Institution of MECHANICAL ENGINEERS

Journal of RISK AND RELIABILITY

Review Article

Proc IMechE Part O:
J Risk and Reliability
2018, Vol. 232(3) 284–307

DOI: 10.1177/1748006X17742776
journals.sagepub.com/home/pio
SAGE

Two-dimensional warranty: A literature review

Xiaolin Wang[1,2] and Wei Xie[3]

Abstract

The warranty policies for capital-intensive products, such as automobiles, heavy equipment, and aircraft engines, typically have two dimensions, that is, the age and usage. During the last two decades, much research has been published on the two-dimensional warranty from many different perspectives. This article reviews and summarizes work and state-of-the-art developments in this research field, with emphasis on both theoretical methodologies and practical applications. In this review, five principal topics are covered: (1) two-dimensional warranty policies, (2) cost analysis of two-dimensional warranty, (3) two-dimensional warranty and engineering/marketing problems, (4) two-dimensional warranty and logistics issues, and (5) two-dimensional long-term warranty. In particular, the practical and application aspects of two-dimensional warranty policies have been surveyed. Overall, 105 journal papers covering 49 journals, published from 1993 to present, are selected and discussed in detail. Finally, some interesting and challenging research topics are presented for the scholars and practitioners to explore new research in this field.

Keywords

Two-dimensional warranty, flexible warranty, failure modeling, warranty data analysis, maintenance, marketing, logistics, long-term warranty

Date received: 9 April 2017; accepted: 25 October 2017

Introduction

Under today's highly competitive market, manufacturers are forced to transform themselves from profit-making entities to customer-centric enterprises. During this transition, good after-sales services gradually become a key factor for the survival of a manufacturer. Warranty is one of the most important elements of after-sale services. Nowadays, almost all products are sold with warranty contracts due to competition, industrial obligations, and/or customer requirements. The primary purpose of a warranty is to offer post-sale remedy, such as repair, replacement, or refund, for the customer when a product fails to perform its intended function within the warranty period. Essentially, most manufacturers are employing various warranty policies as marketing strategy to boost their sales, while the customers regard the warranty as an important factor in their product selection process, especially when the competing brands are nearly identical.[1] As a result, warranty has become increasingly important in commercial transactions and a variety of warranty policies have appeared in the marketplace.

In recent years, product warranty has attracted extensive attention from many different disciplines. A vast amount of literature on warranty can be found from monographs,[2–8] review papers,[1,9–17] and many other journal and conference papers. Interested readers are referred to Xie and Ye,[18] Lee et al.,[19] Limon et al.,[20] Tseng et al.,[21] Park and Pham,[22] Darghouth et al.,[23] Zhao and Xie,[24] and Murthy and Jack[25] for recent progresses on product warranty.

Considerable warranty policies have been developed by researchers and/or adopted by practitioners, and they can be classified into several categories based on different criteria. According to the number of variables covered by the warranty policies, they can be categorized into one-dimensional (1D) and two-dimensional (2D) policies.[26] Specifically, a 1D warranty policy is characterized by an interval with a single variable, for

[1]Department of Systems Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong
[2]City University of Hong Kong Shenzhen Research Institute, Shenzhen, China
[3]School of Business Administration, South China University of Technology, Guangzhou, China

Corresponding author:
Wei Xie, School of Business Administration, South China University of Technology, Guangzhou 510640, Guangdong, China.
Email: bmwxie@scut.edu.cn

example, product age or usage. On the other hand, a 2D warranty policy is represented by a region on the 2D plane, generally with one dimension representing age and the other one denoting usage. Many capital-intensive products, such as automobiles, aircraft engines, and heavy equipment, are sold with a 2D warranty policy. For example, a new automobile may be covered by a 2D warranty up to 3 years or 36,000 miles, whichever occurs first.

Because of the importance in industrial applications, following the seminal work of Murthy and Wilson,[27] Iskandar,[28] Singpurwalla and Wilson,[29] and so on, intensive research on 2D warranty has been conducted for more than two decades. Evidentially, the 2D warranty-related research has become a hot-spot in the field of warranty management. Therefore, a comprehensive review of this research area is needed to summarize the state-of-the-art body of knowledge. To the best of our knowledge, this article is the first identifiable academic overview of 2D warranty. The main purpose of this work is threefold: (1) review the extant literature related to 2D warranty, (2) investigate the insights of existing models and approaches proposed to study 2D warranty, and (3) suggest some new directions for future research.

The rest of this article is organized as follows. In the next section, the framework of this review article is introduced. Then, in the subsequent sections, five principal topics that are closely related to 2D warranty are reviewed comprehensively. Afterward, section "Applications" discusses several practical aspects of the 2D warranty. Finally, the last section concludes this article and discusses some valuable topics for future research.

Review framework

Motivated by real-world problems, the academic studies on 2D warranty cover various topics and aim to address practical issues from different perspectives. Our review of this literature deals with the following topics:

1. 2D warranty policies;
2. 2D warranty cost analysis;
3. 2D warranty and engineering/marketing;
4. 2D warranty and logistics;
5. 2D long-term warranty.

We adopt a similar reviewing framework as that of Murthy and Djamaludin.[1] It is worth mentioning that we have put our effort in bibliography searching throughout the academic journals since 1990s and have found that the references exploring the engineering and marketing-related issues on 2D warranty are really rare. Hence, we combine the two topics into a single section. In addition, 2D long-term warranty, as a special type of 2D warranty, receives relatively limited attention, but its importance in both academic studies and practical applications has been demonstrated. Thus, we believe that it deserves a full-section review.

One thing noteworthy is that this review is only confined to journal papers because the scholars and practitioners in this field usually disseminate new findings or acquire information through the related academic journals. Overall, 105 papers are selected after a comprehensive bibliography search (Table 5). These papers are distributed across 49 journals, as shown in Table 1. The journals are listed by a descending sequence based on the total number of published papers. Note that, in

Table 1. Distribution of the reviewed papers by publication sources.

No.	Journal title	Number of papers	%
1	*Reliability Engineering & System Safety*	19	18.10
2	*Computers & Industrial Engineering*	5	4.76
3	*IIE Transactions/IISE Transactions*[a]	5	4.76
4	*International Journal of Production Research*	5	4.76
5	*European Journal of Operational Research*	4	3.81
6	*International Journal of Reliability and Safety*	4	3.81
7	*IEEE Transactions on Reliability*	3	2.86
8	*International Journal of Quality & Reliability Management*	3	2.86
9	*Journal of Risk and Reliability*	3	2.86
10	*Lifetime Data Analysis*	3	2.86
11	*Naval Research Logistics*	3	2.86
12	*Quality and Reliability Engineering International*	3	2.86
13	*Technometrics*	3	2.86
14	*Communications in Statistics—Theory and Methods*	2	1.90
15	*Engineering Failure Analysis*	2	1.90
16	*International Journal of Operational Research*	2	1.90
17	*International Journal of Product Development*	2	1.90
18	*Mathematical and Computer Modelling*	2	1.90
19	*Quality Technology & Quantitative Management*	2	1.90
20	Others (with only one paper published)	30	28.57
	Total	105	100.00

[a] *IIE Transactions* changed its title to *IISE Transactions* in 2016. Here, we treat them as one journal and combine the number of papers published in both of them together.

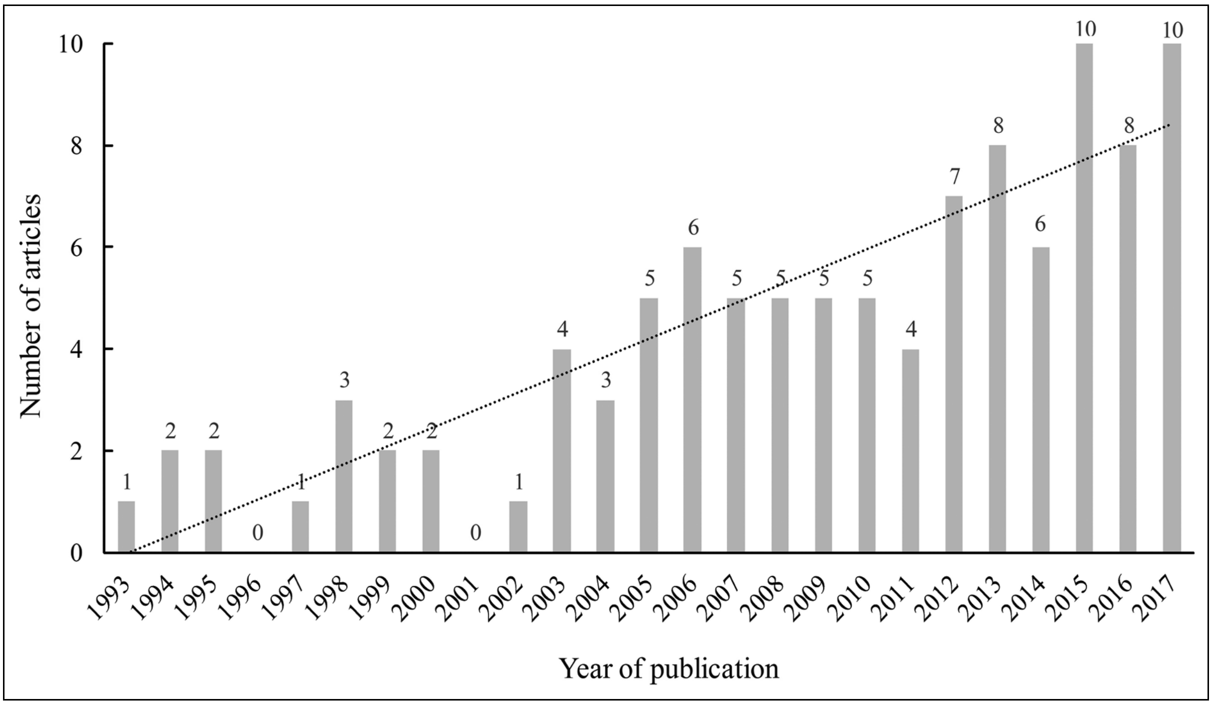

Figure 1. Distribution of the reviewed papers by year of publication (up to July 2017).

the case of a tie in the number of publications, the journals appear in alphabetical order. We also illustrate the number of papers versus year of publication in Figure 1. It can be seen that the published years of the 105 papers range from 1993 to present, and one can also observe a clear upward trend in the number of publications per year (which indicates the growing research attention in this research field).

Moreover, Table 2 identifies the top 15 most productive authors (or coauthors) who contribute at least four publications among the 105 papers reviewed. In the case of a tie in the total number of publications, the authors appear in alphabetical order of their surnames. It can be observed that, from 2010 to present, more than half of these authors are active in this field. In total, more than 150 researchers have made contributions to this field, and around one-third of them published two papers or more.

2D warranty policies

In this section, we review the 2D warranty policies in the literature. Generally, product warranty can be classified into two types, that is, the non-renewable and renewable warranties. Both of them can be further divided into two sub-groups—simple and combination policies. Free repair/replacement warranty (FRW) and pro-rata warranty (PRW) are two typical simple warranty policies. While combination warranty (CW), also called hybrid warranty, synthesizes different simple policies over different phases of the warranty period or combines a simple policy with some additional features. For more details on warranty taxonomy, refer to Blischke and Murthy.[26]

Typically, 2D warranty policies are defined in terms of product age and usage. The usage of different types of products may have different expressions, for example, *output-based usage* (mileages for cars, copies made by photocopier, etc.), *time-based usage* (actual operating time of the electronic devices such as computers and air-conditioners), and *number-based usage* (number of landings of an aircraft, number of times a battery has been recharged, etc.). For example, if one has possessed a car for 2 years and driven it for 850 h and 51,000 miles, then the elapsed time period is 2 years, the time-based usage is 850 h, and the output-based usage is 51,000 miles. These time-dependent measures could be the metrics adopted in a 2D warranty problem. In the literature, most existing 2D warranty policies are defined based on the time and output-based usage, with only a few exceptions based on time-based usage, see Shafiee et al.,[30] Qiu et al.,[31] and Yang et al.[32] for more details.

Typical 2D warranty policies

Generally speaking, a different region forms a different 2D warranty policy. Blischke and Murthy[26] and Singpurwalla and Wilson[29] have suggested and discussed a variety of possible shapes of 2D warranty regions. Once the shape of warranty region is specified, the 2D renewable/non-renewable FRW, PRW, and CW policies can be formulated accordingly. One may notice that Blischke and Murthy[26] have already introduced six 2D warranty policies in detail. For the sake

Table 2. Most productive authors.

No.	Author	Affiliation[a]	Number of publications			
			1993–1999	2000–2009	2010–present	Total
1	Iskandar BP	1. The University of Queensland, Australia 2. Bandung Institute of Technology, Indonesia	2	4	5	11
2	Murthy DNP	The University of Queensland, Australia	2	7	1	10
3	Rai B	1. Wayne State University, USA 2. University of Massachusetts, USA	0	8	0	8
4	Chukova S	Victoria University of Wellington, New Zealand	0	1	6	7
5	Singh N	Wayne State University, USA	0	7	0	7
6	Manna DK	Indian Statistical Institute, India	0	5	1	6
7	Liu Z	Tianjin University, China	0	0	5	5
8	Jack N	University of Abertay Dundee, UK	0	5	0	5
9	Pal S	Indian Statistical Institute, India	0	4	1	5
10	Su C	Southeast University, China	0	0	5	5
11	Huang YS	National Cheng Kung University, Taiwan	0	1	3	4
12	Husniah H	1. Bandung Institute of Technology, Indonesia 2. Langlangbuana University, Indonesia	0	0	4	4
13	Pasaribu US	Bandung Institute of Technology, Indonesia	0	0	4	4
14	Sinha S	Jadavpur University, India	0	3	1	4
15	Wang X	Southeast University, China	0	0	4	4

[a]Iskandar, Rai, and Husniah have two affiliations at different stages, which are presented according to the chronological order of publication.

of self-containment, these policies are re-introduced as follows (see Appendix 2 for some real-world 2D warranty programs).

Policy 1: 2D non-renewing FRW policy (rectangular). Under this policy, the manufacturer agrees to provide free repair or replacement service for failed products with an age limit W and usage limit U, whichever occurs first, from the point of the initial purchase. This policy can be characterized by the following rectangular region

$$\Omega_R(W, U) = \{(t, x) : t \in [0, W] \text{ and } x \in [0, U]\}$$

which is shown in Figure 2(a). This policy allows a customer to access free warranty service within a maximum time period W and a maximum usage U. For a light-usage customer, the warranty will cease by time W (before the cumulative usage reaches U); while for a customer who uses the product frequently, the warranty may expire before time W (the cumulative usage touches its limit U very soon). That is to say, for customers with either light or heavy usage, their actual warranty regions are generally smaller than that of medium-usage customers. In this sense, this policy is not attractive for the customers with light or heavy usage because the warranty coverage is not fully used.[33]

Note that this warranty policy is widely adopted in both academic research and practical applications, for example, it is offered by nearly all automobile manufacturers. For illustrative purpose, the real-world warranty programs of automobile and aircraft companies are presented in Exhibits 1 and 2 (see Appendix 2).

Policy 2: 2D non-renewing FRW policy (L-shaped). This policy is also among the common shapes of 2D warranty regions. Under this policy, the manufacturer will provide free repair/replacement service for the failed products up to a time limit W and up to a cumulative usage limit U, whichever occurs last, from the point of the initial purchase. This policy is defined by an L-shaped region

$$\Omega_L(W, U) = \{(t, x) : t \in [0, W] \text{ or } x \in [0, U]\}$$

as depicted in Figure 2(b). Apparently, this policy assures the sold products will at least be covered by W units of time and U units of usage. For instance, for a heavy-usage customer, the warranty terminates mainly due to the time limit W with the total usage goes well beyond U; while for a light-usage customer, the warranty ceases because of the usage limit U with the time goes well beyond W. In this sense, this policy tends to favor customers with either light or heavy usage[33] because their actual warranty regions are generally larger than that of the medium-usage customers.

However, in real-world situation, it is very difficult to find an application with the L-shaped warranty

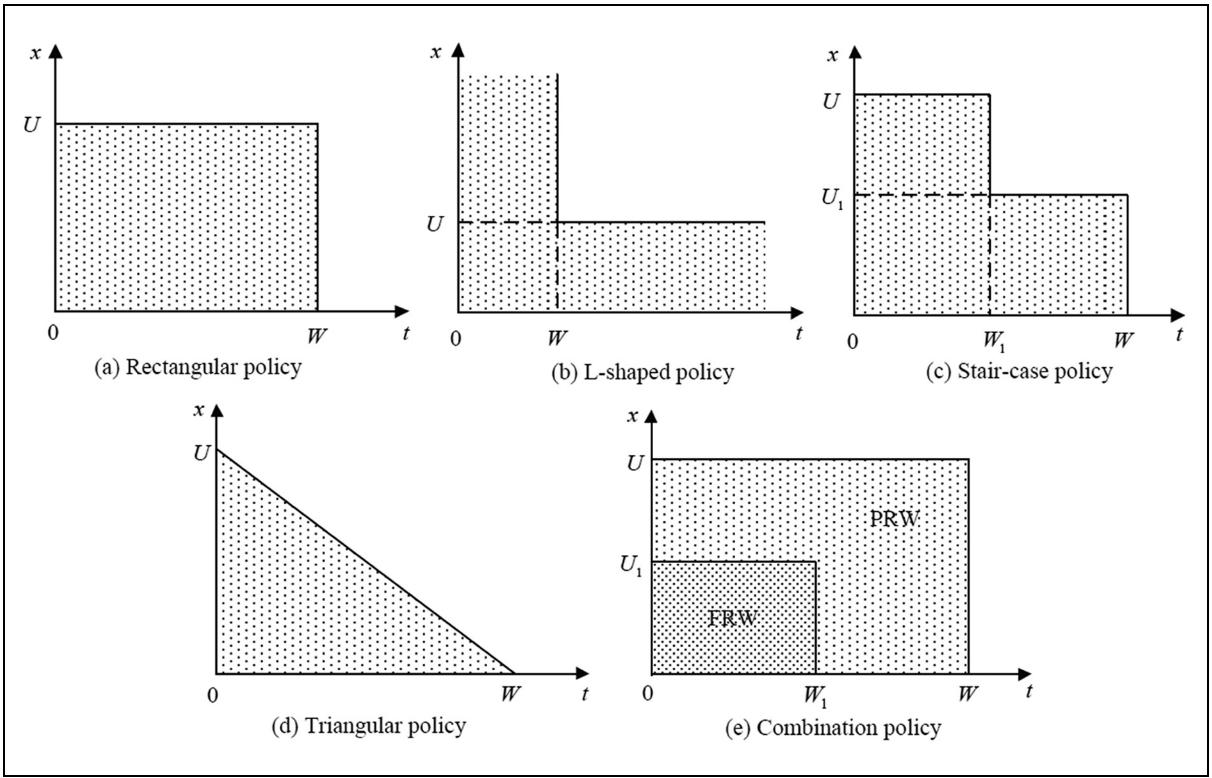

Figure 2. Different shapes of 2D warranty regions: (a) rectangular policy, (b) L-shaped policy, (c) stair-case policy, (d) triangular policy, and (e) combination policy.

policy because the manufacturer would much prefer the rectangular one.[26] Nevertheless, in the literature, several papers can be found to investigate this policy.[30,33–37] Among them, Murthy et al.,[33] Chun and Tang,[34] and Kim and Rao[35] derived the associated cost models for the L-shaped 2D warranty policy under different problem settings.

Policy 3: 2D non-renewing FRW policy (stair-case). As shown in Figure 2(c), this region can be viewed as an L-shaped one with an age limit $W > W_1$ and a usage limit $U > U_1$. Under this policy, the manufacturer promises to provide free repair or replacement services for the failed products up to a time limit W_1 (from the point of the initial purchase) with the total usage at failure less than U, and up to a time limit W with the total usage at failure below U_1. This policy is defined by a stair-case region

$$\Omega_s(W_1, U_1; W, U) = \{(t, x) : (t \in [0, W_1] \text{ and } x \in [0, U]) \cup (t \in [0, W] \text{ and } x \in [0, U_1])\}$$

Under this policy, the sold products will be protected by the warranty program for W_1 units of time and U_1 units of usage at least, and for W units of time and U units of usage at most. The rectangles $[0, W_1] \times [0, U]$ and $[0, W] \times [0, U_1]$ represent a better coverage for heavy-usage and light-usage customers, respectively. Thus, this policy may be attractive for most customers. Furthermore, unlike the L-shaped policy, the upper limits W and U in this policy protect the manufacturer from bearing excessive warranty claims. Therefore, this policy represents a sensible compromise between the rectangular- and L-shaped ones.[26,33]

It should be noted that under this policy, the manufacturer's warranty costs of servicing the light-usage and heavy-usage customers can be set as the same by appropriately choosing the warranty parameters W_1 and U_1. In this case, the manufacturer does not need to differentiate the two categories of customers when estimating the warranty costs. In the literature, Ye and Murthy[38] investigated this problem in the design of 2D warranty menus.

Policy 4: 2D non-renewing FRW policy (triangular). This policy is characterized by a triangular region

$$\Omega_T(W, U) = \{(t, x) : x + (U/W)t \leqslant U\}$$

as shown in Figure 2(d). Under this policy, similarly, all failures within the warranty region will be rectified by the manufacturer free of charge to the customers. The warranty ceases at time t once the cumulative usage x upon time t satisfies the inequality $x + (U/W)t > U$.

In the literature, Murthy et al.[33] and Chun and Tang[34] have investigated the cost analysis of the triangular 2D policy, in addition to the rectangular- and L-shaped policies.

Policy 5: 2D non-renewing PRW policy. Under this policy, unlike the FRW ones, the manufacturer only agrees to cover a fraction of repair/replacement cost or refund a fraction of original purchase price, if the product fails within the warranty period. In particular, the fraction covered/refunded is a function of time and/or usage at the failure. The warranty coverage for this policy is usually given by the rectangular region in Figure 2(a). It is worth noting that the PRW policies for warranty regions shown in Figure 2(b)–(d) can be defined similarly as that of FRW policies, while little research has been conducted to discuss these cases.

As a good application, this warranty policy is usually offered for automobile tires. For example, under the bumper-to-bumper warranty, a defective tire of Ford automobile will be replaced on a pro-rata adjustment basis according to a pre-specified refund plan (see Exhibit 1 in Appendix 2). The academic study of this policy, however, receives only a little attention. Chun and Tang[34] derived the cost model of this policy. Tsai and Fang[39] developed a Bayesian decision model of the pricing and production strategy for this policy.

Policy 6: 2D non-renewing CW policy. To the best of our knowledge, only one journal paper discussed this policy and analyzed the costs of five different 2D CW policies, that is, Iskandar et al.[40] Under this policy (Figure 2(e)), the manufacturer agrees to provide free repair or replacement services for failed products up to time W_1 and usage U_1, while any further failures will be replaced at a pro-rated cost.

Through the above review, we also recognize that little research on renewable 2D warranty policies has been conducted, with only one exception.[40] One possible reason is that the mathematical models of renewable 2D warranty are too complex and not amenable to analysis.

Remark 1. The above-reviewed 2D warranty policies are developed based on two traditional dimensions: age and usage. However, Park and Pham[41,42] adopted two new factors, that is, failure time and *warranty servicing time*, to define a new type of "2D warranty" policy. Under this policy, when a product fails within the warranty period, the minimal repair strategy will be implemented first. If the failed product cannot be restored to a working state within a certain threshold of servicing time, then the product would be replaced with a new one. It should be noted that this warranty policy is not considered in our review framework. Interested readers are referred to Park and Pham[41,42] and their further work for more details.

Flexible 2D warranty policies

In practice, customers are usually offered the same warranty policy regardless of their usage rates or personal preferences. This type of warranty, called fixed warranty, is widely adopted in real life. However, as mentioned before, the 2D warranty policy with a rectangular region favors the medium-usage customers, yet is inequitable to customers with either low or high usage.[38] Besides, it is a common sense that one product can be used for completely different purposes or operated under various conditions/patterns. For example, two identical excavators may be employed for demolition and excavating soft earth, respectively, for which they will be subject to completely different load and wear factors. Thus, different warranty menus may be needed to satisfy various demands.[43] These arguments support the introduction of *flexible warranty* which allows a customer to choose the best-fit warranty plan among the available options based on his or her usage rate and/or preference (see Exhibit 3 in Appendix 2 for Kuhn's flexible warranty program).

Generally speaking, flexible warranty contracts are designed in such a way that the expected warranty costs or failure probabilities are identical given that all customers exactly choose the contracts tailored for them.[36,38,43] To our knowledge, the concept of flexible 2D warranty was first introduced by Moskowitz and Chun.[43] They discussed how to generate the flexible contracts corresponding to a specific warranty price. Then, Gertsbakh and Kordonsky[44] discussed the manufacturer warranty and individual customer warranty by considering two examples—aircraft engines and cars. After that, flexible 2D warranty received some further attentions from the scholars, such as Chun and Tang,[34] Shahanaghi and Heydari,[45] and Manna et al.[36] In a recent paper, Ye and Murthy[38] investigated the warranty menu design for a 2D warranty by dividing the customers into several subgroups. In their case, a tailored warranty region is provided to each subgroup. Specially, when the customers do not know their future usage rates, Ye and Murthy[38] proposed a fixed 2D warranty policy with a stair-case region and showed its equivalence to the flexible policy, which is a quite new and interesting result.

In addition to the aforementioned studies that only considered corrective maintenance over the warranty period, Shahanaghi et al.[46] investigated the flexible 2D extended warranty policies with preventive maintenance (PM). They considered the optimal PM policy during the extended warranty period and employed regression approach to build a warranty cost model with product age and usage as input parameters. They then showed that the flexible extended warranty contracts can be easily designed for any given warranty cost. Recently, Su and Wang[37,47] studied the design of flexible 2D warranty contracts for second-hand/used products.

Remark 2. One drawback of flexible 2D warranty is that a customer has to self-select the best-fit warranty policy for his or her usage pattern among the available options. This raises two problems. First, some customers may be unaware of their usage rates at the time of purchase, especially for the first time purchase. Second, customers may lie to the manufacturer and report a usage rate that is lower than the actual one though Ye and Murthy[38] proved that this is not the best strategy for them. In both cases, the implementation of flexible 2D warranty policy would be hindered. Here, we claim that if the manufacturers are able to obtain sufficient usage rate information of any individual customer, they could design a flexible or even *customized* 2D warranty program for any individual customer. With the development of wireless communication and sensor technologies, this may not be a difficult problem as in the past. Nevertheless, more research efforts should be devoted to the development of this topic.

2D warranty cost analysis

Offering a warranty program will bring significant financial impact to the manufacturers because they have partial or full responsibility to serve eligible warranty claims made by customers. These servicing costs, typically called *warranty costs*, depend on the inherent reliability of the products, warranty terms, usage intensities and environment, service logistics, and so on.[48] In practice, the annual warranty costs for some large companies (such as automobile and computer manufacturers) can run into billions of dollars, and some relevant statistics can be found in "Warranty Week" (www.warrantyweek.com). Therefore, from the manufacturer's point of view, accurate estimation of the warranty costs is essential for managing a warranty program. This section is devoted to the issues related to the cost analysis of 2D warranty.

Warranty cost metrics

The warranty cost metrics that are concerned by manufacturers and customers may be defined in different ways. Blischke and Murthy[48] have introduced several kinds of warranty costs, including:

1. Warranty cost per unit sale;
2. Life cycle cost over a product's lifetime (LCC-I);
3. Life cycle cost over the life cycle of a product (LCC-II);
4. Cost per unit time.

These costs are actually random variables because the warranty claims involve various sources of uncertainty and the cost to rectify each claim can also be random. The warranty cost per unit sale is important for product pricing. As the warranty expense is non-ignorable, to assure the profitability of the manufacturer, the sale price must exceed the sum of warranty cost, production cost, and other major costs. In addition, the LCC-I is the total cost of operating a single product over its lifetime, which consists of elements such as acquisition cost, operating cost, maintenance and repair cost, and disposal cost; thus, it is of particular interest to the customer. Furthermore, the LCC-II depends on the life cycle of a product which is defined as the time period from the point of launching a new product into the market to the instance when it is withdrawn. The total LCC-II is simply given by the product of the number of total sales and the LCC-I. For more discussions on these costs, please refer to Blischke and Murthy.[48]

Remark 3. Several recent works studied the warranty cost analysis from a new perspective, namely, *aggregate warranty cost over an arbitrary time interval*. The importance of this cost is that in practice, manufacturers are more interested in the aggregate warranty cost within a given period of time (which provides more flexibility in dynamically managing the warranty program), such as the next fiscal year, due to the accounting or reserving purpose. To obtain the aggregate warranty cost of all units sold up to an arbitrary time point, the product failure process should be combined with the stochastic sales process, which is usually overlooked in the literature. The readers are referred to Xie and Ye[18] and Xie et al.[49] for more details.

In the context of 2D warranty cost analysis, most existing studies focus on the warranty cost per unit sale, while the associated life cycle costs (LCC-I & LCC-II) receive less attention although Murthy et al.[33] have presented the LCC-II models for the above-mentioned policies 1–4.

2D failure modeling approaches

Essentially, the premise of cost analysis and the subsequent decision-making of 2D warranty are to construct an appropriate failure model for the products, based on the failure physics and/or failure data. Generally speaking, modeling product failures involves two levels: system level and component level. In either level, the modeling process can be done by viewing the system (or component) as a black-box or a white-box, depending on the available information and the goal of the associated problem.[1]

Analysis of 2D warranty policies requires the failure models to be indexed by both age and usage. In the literature, three main approaches have been developed to characterize product failures in this manner, that is, marginal (univariate) approach, bivariate approach, and composite scale approach. These approaches are briefed as follows.

Marginal (univariate) approach. To our knowledge, this approach (also called 1D approach) was first introduced by Murthy and Wilson.[27] In this approach, the 2D problem is effectively reduced to a 1D problem by treating the usage as a random function of age. An assumption adopted here is that the usage rate R varies from customer to customer, but it remains constant for a specific customer. In general, R can be either treated as a discrete variable (e.g. users with light, medium, and heavy usage rates) or as a continuous variable with a distribution function $G(r)$, where $0 \leqslant r < \infty$. Conditioning on $R = r$, the cumulative usage X at time

T is given by $X = rT$. Then, the time to first failure can be modeled by a conditional hazard function $h_r(t)$, which is a non-decreasing function in both t and r. The occurrence of subsequent failures depends on the warranty servicing strategy applied. Suppose that the duration required to perform the rectification is negligible. If the failed products are always replaced by new ones, then the failures will occur as a renewal process associated with the cumulative distribution function (CDF) $F_r(t)$; while if all repairs are minimal, a non-homogenous Poisson process (NHPP) with the conditional intensity function $\lambda_r(t) = h_r(t)$ can be adopted to describe the failure process.

Some papers model the hazard function as a polynomial function (regression model) of time and usage rate.[37,50–56] For example, $h_r(t) = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)t$ and $h_r(t) = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)t^2$ are two simple forms of polynomial hazard functions, where θ_0, θ_1, θ_2, and θ_3 are non-negative parameters. While the model formulation of Moskowitz and Chun[43] and Hamidi et al.[57] is a simplified case of the first form.

On the other hand, other work employs the accelerated failure time (AFT) models to describe the effect of usage rate on the product reliability.[32,38,46,58–64] Let T_0 (T_r) denote the time to first failure under the usage rate r_0 (r), where r_0 is the nominal usage rate set at the design stage, and r is the actual usage rate. The AFT models presume that the time to first failure decreases (increases) as the usage rate increases (decreases), that is, $E[T_r]\downarrow r$.[58] Under the AFT formulation, we have $T_r/T_0 = (r_0/r)^{\gamma}$, where $\gamma > 0$ is the acceleration factor.[59] Then, the CDF $F_r(t)$ and hazard function $h_r(t)$ of T_r can be linked to that of T_0 through $F_r(t) = F_0((r/r_0)^{\gamma}t)$ and $h_r(t) = (r/r_0)^{\gamma}h_0((r/r_0)^{\gamma}t)$, respectively.

Remark 4. It is necessary to note that one appealing advantage of the AFT model is that it can easily incorporate piecewise constant usage rates (Figure 3) into the failure model. For instance, we consider that a product is used for T_1 time units under usage rate r_1, then it starts a new mission with usage rate r_2. In this case, the product's effective age π_1 at the beginning of the second use period can be obtained by solving $F_0((r_1/r_0)^{\gamma}T_1) = F_0((r_2/r_0)^{\gamma}\pi_1)$, which means that using the product under usage rate r_1 for T_1 time units is equivalent to using it under usage rate r_2 for π_1 time units, from the reliability perspective. This further leads to $\pi_1 = (r_1/r_2)^{\gamma}T_1$. In the 2D warranty-related literature, Ye et al.,[62] Wang et al.,[64] and Wang et al.,[65] employed this modeling manner to establish age correspondences in different problem settings.

Bivariate approach. A bivariate approach is a formal 2D approach which models the time to first failure via a bivariate distribution function $F(t, x)$. If all the product failures are replaced by new identical ones and the replacement durations are negligible, then the failures over the warranty region will follow a 2D renewal process associated with the distribution function $F(t, x)$.

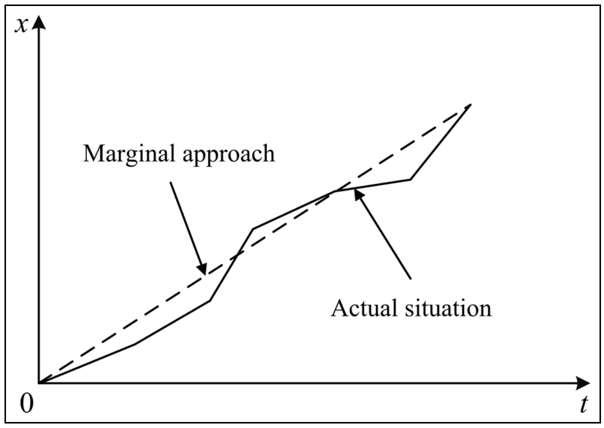

Figure 3. Illustration of the piecewise usage accumulation process.

One typical method is to directly put forward the standard bivariate distribution functions to model product failures under 2D warranty, for example, bivariate Pareto and Beta Stacy distributions,[33,40,66] bivariate exponential distribution,[35,49,67,68] bivariate Weibull distribution,[47,69,70] and bivariate Paulson distribution.[66] On the other hand, some research derives the bivariate density functions by explicitly modeling usage as a function of time. For example, Singpurwalla and Wilson[29,71] and Eliashberg et al.[72] incorporated the effect of usage on time to failure with an additive hazard model.

Note that these bivariate distribution functions should bear the property that $E[X|T = t]$ is an increasing function of t, that is, the expected usage at failure increases as the age at failure increases (however, $E[X|T = t]\uparrow t$ does not imply $E[T_r]\downarrow r$ and vice versa). Manna et al.[58] claimed that some of the existing models, for example, bivariate Pareto distributions and bivariate exponential distribution, fail to meet the necessary condition of the AFT models (i.e. $E[T_r]\downarrow r$).

As a pioneering work, Baik et al.[73,74] presented a framework for modeling 2D failures with minimal repair. Recently, Huang et al.[70] and Su and Wang[47] applied this framework to investigate 2D warranty policies with consideration of PM and upgrade actions, respectively. Nevertheless, unlike the replacement service, the bivariate approach with minimal repair service has not been fully developed and still requires further research.

Composite scale approach. In this approach, the two scales—age T and usage X, are combined together to define a composite scale Z and the product failures can be modeled as a counting process based on this composite scale. Gertsbakh and Kordonsky[44] made the first attempt to use this approach and a new variable $Z = \varepsilon T + (1 - \varepsilon)X$, $\varepsilon \in [0, 1]$ was introduced. Using the warranty claim data, Iskandar and Blischke[75] applied this approach to reliability and warranty analysis of a motorcycle. Other methods of constructing the composite scale can be referred to Ahn et al.[76] and Duchesne and Lawless.[77]

Mathematically specking, the composite scale Z involves the sum of two variables with different dimensions and magnitudes. Nevertheless, by carefully choosing the units of age and usage (e.g. for a vehicle, age can be measured in year and usage in 10^4 km), their magnitudes will be similar and the modeling and analysis will make sense.

Other approaches. In addition to the three approaches above, several other approaches have also been investigated to enrich the literature. Samatlı-Paç and Taner[78] introduced the multiple quasi-renewal process and applied it to model n-dimensional warranties, with 2D warranty as a special case. Majid et al.[79] developed a framework for determining the optimal extended warranty policy using 2D delay time model. Wu[80] and Anderson et al.[81] employed copulas to capture the dependence between age and usage.

Remark 5. The strengths and shortcomings of the 2D failure modeling approaches are summarized and compared as follows:[33,69,80,82]

- Marginal approach requires the specification of the distribution function $G(r)$ for usage rate R. Jung and Bai[69] claim that when the amount of available data is small or the age and usage are weakly correlated, the parameter estimation of $G(r)$ based on marginal approach is crude and usually biased. In this case, some supplementary data are needed. Moreover, another limitation of the marginal approach is that the usage rate of a specific customer is assumed to be constant over time. However, actual usage process may vary over time for an individual customer, and the marginal approach does not allow for this aspect.[83,84] As depicted in Figure 3, the marginal approach only utilizes the information of average usage rate, while the variability of usage rates is lost. As discussed earlier, however, the AFT model can be useful in this case to deal with piecewise constant usage rates. Nevertheless, the assumption of constant usage rate is not far away from the reality and has been supported by practical warranty data analysis. Several studies have revealed that the cumulative usage is approximately linear over the age of a product.[85,86]
- In contrast, the bivariate approach does not suffer from the above limitations. It directly estimates the bivariate function $F(t, x)$ based on the available data. This estimation process is more straightforward and usually more accurate, especially when weak correlation between age and usage is suspected and only limited claim data are available.[69] However, Wu[80] claim that the bivariate approach poses another problem. When the usage is measured as a discrete random variable and the age is described as a continuous random variable, it may be difficult to find a suitable bivariate distribution (in which the two variables are positively correlated) that combines a discrete marginal with a continuous marginal.

Manna et al.[87] summarized the 1D and 2D failure modeling approaches applied to cost analysis for both repairable and non-repairable products under 2D FRW policy, and highlighted the discrepancy in the formulas based on the two approaches. It was concluded that the existing formula by 1D approach for non-repairable products is incorrect, and for repairable products, the formulas under the two approaches are not comparable.

Data-driven models

Besides the model-based analysis, some research intends to develop a data-driven framework to evaluate the expected number or cost of 2D warranty claims. In general, the data-driven warranty analysis can be classified into two categories—parametric and non-parametric analyses.

The techniques, which belong to the first category, estimate the parametric models from warranty data and then apply them to warranty analysis. First of all, we would like to mention the literature that analyzes 2D warranty data with univariate approach. Based on automobile warranty claims data, Lawless and colleagues[85,88] proposed a family of AFT models to estimate reliability indexes and warranty claims with supplementary information about mileage accumulation. Davis[89] dealt with a similar problem using an alternative approach. Lu[90] established a linear relationship between customer median mileage and length of ownership, and then built a parametric model to predict automobile reliability. Baik and Murthy[91] adopted the AFT model to assess component reliability from 2D warranty data by discretizing customer usage rates into several subgroups. Yang et al.[32] employed two generalized renewal process approaches to model the imperfect repair, and applied them to forecast the warranty claims. Besides, a stochastic expectation-maximization algorithm was adopted by Dai et al.[92] to estimate the parameters of reliability model based on 2D warranty data with censoring times. Hong and Meeker[93,94] predicted the field warranty returns according to the failure-time data with auxiliary usage rate or dynamic covariate information. Moreover, Qiu et al.[31] presented a pseudo-likelihood method to estimate the parameters of reliability function for incomplete warranty database, which is lack of assembly dates for some units. Majeske[95] developed an NHPP predictive model for automobile warranty claims with a population size function and a warranty claim rate.

There also exists a group of papers that analyze the 2D warranty data with the bivariate failure modeling approach. Jung and Bai[69] used a bivariate Weibull distribution to investigate the statistically correlated 2D warranty data. Gupta et al.[86,96] addressed some reliability issues, including the estimation of bivariate reliability at different age-usage windows and the construction of usage-rate-based bivariate reliability functions, based on incomplete 2D warranty data. Moreover, Wu[80] introduced asymmetric copulas to capture the tail dependency existing between the age and usage, and showed that the asymmetric copulas fit the data better than the symmetric copulas and the model proposed by Jung and Bai.[69] Anderson et al.[81] also used copulas to model and simulate the 2D warranty data.

While, for the second category, the studies aim at developing non-parametric approach to obtain reliability indexes, for example, empirical hazard rate (with which the warranty claims/costs can be estimated or forecasted), using 2D warranty data. Kleyner and Sandborn[97] presented a warranty forecasting model based on the piecewise statistical distributions and stochastic simulation of the expected warranty returns. Christozov et al.[98] applied a piecewise linear stratification approach to estimate the mean cumulative function of warranty claims and costs. Moreover, Rai and Singh[99] studied the estimation of component-level empirical hazard plots using incomplete and unclean 2D warranty data. Furthermore, to quantify the empirical hazard rate and the associated forecast performance, some characteristic phenomena of automobile warranty data, such as hard failures, customer-rush near the warranty expiration, and maturing data, were investigated by Rai and Singh.[100–104]

In addition, Rai and Singh[105] presented a simple method to assess the impact of new age/mileage warranty limit on the number and cost of warranty claims for the components installed on a new vehicle. Rai[106] attempted to develop a monthly warranty expense forecasting model for subsystem failures by considering the seasonality of calendar month, the working days of authorized service centers, the sales ramp-up, and so on. Anastasiadis et al.[107] investigated the impact of driving-pattern variability on the expected number/cost of vehicle warranty claims, and they found that the average warranty cost increases as the variability gets large.

In fact, one of the major challenges in non-parametric 2D warranty data analysis is to incorporate the effect of warranty usage limit on the warranty claims/costs. To this end, the existing research usually employs usage rate in the product population to derive the product's probability of being on risk at a specific time in service, and then to arrive at the estimation of hazard rate, warrant costs/claims; see Majeske,[95] Kleyner and Sandborn,[97] and Rai and Singh,[104,105] for example.

More discussions on the data-driven framework of 2D warranty analysis can be found in the review papers such as Wu[14,15] and monographs such as Rai and Singh[4] and Blischke et al.[5]

Other miscellaneous issues

In this subsection, we brief the work that addresses other issues of 2D warranty cost analysis.

Approximation of 2D renewal function. The computation of 2D renewal function is very difficult because no explicit form exists. Corbu et al.,[108] Arunachalam and Calvache,[109] and Hadji et al.[110] studied the numerical evaluation of 2D renewal functions, and applied their methods to the 2D warranty cost analysis.

Warranty execution. Most of warranty cost models discussed earlier assume that all product failures would result in warranty claims. However, in reality, customers might not always request a warranty service for a failed product that is still under warranty, which is called *failed-but-not-reported* (FBNR) phenomena.[111,112] Under this circumstance, the warranty cost model should be modified. Mitra and Patankar[113] incorporated the impact of warranty execution on the cost analysis of a 2D warranty policy.

Fuzzy method. Warranty claim data extracted from customer reports may be expressed in a vague way due to subjective and imprecise perception of failures by the customers, imprecise records of warranty data or usage rate, and so on. Lee et al.[114] suggested fuzziness as an alternative to randomness for describing the fuzzy warranty sets.

Multi-component systems. The majority of papers on 2D warranty analyze the warranty cost by viewing the products as one-component or black-box systems; however, only a few exceptions consider the products as complex multi-component systems; see Shahanaghi and Heydari,[45] Manna et al.,[115,116] and Sarada and Mubashirunnissa.[117]

Remark 6. One issue needs to be emphasized is that practical 2D warranty programs for multi-component products may specify different warranty terms (e.g. compensation policy or coverage) for different components. For instance, Ford offers the sold cars a bumper-to-bumper 3 year/36,000 miles warranty, with a 5 year/60,000 miles coverage for powertrain, 5 year/10,000 miles coverage for engine, and so on (see Exhibit 1 in Appendix 2). How to model and analyze the warranty cost for multi-component products under such multi-level warranty contracts is an open and valuable problem.

2D warranty and engineering/marketing

This section reviews the engineering and marketing-related issues for 2D warranty. As stated in Murthy and Djamaludin,[1] engineering focuses on issues prior to the product sales, including product design, development and manufacturing; while marketing deals with the concerns related to the sales, such as the selling price and warranty length.

2D warranty and engineering

The warranty costs can be controlled and reduced by making optimal engineering decisions in the design stage (e.g. design for warranty, and research and development (R&D)) and manufacturing stage (e.g. quality control and burn-in test). These actions will incur additional costs and are worthwhile only if the warranty program receives a significant cost reduction. Therefore, joint optimization of the engineering decisions and warranty policies, which has received much attention under 1D warranty,[1] is of great importance. Surprisingly, our search of the extant literature reveals that rare research has been done regarding the engineering decisions under 2D warranty.

One avenue to improve product reliability is through the R&D programs. Mitra and Patankar[118,119] developed multi-objective decision-making models for 2D warranty policies with product development programs. Besides, some studies, through warranty data analysis, reveal that the product sold with 2D warranty may experience significant infant mortalities.[86,96,120] To deal with this problem, Ye et al.[62] proposed performance-based and cost-based burn-in models for repairable products sold with a 2D warranty, considering two types of failures, that is, normal and defect failures.

2D warranty and marketing

Nowadays, it is generally recognized that both the selling price and warranty policy have significant influences on the customers' buying decisions.[1] Among them, the warranty is employed as a marketing tool to signal the product reliability and boost the sales.

In the literature, many researchers have developed economic decision models to determine the optimal selling price, warranty length, and other relevant decisions. In the context of 1D warranty, Glickman and Berger[121] made an early contribution to quantify the expected sales by connecting it with the product price P and warranty length W, that is, $Q(P, W) = k_1 P^{-a}(W + k_2)^b$ for $k_1 > 0$, $k_2 \geqslant 0$, $a > 1$, and $0 < b < 1$. This implies that a lower sales price and/or a longer warranty period will lead to a higher sales volume. For more discussions on the marketing issues related to 1D warranty, please refer to Murthy and Djamaludin[1] and Thomas and Rao.[9]

However, limited research has been conducted to investigate the optimal marketing decisions for 2D warranty. Singpurwalla and Wilson[29] gave an expository discussion on the determination of optimal price-warranty combination through a game-theoretic formulation. Recently, several studies were conducted to extend or modify Glickman and Berger's log-linear demand model to characterize the additional dimension of product usage.[68,120,122,123] Manna[122] modified the Glickman–Berger model by replacing the warranty period W with the mean warranty coverage $l(W, U) = WG(\eta) + U\int_{\eta}^{\infty}(1/r)\mathrm{d}G(r)$, where $\eta = U/W$. Similarly, Xie[68] adopted WU to replace the term W in the Glickman–Berger model, and then investigated the optimal pricing and warranty policy for a new product sold with 2D warranty. While He et al.[123] proposed a new demand function by introducing an attractiveness index which is defined as the ratio of actual warranty region to the maximal warranty region for customers with heterogeneous usage rates.

Furthermore, Huang and Yen[53] and Huang et al.[70] derived the optimal age and usage limits by maximizing the manufacturer's expected total profit (i.e. the number of sales times the expected profit per unit sold). They assumed that the manufacturer would offer the buyer an extended time limit if the buyer is willing to conduct periodic PM activities within the warranty period. Finally, the new warranty limits can be derived under the condition that the total warranty cost remains unchanged. In Huang et al.,[70] a linear demand function $Q(W, U) = Q_0 + dW + fU$ was adopted, where Q_0 is the expected sales volume without warranty and d and f represent the age limit and usage limit elasticities, respectively.

In summary, more research work is needed to explore other marketing issues for 2D warranty, such as how the 2D warranty affect the customer behavior and how the 2D warranty perform as a promotional tool. Importantly, empirical analyses based on real sales data or customer survey are required to validate the above-reviewed demand models of 2D warranty by investigating the effects of age limit and usage limit on the sales volume.

2D warranty and logistics

In reality, it is impossible that a product will never fail even though the optimal engineering decisions have been implemented to improve its reliability. In this sense, product failures are inevitable. If a product fails within the warranty period, then the manufacturer is obligated to provide a compensation according to the type and terms of the warranty policy, and this incurs additional servicing cost. Generally, this cost can be largely reduced through optimal warranty servicing strategies and effective logistic management.[1]

In 2004, Murthy et al.[10] made an exploratory discussion on warranty logistics issues. They classified the warranty logistic issues into three levels—strategic, tactical, and operational levels. From the strategic perspective, the service center network design, the capacity allocation for the service centers, and the ownership of these centers are three main problems that needed to be addressed. The tactical and operational issues for warranty service usually include optimal repair-replacement decisions, repairmen coordination, spare parts transportation and inventory management, and so on. In the context of 2D warranty, existing literature predominately focuses on tactical and operational issues of warranty logistics. In this section, we summarize some of these issues.

Repair–replacement strategy

When a repairable product fails within the warranty region, the manufacturer has to decide whether to repair or replace it with a new one. This choice has a significant impact on the expected warranty servicing cost per unit sale. For repairable products sold with 1D warranty policies, the repair–replacement strategies have been investigated intensively, see Murthy et al.[10] and Jack et al.,[124] for summaries. The determination of the optimal repair–replacement strategies under a 2D warranty can be viewed as a natural extension of the 1D case. That is, the optimal warranty servicing strategy is typically obtained by dividing the warranty region into multiple disjoint sub-regions and pre-assigning certain servicing strategies (replacement, imperfect repair, or minimal repair) in associated sub-regions.

Iskandar and Murthy[50] proposed two repair–replacement strategies for repairable products sold with warranty region $\Omega = [0, W) \times [0, U)$, see Figure 4(a):

Strategy 1. In region Ω_1, the failed product will be replaced by a new one, while any failure in Ω_2 will only receive a minimal repair.
Strategy 2. For a repairable product, if a failure occurs in Ω_1 (Ω_2), it will be minimally repaired (replaced by a new one).

Similar strategies can be found in Chen and Popova.[125]

Subsequently, Iskandar et al.,[51] Chukova and Johnston,[52] and Jayaraman and Matis[126] further split the warranty region into three disjoint sub-regions, as shown in Figure 4(b). In their models, the first failure in the intermediate region Ω_2 is rectified through replacement and all other failures are rectified through minimal repair. Continuing along this line, Chukova et al.[127] partitioned the warranty region into n disjoint sub-regions and prescribed replacement for the first failure (if any) in each of the intermediate sub-regions $\Omega_2, \ldots, \Omega_{n-1}$; see Figure 4(c). While Jack et al.[59] discussed a new repair–replacement strategy for 2D warranty based on product usage rate, that is, the first failure in a designated region Γ is replaced with a new one and all other failures are minimally rectified; see Figure 4 (d). Furthermore, Husniah et al.[128] investigated a hybrid minimal repair and age replacement policy for repairable products under a 2D non-renewing FRW policy.

In the aforementioned studies,[50–52,126,127] the resulting optimal sub-regions were confined to a rectangular shape; while the policies in Jack et al.[59] and Husniah et al.[128] led to a non-rectangular region which is dependent on the optimal value for each usage rate. In other words, if one confines to the rectangular shape, then this will give a sub-optimal solution. Nevertheless, from a practical perspective, the rectangular shape is much easier to implement than the non-rectangular one.

Subsequently, warranty servicing strategies were further extended to combine minimal repairs with imperfect repair. Under this policy, the first failure within the pre-assigned warranty region will be imperfectly rectified, instead of replacement. For instance, Yun and Kang[129] and Varnosafaderani and Chukova[130,131] extended the work of Iskandar et al.[51] and Chukova et al.,[127] respectively, using the joint minimal and imperfect repair policy. Likewise, Banerjee and Bhattacharjee[60,61] extended the research of Jack et al.[59] Moreover, Husniah et al.[132] proposed a servicing policy that allows more than one imperfect repair over the 2D warranty region. Note that, the resulting optimal sub-regions in studies[129–131] are rectangular, while those in studies[60,61,132] are non-rectangular yet.

Furthermore, several recent work incorporated the variations of repair–replacement servicing strategies into other research problems related to 2D warranty; see Tong et al.,[63] Cheng et al.,[133] and Husniah and Iskandar[134] for more details.

PM optimization

In addition to corrective maintenance upon failure, warranty servicing may also include PM activities (see Exhibit 4 in Appendix 2 for several free vehicle maintenance programs). In recent years, the PM-related problems in the context of 2D warranty have aroused the interests from some researchers. Table 3 summarizes the relevant literature from three different perspectives, namely, PM effect model, PM cost-bearing mechanism, and PM implementation period.

Huang and Yen[53] and Huang et al.[70] developed optimal 2D warranty programs to maximize the manufacturer's profit with periodic PM. While a reliability-based sequential PM policy was examined in Huang et al.[54] to determine the optimal 2D warranty terms. Cheng et al.[133] combined the PM and minimal repair for products sold with 2D warranty, and proposed an optimal warranty servicing strategy, in which both warranty cost and product availability were included. The works of Huang and Yen,[53] Huang et al.,[54] Huang et al.,[70] and Cheng et al.[133] focused on the PM policies for products under 2D basic warranty, while some other papers dealt with the PM optimization by incorporating 2D extended warranty as well. In this regard, Shahanaghi et al.[46] and Huang et al.[138] optimized the

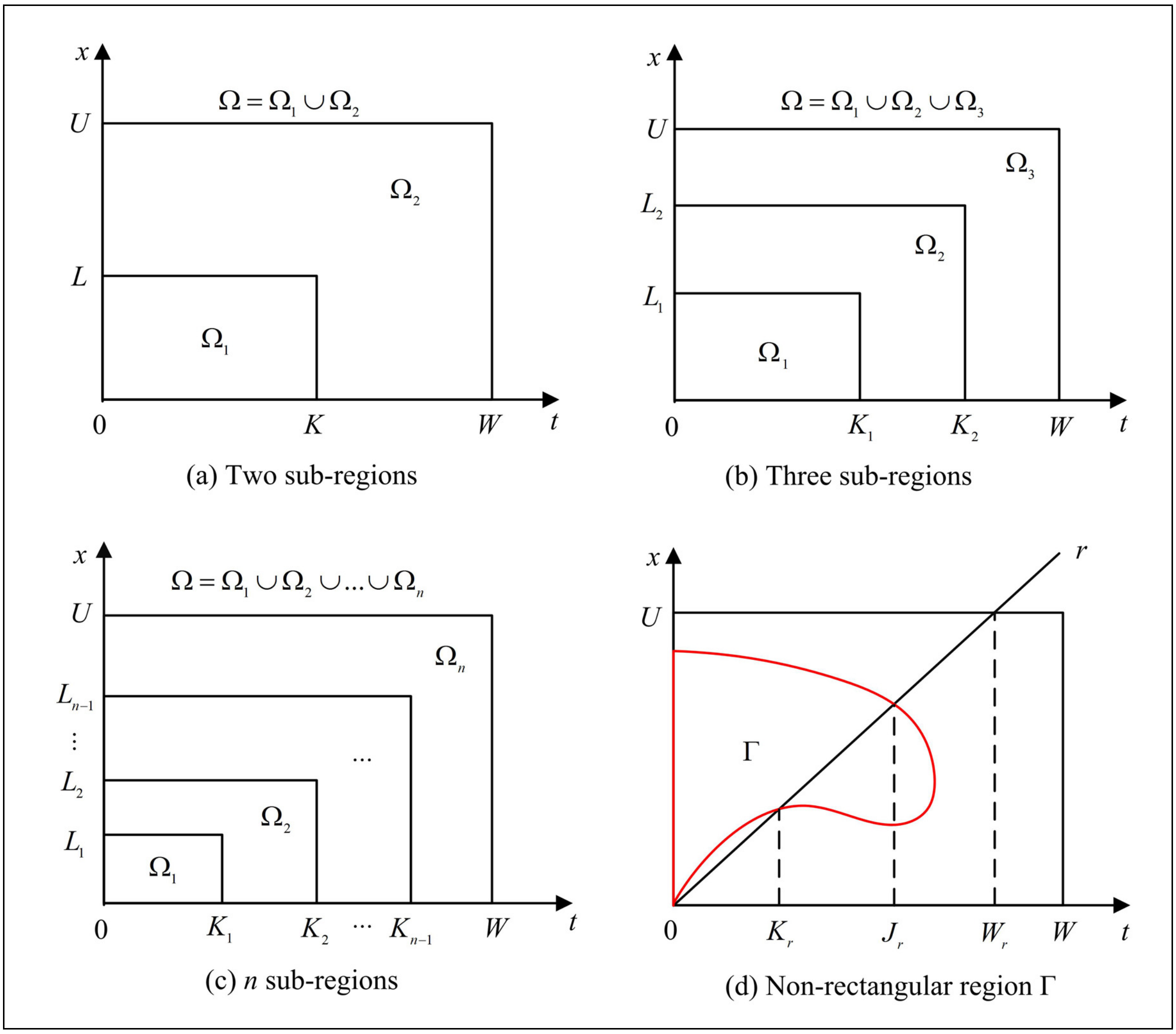

Figure 4. Illustration of various types of sub-regions for warranty servicing strategies: (a) two sub-regions, (b) three sub-regions, (c) n sub-regions, and (d) non-rectangular region Γ.

Table 3. Summary of literature on 2D warranty-related PM problems.

Perspective	Categories	References
PM effect model	Age reduction model	46, 47, 53–57, 64, 70, 133–138
	Failure rate reduction model	37, 56, 139, 140
PM cost-bearing mechanism	Manufacturer side	37, 46, 47, 54, 55, 57, 64, 133–135, 137–142
	Customer side	53, 70
	Both sides (cost-sharing)	56, 136–138
PM implementation period	Basic warranty period	53–57, 70, 133–136, 139, 140
	Extended warranty/post-warranty period	37, 46, 47, 55, 56, 64, 135, 136, 138, 139, 141, 142

PM: preventive maintenance.

PM policies during only the extended warranty period; while Wang et al.,[139] Su and Wang,[55] and Wang et al.[56] pursued the optimal PM decisions by integrating the basic and extended warranty periods.

It is well known that there are various models proposed to describe the effect of imperfect PM actions on the product reliability; see Pham and Wang[143] and Wu and Zuo[144] for references. In the literature, age reduction model and failure rate reduction model are two widely adopted models. Among the papers dealing with PM-related problems under a 2D warranty, Wang et al.,[139] Li et al.,[140] Su and Wang,[37] and Wang et al.[56] are the only four papers that describe imperfect PM (or upgrade) effect through the failure rate reduction models, while the others employ the virtual age reduction models.

Moreover, in practice, the expenses of PM activities within the warranty period may be covered by the manufacturer or customers or both of them. Different manufacturers may have different PM cost-bearing mechanisms, which should be explicitly stated in the warranty/maintenance manuals. In the literature, most studies assumed that the manufacturer should cover the PM expenses under warranty; while only a few ones considered that the PM expenses under warranty should be borne by the customers[53,70] or shared by both sides.[56,136–138]

Remark 7. Two recent progresses on the 2D warranty-related PM optimization are worth being introduced:

- The first achievement is related to the 2D PM strategy, which focuses on the PM scheduling perspective. Traditional PM strategies for products under a 2D warranty are scheduled based on either age or usage (but not both). For example, under the age-(usage-) based PM strategy, the PM activities are carried out every K units of age (every L units of usage), regardless of the product's usage rate. Hu et al.[145] presented a 2D preventive replacement strategy, under which the spare parts should be replaced according to the calendar time and usage, whichever occurs first. However, Hu et al.[145] did not incorporate the 2D warranty policy. Recently, Wang and Su[137] studied a 2D imperfect PM strategy, under which the PM activities are performed every K units of age or every L units of usage, whichever occurs first (see Exhibit 4 in Appendix 2 for real-world applications of this strategy). It was shown that the 2D PM strategy contains age- and usage-based strategies as its special cases, and is superior to them in terms of warranty cost reduction.
- The second one is associated with the 2D PM model, which deals with the PM modeling perspective. Existing research on the PM under a 2D warranty predominately considers its effect on reducing the product's virtual age. For instance, by applying the marginal approach, literature such as Shahanaghi et al.,[46] Huang and Yen,[53] and Wang and Su[137] implicitly assumes that each PM action would have the same influence on the virtual usage as that on the virtual age, which implies that the reduction amount of virtual usage is proportional to that of virtual age. However, in practice, the PM may have different impacts on the age and usage. After each action, the PM may result in a smaller virtual age and a smaller virtual usage, simultaneously. Huang et al.[70] and Su and Wang[47] investigated the 2D PM (upgrade) modeling problem. They utilized the bivariate Weibull process to describe the product failures and considered the effect of periodic PM (upgrade) on age and usage, separately, to develop 2D warranty contracts.

However, there is little research on the 2D PM problem that jointly takes into account these two perspectives.[140] Nevertheless, more efforts are expected to bridge this gap.

Warranty reserve and spare parts provisioning

Thus far, the research on 2D warranty servicing mainly focuses on the repair–replacement or PM strategies, limited work has been done to build the bridges among three interconnected areas, namely, warranty repair demand forecasting, warranty reserve management, and spare parts inventory management.

In the context of 2D warranty, Singpurwalla and Wilson[29] developed a model to forecast the amount of a reserve fund that can be used to support the future warranty claims from the sold products. Eliashberg et al.[72] dealt with a similar warranty reserve problem, in which the size of a reserve should be determined by the manufacturer to meet future warranty claims. Moreover, Rai[106] presented a method to forecast the monthly warranty expense, which is helpful to support the tactical level and operational level decisions, for example, planning an appropriate warranty reserve and prioritizing warranty cost reduction programs.

Recently, a small, but growing, group of researchers have developed the so-called dynamic 2D (warranty) repair demand forecasting technique. For example, Hu et al.[145] presented a mathematical model to forecast the spare part demands under 2D preventive replacement policy. In particular, Xie et al.[49] introduced a new forecasting technique, which can estimate the 2D aggregate warranty repair demand over an arbitrary time interval. Their findings are useful to manage the warranty reserve as well as the spare parts inventory periodically for the manufacturers.

2D long-term warranty

Today, in the global market, the high-speed technological development and fierce competition force manufacturers to extend the lengths of their after-sale services by offering *long-term warranty* policies.[16] According to Rahman and Chattopadhyay,[12] the long-term warranty policies typically appear in the following forms, namely, extended warranty, warranty for used/second-hand products, lifetime warranty, and maintenance service contract (Figure 5).

In the literature, issues on 1D long-term warranty have been well studied with considerable publications. In contrast, the research on 2D long-term warranty, as a conceptual category, is still in its early stage. Our literature review reveals that 22 journal papers have been published in this field. More specifically, 12 of them focus on 2D extended warranty, 6 of them deal with 2D warranty for used products, and the other 4 are related to 2D maintenance contract. Currently, to our knowledge, no paper has reported the issues of 2D

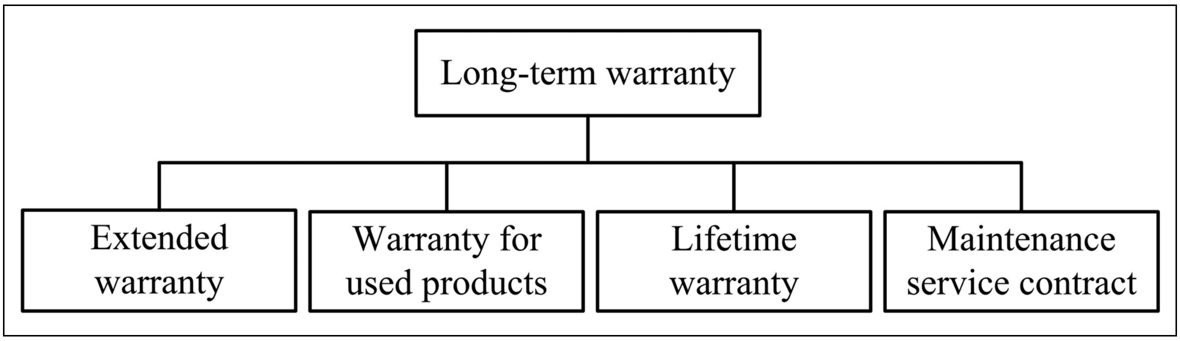

Figure 5. A framework of long-term warranty policies.[12]

lifetime warranty. One possible reason is that the product lifetime in this policy has not been well defined, which raises difficulties in modeling the product failures and warranty costs.

2D extended warranty

In practice, 2D extended warranty policies, similar to 1D extended warranty, have become increasingly important in the markets of capital-intensive products.

Su and Shen[120] made an early attempt to model the expected warranty servicing costs and profits, under 1D and 2D extended warranty policies, by considering different corrective maintenance options. Majid et al.[79] developed a framework to optimize the 2D extended warranty policy using the 2D delay time model. However, they did not provide any detailed mathematical models. Shahanaghi and Heydari[45] derived the expected servicing cost for multi-component series system under 2D extended warranty. In addition, Bouguerra et al.[135] and Jia et al.[136] studied the potential opportunities for the customers as well as the manufacturer when the 2D extended warranty is adopted. Recently, Tong et al.[84] proposed a new corrective maintenance strategy for 2D extended warranty based on dynamic usage rate (Figure 3).

As mentioned before, Shahanaghi et al.[46] and Wang et al.[139] optimized the PM policy incorporating 2D extended warranty under different problem settings. While Su and Wang[55] and Huang et al.[138] investigated optimal customized PM programs with 2D extended warranty by specifying different maintenance policies for different customer categories. Furthermore, Wang et al.[56] studied the consumer's and the manufacturer's optimal strategies for products sold with periodic PM under a 2D warranty policy. They considered that during the extended warranty period, consumers can choose to either accept or reject PM at each time based on their utility.

In practice, from the perspectives of both manufacturers and customers, it is interesting to study the pricing mechanism of 2D extended warranty contracts. Tong et al.[63] presented a designing and pricing model of 2D extended warranty contracts. Their mathematical model was categorized into two parts: the purchase of 2D extended warranty service at the point of sales and at the expiration of the basic warranty. Musakwa[146] studied the pricing of a motor extended warranty by evaluating the effect of usage limit on the risk premium of the extended warranty. For more discussions on 2D extended warranty, please refer to the monograph.[7]

Extended warranty policies are typically optional for customers. In practice, the manufacturer or third-party provider may offer customers several choices with different contract terms (coverage and/or length) and thus different prices (see Exhibit 5 in Appendix 2 for Ford's extended warranty programs). In this regard, how to price flexible or customized 2D extended warranty contracts considering various maintenance strategies deserves an in-depth investigation.

2D warranty for second-hand products

In recent years, the growing market for used/second-hand products facilitates the acceptance of warranty provided by dealers to promote sales and provide assurance for customers (see Exhibit 6 in Appendix 2 for several certified pre-owned vehicle warranty programs). However, unlike the fruitful research on 2D warranty policies for new products, so far, the associated framework for second-hand products has not been well established.[17]

Early studies on 2D warranty policies for used products mainly focus on warranty cost analysis. Chattopadhyay and Yun[147] presented a model to estimate the warranty cost and developed the best pricing strategy for used products considering 2D warranty terms. From the dealers' perspective, Shafiee et al.[30] employed statistical models to predict the expected 2D warranty cost for second-hand products. Based on the characterization of system failure, Sarada and Mubashirunnissa[117] developed two stochastic models to calculate the expected warranty cost for second-hand products sold with 2D FRW and PRW policies.

Statistically, due to the fact that old units have higher initial failure rate, a used/pre-owned product tends to incur more warranty cost than a new one. Currently, to reduce the associated warranty costs, sellers or dealers of used products are employing various reliability improvement actions to restore the used products back to younger states. In the literature, Su and Wang[37,47] have made an early attempt to study the optimal upgrade strategies for used products under 2D warranty policies. Recently, Wang et al.[64] further investigated the joint optimization of upgrade and PM strategies for second-hand products sold with a 2D warranty.

2D maintenance contract

In the manufacturing and service industry, instead of performing in-house maintenance, there is a growing trend toward maintenance outsourcing, which attracts the attention from the academia. Practically, many systems that require professional maintenance services, such as automobiles and dump trucks, deteriorate in two dimensions, that is, age and usage. Thus, when designing the associated maintenance contract, the service provider should take these two dimensions into consideration. For equipment sold with 2D warranty, Iskandar et al.[142] studied the optimal decisions on the post-warranty maintenance service contract. They developed a non-cooperative game to determine the optimal price structure for the manufacturer and the optimal contract option for the owner. Husniah et al.[141] investigated performance-based post-warranty maintenance contracts, which are characterized by both age and usage limits, for a fleet of dump trucks operating in a mining industry.

Instead of owning the equipment and outsourcing maintenance activities, currently, many manufacturers choose to lease the equipment. The maintenance policy and warranty terms (if any) are important elements in the leasing problems. Husniah and Iskandar[134] studied a 2D lease contract with PM and servicing strategy that involves more than one imperfect repair. Recently, Hamidi et al.[57] presented game-theoretic models for usage-based lease contracts with the aim of maximizing the total payoffs for the lessor and lessee. Under the contract, the lessee decides the optimal usage rate and lease duration, and the lessor is responsible for prescribing the optimal PM policies.

One thing noteworthy is that a different region forms a different 2D lease contract. The lease contract in Husniah and Iskandar[134] is a (rectangular) 2D one with time and usage limits, while that in Hamidi et al.[57] and Wang et al.[65] are usage-based ones with lease duration (or time) and usage rate. Due to the popularity of maintenance outsourcing in the heavy equipment industry and the gap of academic studies, 2D maintenance contracts offer a wide scope for future research.

Applications

In this section, application aspects of 2D warranty policies are surveyed in the following two ways. First, real-world 2D warranty programs of several kinds of products are presented and/or compared in Appendix 2, including automobiles, heavy machines, and aircraft engines. Second, case studies provided by some of the references reviewed in this article are identified and summarized in Table 4. Nevertheless, papers related to the applications of 2D warranty in the open academic publications are relatively limited since the warranty data are closely guarded secret for most companies and not available in the public domain, even for academic purposes.

Table 4. Summary of the types of products investigated by the case studies.

Products	References
Automobiles	44–46, 80, 81, 85, 86, 89, 90, 92, 95–108, 120, 146
Heavy machines	32, 69, 71, 72, 123
Others	30, 43, 44, 67, 93, 94

From Table 4, most of the case studies are focused on automobiles/cars, while limited studies deal with other products like dump trucks, excavators, and aircraft engines. This is not surprised, because, up to now, the most successful application of 2D warranty is in the automobile industry. Besides, a closer look at the studies summarized in Table 4 reveals that most of them deal with data-driven warranty analysis using real warranty data, while case studies of other research topics related to 2D warranty are limited and require further investigation.

Nevertheless, it is worth mentioning that the 2D warranty may be appropriate for a large spectrum of products if their usages (time-, output-, or number-based) can be clearly defined and easily measured.

Conclusion and topics for future research

In recent years, 2D warranty is of increasing interest to both researchers and practitioners. Through categorizing the existing literature, highlighting some insights and remarks regarding the current research, and suggesting some interesting topics for future research, this article provided a comprehensive summary and review of the state-of-the-art knowledge related to 2D warranty. We believe that this review is likely to be helpful for researchers in the field of warranty management because almost all of the policies, models, and approaches of 2D warranty are collected at one place for quick reference.

Finally, several new and challenging topics for future research are outlined and listed as follows (and we have also mentioned some research gaps in the main body of this article):

1. Flexible 2D warranty, or even customized 2D warranty, raises some novel issues in warranty menu design. Customers can enjoy the option of selecting or even designing warranty contracts that suits

them best. Under these policies, it would be interesting to model the customer behavior, product reliability, maintenance policies, and other relevant factors, simultaneously.
2. More research should be devoted to component-level failure modeling and cost analysis for multi-component systems under 2D warranty. The system configurations and failure interactions among internal components need to be considered to enhance the applicability of the associated models.
3. Engineering issues related to 2D warranty have received little attention in the literature. In this area, engineering decisions during design stage (e.g. design for 2D warranty, redundancy, and development) and manufacturing stage (e.g. burn-in and quality control) have great potential for further research.
4. The 2D PM model combined with scheduling strategy would be another interesting research topic. In this problem, the product failures should be modeled through a bivariate approach, which can separately capture the effects of age and usage on the product reliability.
5. The logistics of 2D warranty consists of various issues, including choice of service provider, procurement and inventory of spare parts, administration of warranty claims, and so on. The logistics problems of 2D warranty still remain open.
6. 2D long-term warranty is probably the most promising topic for future research, and it delivers a variety of new issues to the academia. Taking 2D warranty for used products as an example. To maximize the expected profit, the dealers have to jointly optimize the product prices, warranty terms, pre-sale upgrade, and post-sale PM actions.
7. Last but not the least, researchers in this field should pay more attention to address practical problems in real-world applications of 2D warranty, such as multi-level warranty menu design and analysis, human factors in warranty claims and services, and coarse warranty data analysis. Excellent case studies of 2D warranty are urgently needed to fill the gap between theoretical models and practical applications.

Acknowledgements

The authors are indebted to the three anonymous referees for their valuable comments and suggestions, which significantly improved the quality and presentation of this paper.

Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This work was supported by the University Grants Council of Hong Kong under a theme-based project grant (No. T32-101/15-R), the National Natural Science Foundation of China (Nos 71532008, 71601079, 71731006, and 71520107001), and the Guangdong Natural Science Foundation (No. 2016 A030310415).

References

1. Murthy DNP and Djamaludin I. New product warranty: a literature review. *Int J Prod Econ* 2002; 79: 231–260.
2. Murthy DNP and Blischke WR. *Warranty management and product manufacture*. London: Springer, 2006.
3. Thomas MU. *Reliability and warranties: methods for product development and quality improvement*. Boca Raton, NY: CRC Press, 2006.
4. Rai B and Singh N. *Reliability analysis and prediction with warranty data: issues, strategies, and methods*. Boca Raton, NY: CRC Press, 2009.
5. Blischke WR, Karim MR and Murthy DNP. *Warranty data collection and analysis*. London: Springer, 2011.
6. González-Prida V and Crespo A. *After-sales service of engineering industrial assets: a reference framework for warranty management*. Switzerland: Springer, 2014.
7. Murthy DNP and Jack N. *Extended warranties, maintenance service and lease contracts: modeling and analysis for decision-making*. London: Springer, 2014.
8. Kurvinen M, Töyrylä I and Murthy DNP. *Warranty fraud management: reducing fraud and other excess costs in warranty and service operations*. New Jersey: John Wiley & Sons, 2016.
9. Thomas MU and Rao SS. Warranty economic decision models: a summary and some suggested directions for future research. *Oper Res* 1999; 47: 807–820.
10. Murthy DNP, Solem O and Roren T. Product warranty logistics: issues and challenges. *Eur J Oper Res* 2004; 156: 110–126.
11. Karim MR and Suzuki K. Analysis of warranty claim data: a literature review. *Int J Qual Reliab Manag* 2005; 22: 667–686.
12. Rahman A and Chattopadhyay G. Review of long-term warranty policies. *Asia Pac J Oper Res* 2006; 23: 453–472.
13. Murthy DNP. Product warranty and reliability. *Ann Oper Res* 2006; 143: 133–146.
14. Wu S. Warranty data analysis: a review. *Qual Reliab Eng Int* 2012; 28: 795–805.
15. Wu S. A review on coarse warranty data and analysis. *Reliab Eng Syst Safe* 2013; 114: 1–11.
16. Shafiee M and Chukova S. Maintenance models in warranty: a literature review. *Eur J Oper Res* 2013; 229: 561–572.
17. Diallo C, Venkatadri U, Khatab A, et al. State of the art review of quality, reliability and maintenance issues in closed-loop supply chains with remanufacturing. *Int J Prod Res* 2017; 55: 1277–1296.

18. Xie W and Ye ZS. Aggregate discounted warranty cost forecast for a new product considering stochastic sales. *IEEE T Reliab* 2016; 65: 486–497.
19. Lee H, Cha JH and Finkelstein M. On information-based warranty policy for repairable products from heterogeneous population. *Eur J Oper Res* 2016; 253: 204–215.
20. Limon S, Yadav OP, Zuo MJ, et al. Reliability estimation considering usage rate profile and warranty claims. *Proc IMechE, Part O: J Risk and Reliability* 2016; 230: 297–308.
21. Tseng ST, Hsu NJ and Lin YC. Joint modeling of laboratory and field data with application to warranty prediction for highly-reliable products. *IIE Trans* 2016; 48: 710–719.
22. Park M and Pham H. Cost models for age replacement policies and block replacement policies under warranty. *Appl Math Model* 2016; 40: 5689–5702.
23. Darghouth MN, Chelbi A and Ait-kadi D. Investigating reliability improvement of second-hand production equipment considering warranty and preventive maintenance strategies. *Int J Prod Res* 2017; 55: 4643–4661.
24. Zhao X and Xie M. Using accelerated life tests data to predict warranty cost under imperfect repair. *Comput Ind Eng* 2017; 107: 223–234.
25. Murthy DNP and Jack N. Game theoretic modelling of service agent warranty fraud. *J Oper Res Soc* 2017; 68: 1399–1408.
26. Blischke WR and Murthy DNP. Product warranty management—I: a taxonomy for warranty policies. *Eur J Oper Res* 1992; 62: 127–148.
27. Murthy DNP and Wilson RJ. Modeling two-dimensional failure free warranties. In: *Proceedings of the fifth symposium on applied stochastic models and data analysis*, Granada, Spain, 23–26 April 1991, pp. 481–492. World Scientific.
28. Iskandar BP. *Modelling and analysis of two dimensional warranties policies*. PhD Thesis, Department of Mechanical Engineering, University of Queensland, QLD, Australia, 1992.
29. Singpurwalla ND and Wilson SP. The warranty problem: its statistical and game-theoretic aspects. *SIAM Rev* 1993; 35: 17–42.
30. Shafiee M, Chukova S, Saidi-Mehrabad M, et al. Two-dimensional warranty cost analysis for second-hand products. *Commun Stat A: Theor* 2011; 40: 684–701.
31. Qiu Y, Nordman DJ and Vardeman SB. A pseudo-likelihood analysis for incomplete warranty data with a time usage rate variable and production counts. *IIE Trans* 2013; 46: 118–130.
32. Yang D, He Z and He S. Warranty claims forecasting based on a general imperfect repair model considering usage rate. *Reliab Eng Syst Safe* 2016; 145: 147–154.
33. Murthy DNP, Iskandar BP and Wilson RJ. Two-dimensional failure-free warranty policies: two-dimensional point process models. *Oper Res* 1995; 43: 356–366.
34. Chun YH and Tang K. Cost analysis of two-attribute warranty policies based on the product usage rate. *IEEE Trans Eng Manag* 1999; 46: 201–209.
35. Kim HG and Rao BM. Expected warranty cost of two-attribute free-replacement warranties based on a bivariate exponential distribution. *Comput Ind Eng* 2000; 38: 425–434.
36. Manna DK, Pal S and Sinha S. Optimal determination of warranty region for 2D policy: a customers' perspective. *Comput Ind Eng* 2006; 50: 161–174.
37. Su C and Wang X. Optimal upgrade policy for used products sold with two-dimensional warranty. *Qual Reliab Eng Int* 2016; 32: 2889–2899.
38. Ye Z-S and Murthy DNP. Warranty menu design for a two-dimensional warranty. *Reliab Eng Syst Safe* 2016; 155: 21–29.
39. Tsai PC and Fang CC. A Bayesian analysis in determining pro-rata warranty decision with two-dimensional deterioration. *Int J Knowl Eng* 2015; 1: 160–164.
40. Iskandar BP, Wilson RJ and Murthy DNP. Two-dimensional combination warranty policies. *RAIRO: Rech Opér* 1994; 28: 57–75.
41. Park M and Pham H. A new warranty policy with failure times and warranty servicing times. *IEEE T Reliab* 2012; 61: 822–831.
42. Park M and Pham H. Warranty cost analysis for k-out-of-n systems with 2-D warranty. *IEEE T Syst Man Cy A* 2012; 42: 947–957.
43. Moskowitz H and Chun YH. A Poisson regression model for two-attribute warranty policies. *Nav Res Log* 1994; 41: 355–376.
44. Gertsbakh IB and Kordonsky KB. Parallel time scales and two-dimensional manufacturer and individual customer warranties. *IIE Trans* 1998; 30: 1181–1189.
45. Shahanaghi K and Heydari M. Two-dimensional extended warranty cost analysis for multi-component repairable products. *Int J Math Model Method Appl Sci* 2013; 3: 247–256.
46. Shahanaghi K, Noorossana R, Jalali-Naini SG, et al. Failure modeling and optimizing preventive maintenance strategy during two-dimensional extended warranty contracts. *Eng Fail Anal* 2013; 28: 90–102.
47. Su C and Wang X. Modeling flexible two-dimensional warranty contracts for used products considering reliability improvement actions. *Proc IMechE, Part O: J Risk and Reliability* 2016; 230: 237–247.
48. Blischke WR and Murthy DNP. *Warranty cost analysis*. Boca Raton, NY: CRC Press, 1994.
49. Xie W, Shen L and Zhong Y. Two-dimensional aggregate warranty demand forecasting under sales uncertainty. *IISE Trans* 2017; 49: 553–565.
50. Iskandar BP and Murthy DNP. Repair-replace strategies for two-dimensional warranty policies. *Math Comput Model* 2003; 38: 1233–1241.
51. Iskandar BP, Murthy DNP and Jack N. A new repair–replace strategy for items sold with a two-dimensional warranty. *Comput Oper Res* 2005; 32: 669–682.
52. Chukova S and Johnston MR. Two-dimensional warranty repair strategy based on minimal and complete repairs. *Math Comput Model* 2006; 44: 1133–1143.
53. Huang YS and Yen C. A study of two-dimensional warranty policies with preventive maintenance. *IIE Trans* 2009; 41: 299–308.
54. Huang YS, Chen E and Ho JW. Two-dimensional warranty with reliability-based preventive maintenance. *IEEE T Reliab* 2013; 62: 898–907.
55. Su C and Wang X. A two-stage preventive maintenance optimization model incorporating two-dimensional extended warranty. *Reliab Eng Syst Safe* 2016; 155: 169–178.
56. Wang J, Zhou Z and Peng H. Flexible decision models for a two-dimensional warranty policy with periodic preventive maintenance. *Reliab Eng Syst Safe* 2017; 162: 14–27.

57. Hamidi M, Liao H and Szidarovszky F. Non-cooperative and cooperative game-theoretic models for usage-based lease contracts. *Eur J Oper Res* 2016; 255: 163–174.
58. Manna DK, Pal S and Sinha S. A use-rate based failure model for two-dimensional warranty. *Comput Ind Eng* 2007; 52: 229–240.
59. Jack N, Iskandar BP and Murthy DNP. A repair-replace strategy based on usage rate for items sold with a two-dimensional warranty. *Reliab Eng Syst Safe* 2009; 94: 611–617.
60. Banerjee R and Bhattacharjee MC. Analysis of a two-dimensional warranty servicing strategy with an imperfect repair option. *Qual Technol Quant M* 2012; 9: 23–33.
61. Banerjee R and Bhattacharjee MC. Warranty servicing with a Brown-Proschan repair option. *Asia Pac J Oper Res* 2012; 29: 1240023.
62. Ye ZS, Murthy DNP, Xie M, et al. Optimal burn-in for repairable products sold with a two-dimensional warranty. *IIE Trans* 2013; 45: 164–176.
63. Tong P, Liu Z, Men F, et al. Designing and pricing of two-dimensional extended warranty contracts based on usage rate. *Int J Prod Res* 2014; 52: 6362–6380.
64. Wang Y, Liu Y, Liu Z, et al. On reliability improvement program for second-hand products sold with a two-dimensional warranty. *Reliab Eng Syst Safe* 2017; 167: 452–463.
65. Wang X, Li L and Xie M. Optimal preventive maintenance strategy for leased equipment under successive usage-based contracts. Working paper, 2017.
66. Tsoukalas MZ and Agrafiotis GK. A new replacement warranty policy indexed by the product's correlated failure and usage time. *Comput Ind Eng* 2013; 66: 203–211.
67. Pal S and Murthy GSR. An application of Gumbel's bivariate exponential distribution in estimation of warranty cost of motor cycles. *Int J Qual Reliab Manag* 2003; 20: 488–502.
68. Xie W. Optimal pricing and two-dimensional warranty policies for a new product. *Int J Prod Res* 2017; 55: 6857–6870.
69. Jung M and Bai DS. Analysis of field data under two-dimensional warranty. *Reliab Eng Syst Safe* 2007; 92: 135–143.
70. Huang Y-S, Gau W-Y and Ho J-W. Cost analysis of two-dimensional warranty for products with periodic preventive maintenance. *Reliab Eng Syst Safe* 2015; 134: 51–58.
71. Singpurwalla ND and Wilson SP. Failure models indexed by two scales. *Adv Appl Probab* 1998; 30: 1058–1072.
72. Eliashberg J, Singpurwalla ND and Wilson SP. Calculating the reserve for a time and usage indexed warranty. *Manag Sci* 1997; 43: 966–975.
73. Baik J, Murthy DNP and Jack N. Two-dimensional failure modeling with minimal repair. *Nav Res Log* 2004; 51: 345–362.
74. Baik J, Murthy DNP and Jack N. Erratum: two-dimensional failure modeling with minimal repair. *Nav Res Log* 2006; 53: 115–116.
75. Iskandar BP and Blischke WR. Reliability and warranty analysis of a motorcycle based on claims data. In: Blischke WR and Murthy DNP (eds) *Case studies in reliability and maintenance*. New Jersey: John Wiley & Sons, 2003, pp.623–656.
76. Ahn CW, Chae KC and Clark GM. Estimating parameters of the power law process with two measures of failure time. *J Qual Technol* 1998; 30: 127–132.
77. Duchesne T and Lawless J. Alternative time scales and failure time models. *Lifetime Data Anal* 2000; 6: 157–179.
78. Samatlı-Paç G and Taner MR. The role of repair strategy in warranty cost minimization: an investigation via quasi-renewal processes. *Eur J Oper Res* 2009; 197: 632–641.
79. Majid HA, Wulandhari LA, Samah AA, et al. A framework in determining extended warranty by using two dimensional delay time model. *Adv Mat Res* 2012; 433–440: 2997–3002.
80. Wu S. Construction of asymmetric copulas and its application in two-dimensional reliability modelling. *Eur J Oper Res* 2014; 238: 476–485.
81. Anderson B, Chukova S and Hirose Y. Using Copulas to model and simulate two-dimensional warranty data. *Qual Reliab Eng Int* 2017; 33: 595–605.
82. Murthy DNP and Blischke WR. Product warranty management—III: a review of mathematical models. *Eur J Oper Res* 1992; 63: 1–34.
83. Murthy DNP, Baik J, Wilson R, et al. Two-dimensional failure modeling. In: Pham H (ed.) *Springer handbook of engineering statistics*. London: Springer, 2006, pp.97–111.
84. Tong P, Song X and Liu Z. A maintenance strategy for two-dimensional extended warranty based on dynamic usage rate. *Int J Prod Res* 2017; 55: 5743–5759.
85. Lawless J, Hu J and Cao J. Methods for the estimation of failure distributions and rates from automobile warranty data. *Lifetime Data Anal* 1995; 1: 227–240.
86. Gupta SK, De S and Chatterjee A. Warranty forecasting from incomplete two-dimensional warranty data. *Reliab Eng Syst Safe* 2014; 126: 1–13.
87. Manna DK, Pal S and Sinha S. A note on calculating cost of two-dimensional warranty policy. *Comput Ind Eng* 2008; 54: 1071–1077.
88. Lawless JF, Crowder MJ and Lee KA. Analysis of reliability and warranty claims in products with age and usage scales. *Technometrics* 2009; 51: 14–24.
89. Davis TP. A simple method for estimating the joint failure time and failure mileage distribution from automobile warranty data. *Ford Tech J* 1999; 2: 1–11.
90. Lu MW. Automotive reliability prediction based on early field failure warranty data. *Qual Reliab Eng Int* 1998; 14: 103–108.
91. Baik J and Murthy DNP. Reliability assessment based on two-dimensional warranty data and an accelerated failure time model. *Int J Reliab Saf* 2008; 2: 190–208.
92. Dai A, He Z, Liu Z, et al. Field reliability modeling based on two-dimensional warranty data with censoring times. *Qual Eng* 2017; 29: 468–483.
93. Hong Y and Meeker WQ. Field-failure and warranty prediction based on auxiliary use-rate information. *Technometrics* 2010; 52: 148–159.
94. Hong Y and Meeker WQ. Field-failure predictions based on failure-time data with dynamic covariate information. *Technometrics* 2013; 55: 135–149.
95. Majeske KD. A non-homogeneous Poisson process predictive model for automobile warranty claims. *Reliab Eng Syst Safe* 2007; 92: 243–251.
96. Gupta SK, De S and Chatterjee A. Some reliability issues for incomplete two-dimensional warranty claims data. *Reliab Eng Syst Safe* 2017; 157: 64–77.
97. Kleyner A and Sandborn P. A warranty forecasting model based on piecewise statistical distributions and stochastic simulation. *Reliab Eng Syst Safe* 2005; 88: 207–214.

98. Christozov DJ, Chukova S and Robinson J. Automotive warranty data: stratification approach for estimating the mean cumulative function. *Int J Prod Dev* 2010; 12: 254–273.
99. Rai B and Singh N. Hazard rate estimation from incomplete and unclean warranty data. *Reliab Eng Syst Safe* 2003; 81: 79–92.
100. Rai B and Singh N. Modeling and analysis of automobile warranty data in presence of bias due to customer-rush near warranty expiration limit. *Reliab Eng Syst Safe* 2004; 86: 83–94.
101. Rai B and Singh N. Customer-rush near warranty expiration limit, and nonparametric hazard rate estimation from known mileage accumulation rates. *IEEE T Reliab* 2006; 55: 480–489.
102. Rai B and Singh N. Forecasting automobile warranty performance in presence of "maturing data" phenomena using multilayer perceptron neural network. *J Syst Sci Syst Eng* 2005; 14: 159–176.
103. Rai B and Singh N. Forecasting warranty performance in the presence of the "maturing data" phenomenon. *Int J Syst Sci* 2005; 36: 381–394.
104. Rai B and Singh N. Non-parametric hazard rate estimation of hard failures with known mileage accumulation rates in vehicle population. *Int J Reliab Saf* 2008; 2: 248–263.
105. Rai B and Singh N. A modeling framework for assessing the impact of new time/mileage warranty limits on the number and cost of automotive warranty claims. *Reliab Eng Syst Safe* 2005; 88: 157–169.
106. Rai B. Warranty spend forecasting for subsystem failures influenced by calendar month seasonality. *IEEE T Reliab* 2009; 58: 649–657.
107. Anastasiadis S, Anderson B and Chukova S. Auto warranty and driving patterns. *Reliab Eng Syst Safe* 2013; 116: 126–134.
108. Corbu D, Chukova S and O'Sullivan J. Product warranty: modelling with 2D-renewal process. *Int J Reliab Saf* 2008; 2: 209–220.
109. Arunachalam V and Calvache Á. Approximation of the bivariate renewal function. *Commun Stat B: Simul* 2015; 44: 154–167.
110. Hadji EM, Kambo NS and Rangan A. Two-dimensional renewal function approximation. *Commun Stat A: Theor* 2015; 44: 3107–3124.
111. Xie W and Liao H. Some aspects in estimating warranty and post-warranty repair demands. *Nav Res Log* 2013; 60: 499–511.
112. Wu S. Warranty claim analysis considering human factors. *Reliab Eng Syst Safe* 2011; 96: 131–138.
113. Mitra A and Patankar JG. Warranty costs for repairable products with a two-attribute policy: impact of warranty execution. *Int J Oper Res* 2006; 1: 363–381.
114. Lee S-H, Lee S-J and Moon K-I. A fuzzy logic-based approach to two-dimensional automobile warranty system. *J Circuit Syst Comp* 2010; 19: 139–154.
115. Manna DK, Surajit P and Kulandaiyan A. Warranty cost estimation of a multi-module product. *Int J Qual Reliab Manag* 2004; 21: 102–117.
116. Manna DK, Pal S and Sinha S. Effect of use-rate on system life and failure models for 2D warranty. *Int J Qual Reliab Manag* 2011; 28: 464–482.
117. Sarada Y and Mubashirunnissa M. Warranty cost analysis for second-hand products using bivariate approach. *Int J Oper Res* 2011; 12: 34–55.
118. Mitra A and Patankar JG. Two-dimensional warranty policies incorporating product development. In: Lawrence KD and Klimberg RK (eds) *Advances in business and management forecasting*. Bingley: Emerald, 2010, pp.3–22.
119. Mitra A and Patankar JG. A multiobjective model for warranty policies integrating product quality, market share, and R&D expenditures. In: Lawrence KD and Klimberg RK (eds) *Advances in business and management Forecasting*. Bingley: Emerald, 2011, pp.43–65.
120. Su C and Shen J. Analysis of extended warranty policies with different repair options. *Eng Fail Anal* 2012; 25: 49–62.
121. Glickman TS and Berger PD. Optimal price and protection period decisions for a product under warranty. *Manag Sci* 1976; 22: 1381–1390.
122. Manna DK. Price-warranty length decision with Glickman-Berger model. *Int J Reliab Saf* 2008; 2: 221–233.
123. He S, Zhang Z, Zhang G, et al. Two-dimensional base warranty design based on a new demand function considering heterogeneous usage rate. *Int J Prod Res* 2017; 55: 7058–7072.
124. Jack N, Murthy DNP and Iskandar BP. Comments on "Maintenance policies with two-dimensional warranty." *Reliab Eng Syst Safe* 2003; 82: 105–109.
125. Chen T and Popova E. Maintenance policies with two-dimensional warranty. *Reliab Eng Syst Safe* 2002; 77: 61–69.
126. Jayaraman R and Matis TI. On prorated servicing costs for two-dimensional warranties with combined repair-replacement strategies. *Int J Prod Dev* 2010; 12: 274–295.
127. Chukova S, Hayakawa Y and Johnston MR. Optimal two-dimensional warranty repair strategy. *Proc IMechE, Part O: J Risk and Reliability* 2007; 221: 265–273.
128. Husniah H, Pasaribu US, Halim AH, et al. Hybrid minimal repair and age replacement policy for two-dimensional warranted products. *Int J Collab Enterp* 2011; 2: 284–301.
129. Yun WY and Kang KM. Imperfect repair policies under two-dimensional warranty. *Proc IMechE, Part O: J Risk and Reliability* 2007; 221: 239–247.
130. Varnosafaderani S and Chukova S. A two-dimensional warranty servicing strategy based on reduction in product failure intensity. *Comput Math Appl* 2012; 63: 201–213.
131. Varnosafaderani S and Chukova S. An imperfect repair strategy for two-dimensional warranty. *J Oper Res Soc* 2012; 63: 846–859.
132. Husniah H, Pasaribu US and Iskandar BP. A new servicing strategy involving imperfect repair for two-dimensional warranties. *Int J Appl Math Stat* 2015; 53: 250–259.
133. Cheng ZH, Yang ZY, Zhao JM, et al. Preventive maintenance strategy optimizing model under two-dimensional warranty policy. *Eksploat Niezawodn* 2015; 17: 365–373.
134. Husniah H and Iskandar BP. Modelling two-dimensional lease contract with preventive maintenance and servicing strategy. *J Teknol* 2016; 78: 19–25.
135. Bouguerra S, Rezg N and Chelbi A. A decision model for adopting an extended two-dimensional warranty considering different maintenance policies with variable customer's behavior. *IFAC Proc Vol* 2012; 45: 968–974.

136. Jia YX, Li XY and Wang HM. Optimal extended warranty policies under different maintenance options. *J Dong Hua Univ* 2014; 31: 890–894.
137. Wang X and Su C. A two-dimensional preventive maintenance strategy for items sold with warranty. *Int J Prod Res* 2016; 54: 5901–5915.
138. Huang Y-S, Huang C-D and Ho J-W. A customized two-dimensional extended warranty with preventive maintenance. *Eur J Oper Res* 2017; 257: 971–978.
139. Wang Y, Liu Z and Liu Y. Optimal preventive maintenance strategy for repairable items under two-dimensional warranty. *Reliab Eng Syst Safe* 2015; 142: 326–333.
140. Li XY, Jia YX and Li Z. Optimal two dimensional preventive maintenance policy based on asymmetric Copula function. *TELKOMNIKA* 2015; 13: 722–729.
141. Husniah H, Pasaribu US, Cakravastia A, et al. Two dimensional maintenance contracts for a fleet of dump trucks used in mining industry. *Appl Mech Mater* 2014; 660: 1026–1031.
142. Iskandar BP, Husniah H and Pasaribu US. Maintenance service contracts for equipment sold with two dimensional warranties. *Qual Technol Quant M* 2014; 11: 321–333.
143. Pham H and Wang H. Imperfect maintenance. *Eur J Oper Res* 1996; 94: 425–438.
144. Wu S and Zuo MJ. Linear and nonlinear preventive maintenance models. *IEEE T Reliab* 2010; 59: 242–249.
145. Hu Q, Bai Y, Zhao J, et al. Modeling spare parts demands forecast under two-dimensional preventive maintenance policy. *Math Probl Eng* 2015; 2015: 728241.
146. Musakwa FT. Pricing a motor extended warranty with limited time and usage cover. *Astin Bull* 2015; 45: 101–125.
147. Chattopadhyay G and Yun WY. Modeling and analysis of warranty cost for 2D-policies associated with sale of second-hand products. *Int J Reliab Appl* 2006; 7: 71–77.
148. Lawless JF and Crowder MJ. Models and estimation for systems with recurrent events and usage processes. *Lifetime Data Anal* 2010; 16: 547–570.
149. Ford. Your Warranty, https://owner.ford.com/tools/account/maintenance/your-warranty.html (accessed 28 July 2017).
150. Embraer. Lineage 1000E warranty, http://www.embraerexecutivejets.com/en-us/customer-support/pages/warranty.aspx (accessed 28 July 2017).
151. Kuhn Group. Flexible warranty, http://www.kuhn.at/en/building-machines/services/customer-services/flexible-warranty (accessed 28 July 2017).
152. Edmunds. Free vehicle maintenance programs, http://static.ed.edmunds-media.com/unversioned/img/pdf/free.vehicle.maintenance.programs/free.vehicle.maint.5.pdf (accessed 28 July 2017).
153. Ford. Extended protect service plan, https://www.lombardfordwarrantys.com/ (accessed 28 July 2017).
154. Edmunds. Certified cars program comparison, https://www.edmunds.com/certified-cars/compare-program.html (accessed 28 July 2017).

Appendix 1

Table 5. Classification of the reviewed papers.

Failure modeling approach	Typical 2D warranty			Flexible 2D warranty	2D long-term warranty		
	Policy 1	Policy 2	Policy 3–6		EW	SHW	MSC
1D	30–32, 34, 36–38, 43, 45, 46, 50–64, 84, 85, 87–95, 97–107, 113, 116, 120, 122–139, 141, 142, 145–148	30, 34, 36–38, 134, 141	34, 38, 39, 126	34, 36–38, 43, 45, 46	45, 46, 55, 56, 63, 84, 120, 135, 136, 138, 139	30, 37, 64, 147	57, 134, 141, 142
2D	29, 33, 35, 47, 49, 66–74, 86, 96, 108–110, 117	33, 35	33, 40	47		47, 117	
Others	44, 77–81, 114, 115, 140, 148			44	79		

EW: extended warranty; SHW: warranty for second-hand (used) products; MSC: maintenance service contract.
Policies 1–6 are referred to section "Typical 2D warranty policies."

Appendix 2

Exhibit 1: 2017 model year Ford warranty (in USA) (adapted from Ford[149])

New vehicle limited warranty coverage

The following chart (Figure 6) gives a general summary of warranty coverage provided by Ford Motor Company under the New Vehicle Limited Warranty. For each type of coverage, the chart shows two measures, that is, years in service and miles driven (whichever occurs first).

Under the New Vehicle Limited Warranty, if your Ford vehicle is properly operated and maintained and was taken to a Ford dealership for a warranted repair during the warranty period, then authorized Ford dealers will repair, replace, or adjust all parts on your

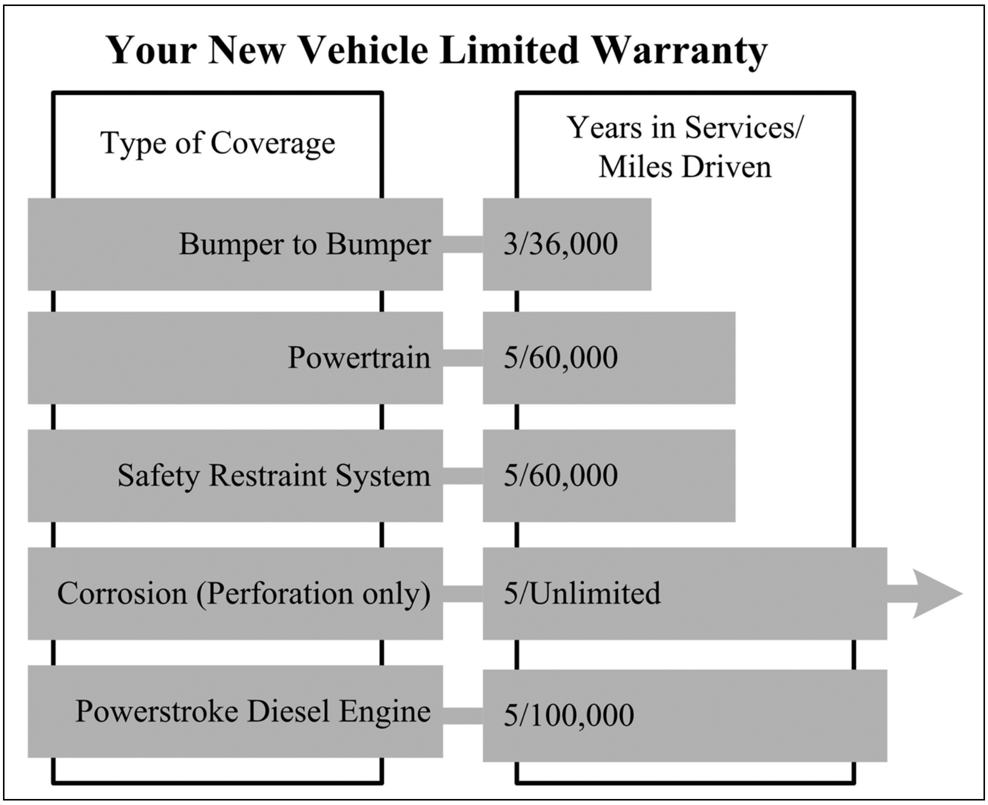

Figure 6. Warranty package for 2017 Ford new vehicles.

Table 6. Mileage-based reimbursement schedule for Ford vehicle tires.

Miles driven	Percentage of parts covered by Ford (%)
1–12,000	100
12,001–24,000	60
24,001–36,000	30

vehicle for free if the components malfunction or fail during normal use within the applicable coverage period due to a manufacturing defect in factory-supplied materials or factory workmanship.

Tire warranty

For vehicles within the New Vehicle Limited Warranty time in service and mileage coverage period, defective tires will be replaced on a pro-rata adjustment basis according to the following mileage-based Reimbursement Schedule (Table 6).

The tire manufacturer also provides a separate tire warranty that may extend the New Vehicle Limited Warranty coverage. You have the option of having a tire warranty repair performed by the tire manufacturer's authorized service center. If you go to a tire service center for a repair covered by the New Vehicle Limited Warranty, you may be charged a pro-rated amount for wear or other charges. If so, you should present your

Table 7. Warranty package for Embraer's Lineage 1000E.

Covered parts	Warranty terms
Parts manufactured by Embraer or its subcontractors (including structural parts)	60 months/5000 flight hours
Parts manufactured by Embraer's vendors	60 months/5000 flight hours
Avionics and avionics accessories	60 months/5000 flight hours
Interior items	24 months
Exterior Paint	24 months/1000 flight hours
APU and APU accessories	60 months/3000 flight hours or 6000 flight cycles[a]
Engines and engines accessories—covered directly by GE Company	60 months/2500 flight hours
AD (parts only)	36 months

[a]At 3000 APU hours or 6000 APU cycles, whichever occurs first, warranty shall be applied on a pro-rated basis starting at 100% and down to 0% at 4000 h or 8000 cycles.

Table 8. Comparison of several free vehicle maintenance programs.

Automaker	Year covered	Miles covered	Maintenance plan
Audi	1	5000	The first scheduled maintenance provided by Audi is free and will be performed at 5,000 miles or 12 months, whichever occurs first.
BMW	4	50,000	The BMW Ultimate Service Program covers all factory recommended maintenance, oil service and fluid service, as well as items that need replacement due to normal wear and tear, such as brake pads, brake rotors and wiper blade inserts.
Buick	2	24,000	Cover scheduled oil changes with filter and tire rotations. According to your new vehicle's recommended maintenance schedule, the coverage includes a maximum of 4 services within 2 years or 24,000 miles, whichever comes first.
Jaguar	5	60,000	Cover factory recommended scheduled maintenance for five years or 60,000 miles, whichever occurs first. A Service Interval Indicator appears every 16,000 miles or every 12 months, whichever occurs first, to tell you when the vehicle maintenance needs to be scheduled.
Lincoln	2	24,000	Cover oil changes, tire rotations and regularly scheduled maintenances for two years or 24,000 miles. Coverage includes a maximum of four regularly scheduled maintenance services.

Table 9. Compare coverage of different Ford extended service plans.

PowertrainCare	BaseCare	ExtraCare	PremiumCare
Engine*	Engine*	Engine*	Engine
Transmission*	Transmission*	Transmission*	Transmission
Rear-Wheel Drive*	Rear-Wheel Drive*	Rear-Wheel Drive	Rear-Wheel Drive
Front-Wheel Drive*	Front-Wheel Drive*	Front-Wheel Drive	Front-Wheel Drive
Covers 29 components	Steering	Steering	Steering
	Brakes*	Brakes	Brakes
	F&R Suspension*	F&R Suspension*	F&R Suspension
	AC & Heating*	AC & Heating*	AC & Heating
	Electrical*	Electrical*	Electrical
	Covers 84 components	High-Tech*	High-Tech
		Covers 113 components	Emissions
			Ford Audio
			Safety
			Covers 1000 components

For a system with symbol "*", only a proportion of its components are covered by the extended service plan.

paid invoice detailing the nature of the charges to any Ford Motor Company dealership for refund consideration.

Exhibit 2: limited warranty for Embraer's Lineage 1000E (adapted from Embraer[150])

The warranty package for Embraer's Lineage 1000E is shown in Table 7.

Exhibit 3: Kuhn's flexible warranty program (adapted from Kuhn Group[151])

Different options and lengths

The flexible warranty from Kuhn offers various options and lengths:

- Full coverage;
- Full coverage, parts only;
- Power line, full coverage;
- Power line, parts only.

In addition, Kuhn's customers also benefit from flexibility regarding lengths, for example, options include 1500 h/12 months or 6000 h/48 months. For large Kuhn production machines, special extended lengths (up to 10,000 h/72 months) are available.

Exhibit 4: free vehicle maintenance programs (adapted from Edmunds[152])

Many automakers provide customers with a variety of free maintenance programs. Table 8 summarizes the free maintenance programs of five automobile manufacturers. For more vehicle maintenance programs, please visit Edmunds (www.edmunds.com).

Exhibit 5: Ford protect extended service plans (adapted from Ford[153])

Ford Protect Extended Service Plan (ESP) includes PremiumCare, ExtraCare, BaseCare, PowertrainCare,

Table 10. Comparison of three CPO vehicle warranty programs.

	Ford	Chrysler	Lexus
Maximum age/mileage	Only sell vehicles that are 6 years old or less, with less than 80,000 miles	Only sell vehicles that are 5 years old or less, with less than 75,000 miles	Only sell vehicles that are 6 years old or less, with less than 70,000 miles
Warranty terms	12 month/12,000 mile limited warranty and 7 year/100,000 mile limited powertrain warranty	3 month/3000 mile limited warranty and 7 year/100,000 mile limited powertrain warranty	Balance of new car warranty (4 year/50,000 miles) plus 2 year/unlimited mileage warranty
Deductible	$100	$100	No deductible
Transferability	Warranty is transferable	Warranty is transferable	Warranty is non-transferable
Dealer certification required	172 point vehicle inspection	125 point vehicle inspection	161 point vehicle inspection
Service/maintenance	N/A	N/A	Covers the first four basic factory scheduled maintenance services for 2 years or 20,000 miles

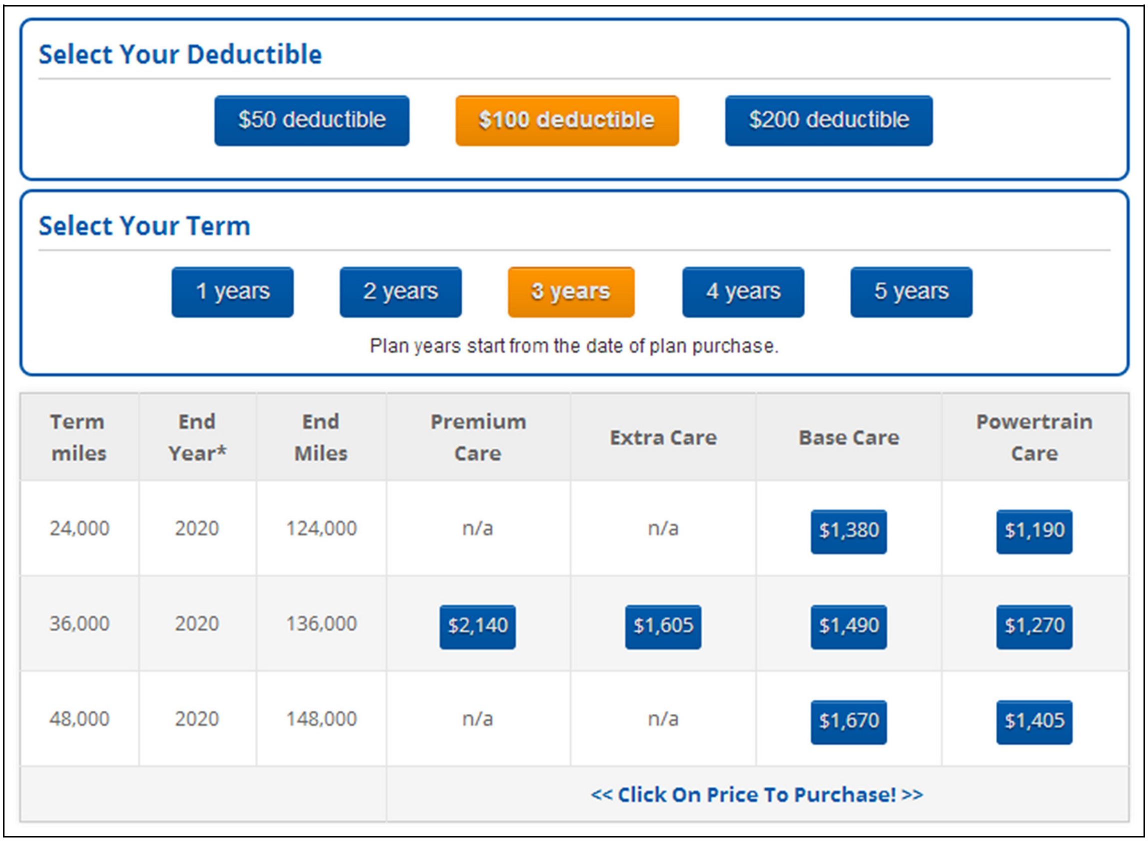

Figure 7. Ford Protect pricing scheme.

among others. Various ESP plans are available to suit every driving, vehicle, and budgetary need. Table 9 compares the coverage of four Ford ESP plans. To make the rate comparison process quick and easy, Ford has organized prices by deductible and contract term length (see Figure 7 for example). A vehicle owner can easily select his or her preferred deductible and term to see the plan price.

Exhibit 6: CPO vehicle warranty programs (adapted from Edmunds[154])

The Certified Pre-Owned (CPO) vehicle warranty programs of Ford, Chrysler, and Lexus are compared in Table 10 with regard to maximum age/mileage, warranty terms, transferability, and scheduled maintenance, among others. For a comprehensive comparison of more CPO vehicle warranty programs, please visit Edmunds (www.edmunds.com).