem shows that there exists a unique optimal upgrade degree p^* such that $C_R(W, U, p)$ is minimized.

Theorem 1

When $\varphi > 1$, the following results hold for the rectangular warranty policy.

(i) If $\varphi C_v A^{\xi} B^{\zeta} - C_r[\lambda(A|r_1) - \lambda(0|r_1)]\left[\int_0^{\eta} W \mathrm{d}G(r) + \int_{\eta}^{\infty} (U/r)\mathrm{d}G(r)\right] \le 0$, then $p^* = 1$.

(ii) If $\varphi C_v A^{\xi} B^{\zeta} - C_r[\lambda(A|r_1) - \lambda(0|r_1)]\left[\int_0^{\eta} W \mathrm{d}G(r) + \int_{\eta}^{\infty} (U/r)\mathrm{d}G(r)\right] > 0$, then there exists a unique solution p^* within (0, 1), which is given by (12).

Proof

When $\varphi > 1$, $\partial^2 C_R(W, U, p)/\partial p^2$ is positive for all p, then we know that $\partial C_R(W, U, p)/\partial p$ is an increasing function of $p \in [0, 1]$, and $C_R(W, U, p)$ is a convex function. Also, we know that $\lim_{p \to 0} \varphi C_v p^{\varphi-1} A^{\xi} B^{\zeta} = 0$, thus $\partial C_R(W, U, p)/\partial p|_{p=0}$ is always negative.

(i) If $\varphi C_v A^{\xi} B^{\zeta} - C_r[\lambda(A|r_1) - \lambda(0|r_1)]\left[\int_0^{\eta} W \mathrm{d}G(r) + \int_{\eta}^{\infty} (U/r)\mathrm{d}G(r)\right] \le 0$, then $\partial C_R(W, U, p)/\partial p|_{p=1} \le 0$. This implies that $\partial C_R(W, U, p)/\partial p$ is always non-positive, thus $C_R(W, U, p)$ is a non-increasing function of $p \in [0, 1]$. Therefore, the optimal upgrade degree is $p^* = 1$.

(ii) If $\varphi C_v A^{\xi} B^{\zeta} - C_r[\lambda(A|r_1) - \lambda(0|r_1)]\left[\int_0^{\eta} W \mathrm{d}G(r) + \int_{\eta}^{\infty} (U/r)\mathrm{d}G(r)\right] > 0$, then $\partial C_R(W,U,p)/\partial p$ changes its sign exactly once in the interval (0, 1) because $\partial C_R(W, U, p)/\partial p$ is a strictly increasing function of p. This implies that there exists a unique solution p^* within (0, 1), such that $C_R(W, U, p)$ can be minimized. The solution is given by (12). □

The optimization procedure in the case of L-shaped policy is quite similar to that of rectangular policy; hence, we present the following theorem without further proof.

Theorem 2
When $\varphi > 1$, the following results hold for the L-shaped warranty policy.

(i) If $\varphi C_v A^{\xi} B^{\zeta} - C_r[\lambda(A|r_1) - \lambda(0|r_1)]\left[\int_0^{\eta}(U/r)\mathrm{d}G(r) + \int_{\eta}^{\infty} W\mathrm{d}G(r)\right] \le 0$, then $p^* = 1$.

(ii) If $\varphi C_v A^{\xi} B^{\zeta} - C_r[\lambda(A|r_1) - \lambda(0|r_1)]\left[\int_0^{\eta}(U/r)\mathrm{d}G(r) + \int_{\eta}^{\infty} W\mathrm{d}G(r)\right] > 0$, then there exists a unique solution p^* within (0, 1), which is given by (14) below.

$$p_L = \left[C_r[\lambda(A|r_1) - \lambda(0|r_1)]\frac{\int_0^{\eta}(U/r)\mathrm{d}G(r) + \int_{\eta}^{\infty} W\mathrm{d}G(r)}{\varphi C_v A^{\xi} B^{\zeta}}\right]^{\frac{1}{\varphi-1}} \tag{14}$$

3. Determining flexible warranty policies

In practice, customers are usually offered warranty contracts with fixed region regardless of their usage rates. This is typical in the automobile market today. Moskowitz and Chun[29] argue that the usage pattern of an automobile driven by a grandmother is often quite different from that driven by a travelling salesman, thus it would be more wise to offer different warranty regions to them. This argument supports the utilization of flexible warranty policies. The flexible warranty policies, which allow customers to choose any warranty plans among the available options offered by the manufacturer or the dealer, can help to satisfy the needs of different customers without incurring any additional warranty costs.[30]

Here, we focus on deriving flexible warranty contracts under the condition that the total expected servicing cost is specified. Let us denote the specified total expected servicing cost per unit sale by C_0. Then, for given A, B and p, the iso-cost curves for rectangular and L-shaped policies can be generated by the following equation:

$$C_u(p) + E[C_{wi}(W, U, p)] = C_0, \text{ for } i = R, L \tag{15}$$

For given A and B, the following five-step approach is proposed to draw the iso–cost curves:

Step 1 Set N discrete values of η within (0, ∞), namely, $\eta_1, \eta_2, \ldots, \eta_N$, in an ascending order. For the sake of simplicity, the increment $\Delta\eta$ is fixed in this study, that is $\eta_2 - \eta_1 = \eta_3 - \eta_2 = \ldots = \Delta\eta$.
Step 2 Here, for upgrade degree p, its region [0, 1] is discretized into $M + 2$ equidistant values, that is, 0, $p_1, \ldots, p_M$, 1, with $p_1 - 0 = p_2 - p_1 = \ldots = 1 - p_M = \Delta p$.
Step 3 For any given η_n, $n = 1, 2, \ldots, N$, obtain W_{nm} (the corresponding values of W) for any $p_m \in [0, 1]$, by solving $C_u(p_m) + E[C_{wi}(W_{nm}, \eta_n W_{nm}, p_m)] = C_0$, for $i = R, L$.
Step 4 Set $W_n = \max_{m=0,1,\ldots,M+1}\{W_{nm}\}$, and this gives a single point $(W_n, U_n = \eta_n W_n)$ on the iso–cost curve.
Step 5 By connecting all the points $(W_n, U_n = \eta_n W_n)$, for $n = 1, 2, \ldots, N$, the curve can be obtained.

As can be seen, for a specified total expected servicing cost C_0, all possible combinations of time limit and usage limit can be obtained by the proposed approach. As expected, plotting these iso–cost curves can help the dealer to offer a variety of contracts with the same total servicing cost.

It is worth pointing out that the aforementioned five-step approach is a modification of the approach proposed by Manna *et al.*[30] One possible alternative is the regression approach proposed by Shahanaghi *et al.*,[31] which is simple and practical at the expense of potential precision loss.

4. Numerical example

Since 2D warranty policies are most often applied in the automobile industry, this study presents an application of the proposed model to a problem faced by a used automobile dealer. Here, suppose that the item under consideration is a repairable component of an automobile sold with a 2D free repair warranty policy, which can be restored to a better working state by upgrade and/or maintenance.

4.1. Optimal upgrade policies

In this study, the time is measured in unit of years and the usage in 10^4 km, and the unit for cost is US Dollar ($). Following Iskandar and Murthy,[26] the intensity function in (2) is given by $\lambda(t|r) = 0.1 + 0.2r + (0.3 + 0.3r)t$. Besides, the usage rate R is assumed to be uniformly distributed with a lower bound $r_l = 0.5$ and upper bound $r_u = 2.5$. Suppose that a used automobile with past age $A = 2$ years and past usage $B = 2 \times 10^4$ km is subjected to an instantaneous upgrade action with cost parameters: $C_s = \$20$, $C_v = \$100$, $\varphi = 4$,

$\xi = 0.45$, $\zeta = 0.35$. After upgrade, the item is sold with 2D free repair warranty policies with $W = 3$ years and $U = 3 \times 10^4$ km. Furthermore, the average minimal repair cost is set as $C_r = \$40$. Under the aforementioned parameter setting, the optimal upgrade policies are shown in Figure 2.

It can be seen that under the rectangular policy, the optimal upgrade degree p^* is 0.53 and the dealer's total expected cost is \$177.23; while under the L-shaped policy, the optimal upgrade degree p^* is 0.61, and the total expected cost is \$313.44. As can be seen, the optimal upgrade degree and the corresponding servicing cost under the L-shaped policy are higher than those under the rectangular policy. The reason behind this observation is intuitive; that is, with fixed W and U, the rectangular warranty region is a subset of the L-shaped one.

4.1.1. Sensitivity analysis of customer usage types It is intuitively known that different usage types (i.e., usage intensities) may result in different upgrade policies. Here, we consider three sets of values for r_l and r_u, which broadly correspond to light, medium, and heavy users, respectively. Following Iskandar and Murthy,[26] these values and the corresponding mean usage rates ($E[R]$) are listed in Table I. Under different usage types (other parameters remain unchanged), the optimal upgrade policies and the corresponding total expected costs are shown in Table I.

As can be seen, under the rectangular policy, the total cost for customers with medium usage intensity is the highest, while those for light and heavy usage intensities are lower, which is somewhat counterintuitive. This is due to a unique property of the rectangular policy, that is, for customers with both heavy and light usage intensities, this warranty policy tends to favor the dealer (Section 2.3). For such customers, their actual warranty regions are normally smaller than those for customers with medium usage intensity. Smaller warranty region results in lower warranty cost. On the contrary, under the L-shaped policy, the total cost for customers with medium usage intensity is the lowest, while those for light and heavy usage intensities are higher. This observation is consistent with the argument—L-shaped policy tends to favor customers with both low and high usage intensities. However, this pattern will change depending on the parameter values and the structure of the conditional intensity function.

4.1.2. Sensitivity analysis of warranty limits The time limit and usage limit of 2D warranty policies are also key factors that can affect the optimal upgrade policies. Longer warranty period increases the warranty servicing cost; thus, the optimal upgrade degree may increase accordingly. The effect of warranty limits on the optimal upgrade policies and the corresponding total costs under rectangular and L-shaped policies are summarized in Tables II and III, respectively.

It can be observed from Tables II and III that under both rectangular and L-shaped policies, the optimal upgrade degree and the dealer's total servicing cost tend to increase as the time limit and/or usage limit increases.

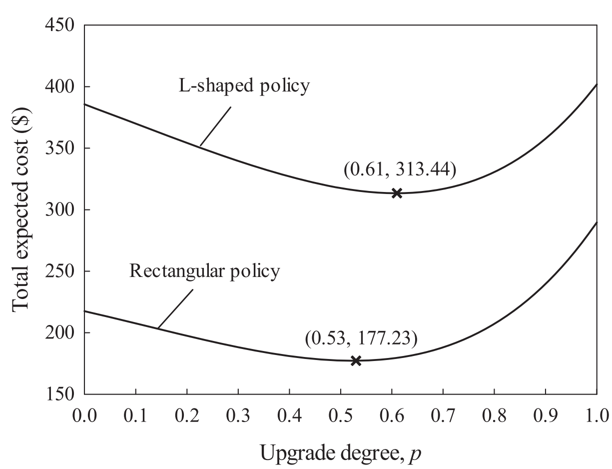

Figure 2. Illustration of the existence of optimal upgrade policies

Table I. Optimal upgrade policies under various usage types

Usage type	r_l	r_u	$E[R]$	Rectangular policy p^*	Rectangular policy C_R	L-shaped policy p^*	L-shaped policy C_L
Light	0.1	0.9	0.5	0.59	205.14	0.83	975.13
Medium	0.7	1.3	1.0	0.58	224.17	0.61	273.23
Heavy	1.1	2.9	2.0	0.48	146.04	0.59	322.14

Table II. The impact of warranty limits on the result under the rectangular policy

	$W=1$		$W=2$		$W=3$		$W=4$		$W=5$	
	p^*	C_R	p^*	C_R	p^*	C_R	p^*	C_R	p^*	C_R
$U=1$	0.37	62.18	0.38	68.43	0.38	68.43	0.38	68.43	0.38	68.43
$U=2$	0.41	81.87	0.46	113.87	0.48	126.78	0.48	130.37	0.48	130.37
$U=3$	0.41	84.24	0.50	145.32	0.53	177.23	0.54	195.65	0.55	205.39
$U=4$	0.41	84.24	0.51	164.35	0.56	218.03	0.58	252.92	0.59	276.54
$U=5$	0.41	84.24	0.52	170.81	0.58	249.33	0.61	302.54	0.62	341.31

Table III. The impact of warranty limits on the result under the L-shaped policy

	$W=1$		$W=2$		$W=3$		$W=4$		$W=5$	
	p^*	C_L	p^*	C_L	p^*	C_L	p^*	C_L	p^*	C_L
$U=1$	0.42	90.29	0.52	170.81	0.59	283.14	0.65	422.28	0.70	588.80
$U=2$	0.48	132.64	0.53	186.80	0.59	286.52	0.65	422.28	0.70	588.80
$U=3$	0.55	208.39	0.57	233.28	0.61	313.44	0.66	434.46	0.70	591.63
$U=4$	0.61	303.28	0.61	309.50	0.63	367.30	0.67	471.38	0.71	614.58
$U=5$	0.65	415.48	0.65	415.48	0.66	448.43	0.69	533.64	0.72	661.27

However, it is interesting to observe from Table II that in the cases of $W=1$ and $U\geq 3$, p^* and the total cost remain unchanged. This is because for a specific time limit, the increase of usage limit would not affect p^* and the total cost any more when $\eta > r_u$, where $\eta = U/W$. On the other hand, in the cases of $U=1$ and $W\geq 2$, and $U=2$ and $W\geq 4$, p^* and the total cost also keep unchanged. This is because for a specific usage limit, the increase of time limit would not affect p^* and the total cost any more when $\eta < r_l$.

Similar phenomena can be seen from Table III. In the cases of $W=4$ and $U\leq 2$, and $W=5$ and $U\leq 2$, p^* and the total cost keep unchanged. This observation can be explained by the fact that for a specific time limit, the decrease of usage limit would not affect p^* and the total cost any more when $\eta < r_l$. On the other hand, in the cases of $U=5$ and $W\leq 2$, p^* and the total cost also remain unchanged. This is because for a specific usage limit, the decrease of time limit would not affect p^* and the total cost any more when $\eta > r_u$.

It should be noted that the aforementioned findings result from the assumption of uniform usage rate distribution and the characteristics of rectangular and L-shaped policies, that is, customers are guaranteed with maximum warranty coverage under the rectangular policy, while they are guaranteed a minimum coverage under the L-shaped policy.

4.1.3. Sensitivity analysis of minimal repair cost Table IV shows how the optimal upgrade degree and the dealer's total expected cost change as the minimal repair cost changes.

It shows that under both the two warranty policies, p^* and the total cost increase as C_r increases. This is because that higher failure repair cost will lead to higher warranty cost, thus more upgrade efforts are needed to reduce the failure rate, and thus to mitigate the increase of warranty cost.

In summary, the previous sensitivity analyses demonstrate that the decision on optimal upgrade policy depends on various factors related to used products, for example, customers' usage type, warranty limits, and minimal repair cost.

4.2. Flexible warranty contracts

Using the presented approach in Section 3, this section illustrates how the dealer can design flexible and optimized 2D warranty contracts for used products. The iso–cost curves for rectangular and L-shaped warranty regions are illustrated in Figures 3 and 4, respectively, with the parameter values following directly from Section 4.1 except that W and U are to be determined here.

Table IV. The impact of minimal repair cost on the result

	Rectangular region			L-shaped region	
C_r	p^*	C_R	C_r	p^*	C_L
20	0.42	102.77	20	0.48	174.17
40	0.53	177.23	40	0.61	313.44
60	0.60	247.09	60	0.70	444.48
80	0.66	313.50	80	0.77	569.34
100	0.72	377.07	100	0.83	689.11

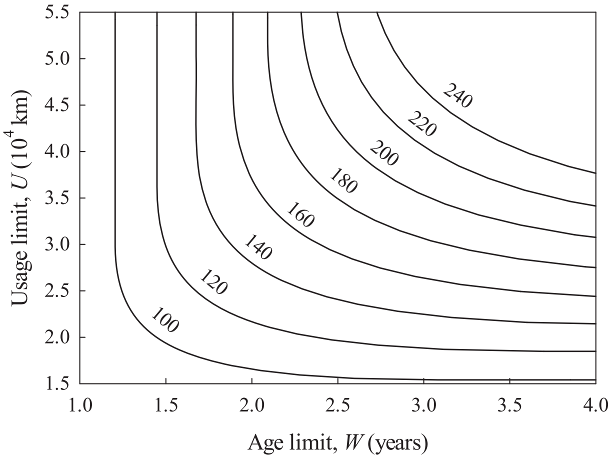

Figure 3. Iso–cost curves of the rectangular warranty policy for used products

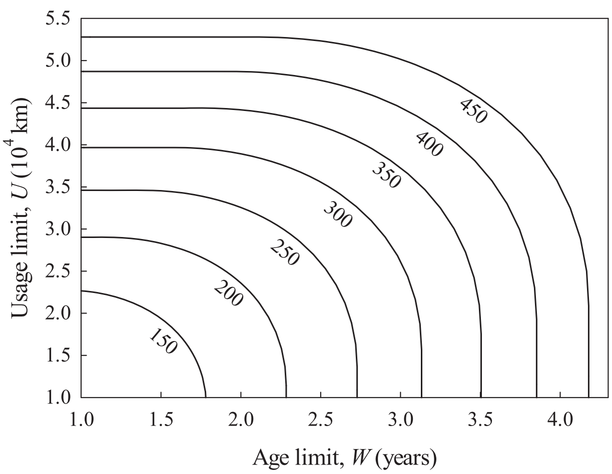

Figure 4. Iso–cost curves of the L-shaped warranty policy for used products

In Figures 3 and 4, the minimum expected servicing costs are shown beside the corresponding curves. As can be seen, for any given total servicing cost C_0, all combinations of W and U that lie on the corresponding contour can be considered as possible warranty contracts. Note that the shapes of iso–cost curves in Figures 3 and 4 are consistent with those in Figure 3 of Manna *et al.*,[30] respectively.

It is interesting to observe that at the upper left and lower right corners of Figure 3, the iso–cost curves become vertical and horizontal lines, respectively. Likewise, in Figure 4, the iso–cost curves become horizontal and vertical lines at the upper left and lower right corners, respectively. The reasons behind these phenomena are similar to those for Tables II and III. For instance, the upper left corner corresponds to high usage limit-to-time limit ratio, that is, $\eta = U/W$. Under the rectangular policy, for a fixed total cost C_0, when η is greater than r_u, further increase of η would not affect the outcome of time limit. Other cases can be explained similarly.

5. Conclusions

With the steady increase of used products' market, offering warranty for used products is becoming popular from the dealer's perspective. However, used products usually incur higher warranty servicing cost to dealers because they have more breakdowns than new products. Modeling and optimizing reliability improvement strategies, such as upgrade and overhaul, for used products has emerged as an effective means to improve product reliability. In this paper, for repairable used products sold with 2D rectangular and L-shaped warranty contracts, a stochastic model is proposed to determine the optimal upgrade policies in order to minimize the dealer's total expected servicing costs. In addition, how to design flexible and optimized warranty contracts for used products is dealt with, and a five-step approach is presented to provide dealers of used products with useful guidelines.

For the same warranty limits (i.e., W and U), the optimal upgrade degree and the corresponding total servicing cost under the L-shaped policy are much higher than those under the rectangular policy, as to be expected. This is possibly why the rectangular policy is the only one offered in the marketplace. Numerical results of this study also support the statement that the rectangular policy tends to favor customers with medium usage intensity, whereas the L-shaped policy tends to favor customers with both light and heavy usage intensities.

There are several possible directions for further research. In this study, the customers' usage rates are assumed to be uniformly distributed, which is not always convinced. Weibull, lognormal, and gamma distributions are also of potential interest. Moreover, developing models to incorporate past usage and maintenance history of used products by using bivariate failure modeling approach is an open problem. Finally, because different warranty regions and upgrade policies can affect the demand of customers, investigating the dealer's expected profit considering these factors is another valuable direction for future research.

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