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Cost analysis of a piece-wise renewing free replacement warranty policy

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ABSTRACT

Nowadays, fierce competition and increasing customer requirements force manufacturers to continuously provide better after-sales services and supports. One such kind of after-sales services is product warranty which has been offered for almost all products in today's market. In this paper, we study a new warranty policy, called piece-wise renewing free replacement warranty. Under this policy, the whole warranty period is divided into two sub-periods and once an item fails in a specific sub-period, it will be replaced by a new identical one and the warranty period is *fully* or *partially* renewed. The expected warranty cost and warranty cycle of this policy are derived from the manufacturer's perspective. The proposed model is then modified by involving three sub-periods and a failure limit, respectively. In the latter scenario, when the number of item failures over a warranty cycle exceeds a pre-specified threshold, the manufacturer has to refund the item's purchase price, and consequently the warranty ceases. A real-world case study is presented using warranty data of a battery manufacturer. It is found that for a given warranty period, the piece-wise renewing free replacement policy has a better performance than traditional fully renewing policy, in terms of expected warranty cost, warranty cycle, and cost rate.

1. Introduction

Warranty is a contractual agreement attached to an item that requires the manufacturer to offer compensation to the consumer according to pre-specified terms when the warranted item fails to perform its intended functions within the warranty period (Blischke & Murthy, 1992). The past decades witnessed significant growth of employing product warranties as a strategic marketing tool to attract potential consumers. Meanwhile, warranty management has gained an extensive interest from different perspectives and disciplines, as summarized and reviewed in Murthy and Djamaludin (2002), Murthy, Solem, and Roren (2004), Shafiee and Chukova (2013), Wang and Xie (2018), and Wu (2013), for example.

Generally speaking, warranty policies can be classified into various categories according to different criteria. Based on whether the policy is renewing or non-renewing, warranties can be categorized into renewing and non-renewing warranty policies (Blischke & Murthy, 1992). Under a renewing policy, whenever an item fails within the warranty period, it is replaced by a new identical one, and the new replacement is attached with a new warranty; while the repair or replacement of a failed item protected by a non-renewing warranty does not alter the original warranty. Both of renewing and non-renewing policies can be further divided into two groups—simple and combination policies. Free replacement warranty and pro-rata warranty are two typical simple policies. Under the former policy, all failures within the warranty period will be replaced by the manufacturer at no cost to consumers, while under the latter policy, the manufacturer only agrees to cover a fraction of replacement cost or to refund a fraction of original purchase price. A combination policy (also called hybrid policy), however, either applies different simple policies during different stages of the warranty period or combines a simple policy with additional features (Blischke & Murthy, 1992). In this paper, we propose and study a new type of renewing free replacement warranty policy.

However, offering warranty is by no means free for manufacturers, given that warranty is a kind of post-sales service. The warranty cost—resulting from the servicing of warranty claims in the field—is a huge financial burden to manufacturers, and can account for as much as 15% of net sales (Murthy & Djamaludin, 2002). As a result, accurate warranty cost evaluation is indispensable to manufacturers, particularly for designing a new warranty policy. Either underestimating or overestimating warranty servicing costs would result in economic losses for manufacturers. Specifically, overestimation may lead to high product price and thus low sales, while underestimation increases liquidity risks. It is therefore not surprising that warranty cost modeling and

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analysis have long been an important research topic; see Eliashberg, Singpurwalla, and Wilson (1997), Frees and Nam (1988), Mi (1999), Murthy, Iskandar, and Wilson (1995), and Thomas and Rao (1999) for some early references. In particular, the monograph (Blischke & Murthy, 1993) presents and summarizes detailed cost models for various warranty policies.

The existing studies on warranty cost modeling and analysis focus predominately on renewing free replacement warranty, non-renewing free repair/replacement warranty, and hybrid or combination warranty, among others. Dimitrov, Chukova, and Khalil (2004) analyzed expected warranty servicing costs for repairable products sold with free replacement (non-renewing and renewing) and linear pro-rata warranties, with age-dependent failure/repair models. Bai and Pham (2006) tackled cost analysis on renewable full-service warranties for repairable multi-component systems. Liu, Wu, and Xie (2015) developed a warranty cost model for a renewing free replacement policy by considering failure interactions in multi-component systems. Marshall, Arnold, Chukova, and Hayakawa (2018) proposed two new cost analysis models for renewing and non-renewing free repair warranties with non-zero increasing repair times. Zhang, Fouladirad, and Barros (2018) evaluated warranty cost for a two-component series system with type I stochastic dependence between the components, under both non-renewing and renewing free replacement policies. For both repairable and non-repairable products, Manna, Pal, and Sinha (2008) discussed the discrepancy in two-dimensional warranty cost models based on univariate and bivariate approaches, respectively. Su and Shen (2012) analyzed expected costs and profits for extended warranties with three corrective repair options. By coupling stochastic sales process with product failure process, Xie, Shen, and Zhong (2017) presented a general aggregate repair demand forecasting model for a two-dimensional free repair warranty policy. Zhao and Xie (2017) and Zhao, He, and Xie (2018) developed warranty cost optimization frameworks, based on life/degradation testing data, for repairable products under imperfect repairs. Furthermore, Chien and Zhang (2015) studied a hybrid warranty policy with a renewable free replacement period and a rebate period for discrete-time operating products. Zhang and Wang (2019) introduced a geometric process repair model to derive the expected warranty cost for a non-renewing combination policy. In recent years, warranty cost analysis has attracted increasing attentions in terms of supporting, e.g., warranty contract design (Tong, Liu, Men, & Cao, 2014; Ye & Murthy, 2016; Zhang, He, He, & Dai, 2019), warranty reserve management (Wang, Xie, Ye, & Tang, 2017; Xie & Ye, 2016), and maintenance decision-making (Chien, 2019; Su & Wang, 2016; Wang, Xie, & Li, 2019), etc.

This paper deals with cost analysis of a new piece-wise renewing free replacement warranty policy, which has not been studied in the existing literature. Under this policy, the entire warranty period is divided into two sub-periods and different warranty renewing mechanisms, following failure replacements, are applied in these periods. The expected warranty cost and warranty cycle of this policy are derived from the manufacturer's perspective. After that, the proposed model is modified by considering one more sub-period and a failure limit within the warranty cycle, respectively. In the latter scenario, the manufacturer has to return original purchase price to the consumer should the number of failures over a warranty cycle exceed a pre-specified threshold, and consequently the warranty is terminated. This failure-limit policy is in line with the spirit of "*Lemon Law*" for automobiles sold in the USA (Coffinberger & Samuels, 1985). It is worth mentioning that this study is motivated by the warranty policy adopted by a battery manufacturer, and a real-world case study is also presented using the warranty claim data collected by this company.

The remainder of this paper is organized as follows. Section 2 describes the piece-wise renewing free replacement warranty policy in detail, and derives its expected warranty cost and warranty cycle. This warranty policy is then extended in Section 3 by considering three sub-periods and a pre-specified threshold on the number of item failures, respectively. Section 4 presents a case study to demonstrate the implementation of warranty cost models using battery warranty claim data. Finally, Section 5 concludes this paper with some suggestions on future research topics.

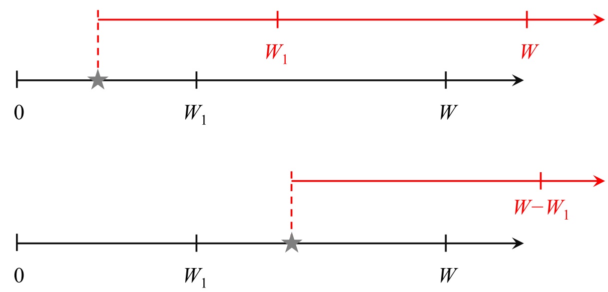

Fig. 1. Graphical illustration of the warranty renewing mechanisms with $W_1 \leqslant W$.

2. Piece-wise renewing free replacement warranty

2.1. Warranty policy description

Under the proposed piece-wise renewing free replacement warranty, the whole warranty period $[0, W]$ is divided into two sub-periods—$[0, W_1]$ and $(W_1, W]$, as illustrated in Fig. 1. Different warranty renewing mechanisms, following failure replacements, are applied in these periods. More specifically,

- if an item fails in $[0, W_1]$, then it will be replaced by a new one, and the warranty period is totally renewed, i.e., a new identical warranty of length W is attached to the new replacement;
- if an item fails in $(W_1, W]$, then it will be replaced by a new one, with a partially renewing warranty of length $W - W_1$.

As can be seen, the proposed piece-wise renewing free replacement policy is different from traditional fully-renewing free replacement policy (Blischke & Murthy, 1993, 1992), due to its piece-wise renewing nature. In this manner, a warranted item (or its subsequent replacements) would survive the piece-wise renewing warranty periods more easily than a fully renewing warranty period, especially when the item reliability is relatively low.

In this study, we assume that (*i*) all items are statistically identical, (*ii*) all warranty claims are executed and all claims are valid, and (*iii*) the replacement duration is much shorter than the mean time to failure (MTTF) and thus is negligible. In Sections 2.2 and 2.3, we derive the expected warranty servicing cost and warranty cycle, respectively, for the piece-wise renewing warranty policy.

2.2. Expected warranty cost

Let random variable $X_i \geqslant 0$ represent the ith item's lifetime with a common probability distribution function $F(x)$, density function $f(x) = \mathrm{d}F(x)/\mathrm{d}x$, and survival function $\overline{F}(x) = 1 - F(x)$. Let random variable ξ denote the number of item failures until the first item survives the first renewing period $[0, W_1]$. It is well known that ξ obeys a geometric probability distribution, i.e., for any $i \geqslant 0$,

$$P(\xi = i) = [F(W_1)]^i \overline{F}(W_1), \quad i = 0, 1, \ldots,$$

and its mean is thus

$$E[\xi] = \sum_{i=0}^{\infty} iP\left(\xi = i\right) = \frac{F(W_1)}{\overline{F}(W_1)}. \tag{1}$$

Let $z(W_1)$ represent the random cost to the manufacturer for having a survived item in the first renewing period. Then, $z(W_1)$ can be expressed as $z(W_1) = c_r\xi$ for $\xi = 0, 1, \ldots$, and its expectation is given by

$$E\left[z(W_1)\right] = c_r E[\xi] = c_r \frac{F(W_1)}{\overline{F}(W_1)}, \tag{2}$$

where c_r is the replacement cost per failure. Note that we exclude the cost of manufacturing the original item.

It is known that a survived item will have remaining lifetime $\mathcal{Y} = X_{\xi+1} - W_1$ which has a survival distribution $\overline{F}_{W_1}(\cdot)$, where $\overline{F}_{W_1}(x) = \overline{F}(W_1 + x)/\overline{F}(W_1)$ for any $x \geqslant 0$. It should be pointed out that for all items that have survived the first renewing period, their remaining lifetimes $\mathcal{Y}$ are independent and identically distributed (i.i.d).

Define ϕ as the number of item failures in the second renewing period until an item has survived this period, and let $C_{\mathcal{P}}(W_1, W)$ be the cumulative servicing cost to the manufacturer after the second renewing period. Then, we have $C_{\mathcal{P}}(W_1, W) = z(W_1) + c_r\phi$ for $\phi = 0, 1, \ldots$. The expected number of item failures in the second renewing period is derived as

$$E[\phi] = \frac{F_{W_1}(W - W_1)}{\overline{F}(W - W_1)}, \tag{3}$$

and the expected cumulative servicing cost to the manufacturer after the second renewing period, i.e., the expected total warranty cost, is given by

$$E\left[C_{\mathcal{P}}\left(W_1, W\right)\right] = c_r \frac{F(W_1)}{\overline{F}(W_1)} + c_r \frac{F_{W_1}(W - W_1)}{\overline{F}(W - W_1)}. \tag{4}$$

The detailed derivation of Eqs. (3) and (4) and especially subsequent results are tedious and thus presented in Appendix A, for the sake of readability. It is worth mentioning that from Eq. (4), we can easily obtain the expected total number of item failures during the warranty cycle, i.e., $E[\mathcal{N}_{\mathcal{P}}(W_1, W)] = E[C_{\mathcal{P}}(W_1, W)]/c_r$. Alternatively, this quantity can also be obtained by $E[\mathcal{N}_{\mathcal{P}}(W_1, W)] = E[\xi] + E[\phi]$, where $E[\xi]$ and $E[\phi]$ are given by (1) and (3), respectively. In practice, $E[\mathcal{N}_{\mathcal{P}}(W_1, W)]$ is an essential quantity for spare part management.

2.3. Expected warranty cycle

In this subsection, we derive the expected warranty cycle of the piece-wise renewing warranty policy. It is necessary to distinguish the warranty cycle $\mathcal{W}_{\mathcal{P}}$, which is a random variable, from the original warranty period W, which is a pre-determined constant. The warranty cycle $\mathcal{W}_{\mathcal{P}}$ is defined as the period from the purchase of a new item to the warranty expiry. More specifically, the warranty cycle of the piece-wise renewing policy consists of failure intervals and the original warranty period from the last failure instant (Blischke & Murthy, 1993). As a consequence, the warranty cycle that a consumer experiences is generally greater than or equal to the original warranty period, i.e., $\mathcal{W}_{\mathcal{P}} \geqslant W$.

Recall that X_i is the lifetime of the ith item and ξ is the number of item failures until the first item survives for W_1. Then, actual warranty length of the first renewing period is given by

$$l(W_1) = \sum_{i=1}^{\xi} X_i + W_1,$$

and its expectation is thus

$$E\left[l(W_1)\right] = E[\xi]E\left[X_1\middle|_{X_1 < W_1}\right] + W_1 = \frac{\mu_{W_1}}{\overline{F}(W_1)} + W_1, \tag{5}$$

where $\mu_{W_1} = \int_0^{W_1} xf(x)\mathrm{d}x$.

Moreover, let us denote the total warranty cycle after the second renewing period as $\mathcal{W}_{\mathcal{P}}(W_1, W)$. The mean of $\mathcal{W}_{\mathcal{P}}(W_1, W)$ is given by

$$E\left[\mathcal{W}_{\mathcal{P}}\left(W_1, W\right)\right] = \frac{\mu_{W_1}}{\overline{F}(W_1)} + \mu_{W|W_1} + \frac{F_{W_1}(W - W_1)}{\overline{F}(W - W_1)}\mu_{W-W_1} + W, \tag{6}$$

where $\mu_{W|W_1} = \int_0^{W-W_1} x\mathrm{d}F_{W_1}(x)$ and $\mu_{W-W_1} = \int_0^{W-W_1} xf(x)\mathrm{d}x$.

For a (piece-wise) renewing warranty policy, manufacturers might be interested in the expected warranty servicing cost per unit time (i.e., the expected cost rate) for warranty reserve management purposes. In this study, the expected cost rate for servicing this piece-wise renewing free replacement warranty is given by

$$E[\mathcal{CR}_{\mathcal{P}}(W_1, W)] = E[C_{\mathcal{P}}(W_1, W)]/E[\mathcal{W}_{\mathcal{P}}(W_1, W)],$$

where $E[C_{\mathcal{P}}(W_1, W)]$ is given by (4) and $E[\mathcal{W}_{\mathcal{P}}(W_1, W)]$ by (6).

2.4. Further discussions

2.4.1. Optimal W_1 for a given W

Sections 2.2 and 2.3 have evaluated the expected warranty cost and warranty cycle when W_1 and W are given for the piece-wise renewing policy. In real applications, how to determine an optimal W_1 for a given W is an interesting problem.

One way of identifying the optimal W_1 is through minimizing the expected warranty cost in Eq. (4), namely,

$$\min_{W_1} E\left[C_{\mathcal{P}}\left(W_1, W\right)\right] \quad s.\ t.\quad 0 \leqslant W_1 \leqslant W. \tag{7}$$

In order to solve this optimization problem, we take the first derivative of $E[C_{\mathcal{P}}(W_1, W)]$ with respect to W_1:

$$\begin{aligned}\frac{\partial E[C_{\mathcal{P}}(W_1, W)]}{\partial W_1} &= c_r \frac{1}{[\overline{F}(W_1)\overline{F}(W - W_1)]^2} \times \{\overline{F}(W - W_1)f(W_1)[\overline{F}(W - W_1) - \overline{F}(W)] \\ &\quad - \overline{F}(W_1)f(W - W_1)[\overline{F}(W_1) - \overline{F}(W)]\}.\end{aligned}$$

It is easy to verify that $W_1 = W/2$ is a solution to $\partial E[C_{\mathcal{P}}(W_1, W)]/\partial W_1 = 0$. Unfortunately, it is difficult, if not impossible, to rigorously prove that $W_1 = W/2$ is the unique optimal solution since the convexity of $E[C_{\mathcal{P}}(W_1, W)]$ is unclear. Nonetheless, the optimal W_1^* can be easily obtained by grid search method. Numerical results in Section 4 show that $W_1^* = W/2$ is indeed the optimal solution for all parameter settings considered.

2.4.2. Comparison with fully renewing free replacement policy

In practice, it is of interest to compare the proposed piece-wise renewing free replacement policy with the *fully* renewing policy in terms of expected warranty cost and warranty cycle.

Under the fully renewing free replacement policy, a failed item will be replaced by a new identical one if it is under warranty, and the warranty period is fully renewed (Blischke & Murthy, 1993). The warranty policy ceases only when the original item or an item supplied as a replacement survives for a period W. Under this policy, the manufacturer's expected warranty cost is

$$E\left[C_{\mathcal{F}}(W)\right] = c_r \frac{F(W)}{\overline{F}(W)}, \tag{8}$$

and the expected warranty cycle is given by

$$E\left[\mathcal{W}_{\mathcal{F}}(W)\right] = \frac{1}{\overline{F}(W)}\mu_W + W, \tag{9}$$

where $\mu_W = \int_0^W xf(x)\mathrm{d}x$. Therefore, the expected cost rate of the fully renewing free replacement policy is $E[\mathcal{CR}_{\mathcal{F}}(W)] = E[C_{\mathcal{F}}(W)]/E[\mathcal{W}_{\mathcal{F}}(W)]$.

For any given $0 \leqslant W_1 \leqslant W$, we can obtain the following result regarding the comparison of the two warranty policies.

Proposition 1. $E[C_{\mathcal{P}}(W_1, W)] \leqslant E[C_{\mathcal{F}}(W)]$, and $E[\mathcal{W}_{\mathcal{P}}(W_1, W)] \leqslant E[\mathcal{W}_{\mathcal{F}}(W)]$.

All proofs can be found in Appendix B. Proposition 1 shows that for any value of W_1 (including the optimal W_1^*), the expected warranty cost and warranty cycle of the piece-wise renewing free replacement policy are always less than or equal to those of the fully renewing free replacement policy, respectively. This is to be expected since the fully renewing policy expires only when a new item or its subsequent replacements successfully survive the entire warranty period $[0, W]$, which is more difficult than surviving for two short, successive periods. In addition to warranty costs and cycles, we are also interested in the comparison of their cost rates. However, it is difficult to obtain analytical property regarding $E[\mathcal{CR}_{\mathcal{P}}(W_1, W)]$ and $E[\mathcal{CR}_{\mathcal{F}}(W)]$. We will investigate this issue in Section 4.

2.4.3. Cycle-equivalent warranty policies

The proposed warranty policy has been shown to incur lower warranty servicing cost than the fully renewing free replacement policy for a fixed W, due to its piece-wise renewing nature. This is because the expected length of warranty period experienced by a consumer becomes shorter under the piece-wise renewing policy. In other words, the warranty cost reduction is attained at potential cost of customer dissatisfaction.

In order to have a comprehensive understanding of the piece-wise renewing policy, we compare it with the fully renewing policy in a *cycle-equivalent* manner. That is, the piece-wise renewing policy with parameters W_1^* and $W_{\rm P}$ will have the same expected warranty cycle as the fully renewing policy with parameter $W_{\rm F}$. Mathematically, the values of W_1^* and $W_{\rm P}$ are obtained by solving $E[\mathcal{W}_{\mathcal{P}}(W_1^*, W_{\rm P})] = E[\mathcal{W}_{\mathcal{F}}(W_{\rm F})]$ for a fixed $W_{\rm F}$. In this way, we are able to compare their corresponding expected warranty costs $E[C_{\mathcal{P}}(W_1^*, W_{\rm P})]$ and $E[C_{\mathcal{F}}(W_{\rm F})]$ to examine which policy is more cost-efficient.

3. Model extensions

In this section, two extensions are presented to enhance the applicability of the piece-wise renewing free replacement warranty. The first extension specifies one more sub-period, i.e., there are totally three renewing periods during the warranty period; while the second one considers the Lemon Law—when the number of failures over a warranty cycle exceeds a pre-specified threshold, the manufacturer has to refund the item's purchase price.

3.1. Piece-wise renewing warranty with three sub-periods

In the first extension, three renewing periods are specified in the warranty period (see Fig. 2), i.e., one more renewing period is considered in comparison to the original piece-wise renewing policy in Section 2.

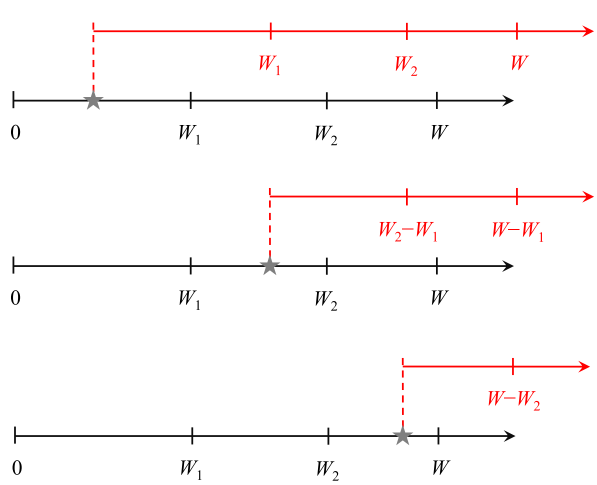

Fig. 2. Graphical illustration of the warranty renewing mechanisms within the three renewing periods ($W_1 \leqslant W_2 \leqslant W$).

3.1.1. Expected warranty cost and cycle

In the first renewing period $[0, W_1]$, a failed item will be replaced by a new one, and the warranty period is fully renewed. The expected number of item failures, $E[\xi]$, and the expected warranty cost, $E[z(W_1)]$, within the first period are thus given by Eqs. (1) and (2), respectively. If an item fails in the second renewing period $(W_1, W_2]$, then it will be replaced by a new one, with a partially renewing warranty of length $W_2 - W_1$. In this period, the expected number of item failures, $E[\widetilde{\phi}]$, and the expected cumulative warranty cost, $E[Z(W_1, W_2)]$, are given by Eqs. (3) and (4), respectively, with W_2 replacing W.

Based on whether there are item failures or not in the second renewing period, the remaining lifetime, $\mathcal{H}$, of the item that survives this period can be expressed as

$$\mathcal{H} = \begin{cases} \mathcal{Y} - (W_2 - W_1), & \text{if } \widetilde{\phi} = 0, \\ X_{\xi+\widetilde{\phi}+1} - (W_2 - W_1), & \text{if } \widetilde{\phi} \geqslant 1. \end{cases} \tag{10}$$

Lemma 1. *The conditional survival function of $\mathcal{H}$ is given by*

$$P\left(\mathcal{H} > x \middle| \widetilde{\phi}\right) = \begin{cases} \overline{F}_{W_2}(x), & \text{if } \widetilde{\phi} = 0, \\ \overline{F}_{W_2-W_1}(x), & \text{if } \widetilde{\phi} \geqslant 1, \end{cases} \tag{11}$$

where $\overline{F}_{W_2}(x) = \overline{F}(W_2 + x)/\overline{F}(W_2)$ and $\overline{F}_{W_2-W_1}(x) = \overline{F}(W_2 - W_1 + x)/\overline{F}(W_2 - W_1)$ for any $x \geqslant 0$.

Let us define τ as the number of item failures within the third renewing period until an item survives this period. Further let $\widetilde{C}_{\mathcal{P}}(W_1, W_2, W)$ represent the total cost for servicing the piece-wise renewing free replacement warranty with three sub-periods. Then, it can be expressed as $\widetilde{C}_{\mathcal{P}}(W_1, W_2, W) = Z(W_1, W_2) + c_r\tau$ for $\tau = 0, 1, \ldots$, and the expected number of item failures within the third renewing period is given by

$$E[\tau] = \frac{\Psi}{\overline{F}(W - W_2)}, \tag{12}$$

and the expected warranty cost per unit sold is

$$E\left[\widetilde{C}_{\mathcal{P}}\left(W_1, W_2, W\right)\right] = c_r\frac{F(W_1)}{\overline{F}(W_1)} + c_r\frac{F_{W_1}(W_2 - W_1)}{\overline{F}(W_2 - W_1)} + c_r\frac{\Psi}{\overline{F}(W - W_2)}, \tag{13}$$

where $\Psi = \overline{F}_{W_1}(W_2 - W_1)F_{W_2}(W - W_2) + F_{W_1}(W_2 - W_1)F_{W_2-W_1}(W - W_2)$.

Moreover, we derive the expected warranty cycle of the three-period piece-wise renewing policy. Similar to the two-period policy, the expected length of the first renewing period, $E[l(W_1)]$, is given by (5) and the expected cumulative warranty length after the second renewing period, $E[L(W_1, W_2)]$, is given by (6) with W_2 replacing W. The expected length of the whole warranty cycle after the third renewing period, i.e., $E[\widetilde{\mathcal{W}}_{\mathcal{P}}(W_1, W_2, W)]$, is given by

$$\begin{aligned} E\left[\widetilde{\mathcal{W}}_{\mathcal{P}}\left(W_1, W_2, W\right)\right] &= \frac{1}{\overline{F}(W_1)}\mu_{W_1} + \mu_{W_2|W_1} + \frac{F_{W_1}(W_2 - W_1)}{\overline{F}(W_2 - W_1)}\mu_{W_2-W_1} + \overline{F}_{W_1}\left(W_2 - W_1\right) \\ &\quad \mu_{W|W_2} + F_{W_1}\left(W_2 - W_1\right)\mu_{W-W_1|W_2-W_1} + \frac{\Psi}{\overline{F}(W - W_2)}\mu_{W-W_2} + W, \end{aligned} \tag{14}$$

where $\mu_{W|W_2} = \int_0^{W-W_2} x\mathrm{d}F_{W_2}(x)$, $\mu_{W-W_1|W_2-W_1} = \int_0^{W-W_2} x\mathrm{d}F_{W_2-W_1}(x)$, and $\mu_{W-W_2} = \int_0^{W-W_2} xf(x)\mathrm{d}x$.

Therefore, the expected cost rate of the three-period piece-wise renewing policy is given by

$$E\left[\widetilde{CR}_{\mathcal{P}}\left(W_1, W_2, W\right)\right] = \frac{E[\widetilde{C}_{\mathcal{P}}(W_1, W_2, W)]}{E[\widetilde{\mathcal{W}}_{\mathcal{P}}(W_1, W_2, W)]},$$

where $E[\widetilde{C}_{\mathcal{P}}(W_1, W_2, W)]$ is given by (13) and $E[\widetilde{\mathcal{W}}_{\mathcal{P}}(W_1, W_2, W)]$ by (14).

3.1.2. Optimal W_1 and W_2 for a given W

We look at a simple optimization problem to identify optimal W_1 and W_2 for a given W. As before, optimal W_1 and W_2 are determined by minimizing the expected warranty cost in (13):

$$\min_{W_1, W_2} E\left[\widetilde{C}_{\mathcal{P}}\left(W_1, W_2, W\right)\right] s.\ t.\ \ 0 \leqslant W_1 \leqslant W, W_1 \leqslant W_2 \leqslant W. \tag{15}$$

It is also difficult to derive a closed-form solution to this optimization problem due to the complex structure of $E[\widetilde{C}_{\mathcal{P}}(W_1, W_2, W)]$. Fortunately, the optimal parameters $\widetilde{W}_1^*$ and $\widetilde{W}_2^*$ should be finite, thus grid search method can be adopted to obtain them.

3.1.3. Policy comparisons

We further compare the two-period/three-period piece-wise renewing policy with the fully renewing policy in terms of expected warranty cost and warranty cycle.

For any given $0 \leqslant W_1 \leqslant W_2 \leqslant W$, we have the following result.

Proposition 2. $E[\widetilde{C}_{\mathcal{P}}(W_1, W_2, W)] \leqslant E[C_{\mathcal{P}}(W_1, W)] \leqslant E[C_{\mathcal{F}}(W)]$ *and* $E[\widetilde{\mathcal{W}}_{\mathcal{P}}(W_1, W_2, W)] \leqslant E[\mathcal{W}_{\mathcal{P}}(W_1, W)] \leqslant E[\mathcal{W}_{\mathcal{F}}(W)]$.

Proposition 2 indicates that for any fixed W_1, W_2 and W, the expected warranty cost and warranty cycle of the three-period piece-wise renewing policy are always less than or equal to those of the two-period piece-wise renewing policy, which in turn less than or equal to those of the fully renewing free replacement policy, respectively. From Proposition 2, it is straightforward to obtain the following corollary.

Corollary 1. $E[\widetilde{C}_{\mathcal{P}}(\widetilde{W}_1^*, \widetilde{W}_2^*, W)] \leqslant E[C_{\mathcal{P}}(W_1^*, W)] \leqslant E[C_{\mathcal{F}}(W)]$.

Instead, Corollary 1 compares the expected warranty costs of the three policies under their optimal parameter settings. That is, the *minimal* expected warranty cost of the three-period piece-wise renewing policy with its optimal parameters $\widetilde{W}_1^*$ and $\widetilde{W}_2^*$ is less than or equal to that of the two-period piece-wise renewing policy with its optimal parameter W_1^*, which in turn less than or equal to that of the fully renewing policy. It is noteworthy that the same argument is not necessarily preserved for the expected warranty cycle (though we find it is well preserved in our case study) because $\widetilde{W}_1^*$ and $\widetilde{W}_2^*$ are obtained by minimizing the expected warranty cost, rather than the expected warranty cycle.

Based on the results above, an intuitive yet reasonable conjecture is that specifying more sub-periods for the piece-wise renewing policy could result in a lower warranty servicing cost, since the warranty period is partially renewed in each sub-period. Rigorous proof is left for future research. In practice, however, it may be not convenient to specify too many renewing sub-periods for warranty management considerations, and two or three sub-periods are generally enough.

3.1.4. Cycle-equivalent warranty policies

Following Section 2.4.3, we compare the three-period piece-wise renewing policy with the fully renewing policy in a cycle-equivalent manner.

The warranty parameters $\widetilde{W}_1^*$, $\widetilde{W}_2^*$, and W_P of the three-period piece-wise renewing policy are obtained by solving $E[\widetilde{\mathcal{W}}_{\mathcal{P}}(\widetilde{W}_1^*, \widetilde{W}_2^*, W_P)] = E[\mathcal{W}_{\mathcal{F}}(W_F)]$ for a fixed W_F. Then, we compare their corresponding expected warranty costs $E[\widetilde{C}_{\mathcal{P}}(\widetilde{W}_1^*, \widetilde{W}_2^*, W_P)]$ and $E[C_{\mathcal{F}}(W_F)]$ to examine which policy is more cost-efficient. One thing particularly noteworthy is that $\widetilde{W}_1^*$, $\widetilde{W}_2^*$ are the optimal parameters that minimize $E[\widetilde{C}_{\mathcal{P}}(W_1, W_2, W_P)]$. The procedure of searching $\widetilde{W}_1^*$, $\widetilde{W}_2^*$, and W_P for a given W_F is presented in Algorithm 1.

Algorithm 1. Determining $\widetilde{W}_1^*$, $\widetilde{W}_2^*$, and W_P for a given W_F

Input: $F(t)$, $W_1^{(1)}$, $W_2^{(1)}$, W_F
1: Set $k = 1$, $W_1^{(k)} = W_1^{(1)}$, $W_2^{(k)} = W_2^{(1)}$
2: Determine $W^{(k)}$ by solving $E[\widetilde{C}_{\mathcal{P}}(W_1^{(k)}, W_2^{(k)}, W^{(k)})] = E[C_{\mathcal{F}}(W_F)]$
3: Set $W = W^{(k)}$, and identify $W_1^{(k+1)}$ and $W_2^{(k+1)}$ by solving (15)
4: **if** $W_1^{(k+1)} = W_1^{(k)}$ and $W_2^{(k+1)} = W_2^{(k)}$ **then**
5: $\widetilde{W}_1^* = W_1^{(k)}$, $\widetilde{W}_2^* = W_2^{(k)}$, and $W_P = W^{(k)}$
6: **else**
7: Set $k = k + 1$, and return to 2
8: **end if**
Output: $\widetilde{W}_1^*$, $\widetilde{W}_2^*$, W_P

3.2. Piece-wise renewing warranty with a failure limit

Under the piece-wise renewing policy discussed in the previous section, the number of item failures within the warranty cycle may be large if the item reliability is relatively low. In this case, item users may lose confidence and customer satisfaction is thus jeopardized. In some extreme situations, consumers may expect cash rebates or refunds to compensate their losses.

In this subsection, we attempt to modify the original piece-wise renewing policy by considering a threshold point m on the number of item failures. That is, in case there are more than m item failures over the warranty cycle, the consumer will receive a refund of the original purchase price, and consequently the warranty expires. Such a failure-limit policy is in line with the spirit of "*Lemon Law*" for automobiles sold in the USA (Coffinberger & Samuels, 1985), and it is potentially desirable for both manufacturers and consumers. On the one hand, consumers should prefer such a policy to a pure piece-wise renewing policy because there are chances that they could receive a 100% refund if the product quality is low. On the other hand, from the manufacturers' perspective, this policy offers extra marketing incentives, and protects them from high after-sales servicing expenses due to excessive failures. Note that there is another type of Lemon Law policy in practice, namely, if the total time taken to repair a failed item under warranty exceeds a preset threshold, the item will be replaced by a new one; refer to, e.g., Park and Pham (2016) and Park, Jung, and Park (2018).

The expected warranty servicing cost and warranty cycle of the failure-limit piece-wise renewing policy are examined in Sections 3.2.1 and 3.2.2, respectively.

3.2.1. Expected warranty cost

Under the failure-limit policy, let us define ξ' as the number of item failures until the first item survives the first renewing period or the warranty expires due to the occurrence of the $(m+1)$th failure during this period. As this policy specifies, if there are less than or equal to m failures over this period, then the failed item will be replaced by a new identical one; while if the $(m+1)$ th failure occurs, then the consumer will receive a refund of the original purchase price and the warranty ceases prematurely. In other words, the failure-limit piece-wise renewing policy, in essence, specifies a *censoring rule* on the number of item failures, i.e.,

$$\xi' = \begin{cases} \xi, & \text{if } \xi = 0, 1, \ldots, m, \\ m+1, & \text{if } \xi = m+1, \ldots, \end{cases} \tag{16}$$

where ξ is defined in Section 2.2. As a result, ξ' has a geometric distribution censored above $m+1$:

$$P(\xi' = i) = [F(W_1)]^i \overline{F}(W_1), \quad i = 0, 1, \ldots, m,$$

$$P\left(\xi' = m+1\right) = 1 - \sum_{i=0}^{m} [F(W_1)]^i \overline{F}(W_1) = [F(W_1)]^{m+1}.$$

It is important to clarify that under the censoring rule (16), the threshold point m has no effect on the probability of the number of failures, except when it equals to $m+1$ (Bai & Pham, 2005). The mean of ξ' is thus given by

$$E[\xi'] = \sum_{i=0}^{m+1} iP\left(\xi' = i\right) = \frac{F(W_1)}{\overline{F}(W_1)}(1 - [F(W_1)]^{m+1}). \tag{17}$$

Let $z'(W_1, m)$ represent the random servicing cost in the first renewing period of the failure-limit policy, with threshold point m. Under this policy, $z'(W_1, m)$ is expressed as

$$z'\left(W_1, m\right) = \begin{cases} c_r \xi', & \text{if } \xi' = 0, \ldots, m, \\ c_r m + c_b, & \text{if } \xi' = m+1, \end{cases}$$

where c_b is the selling price of an item. The expectation of $z'(W_1, m)$ is given by

$$\begin{aligned} E\left[z'\left(W_1, m\right)\right] &= \sum_{i=0}^{m+1} E\left[z'\left(W_1, m\right)\middle|\xi' = i\right]P\left(\xi' = i\right) \\ &= c_r E[\xi'] + (c_b - c_r)P(\xi' = m+1) \\ &= c_r \frac{F(W_1)}{\overline{F}(W_1)}(1 - [F(W_1)]^{m+1}) + \left(c_b - c_r\right)[F(W_1)]^{m+1}. \end{aligned} \tag{18}$$

Define $\mathcal{Y}'$ as the remaining lifetime of an item surviving the first renewing period, then we have

$$\mathcal{Y}' = \begin{cases} X_{\xi'+1} - W_1, & \text{if } \xi' \leqslant m, \\ 0, & \text{if } \xi' = m+1. \end{cases}$$

The conditional survival function of $\mathcal{Y}'$ is thus given by

$$P\left(\mathcal{Y}' > x\middle|\xi'\right) = \begin{cases} \overline{F}_{W_1}(x), & \text{if } \xi' \leqslant m, \\ 0, & \text{if } \xi' = m+1. \end{cases}$$

Similar to $\mathcal{Y}$ defined in Section 2.2, the remaining lifetimes $\mathcal{Y}'$ for all items that have survived the first renewing period are also i.i.d.

Next, we consider the second renewing period of the failure-limit policy. Define ϕ' as the number of item failures in the second renewing period until an item survives this period or the warranty ceases due to the occurrence of the $(m+1)$th failure during this period. Also, let $C'_{\mathcal{P}}(W_1, W, m)$ be the total warranty cost to the manufacturer after the second renewing period of the failure-limit policy. The expectations of ϕ' and $C'_{\mathcal{P}}(W_1, W, m)$, respectively, are given by

$$E[\phi'] = \frac{F_{W_1}(W - W_1)}{\overline{F}(W - W_1)}(1 - [F(W_1)]^{m+1} - F(W - W_1)\Upsilon), \tag{19}$$

and

$$\begin{aligned} E\left[C'_{\mathcal{P}}\left(W_1, W, m\right)\right] &= \left(1 - [F(W_1)]^{m+1}\right)E\left[C_{\mathcal{P}}\left(W_1, W\right)\right] \\ &\quad + \left(c_b - c_r\right)[F(W_1)]^{m+1} - c_r \frac{F_{W_1}(W - W_1)}{\overline{F}(W - W_1)}F \\ &\quad \left(W - W_1\right)\Upsilon + \left(c_b - c_r\right)F_{W_1}\left(W - W_1\right)\Upsilon, \end{aligned} \tag{20}$$

where $\Upsilon = \overline{F}(W_1)\frac{[F(W-W_1)]^{m+1} - [F(W_1)]^{m+1}}{F(W-W_1) - F(W_1)}$ and $E[C_{\mathcal{P}}(W_1, W)]$ is given by (4).

Likewise, under the failure-limit policy, the expected number of item failures during the entire warranty cycle is $E[\mathcal{N}'_{\mathcal{P}}(W_1, W, m)] = E[\xi'] + E[\phi']$, where $E[\xi']$ and $E[\phi']$ are given by (17) and (19), respectively. One can also obtain this quantity by $E[\mathcal{N}'_{\mathcal{P}}(W_1, W, m)] = E[C'_{\mathcal{P}}(W_1, W, m)|_{c_b = c_r}]/c_r$, where $E[C'_{\mathcal{P}}(W_1, W, m)|_{c_b=c_r}]$ is the expected warranty servicing cost given $c_b = c_r$.

3.2.2. Expected warranty cycle

We then derive the expected warranty cycle of the failure-limit piece-wise renewing warranty policy. Recall that ξ' denotes the number of item failures until the first item survives the first renewing period or the warranty expires due to the occurrence of the $(m+1)$th failure within this period. Then, the actual warranty length of the first renewing period, under the failure-limit renewing policy, is given by

$$l'\left(W_1, m\right) = \begin{cases} \sum_{i=1}^{\xi'} X_i + W_1, & \text{if } \xi' = 0, \ldots, m, \\ \sum_{i=1}^{\xi'} X_i, & \text{if } \xi' = m+1, \end{cases}$$

and its mean is thus

$$\begin{aligned} E\left[l'\left(W_1, m\right)\right] &= \sum_{i=0}^{m+1} E\left[l'\left(W_1, m\right)\middle|\xi' = i\right]P\left(\xi' = i\right) \\ &= E[\xi']\frac{\mu_{W_1}}{F(W_1)} + W_1 \sum_{i=0}^{m} P\left(\xi' = i\right) \\ &= (1 - [F(W_1)]^{m+1})E\left[l(W_1)\right], \end{aligned} \tag{21}$$

where $E[l(W_1)]$ is given by (5).

Now, we examine the warranty cycle, $\mathcal{W}'_{\mathcal{P}}(W_1, W, m)$, of the failure-limit policy. Under the failure-limit policy, the expected warranty cycle is given by

$$\begin{aligned} E\left[\mathcal{W}'_{\mathcal{P}}\left(W_1, W, m\right)\right] &= (1 - [F(W_1)]^{m+1})E\left[\mathcal{W}_{\mathcal{P}}\left(W_1, W\right)\right] \\ &\quad - F_{W_1}\left(W - W_1\right)\left(\frac{\mu_{W-W_1}}{\overline{F}(W - W_1)} + W - W_1\right)\Upsilon, \end{aligned} \tag{22}$$

where $E[\mathcal{W}_{\mathcal{P}}(W_1, W)]$ is given by (6).

Therefore, the expected cost rate of the failure-limit piece-wise renewing free replacement policy is given by

$$E\left[\mathcal{CR}'_{\mathcal{P}}\left(W_1, W, m\right)\right] = \frac{E[C'_{\mathcal{P}}(W_1, W, m)]}{E[\mathcal{W}'_{\mathcal{P}}(W_1, W, m)]},$$

where $E[C'_{\mathcal{P}}(W_1, W, m)]$ is given by (20) and $E[\mathcal{W}'_{\mathcal{P}}(W_1, W, m)]$ by (22).

The complex expressions of the expected warranty cost $E[C'_{\mathcal{P}}(W_1, W, m)]$ and warranty cycle $E[\mathcal{W}'_{\mathcal{P}}(W_1, W, m)]$ hinder us from analytically comparing them with those of the original piece-wise renewing policy and the fully renewing policy. Alternatively, the comparison is performed in a numerical way in Section 4. Nevertheless, it is easy to obtain the following observation.

Remark 1. It can be easily verified that when $m \to \infty$, we have $E[C'_{\mathcal{P}}(W_1, W, m)] = E[C_{\mathcal{P}}(W_1, W)]$, $E[\mathcal{W}'_{\mathcal{P}}(W_1, W, m)] = E[\mathcal{W}_{\mathcal{P}}(W_1, W)]$ and thus $E[\mathcal{CR}'_{\mathcal{P}}(W_1, W, m)] = E[\mathcal{CR}_{\mathcal{P}}(W_1, W)]$. That is to say, when m approaches infinity, the failure-limit piece-wise renewing policy reduces to the original piece-wise renewing policy studied in Section 2.

4. Case study

In this section, we demonstrate the proposed warranty cost models by applying them to a lead-acid battery pack model manufactured by a Chinese battery company. The battery pack is mainly used to drive twowheel or three-wheel electric bikes, and it contains four 12 V lead-acid cells in series to produce 48 V nominal voltage. Good battery performance is essential because quality and reliability problems during the warranty period will damage customer satisfaction and result in economic losses for the company. Each battery chemistry behaves differently in terms of aging and wear. Generally speaking, the lifetime of a battery pack is associated with the charge methods, depth of discharge, environmental conditions, and maintenance procedures, etc. In this study, the *black-box* approach is adopted to model battery lifetimes with an appropriate probability distribution, without the information on batteries' physical mechanisms.

The battery packs of interest are sold with the piece-wise renewing free replacement warranty policy. The whole warranty period for sold packs is 15 months, of which the lengths of the two renewing sub-periods are 8 and 7 months, respectively. This corresponds to $W_1 = 8$ months and $W = 15$ months. The main concern of the company is on a clear assessment of the warranty servicing cost of this policy. To this end, the company keeps track of warranty data with respect to the number of packs shipped and number of packs returned from the shipments in subsequent time periods. Since the shipments are made at different dates and their corresponding returns/claims are noted, the collected data are stored in a quasi-life table (Table 1); refer to Wu (2013) for more details on the quasi-life table.

4.1. The data and parameter estimation

As shown in Table 1, the starting point of the sales-claim process is denoted as day one. The shipment volume at the ith shipment date is S_i, $i = 1, 2, ..., p$. Also, $r_{i,j}$ represents the number of warranty claims received on the $(i + j - 1)$ th day and the claimed items are sold on the ith day, $j = 1, 2, \dots, q$ $(q > p)$. Then, the total number of items shipped during this period is $\boldsymbol{S} = \sum_{i=1}^{p} S_i$ and the total number of warranty claims received on the jth day is $\boldsymbol{r_j} = \sum_{i=1}^{\min\{j,p\}} r_{i,j-i+1}$. Though the quasi-life table is convenient to use, one needs to convert it to a common time-to-failure format in order to facilitate warranty data analysis. Specifically, let $\boldsymbol{n_j}$ represent the number of items that have failed at age j, which is an essential quantity for field reliability analysis. Using the quasi-life table, we have $\boldsymbol{n_j} = \sum_{i=1}^{\min\{p,q-j+1\}} r_{i,j}$, $j = 1, 2, \dots, q$. In addition, warranty claims data are usually right-censored because they are collected only until the end of warranty period or even before the warranty expiry. One can thus observe $\boldsymbol{n}_{j^+}$ items that are censored at age j^+, $j^+ = q - p + 1, ..., q$. In this sense, the warranty claims data available are multiply censored.

In this study, for confidential issues, we only extract a small proportion of warranty data set to illustrate the warranty cost models. The simplified data set contains the shipment data of a specific four-cell battery pack model from 14 February 2017 to 20 February 2017 (i.e., $p = 7$ days), and the total shipment volume is $\boldsymbol{S} = 337,597$ battery packs. The warranty claims between 14 February 2017 and 27 August 2017 are collected (i.e., $q = 195$ days), and there are $\sum_{j=1}^{195} \boldsymbol{r_j} = 5101$ claims in total, while the others are censored at different ages. In this case, we need to predict the warranty servicing cost over the whole warranty period with claims data collected only in a short period of time.

The maximum likelihood estimate (MLE) method is then applied to analyze this data set and fit various probability distributions such as exponential, Weibull, and log-normal distributions. It is found that the Weibull distribution performs the best for this data set (see Appendix C for Weibull MLE with multiply censored data). The Weibull parameters are estimated as $\hat{\beta} = 3.201$, $\hat{\alpha} = 23.460$ months, and their 95% confidence intervals are [3.114, 3.291] and [22.602, 24.350], respectively. The corresponding MTTF is 21.012 months, and its 95% confidence interval is [20.270, 21.782]. Moreover, the selling price of a battery pack is about 450 Chinese Yuan (CNY) and its replacement cost is 380 CNY on average. The units for time and money are months and CNY in this work, unless noted specifically.

Table 1
Illustration of the quasi-life table.

Shipment date	Shipment volume	Number of warranty claims received						
		1	2	…	p	…	$q-1$	q
1	S_1	$r_{1,1}$	$r_{1,2}$	…	$r_{1,p}$	…	$r_{1,q-1}$	$r_{1,q}$
2	S_2		$r_{2,1}$	…	$r_{2,p-1}$	…	$r_{2,q-2}$	$r_{2,q-1}$
…	…			…	…	…	…	…
$p-1$	S_{p-1}			…	$r_{p-1,2}$	…	$r_{p-1,q-p+1}$	$r_{p-1,q-p+2}$
p	S_p				$r_{p,1}$	…	$r_{p,q-p}$	$r_{p,q-p+1}$
Total	$\boldsymbol{S}$	$\boldsymbol{r_1}$	$\boldsymbol{r_2}$	…	$\boldsymbol{r_p}$	…	$\boldsymbol{r_{q-1}}$	$\boldsymbol{r_q}$

4.2. Warranty cost analyses and comparisons

We first look at the piece-wise renewing free replacement policy currently adopted by the company, i.e., $W_1 = 8$ and $W = 15$ months. The expected warranty servicing cost per unit sold is evaluated as 84.875 CNY, and the expected warranty cycle is 15.992 months, which leads to a cost rate of 5.307 CNY/month. As can be seen, the expected warranty cycle of this policy is slightly longer than the warranty period, due to its (piece-wise) renewing property. More importantly, the expected warranty servicing cost accounts for 18.861% of the selling price. Such an unacceptably high warranty cost severely damages customer satisfaction and the company's profitability. This is largely due to the insufficiency of battery reliability. Currently, the company is seeking and implementing effective ways to improve battery quality and reliability.

Moreover, Table 2 presents the expected warranty costs, warranty cycles, and cost rates for the two-period piece-wise renewing policy, three-period piece-wise renewing policy, and fully renewing policy under their optimal parameter settings. Fig. 3 illustrates the existence of optimal parameters under the two-period and three-period policies. It can be seen that $W_1^* = W/2 = 7.5$ months for the two-period policy, which is consistent with our earlier analysis. The warranty parameter $W_1 = 8$ that is currently adopted by the company is close to the optimal value. Also, the corresponding expected warranty cost, warranty cycle, and cost rate are quite close to those under the optimal setting. In Table 2, $\widetilde{W}_1^*$ and $\widetilde{W}_2^*$ of the three-period policy are searched with a step length of 0.5 month, for practical implementation considerations. By comparing the three policies under their optimal parameter settings, we observe that the expected warranty cost and warranty cycle of the fully renewing policy are much larger than those of the two-period policy, which in turn slightly higher than those of the three-period policy (Proposition 1 and Corollary 1). This is to be expected since the fully renewing policy ceases only when a new battery or its subsequent replacements successfully survive for 15 months, which is more difficult than surviving for two or three short periods, especially for low-reliability batteries. It is surprising, however, that the expected cost rate of the three-period policy is even higher than that of the two-period policy. This is because the relative reduction of warranty cost is less than that of warranty cycle, which results in a even higher cost rate. In Table 2, we also list the upper and lower bounds of the expected warranty costs, warranty cycles, and cost rates by using the 95% confidence interval of Weibull distribution parameters. The upper and lower bounds provide the company with useful information on the uncertainty of warranty servicing cost.

Furthermore, we explore the (two-period) failure-limit piece-wise renewing free replacement policy. Table 3 summarizes the means, upper bounds, and lower bounds of the warranty cost, warranty cycle, and cost rate for this policy under different threshold point m. Note that $W_1^* = 7.5$ months always holds for all values of m. It is observed that as the threshold point m increases, the expected warranty servicing cost (resp. warranty cycle) of the failure-limit policy decreases (resp.

Table 2
Comparison of the three warranty policies under their optimal parameter settings (W = 15 months).

	Fully renewing policy			Two-period piece-wise policy				Three-period piece-wise policy				
	$E[C_{\mathcal{F}}]$	$E[\mathcal{W}_{\mathcal{F}}]$	$E[\mathcal{CR}_{\mathcal{F}}]$	W_1^*	$E[C_{\mathcal{P}}]$	$E[\mathcal{W}_{\mathcal{P}}]$	$E[\mathcal{CR}_{\mathcal{P}}]$	$\widetilde{W}_1^*$	$\widetilde{W}_2^*$	$E[\widetilde{C}_{\mathcal{P}}]$	$E[\widetilde{\mathcal{W}}_{\mathcal{P}}]$	$E[\widetilde{\mathcal{CR}}_{\mathcal{P}}]$
Mean	102.552	18.034	5.686	7.5	84.802	16.043	5.286	5.5	9.5	83.054	15.653	5.306
Upper Bound	112.552	18.349	6.134	7.5	91.286	16.127	5.661	5.5	9.5	89.315	15.705	5.687
Lower Bound	94.079	17.768	5.295	7.5	79.108	15.969	4.954	5.5	9.5	77.549	15.608	4.969

increases), and hence the corresponding cost rate decreases (see Fig. 4). More importantly, all of these three quantities converge to those of the original piece-wise renewing policy, respectively, as m increases (Remark 1). The failure-limit piece-wise renewing policy produces comparable results—which are not substantially higher—when compared with the original piece-wise renewing policy.

The warranty cost comparison above is performed under the same W. We end this subsection with a demonstration on the design and comparison of cycle-equivalent warranty policies. Table 4 shows warranty parameters and performance measures of the three warranty policies in a cycle-equivalent manner, with W_F = 15 months. Surprisingly, it is observed that the cycle-equivalent piece-wise renewing policy (either with two sub-periods or three sub-periods) results in a higher warranty cost and cost rate than the fully renewing policy; and the cost performance of the three-period policy is even worse than the two-period policy. This observation is reflected in the fact that the expected number of item failures under the piece-wise renewing policy is larger than that under the fully renewing policy (see Table 4). It is because that the piece-wise renewing policy requires a longer warranty period W and more renewals (i.e., number of failures) to attain the same expected warranty cycle as the fully renewing policy, due to its piece-wise renewing characteristic. This finding implies that although the piece-wise renewing policy is able to reduce warranty servicing cost for a given warranty period, it is indeed not as cost-efficient as the fully renewing policy in a cycle-equivalent setting. Nevertheless, consumers usually perceive a warranty policy through its length and renewing mechanism (Expected warranty cycle is not accessible to consumers because they have little knowledge on product reliability). In this sense, the piece-wise renewing policy might be more attractive to consumers in a cycle-equivalent setting. In real applications, however, it is not easy to implement cycle-equivalent warranty policies as W_1 (and W_2) should be integers for the sake of effective management. It is thus difficult for two policies to attain exact cycle equivalence.

4.3. Sensitivity analyses

In order to obtain more insights on these warranty policies, comprehensive sensitivity analyses are conducted by varying one or two parameters at a time while keeping the others unchanged. It is worth mentioning that the policy comparisons below are performed for a fixed W, rather than in a cycle-equivalent manner, to keep consistent with the current practice.

We first consider the effect of Weibull distribution parameters (α and β) on the performance measures of the three warranty policies. Tables 5 and 6 summarize the results under various combinations of α and β. The following observations can be made:

- For all of the four warranty policies, their expected warranty costs, warranty cycles, and cost rates decrease as α and/or β increases. This implies that a large α or β is beneficial for warranty cost reduction. Among these three performance measures, the expected warranty cost and cost rate depend quite significantly on the value of α and β. Hence, the company has to estimate α and β as accurately as possible, and also take into account parameter uncertainty.
- The result in Corollary 1 well holds for all combinations of α and β, i.e., $E[\widetilde{C}_{\mathcal{P}}(\widetilde{W}_1^*, \widetilde{W}_2^*, W)] \leqslant E[C_{\mathcal{P}}(W_1^*, W)] \leqslant E[C_{\mathcal{F}}(W)]$ and $E[\widetilde{\mathcal{W}}_{\mathcal{P}}(\widetilde{W}_1^*, \widetilde{W}_2^*, W)] \leqslant E[\mathcal{W}_{\mathcal{P}}(W_1^*, W)] \leqslant E[\mathcal{W}_{\mathcal{F}}(W)]$. On the other hand, the cost rates of the three policies exhibit different orders from warranty costs and cycles, i.e., $E[\widetilde{\mathcal{CR}}_{\mathcal{P}}(\widetilde{W}_1^*, \widetilde{W}_2^*, W)] \leqslant E[\mathcal{CR}_{\mathcal{P}}(W_1^*, W)] \leqslant E[\mathcal{CR}_{\mathcal{F}}(W)]$ for $\beta = 1.5$ and 2; while $E[\mathcal{CR}_{\mathcal{P}}(W_1^*,W)] \leqslant E[\widetilde{\mathcal{CR}}_{\mathcal{P}}(\widetilde{W}_1^*, \widetilde{W}_2^*, W)] \leqslant E[\mathcal{CR}_{\mathcal{F}}(W)]$ for $\beta = 3$ and 4. This demonstrates the advantage of piece-wise renewing policies, in terms of warranty cost reduction, for a fixed warranty period W.
- For the failure-limit piece-wise renewing policy, its warranty cost, warranty cycle, and cost rate always converge to the corresponding values of the original piece-wise renewing policy (marked as bold in Table 6) as the threshold point m increases, and the convergence rate is faster for larger α and β. More interestingly, $E[C'_{\mathcal{P}}(W_1^*, W, m)]$ increases as m increases when $\beta = 1.5$, while it decreases with m in most of the other cases, except for $\alpha = 20$ and $\beta = 2$ (see Fig. 5). This is because the risk of battery failure in its early age is relatively high when $\alpha = 20$ and $\beta = 1.5$, it is thus better to refund the purchase price as early as possible. So a small m should be chosen in this case. This further suggests that α and β should be relatively large for well designed and manufactured

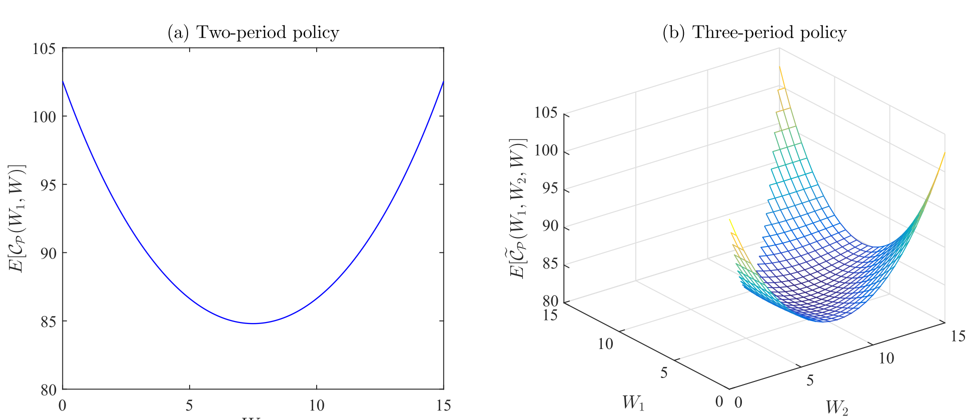

Fig. 3. Existence of optimal parameters under two-period and three-period piece-wise renewing policies.

Table 3
Warranty cost analysis of the (two-period) failure-limit policy ($W_1^* = 7.5$ months).

	Mean			Upper bound			Lower bound		
	$E[C'_{\mathcal{P}}]$	$E[\mathcal{W}'_{\mathcal{P}}]$	$E[\mathcal{CR}'_{\mathcal{P}}]$	$E[C'_{\mathcal{P}}]$	$E[\mathcal{W}'_{\mathcal{P}}]$	$E[\mathcal{CR}'_{\mathcal{P}}]$	$E[C'_{\mathcal{P}}]$	$E[\mathcal{W}'_{\mathcal{P}}]$	$E[\mathcal{CR}'_{\mathcal{P}}]$
$m = 1$	85.367	15.787	5.408	91.905	15.835	5.804	79.628	15.743	5.058
$m = 2$	84.824	15.990	5.305	91.311	16.062	5.685	79.128	15.926	4.968
$m = 3$	84.802	16.032	5.290	91.287	16.112	5.666	79.109	15.961	4.957
$m = 4$	84.802	16.040	5.287	91.286	16.123	5.662	79.108	15.967	4.954
$m = 5$	84.802	16.042	5.286	91.286	16.126	5.661	79.108	15.969	4.954
$m = 6$	84.802	16.043	5.286	91.286	16.126	5.661	79.108	15.969	4.954
$m = 7$	84.802	16.043	5.286	91.286	16.127	5.661	79.108	15.969	4.954
$m = 8$	84.802	16.043	5.286	91.286	16.127	5.661	79.108	15.969	4.954

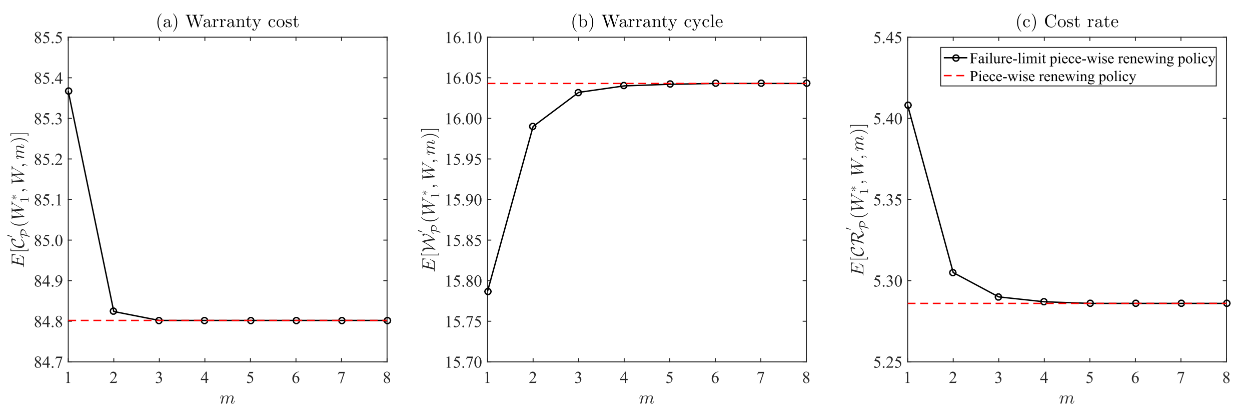

Fig. 4. Expected warranty cost, warranty cycle, and cost rate of the two-period failure-limit policy with $W_1^* = 7.5$ months.

Table 4
Comparison of the three warranty policies in a cycle-equivalent manner.

	W_1	W_2	W	$E[\mathcal{N}]$	$E[C]$	$E[\mathcal{W}]$	$E[\mathcal{CR}]$
Fully renewing policy	–	–	15	0.270	102.552	**18.034**	5.686
Two-period piece-wise policy	8.258	–	16.516	0.296	112.722	**18.034**	6.251
Three-period piece-wise policy	6.130	10.856	16.986	0.312	118.622	**18.034**	6.578

Table 5
Sensitivity analysis of the Weibull parameters for the three warranty policies ($W = 15$ months).

		Fully renewing policy			Two-period piece-wise policy				Three-period piece-wise policy				
α	β	$E[C_{\mathcal{F}}]$	$E[\mathcal{W}_{\mathcal{F}}]$	$E[\mathcal{CR}_{\mathcal{F}}]$	W_1^*	$E[C_{\mathcal{P}}]$	$E[\mathcal{W}_{\mathcal{P}}]$	$E[\mathcal{CR}_{\mathcal{P}}]$	$\widetilde{W}_1^*$	$\widetilde{W}_2^*$	$E[\widetilde{C}_{\mathcal{P}}]$	$E[\widetilde{\mathcal{W}}_{\mathcal{P}}]$	$E[\widetilde{\mathcal{CR}}_{\mathcal{P}}]$
20	1.5	347.556	22.561	15.405	7.5	262.022	17.784	14.734	5.0	10.0	244.324	16.602	14.717
	2	286.921	22.122	12.970	7.5	207.916	17.346	11.987	5.0	10.0	195.400	16.337	11.960
	3	199.431	20.724	9.623	7.5	144.224	16.736	8.618	5.5	9.5	138.800	16.054	8.646
	4	141.431	19.387	7.295	7.5	107.077	16.366	6.542	5.5	9.5	104.986	15.863	6.618
30	1.5	161.165	18.648	8.643	7.5	138.574	16.491	8.403	5.0	10.0	133.348	15.903	8.385
	2	107.930	17.769	6.074	7.5	93.667	16.068	5.829	5.5	9.5	91.038	15.662	5.813
	3	50.596	16.484	3.069	7.5	45.974	15.563	2.954	5.5	9.5	45.434	15.357	2.958
	4	24.508	15.771	1.554	7.5	23.198	15.302	1.516	5.5	9.5	23.105	15.196	1.520
40	1.5	98.096	17.256	5.685	7.5	88.997	15.962	5.575	5.0	10.0	86.791	15.596	5.565
	2	57.377	16.489	3.480	7.5	52.995	15.607	3.396	5.5	9.5	52.144	15.385	3.389
	3	20.577	15.607	1.318	7.5	19.762	15.243	1.296	5.5	9.5	19.663	15.156	1.297
	4	7.589	15.239	0.498	7.5	7.459	15.097	0.494	5.5	9.5	7.449	15.063	0.495
50	1.5	67.863	16.574	4.095	7.5	63.315	15.687	4.036	5.0	10.0	62.185	15.430	4.030
	2	35.786	15.933	2.246	7.5	34.015	15.390	2.210	5.5	9.5	33.662	15.250	2.207
	3	10.400	15.307	0.679	7.5	10.187	15.125	0.673	5.5	9.5	10.161	15.081	0.674
	4	3.090	15.098	0.205	7.5	3.069	15.040	0.204	5.5	9.5	3.067	15.026	0.204

Table 6
Sensitivity analysis of the Weibull parameters for the failure-limit piece-wise renewing policy ($W_1^* = 7.5$ months).

		$\beta = 1.5$			$\beta = 2$			$\beta = 3$			$\beta = 4$		
α	m	$E[C'_{\mathcal{P}}]$	$E[\mathcal{W}'_{\mathcal{P}}]$	$E[\mathcal{CR}'_{\mathcal{P}}]$	$E[C'_{\mathcal{P}}]$	$E[\mathcal{W}'_{\mathcal{P}}]$	$E[\mathcal{CR}'_{\mathcal{P}}]$	$E[C'_{\mathcal{P}}]$	$E[\mathcal{W}'_{\mathcal{P}}]$	$E[\mathcal{CR}'_{\mathcal{P}}]$	$E[C'_{\mathcal{P}}]$	$E[\mathcal{W}'_{\mathcal{P}}]$	$E[\mathcal{CR}'_{\mathcal{P}}]$
20	1	250.796	16.297	15.389	206.532	16.352	12.631	145.515	16.148	9.011	107.677	15.974	6.741
	2	259.397	17.278	15.013	207.799	17.007	12.218	144.328	16.547	8.723	107.094	16.261	6.586
	3	261.417	17.584	14.867	207.909	17.212	12.079	144.231	16.671	8.651	107.077	16.338	6.554
	4	261.885	17.697	14.798	207.916	17.290	12.025	144.224	16.714	8.629	107.077	16.359	6.546
	5	261.991	17.744	14.765	207.916	17.322	12.003	144.224	16.728	8.622	107.077	16.364	6.543
	6	262.015	17.765	14.749	207.916	17.335	11.994	144.224	16.733	8.619	**107.077**	**16.366**	**6.542**
	7	262.021	17.775	14.741	207.916	17.341	11.990	**144.224**	**16.736**	**8.618**			
	8	262.022	17.780	14.737	207.916	17.344	11.988						
	9	262.022	17.782	14.735	**207.916**	**17.346**	**11.987**						
	10	**262.022**	**17.784**	**14.734**									
30	1	138.450	15.917	8.698	94.465	15.779	5.987	46.183	15.478	2.984	23.229	15.277	1.520
	2	138.608	16.361	8.472	93.742	16.013	5.854	45.979	15.553	2.956	23.198	15.300	1.516
	3	138.584	16.457	8.421	93.673	16.057	5.834	45.974	15.562	2.954	**23.198**	**15.302**	**1.516**
	4	138.576	16.481	8.408	93.667	16.066	5.830	**45.974**	**15.563**	**2.954**			
	5	138.575	16.488	8.405	93.667	16.067	5.830						
	6	**138.574**	**16.490**	**8.404**	**93.667**	**16.068**	**5.829**						
40	1	89.632	15.688	5.713	53.392	15.501	3.445	19.804	15.226	1.301	7.462	15.095	0.494
	2	89.075	15.919	5.596	53.015	15.595	3.400	19.763	15.242	1.297	**7.459**	**15.097**	**0.494**
	3	89.005	15.954	5.579	52.996	15.605	3.396	**19.762**	**15.243**	**1.296**			
	4	88.997	15.961	5.576	**52.995**	**15.607**	**3.396**						
	5	**88.997**	**15.962**	**5.575**									
50	1	63.844	15.536	4.109	34.207	15.343	2.229	10.198	15.121	0.674	**3.069**	**15.040**	**0.204**
	2	63.359	15.669	4.044	34.021	15.387	2.211	10.187	15.125	0.674			
	3	63.318	15.684	4.037	**34.015**	**15.390**	**2.210**	**10.187**	**15.125**	**0.673**			
	4	63.315	15.686	4.036									
	5	**63.315**	**15.687**	**4.036**									

Note: The bold numbers are converged results with m^o, in which $E[C'_{\mathcal{P}}(W_1^*, W, m^o)] = E[C_{\mathcal{P}}(W_1^*, W)]$, $E[\mathcal{W}'_{\mathcal{P}}(W_1^*, W, m^o)] = E[\mathcal{W}_{\mathcal{P}}(W_1^*, W)]$, and $E[\mathcal{CR}'_{\mathcal{P}}(W_1^*, W, m^o)] = E[\mathcal{CR}_{\mathcal{P}}(W_1^*, W)]$.

batteries.

- Nevertheless, it seems that the impact of threshold point m on the results is very small for all values of α and β considered. This is because (i) the price of the battery pack of interest is not much higher than its replacement cost, and (ii) the probability that multiple failures occur over a single warranty cycle is very small. As a result, the company could simply choose $m = 1$ or $m = 2$ in real applications, as long as the battery reliability is reasonably high (say, $\alpha > 30$ and/or $\beta > 3$).

In addition to the parameters of failure distribution, the replacement cost and selling price are also important factors that may impact the warranty servicing cost. For the fully renewing and piece-wise renewing policies, it is straightforward to know that their warranty costs are increasing functions of the replacement cost. Here, we focus primarily on the impact of c_r and c_b on the warranty cost of the failure-limit policy. Table 7 and Fig. 6 show the results for various combinations of c_r and c_b/c_r, with other parameters keeping unchanged. In Table 7, we solely present the expected warranty servicing cost, since the expected warranty cycle is independent of the replacement cost and/or selling price.

The following observations can be drawn:

- The expected warranty cost of the failure-limit piece-wise renewing policy shows upward trend as the replacement cost c_r and/or the selling price c_b increases. Also, it seems that the impact of c_r is more significant than that of c_b. This is reasonable since c_b affects the warranty cost only when there are $m + 1$ failures within the warranty cycle, whose probability of occurrence is not high.
- It is interesting to observe that when $c_b/c_r = 0.8$ and 1.0, the

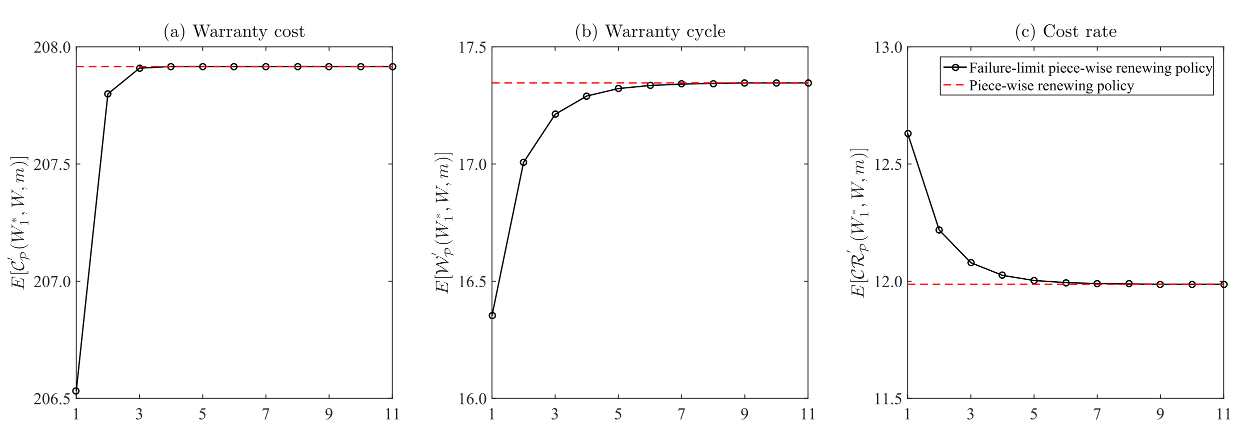

Fig. 5. Expected warranty cost, warranty cycle, and cost rate of the failure-limit piece-wise renewing policy when $\alpha = 20$ and $\beta = 2$.

Table 7
Sensitivity analysis of c_r and c_b/c_r for the failure-limit piece-wise renewing policy ($W_1^* = 7.5$ months).

c_r	m	$E[C_{\mathcal{P}}]$	$E[C'_{\mathcal{P}}(W_1^*, W, m)]$			
			$\frac{c_b}{c_r} = 0.8$	1.0	1.5	2.0
200	1	44.632	44.143	44.553	45.577	46.601
	2		44.614	44.630	44.668	44.707
	3		**44.632**	**44.632**	44.634	44.635
	4				**44.632**	44.633
	5					**44.632**
300	1	66.949	66.214	66.829	68.365	69.902
	2		66.921	66.945	67.002	67.060
	3		66.948	**66.949**	66.950	66.952
	4		**66.949**		**66.949**	**66.949**
400	1	89.265	88.286	89.105	91.154	93.202
	2		89.229	89.260	89.337	89.414
	3		89.264	**89.265**	89.267	89.270
	4		**89.265**		**89.265**	**89.265**
500	1	111.581	110.357	111.382	113.942	116.503
	2		111.536	111.574	111.671	111.767
	3		111.580	**111.581**	111.584	111.587
	4		**111.581**		**111.581**	**111.581**

Note: The bold numbers are converged results with m^o, in which $E[C'_{\mathcal{P}}(W_1^*, W, m^o)] = E[C_{\mathcal{P}}(W_1^*, W)]$.

expected warranty cost increases with the threshold point m, whereas when $c_b/c_r = 1.5$ and 2.0, the expected warranty cost decreases with m (see Fig. 6). This can be explained by the fact that when the selling price c_b is lower than the replacement cost c_r, refunding the purchase price as early as possible could avoid high replacement expenses due to excessive battery failures. In contrast, it is beneficial to choose a large m to avoid price refund if the selling price is higher than the replacement cost.

- Again, the expected warranty servicing cost is insensitive to the threshold point m, as discussed before.

5. Concluding remarks

Warranty designers, such as product manufacturers, are constantly seeking novel ideas to promote their products due to more-than-ever fierce market competition and high consumer requirements. In this paper, we investigate the cost analysis of a new piece-wise renewing free replacement warranty as well as its two modifications with three sub-periods and a threshold point on the number of item failures, respectively. The expressions of expected total warranty costs, warranty cycles, and cost rates have been derived for the warranty policies, from the manufacturer's perspective. A real-world case study is presented to demonstrate the implementation of warranty cost models, by using warranty claim data of a battery pack model. The piece-wise renewing warranty policies are also compared with traditional fully renewing free replacement policy in terms of expected warranty cost, warranty cycle, and cost rate.

It is observed that the expected warranty cost and cycle of the piece-wise renewing policy are always lower than those of the fully renewing free replacement policy; while the two-period piece-wise renewing policy leads to higher warranty cost/cycle and lower cost rate against the three-period piece-wise policy. Moreover, the failure-limit piece-wise renewing policy produces comparable results—which are not substantially higher or lower—when compared with the original piece-wise renewing policy. Surprisingly, when comparing the warranty policies in a cycle-equivalent manner, we find that the piece-wise renewing policies are not as cost-efficient as the fully renewing policy. As a consequence, the manufacturers should carefully evaluate and compare these warranty policies before making warranty-related decisions, with the help of field return data. Nevertheless, keep in mind that one of the most effective ways of reducing warranty cost is to improve product reliability.

There are several open problems on the proposed warranty policy. A possible extension is to introduce a piece-wise renewing combination warranty policy, under which the last period bears a renewing pro-rata nature. This would reduce warranty cost without decreasing customer satisfaction, to some extent. One possible way of further reducing warranty cost is to apply imperfect repairs for failed items, since the repair cost would be much lower than the replacement cost. Finally,

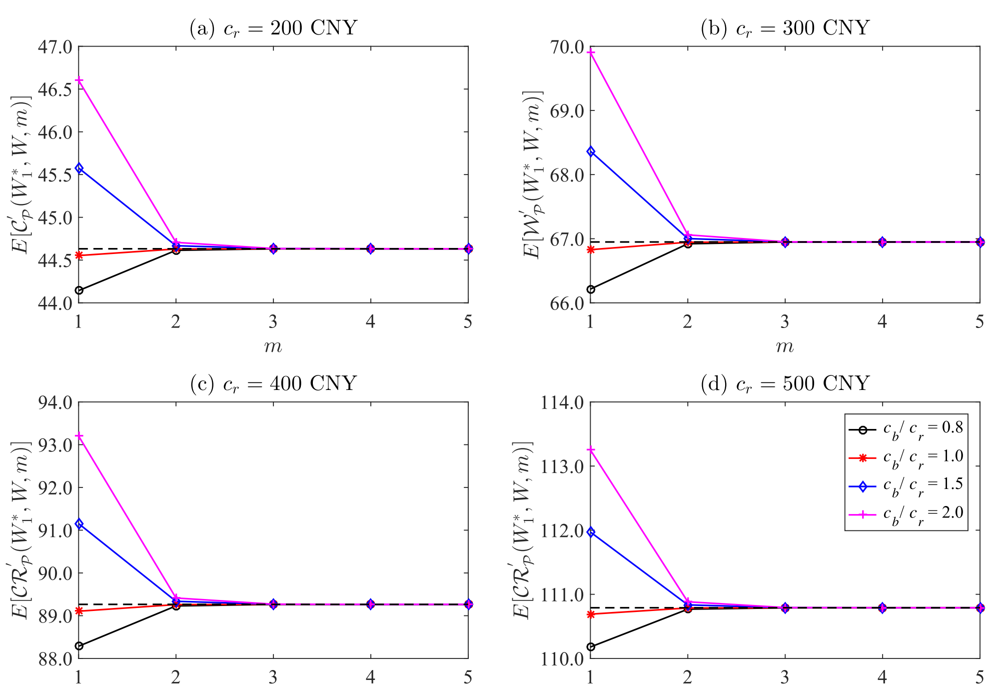

Fig. 6. Sensitivity analysis of c_r and c_b/c_r for the failure-limit piece-wise renewing policy.

one of the limitations of this study is that we focus solely on warranty cost analysis without taking into account product sales and profit. It is thus desirable to empirically study the impacts of piece-wise renewing warranty policies on product sales and profit, which would help manufacturers make better decisions.

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Appendix A. Technical derivations

Derivation of Eqs.(3) and (4). It is straightforward to know that there are two cases where an item survives the second renewing period:

(a) the $(\xi+1)$th item, that has survived for W_1, also survives for its age W, with probability $\overline{F}_{W_1}(W-W_1)$; and
(b) the $(\xi+1)$th item fails within $(W_1, W]$, and a subsequent replacement survives for its age $W-W_1$, with probability $F_{W_1}(W-W_1)[F(W-W_1)]^{\phi-1}\overline{F}(W-W_1),\ \phi=1,2,\cdots$.

Hence, ϕ has a geometric distribution with probability:

$$P\left(\phi=j\right)=\begin{cases}\overline{F}_{W_1}(W-W_1), & j=0,\\ F_{W_1}(W-W_1)[F(W-W_1)]^{j-1}\overline{F}(W-W_1), & j=1,2,...,\end{cases}$$

and its mean is thus

$$E[\phi]=\sum_{j=0}^{\infty}jP\left(\phi=j\right)=\frac{F_{W_1}(W-W_1)}{\overline{F}(W-W_1)}.$$

The expected warranty servicing cost can be derived by using the conditional expectation formula: $E[C_{\mathcal{P}}(W_1,W)]=E[E[C_{\mathcal{P}}(W_1,W)|\phi]]=E[z(W_1)]+c_rE[\phi]$. □

Derivation of Eq.(6). After the second renewing period, based on whether there are item failures or not in this period, the warranty cycle can be expressed as

$$\mathcal{W}_{\mathcal{P}}\left(W_1,W\right)=\begin{cases}l(W_1)+W-W_1, & \text{if } \phi=0,\\ l(W_1)+\mathcal{Y}+\sum\limits_{e=\xi+2}^{\xi+\phi}X_e+W-W_1, & \text{if } \phi=1,\ ...\end{cases}$$

This implies that if no failure occurs in the second renewing period, then $\mathcal{W}_{\mathcal{P}}(W_1,W)=l(W_1)+W-W_1$; while if there are ϕ ($\phi=1,2,...$) failures in this period, then $\mathcal{W}_{\mathcal{P}}\left(W_1,W\right)=l(W_1)+\mathcal{Y}+\sum_{e=\xi+2}^{\xi+\phi}X_e+W-W_1$, where $\mathcal{Y}+\sum_{e=\xi+2}^{\xi+\phi}X_e$ is the summation of intermediate failure intervals and $W-W_1$ is the length of the second renewing period after the $(\xi+\phi)$ th failure instant.

By using the conditional expectation technique, the mean of $\mathcal{W}_{\mathcal{P}}(W_1,W)$ can be derived as

$$\begin{aligned}E[\mathcal{W}_{\mathcal{P}}(W_1,W)]&=\sum_{j=0}^{\infty}E\left[\mathcal{W}_{\mathcal{P}}\left(W_1,W\right)\middle|\phi=j\right]P\left(\phi=j\right)\\ &=E[l(W_1)]+W-W_1+E\left[\mathcal{Y}\middle|_{\mathcal{Y}<W-W_1}\right]+E\left[X_e\middle|_{X_e<W-W_1}\right]\sum_{j=1}^{\infty}\left(j-1\right)P\left(\phi=j\right)\quad\square\\ &=\tfrac{1}{\overline{F}(W_1)}\mu_{W_1}+\mu_{W|W_1}+\tfrac{F_{W_1}(W-W_1)}{\overline{F}(W-W_1)}\mu_{W-W_1}+W.\end{aligned}$$

Derivation of Eqs.(12) and (13). To obtain the expectation of τ, one needs to consider two cases of $\widetilde{\phi}$, i.e., $\widetilde{\phi}=0$ and $\widetilde{\phi}\geqslant 1$.

Case 1: $\widetilde{\phi}=0$. In this case, there are two cases where an item survives the third renewing period:

(a-1) the $(\xi+1)$th item that has survived for W_2, also survives for another period of $W-W_2$, with conditional probability $\overline{F}_{W_2}(W-W_2)$; and
(a-2) the $(\xi+1)$th item fails within (W_2, W) and a subsequent replacement survives for its age $W-W_2$, with conditional probability $F_{W_2}(W-W_2)[F(W-W_2)]^{\tau-1}\overline{F}(W-W_2),\ \tau=1,2,\cdots$.

As a result, conditioning on $\widetilde{\phi}=0$, τ follows a geometric distribution with conditional probability:

$$P\left(\tau=k\middle|\widetilde{\phi}=0\right)=\begin{cases}\overline{F}_{W_2}(W-W_2), & k=0,\\ F_{W_2}(W-W_2)[F(W-W_2)]^{\tau-1}\overline{F}(W-W_2), & k=1,2,...,\end{cases}$$

and its corresponding conditional mean is

$$E\left[\tau\middle|\widetilde{\phi}=0\right]=\sum_{k=0}^{\infty}kP\left(\tau=k\middle|\widetilde{\phi}=0\right)=\frac{F_{W_2}(W-W_2)}{\overline{F}(W-W_2)}.$$

Case 2: $\widetilde{\phi}\geqslant 1$. In this case, there are the following two cases where an item survives the third renewing period:

(b-1) the $(\xi+\widetilde{\phi}+1)$th item that has survived for W_2-W_1, survives for another period of $W-W_2$, with conditional probability $\overline{F}_{W_2-W_1}(W-W_2)$; and
(b-2) the $(\xi+\widetilde{\phi}+1)$th item fails within $(W_2-W_1, W-W_1)$ and a subsequent replacement survives for its age $W-W_2$, with conditional probability $F_{W_2-W_1}(W-W_2)[F(W-W_2)]^{\tau-1}\overline{F}(W-W_2),\ \tau=1,2,\cdots$.

Likewise, conditioning on $\widetilde{\phi}\geqslant 1$, τ obeys a geometric distribution with conditional probability:

$$P\left(\tau=k\middle|\widetilde{\phi}\geqslant 1\right)=\begin{cases}\overline{F}_{W_2-W_1}(W-W_2), & k=0,\\ F_{W_2-W_1}(W-W_2)[F(W-W_2)]^{\tau-1}\overline{F}(W-W_2), & k=1,2,\ldots,\end{cases}$$

and the conditional mean of τ is thus

$$E\left[\tau\middle|\widetilde{\phi}\geqslant 1\right]=\sum_{k=0}^{\infty}kP\left(\tau=k\middle|\widetilde{\phi}\geqslant 1\right)=\frac{F_{W_2-W_1}(W-W_2)}{\overline{F}(W-W_2)}.$$

Through the conditional expectation formula, $E[\tau]$ in (12) can be easily determined as follows:

$$E[\tau]=\sum_{j=0}^{\infty}E\left[\tau\middle|\widetilde{\phi}=j\right]P\left(\widetilde{\phi}=j\right)=\frac{\Psi}{\overline{F}(W-W_2)}.$$

Finally, the expected warranty cost of the three-period piece-wise renewing policy can be derived by $E[\widetilde{C}_{\mathcal{P}}(W_1, W_2, W)]=E[E[\widetilde{C}_{\mathcal{P}}(W_1, W_2, W)|\widetilde{\phi}]]=E[Z(W_1, W_2)]+c_rE[\tau]$. □

Derivation of Eq.(14). In order to derive the expected total warranty cycle, we also consider the following two cases of $\widetilde{\phi}$.

Case 1: $\widetilde{\phi}\geqslant 1$. Conditioning on $\widetilde{\phi}\geqslant 1$, the length of the whole warranty cycle is

$$\widetilde{W}_{\mathcal{P}}\left(W_1, W_2, W\middle|\widetilde{\phi}\geqslant 1\right)=\begin{cases}L(W_1, W_2)+W-W_2, & \text{if}\tau=0,\\ L\left(W_1, W_2\right)+\mathcal{H}+\sum_{e=\xi+\widetilde{\phi}+2}^{\xi+\widetilde{\phi}+\tau}X_e+W-W_2, & \text{if}\tau=1,\ldots,\end{cases}\tag{A.1}$$

with $\mathcal{H}=X_{\xi+\widetilde{\phi}+1}-(W_2-W_1)$, and its conditional mean is thus

$$\begin{aligned}E[\widetilde{W}_{\mathcal{P}}(W_1, W_2, W)|\widetilde{\phi}\geqslant 1]&=\sum_{k=0}^{\infty}E\left[\widetilde{W}_{\mathcal{P}}\left(W_1, W_2, W\right)\middle|\widetilde{\phi}\geqslant 1, \tau=k\right]P\left(\tau=k\middle|\widetilde{\phi}\geqslant 1\right)\\ &=E\left[L\left(W_1, W_2\right)\right]+W-W_2+\mu_{W-W_1|W_2-W_1}+\frac{F_{W_2-W_1}(W-W_2)}{\overline{F}(W-W_2)}\mu_{W-W_2}.\end{aligned}$$

Case 2: $\widetilde{\phi}=0$. While conditioning on $\widetilde{\phi}=0$, the length of the whole warranty cycle is given by Eq. (A.1), with $\mathcal{H}=\mathcal{Y}-(W_2-W_1)$ and $\widetilde{\phi}=0$. Hence, its conditional mean is given by

$$E\left[\widetilde{W}_{\mathcal{P}}\left(W_1, W_2, W\right)\middle|\widetilde{\phi}=0\right]=E\left[L\left(W_1, W_2\right)\right]+W-W_2+\mu_{W|W_2}+\frac{F_{W_2}(W-W_2)}{\overline{F}(W-W_2)}\mu_{W-W_2}.$$

Therefore, the expected warranty cycle of the three-period piece-wise renewing free replacement warranty policy in (14) can be derived by removing the conditioning on $\widetilde{\phi}$:

$$\begin{aligned}E[\widetilde{W}_{\mathcal{P}}(W_1, W_2, W)]&=\sum_{j=0}^{\infty}E\left[\widetilde{W}_{\mathcal{P}}\left(W_1, W_2, W\right)\middle|\widetilde{\phi}=j\right]P\left(\widetilde{\phi}=j\right)\\ &=\frac{1}{\overline{F}(W_1)}\mu_{W_1}+\mu_{W_2|W_1}+\frac{F_{W_1}(W_2-W_1)}{\overline{F}(W_2-W_1)}\mu_{W_2-W_1}+\overline{F}_{W_1}\left(W_2-W_1\right)\mu_{W|W_2}\quad\square\\ &+F_{W_1}\left(W_2-W_1\right)\mu_{W-W_1|W_2-W_1}+\frac{\Psi}{\overline{F}(W-W_2)}\mu_{W-W_2}+W.\end{aligned}$$

Derivation of Eqs.(19) and (20). It is known that when $\xi'=m+1$, the warranty has already ceased in the first renewing period, with probability $[F(W_1)]^{m+1}$. Otherwise, there are three cases where an item survives the second renewing period or the warranty expires during this period:

(a) the $(\xi'+1)$th item, that has survived for W_1, also survives for its age W, with conditional probability $\overline{F}_{W_1}(W-W_1)$;
(b) the $(\xi'+1)$th item fails within $(W_1, W]$, and a subsequent replacement survives for its age $W-W_1$, with conditional probability $F_{W_1}(W-W_1)[F(W-W_1)]^{\phi'-1}\overline{F}(W-W_1)$, $\phi'=1,\ldots,m-\xi'$; and
(c) the $(\xi'+1)$th item fails within $(W_1, W]$, and totally there are at least $m-\xi'+1$ failures in the second renewing period, with conditional probability $F_{W_1}(W-W_1)[F(W-W_1)]^{m-\xi'}$.

Hence, conditioning on $\xi' \leqslant m$, ϕ' has a geometric distribution censored above $m - \xi' + 1$, with conditional probability:

$$P\left(\phi' = j \middle| \xi' \leqslant m\right) = \begin{cases} \overline{F}_{W_1}(W_2 - W_1), & j = 0, \\ F_{W_1}(W_2 - W_1)[F(W_2 - W_1)]^{j-1}\overline{F}(W_2 - W_1), & j = 1, \dots, m - \xi', \\ F_{W_1}(W_2 - W_1)[F(W_2 - W_1)]^{m-\xi'}, & j = m - \xi' + 1, \end{cases}$$

where $P\left(\phi' = m - \xi' + 1 \middle| \xi' \leqslant m\right) = 1 - \sum_{j=0}^{m-\xi'} P\left(\phi' = j \middle| \xi' \leqslant m\right)$, according to the censoring rule.

Consequently, conditioning on $\xi' \leqslant m$, the conditional mean of ϕ' is

$$E\left[\phi' \middle| \xi' \leqslant m\right] = \sum_{j=0}^{m-\xi'+1} jP\left(\phi' = j \middle| \xi' \leqslant m\right) = \frac{F_{W_1}(W_2 - W_1)}{\overline{F}(W_2 - W_1)}(1 - [F(W_2 - W_1)]^{m-\xi'+1}).$$

By removing the conditioning on ξ', $E[\phi']$ can be derived as

$$E[\phi'] = \sum_{i=0}^{m+1} E\left[\phi' \middle| \xi' = i\right] P\left(\xi' = i\right) = \frac{F_{W_1}(W_2 - W_1)}{\overline{F}(W_2 - W_1)}[1 - [F(W_1)]^{m+1} - F(W_2 - W_1)\Upsilon].$$

Notice that $E[\phi'|\xi' = m + 1] = 0$ in the expression above.

Now, we derive the expression of $E[C'_{\mathcal{P}}(W_1, W, m)]$ in (20). It is clear that when $\xi' = m + 1$, $C'_{\mathcal{P}}(W_1, W, m) = z'(W_1, m)$. In contrast, when $\xi' \leqslant m$, $C'_{\mathcal{P}}(W_1, W, m)$ can be expressed as

$$C'_{\mathcal{P}}\left(W_1, W, m \middle| \xi' \leqslant m\right) = \begin{cases} z'(W_1, m) + c_r\phi', & \phi' = 0, \dots, m - \xi', \\ z'(W_1, m) + c_r(m - \xi') + c_b, & \phi' = m - \xi' + 1, \end{cases}$$

and its conditional expectation is given by

$$\begin{aligned} E[C'_{\mathcal{P}}(W_1, W, m)|\xi' \leqslant m] &= \sum_{j=0}^{m-\xi'+1} E\left[C'_{\mathcal{P}}\left(W_1, W, m\right) \middle| \xi' \leqslant m, \phi' = j\right] P\left(\phi' = j \middle| \xi' \leqslant m\right) \\ &= E[z'(W_1, m)] + c_r E[\phi'|\xi' \leqslant m] + (c_b - c_r)F_{W_1}(W - W_1)[F(W - W_1)]^{m-\xi'}. \end{aligned}$$

Thus, by removing the conditioning on ξ', $E[C'_{\mathcal{P}}(W_1, W, m)]$ in (20) can be determined by

$$E\left[C'_{\mathcal{P}}\left(W_1, W, m\right)\right] = \sum_{i=0}^{m+1} E\left[C'_{\mathcal{P}}\left(W_1, W, m\right) \middle| \xi' = i\right] P\left(\xi' = i\right) = E\left[z'\left(W_1, m\right)\right] + c_r E[\phi'] + \left(c_b - c_r\right) F_{W_1}\left(W - W_1\right)\Upsilon,$$

where $E[z'(W_1, m)]$ and $E[\phi']$ are given by (18) and (19), respectively. □

Derivation oDerivation of Eq.(22). It is known that if $\xi' = m + 1$, then $\mathcal{W}'_{\mathcal{P}}(W_1, W, m) = l'(W_1, m)$. Otherwise, conditioning on $\xi' \leqslant m$, the cumulative warranty length after the second renewing period is given by

$$\mathcal{W}'_{\mathcal{P}}\left(W_1, W, m \middle| \xi' \leqslant m\right) = \begin{cases} l'(W_1, m) + W - W_1, & \text{if } \phi' = 0, \\ l'\left(W_1, m\right) + \mathcal{Y}' + \sum_{e=\xi'+2}^{\xi'+\phi'} X_e + W - W_1, & \text{if } \phi' = 1, \dots, m - \xi', \\ l'\left(W_1, m\right) + \mathcal{Y}' + \sum_{e=\xi'+2}^{\xi'+\phi'} X_e, & \text{if } \phi' = m - \xi' + 1, \end{cases}$$

and the conditional mean of $\mathcal{W}'_{\mathcal{P}}(W_1, W, m)$ is

$$\begin{aligned} E[\mathcal{W}'_{\mathcal{P}}(W_1, W, m)|\xi' \leqslant m] &= \sum_{j=0}^{m-\xi'+1} E\left[\mathcal{W}'_{\mathcal{P}}\left(W_1, W, m\right) \middle| \xi' \leqslant m, \phi' = j\right] P\left(\phi' = j \middle| \xi' \leqslant m\right) \\ &= E\left[l'\left(W_1, m\right)\right] + \mu_{W|W_1} + (E[\phi'|\xi' \leqslant m] - F_{W_1}(W - W_1))\frac{\mu_{W-W_1}}{F(W - W_1)} \\ &+ (W - W_1)(1 - F_{W_1}(W - W_1)[F(W - W_1)]^{m-\xi'}). \end{aligned}$$

Hence, by removing the conditioning on ξ', the mean of $\mathcal{W}'_{\mathcal{P}}(W_1, W, m)$ in (22) can be derived by $E[\mathcal{W}'_{\mathcal{P}}(W_1, W, m)] = \sum_{i=0}^{m+1} E[\mathcal{W}'_{\mathcal{P}}(W_1, W, m)|\xi' = i]P(\xi' = i)$. □

Appendix B. Proofs

Proof of Proposition 1. *Part I:* First of all, notice that $E[C_{\mathcal{F}}(W)]$ in (8) can be rewritten as

$$E\left[C_{\mathcal{F}}(W)\right] = c_r\frac{F(W_1)}{\overline{F}(W_1)} + c_r\left(\frac{1}{\overline{F}(W)} - \frac{1}{\overline{F}(W_1)}\right).$$

Then, by subtracting $E[C_{\mathcal{P}}(W_1, W)]$ from $E[C_{\mathcal{F}}(W)]$ (and keeping in mind that the survival function $\overline{F}(x)$ is a decreasing function of x), we have

$$E\left[C_{\mathcal{F}}(W)\right]-E\left[C_{\mathcal{P}}\left(W_1,W\right)\right]=c_r\left(1-\frac{\overline{F}(W)}{\overline{F}(W-W_1)}\right)\left(\frac{1}{\overline{F}(W)}-\frac{1}{\overline{F}(W_1)}\right)\geqslant 0.$$

The inequality is due to $0<\overline{F}(W)\leqslant\overline{F}(W-W_1)$ and $0<\overline{F}(W)\leqslant\overline{F}(W_1)$.

Part II: Following the same procedure of deriving $E[\mathcal{W}_{\mathcal{P}}(W_1,W)]$, $E[\mathcal{W}_{\mathcal{F}}(W)]$ in (9) can be rewritten as

$$E\left[\mathcal{W}_{\mathcal{F}}(W)\right]=\frac{1}{\overline{F}(W_1)}\mu_{W_1}+W_1F_{W_1}\left(W-W_1\right)+\mu_{W|W_1}+\frac{F_{W_1}(W-W_1)}{\overline{F}(W)}\mu_W+W.$$

Then, by subtracting $E[\mathcal{W}_{\mathcal{P}}(W_1,W)]$ from $E[\mathcal{W}_{\mathcal{F}}(W)]$, we have

$$E\left[\mathcal{W}_{\mathcal{F}}(W)\right]-E\left[\mathcal{W}_{\mathcal{P}}\left(W_1,W\right)\right]=F_{W_1}\left(W-W_1\right)\left(W_1+\frac{\mu_W}{\overline{F}(W)}-\frac{\mu_{W-W_1}}{\overline{F}(W-W_1)}\right)\geqslant 0.$$

The inequality is due to $\mu_W\geqslant\mu_{W-W_1}\geqslant 0$, $\overline{F}(W-W_1)\geqslant\overline{F}(W)>0$, and thus $\mu_W/\overline{F}(W)\geqslant\mu_{W-W_1}/\overline{F}(W-W_1)$. This completes the proof. □

Proof of Lemma 1. We need to consider the following two cases of $\widetilde{\phi}$ when deriving the conditional survival function of $\mathcal{H}$: $\widetilde{\phi}=0$ and $\widetilde{\phi}\geqslant 1$.

Case 1:$\widetilde{\phi}=0$. In this case, there is no item failure in the second renewing period, which implies that the $(\xi+1)$ th item that has survived for W_1, still survives for another period of W_2-W_1. One can know that conditioning on $\widetilde{\phi}=0$, $\mathcal{H}$ has conditional survival distribution:

$$\begin{aligned}P(\mathcal{H}>x|\widetilde{\phi}=0)&=P(\mathcal{Y}-(W_2-W_1)>x|\mathcal{Y}-(W_2-W_1)>0)\\&=\tfrac{P(\mathcal{Y}>W_2-W_1+x)}{P(\mathcal{Y}>W_2-W_1)}=\tfrac{\overline{F}_{W_1}(W_2-W_1+x)}{\overline{F}_{W_1}(W_2-W_1)}=\overline{F}_{W_2}(x).\end{aligned}$$

Case 2:$\widetilde{\phi}\geqslant 1$. In this case, there are at least one failure in the second renewing period. We then have

$$P\left(\mathcal{H}>x\middle|\widetilde{\phi}\geqslant 1\right)=\sum_{\widetilde{\phi}=1}^{\infty}P(X_{\xi+\widetilde{\phi}+1}>W_2-W_1+x|\mathcal{Y}-(W_2-W_1)<0)$$

$$=\sum_{\widetilde{\phi}=1}^{\infty}P(\mathcal{Y}<W_2-W_1,X_{\xi+2}<W_2-W_1,\ldots,X_{\xi+\widetilde{\phi}}<W_2-W_1,X_{\xi+\widetilde{\phi}+1}>W_2-W_1+x|\mathcal{Y}<W_2-W_1)$$

$$=\sum_{\widetilde{\phi}=1}^{\infty}P(X_1<W_2-W_1)^{\widetilde{\phi}-1}P\left(X_1>W_2-W_1+x\right)=\frac{P(X_1>W_2-W_1+x)}{P(X_1>W_2-W_1)}=\overline{F}_{W_2-W_1}(x),\ x\geqslant 0.$$

Therefore, the conditional survival function of $\mathcal{H}$ in (11) holds. □

Proof of Proposition 2. *Part I:* Notice that following the same procedure of deriving $E[\widetilde{C}_{\mathcal{P}}(W_1,W_2,W)]$, $E[C_{\mathcal{P}}(W_1,W)]$ in (4) can be rewritten as

$$E\left[C_{\mathcal{P}}\left(W_1,W\right)\right]=c_r\frac{F(W_1)}{\overline{F}(W_1)}+c_r\frac{F_{W_1}(W_2-W_1)}{\overline{F}(W_2-W_1)}+c_r\frac{\Psi}{\overline{F}(W-W_1)}.$$

It is thus easy to have $E[\widetilde{C}_{\mathcal{P}}(W_1,W_2,W)]\leqslant E[C_{\mathcal{P}}(W_1,W)]$, since $\overline{F}(W-W_1)\leqslant\overline{F}(W-W_2)$. Combining the result in Proposition 1, we obtain $E[\widetilde{C}_{\mathcal{P}}(W_1,W_2,W)]\leqslant E[C_{\mathcal{P}}(W_1,W)]\leqslant E[C_{\mathcal{F}}(W)]$.

Part II: Likewise, $E[\mathcal{W}_{\mathcal{P}}(W_1,W)]$ in (6) can be rewritten as

$$E\left[\mathcal{W}_{\mathcal{P}}\left(W_1,W\right)\right]$$
$$=\frac{1}{\overline{F}(W_1)}\mu_{W_1}+\mu_{W_2|W_1}+\frac{F_{W_1}(W_2-W_1)}{\overline{F}(W_2-W_1)}\mu_{W_2-W_1}+\overline{F}_{W_1}\left(W_2-W_1\right)\mu_{W|W_2}+F_{W_1}\left(W_2-W_1\right)\mu_{W-W_1|W_2-W_1}+\frac{\Psi}{\overline{F}(W-W_1)}\mu_{W-W_1}+W+\left(W_2-W_1\right)\Psi.$$

Then, by subtracting $E[\widetilde{\mathcal{W}}_{\mathcal{P}}(W_1,W_2,W)]$ from $E[\mathcal{W}_{\mathcal{P}}(W_1,W)]$, we obtain

$$E\left[\mathcal{W}_{\mathcal{P}}\left(W_1,W\right)\right]-E\left[\widetilde{\mathcal{W}}_{\mathcal{P}}\left(W_1,W_2,W\right)\right]=\Psi\left(W_2-W_1+\frac{\mu_{W-W_1}}{\overline{F}(W-W_1)}-\frac{\mu_{W-W_2}}{\overline{F}(W-W_2)}\right)\geqslant 0.$$

Therefore, it is straightforward to have $E[\widetilde{\mathcal{W}}_{\mathcal{P}}(W_1,W_2,W)]\leqslant E[\mathcal{W}_{\mathcal{P}}(W_1,W)]\leqslant E[\mathcal{W}_{\mathcal{F}}(W)]$. This completes the proof. □

Proof of Corollary 1. From Proposition 2, we know that $E[\widetilde{C}_{\mathcal{P}}(W_1,W_2,W)]\leqslant E[C_{\mathcal{P}}(W_1,W)]\leqslant E[C_{\mathcal{F}}(W)]$ for any fixed W_1, W_2, and W. Then, it is clear that $E[C_{\mathcal{P}}(W_1^*,W)]\leqslant E[C_{\mathcal{F}}(W)]$ since $E[C_{\mathcal{P}}(W_1^*,W)]\leqslant E[C_{\mathcal{P}}(W_1,W)]$ for any $W_1\in[0,W]$. On the other hand, Proposition 2 also implies that $E[\widetilde{C}_{\mathcal{P}}(W_1^*,W_2,W)]\leqslant E[C_{\mathcal{P}}(W_1^*,W)]$. Thus, we have $E[\widetilde{C}_{\mathcal{P}}(\widetilde{W}_1^*,\widetilde{W}_2^*,W)]\leqslant E[C_{\mathcal{P}}(W_1^*,W)]$ since $E[\widetilde{C}_{\mathcal{P}}(\widetilde{W}_1^*,\widetilde{W}_2^*,W)]\leqslant E[\widetilde{C}_{\mathcal{P}}(W_1,W_2,W)]$ for any $0\leqslant W_1\leqslant W_2\leqslant W$. □

Appendix C. Weibull MLE with multiply censored data

The objective of MLE method is to determine the optimal parameter value to maximize the likelihood (or, equivalently, log-likelihood) function (Blischke, Karim, & Murthy, 2011; Chen & Ye, 2017; Dai, He, Liu, Yang, & He, 2017). When multiply censored data are present, the likelihood function can be written as

$$\mathcal{L}(\theta) = \prod_{j=1}^{q} f(j;\theta)^{\boldsymbol{n_j}} \prod_{j^+=q-p+1}^{q} \overline{F}(j^+;\theta)^{\boldsymbol{n}_{j^+}},$$

where θ is the unknown parameter(s), and other notations follow directly from Section 4.1.

For Weibull distribution with scale parameter α and shape parameter β, the likelihood function can be written as

$$\mathcal{L}\left(\alpha, \beta\right) = \prod_{j=1}^{q} \left[\frac{\beta}{\alpha}\left(\frac{j}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{j}{\alpha}\right)^{\beta}\right\}\right]^{\boldsymbol{n_j}} \prod_{j^+=q-p+1}^{q} \left[\exp\left\{-\left(\frac{j^+}{\alpha}\right)^{\beta}\right\}\right]^{\boldsymbol{n}_{j^+}}.$$

By solving $\partial\ln\mathcal{L}(\alpha, \beta)/\partial\alpha = 0$ and $\partial\ln\mathcal{L}(\alpha, \beta)/\partial\beta = 0$, simultaneously, the MLE estimates of Weibull distribution parameters, i.e., $\hat{\alpha}$ and $\hat{\beta}$, can be determined by

$$\hat{\alpha} = \left[\frac{\sum_{j=1}^{q} \boldsymbol{n_j} j^{\hat{\beta}} + \sum_{j^+=q-p+1}^{q} \boldsymbol{n}_{j^+}(\boldsymbol{j}^+)^{\hat{\beta}}}{\sum_{j=1}^{q} \boldsymbol{n_j}}\right]^{1/\hat{\beta}}, \tag{C.1}$$

and

$$\frac{\sum_{j=1}^{q} \boldsymbol{n_j}\left[\sum_{j=1}^{q} \boldsymbol{n_j} j^{\hat{\beta}} \ln(j) + \sum_{j^+=q-p+1}^{q} \boldsymbol{n}_{j^+}(\boldsymbol{j}^+)^{\hat{\beta}} \ln(\boldsymbol{j}^+)\right]}{\sum_{j=1}^{q} \boldsymbol{n_j} j^{\hat{\beta}} + \sum_{j^+=q-p+1}^{q} \boldsymbol{n}_{j^+}(\boldsymbol{j}^+)^{\hat{\beta}}} - \frac{1}{\hat{\beta}} \sum_{j=1}^{q} \boldsymbol{n_j} = \sum_{j=1}^{q} \boldsymbol{n_j} \ln j. \tag{C.2}$$

¶It should be pointed out that Eq. (C.2) must be solved numerically. The Newton-Raphson method is a good candidate for solving such a nonlinear equation. Alternatively, since the left-hand side is a monotonically increasing function of β, this equation can be solved by the direct-search or bisection search technique as well. In addition, some statistical software, such as Weibull+ + and Minitab, also contain life data analysis module which can be employed to perform MLE for multiply censored warranty data.

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