

Imperfect Preventive Maintenance Policies With Unpunctual Execution

Xiaolin Wang , *Student Member, IEEE*, Hang Zhou , Ajith Kumar Parlikad , *Member, IEEE*, and Min Xie , *Fellow, IEEE*

Abstract—Traditional maintenance planning problems usually presume that preventive maintenance (PM) policies will be executed exactly as planned. In reality, however, maintainers often deviate from the intended PM policy, resulting in unpunctual PM executions that may reduce maintenance effectiveness. This article studies two imperfect PM policies with unpunctual executions for infinite and finite planning horizons, respectively. Under the former policy, imperfect PM actions are periodically performed and the system is preventively replaced at the last PM instant. The objective is to determine the optimal number of PM actions and associated PM interval so as to minimize the long-run average cost rate. However, the latter policy specifies that a system is subject to periodic PM activities within a finite planning horizon and there is no PM activity at the end of the horizon. The aim is then to identify the optimal number of PM activities to minimize the expected total maintenance cost. In this article, we discuss the modeling and optimization of the two unpunctual PM policies and then explore the impact of unpunctual executions on the optimal PM decisions and corresponding maintenance expenses in an analytical or numerical way. The resulting insights are helpful for practitioners to adjust their PM plans when unpunctual executions are anticipated.

Index Terms—Finite horizon, hazard rate adjustment, imperfect preventive maintenance (PM), infinite horizon, unpunctual execution.

I. INTRODUCTION

PREVENTIVE maintenance (PM) is a set of activities to be performed before system failures, aiming at keeping the system in a good working state and reducing its operational expenditure [1], [2]. However, it is not wise to maintain systems too frequently as this would incur excessive maintenance expenses. For this reason, determining an optimal PM policy (in terms of number of PM actions, PM intervals, etc.) to balance

the benefits and costs of PM is an essential decision-making problem. The study of PM starts from the seminal work of Barlow and Hunter [3] and is still a vibrant research topic; see [4]–[9] for recent progresses.

In practice, PM actions can be periodically or sequentially scheduled [10]. Under a periodic PM policy, a system is preventively maintained at integer multiples of a certain fixed interval, whereas a sequential policy specifies that a system is maintained at a sequence of unequal-length intervals. Generally, periodic PM is convenient to implement, whereas sequential PM is more effective for aging systems that require more frequent maintenance. On the other hand, the effect of a maintenance action on system reliability could be perfect, minimal, or imperfect [11]. Perfect and minimal maintenance actions are two extremes: A perfect maintenance action is assumed to restore a failed system to an “as-good-as-new” state, whereas a minimal repair makes a failed system operational, with its state being that just before failure, i.e., the system is “as-bad-as-old” after a minimal repair [3]. A more realistic consideration is that the system state after maintenance is between “as-good-as-new” and “as-bad-as-old”. This leads to the concept of imperfect maintenance [2], [11], [12]. Up to now, numerous models have been developed to formulate imperfect maintenance effects. In principle, existing models can be categorized by the measures that maintenance models manipulate: Methods modifying the lifetime distribution (e.g., geometric process models [13], [14]), methods relating to the failure rate/intensity function (e.g., failure intensity reduction models [12] and hazard rate adjustment models [15], [16]), methods reducing the system age [12], [17], and hybrid models [2], [18]. In this article, we focus on periodic imperfect PM policies and adopt the hazard rate adjustment model.

In the literature, many studies deal with the optimization of imperfect PM policies for an infinite planning horizon. The objective is to determine optimal number of PM actions and PM interval (or a sequence of PM intervals) to minimize the long-run average cost rate. Lin *et al.* [18] developed sequential imperfect PM models with two failure modes—maintainable and nonmaintainable. Wu and Clements-Croome [19] investigated two periodic imperfect PM policies with random PM quality. Zequeira and Bérenguer [20] studied an optimal periodic imperfect PM policy for a system with maintainable and nonmaintainable failure modes. Sheu and Chang [21] developed an extended periodic imperfect PM model with age-dependent failure types—Type I (minor) and Type II (catastrophic). Cha

Manuscript received September 23, 2019; revised February 12, 2020; accepted March 22, 2020. Date of publication April 16, 2020; date of current version November 30, 2020. This work was supported in part by the National Natural Science Foundation of China under Grant 71532008 and Grant 71971181, in part by the Research Grants Council of Hong Kong under a Theme-Based Research Fund under Grant T32-101/15-R, and in part by the General Research Fund under Grant CityU 11203519. Associate Editor: Ming Jian Zuo. (Corresponding author: Xiaolin Wang.)

Xiaolin Wang and Min Xie are with the Department of Systems Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong, SAR, and also with the City University of Hong Kong Shenzhen Research Institute, Shenzhen, China (e-mail: xlwang28-c@my.cityu.edu.hk; minxie@cityu.edu.hk).

Hang Zhou and Ajith Kumar Parlikad are with the Department of Engineering, Institute for Manufacturing, University of Cambridge, CB30FS Cambridge, U.K. (e-mail: hz356@cam.ac.uk; aknp2@cam.ac.uk).

Color versions of one or more of the figures in this article are available online at <http://ieeexplore.ieee.org>.

Digital Object Identifier 10.1109/TR.2020.2983415

and Finkelstein [22] studied optimal imperfect periodic and age-based maintenance with long-run asymptotic virtual age. Sun *et al.* [23] developed a novel periodic PM model that involves the saturation effect of imperfect maintenance. Zhao *et al.* [24] investigated a condition-based inspection-replacement policy for degrading systems. Moreover, there are also a few publications addressing both periodic and sequential imperfect PM policies. Nakagawa [10] studied both periodic and sequential PM policies based on the hazard rate adjustment PM model. Wang and Yam [13] and Wang *et al.* [14] investigated periodic and sequential PM policies based on generalized geometric processes.

In reality, the working spans of most systems are finite [16]. The maintenance planning horizon in this scenario is thus finite, and the decision-making problem is to identify the optimal number of PM actions (and PM degree) to minimize the total expected maintenance cost over the finite horizon of interest. In the literature, there are quite a few studies on imperfect PM policies for a finite planning horizon, e.g., warranty period, lease period, and maintenance outsourcing period [25]. Nakagawa [16] provided an excellent summary of maintenance policies (including replacement, imperfect PM, and inspection policies) for a finite horizon in a general context. Chien [26] developed an imperfect PM model for warranted systems with “worse-than-minimal” repair upon a failure. Shang *et al.* [27] studied optimal condition-based warranty design and post-warranty imperfect PM optimization for products subject to stochastic degradation. Su and Wang [28] and Wang *et al.* [29] investigated quasi-sequential imperfect PM strategies over warranty/lease periods: PM actions are periodically performed within a single phase and the PM intervals in different phases differ. Qiu *et al.* [30] developed a novel maintenance model for energy generation systems under a power purchase agreement so as to maximize the expected net revenue. Recently, Petchrompo *et al.* [31] proposed a value-based approach to optimizing maintenance plans for a multi-asset k -out-of- N system over a long planning horizon.

The aforementioned studies, either in an infinite or finite planning-horizon setting, implicitly assume that scheduled PM activities are punctually implemented exactly as planned. In practice, however, there exist quite a few cases in which PM executions are not punctual. For instance, scheduled PM activities may be advanced by breaks in operations or by forecasted bad weather (e.g., for offshore wind turbines); conversely, maintenance actions may be postponed by busy operational schedules or by unavailability of maintenance resources. de Jonge *et al.* [32] showed that strategically postponing a PM action helps collect more information to reduce the lifetime distribution uncertainty, leading to potential cost benefits in the long run. Yang *et al.* [33] developed a two-phase PM model, where preventive replacement in the second phase is delayed to sufficiently utilize the system’s remaining lifetime and facilitate replacement preparations. Scarf *et al.* [34] found that for a critical system, opportunistic inspections may offer an economic advantage against periodic inspections in certain cases. On the other hand, the working cycle of a system that executes successive jobs might be variable. In this case, it is better to perform PM (that is originally scheduled during a working cycle) after the job

is completed. This motivates the so-called random replacement policies in, e.g., [35] and [36]. In a similar setting, Zhu *et al.* [37] studied a PM rescheduling problem in which the equipment may not be available for planned PM due to busy operational schedules.

In the PM studies above, the maintenance planner and executor are, in essence, the same and the planner can adjust (mostly postpone) maintenance activities according to operational schedules, weather conditions, resource availability, etc. In practice, however, maintenance planning and execution might be separate in certain scenarios [38]. For example, in vehicle warranty and maintenance contexts, optimal PM schedule within the warranty period is recommended by the manufacturer, whereas the PM implementation relies on vehicle owners to return their vehicles to authorized maintenance centers [39]. The vehicle owners, however, may be unpunctual so that the PM activities are performed earlier or later than recommended. A similar scenario is that a maintenance planner specifies an optimal PM policy but relies on a group of crews to execute the maintenance operations [38]. In such scenarios, the potential unpunctuality—stemming from the separation of policy specification and implementation—is independent of the PM policy specified by the maintenance planner. As a consequence, it is better for maintenance planners to prescribe their PM policies in anticipation of unpunctual executions. The work of He *et al.* [38] is, to the authors’ knowledge, the first attempt in this context. Considering traditional age and periodic replacement policies, He *et al.* [38] determined optimal planned replacement instants by minimizing long-run average cost rates, in anticipation of replacement unpunctuality. Recently, Wang *et al.* [39] studied an unpunctual PM policy for repairable products sold with a 2-D warranty.

In this article, we aim to extend the investigation of unpunctual replacement policies in [38] to a more general imperfect PM scenario. The potential unpunctuality of maintainers is what causes actual PM instants to randomly deviate from the planned instants. The hazard rate adjustment model in [15], [16], and [19] is adopted to formulate imperfect maintenance effects. We first discuss the modeling and optimization of imperfect PM policies in infinite and finite planning-horizon settings, and then explore the impact of unpunctual executions on the optimal PM decisions and corresponding maintenance expenses in an analytical or numerical way. Some managerial insights are obtained accordingly.

Overall, the main contributions of this article are fourfold, which are as follows.

- 1) From the practical perspective, imperfect PM policies are studied in both infinite and finite planning-horizon settings, unlike Wang *et al.* [39] that only considers a finite warranty period.
- 2) From the mathematical modeling perspective, the unpunctual PM models generalize their punctual counterparts by assuming that the maintenance unpunctuality is governed by a specific probability distribution.
- 3) From the analysis perspective, the impacts of unpunctual executions on the optimal number and interval of imperfect PM actions are discussed in an analytical or numerical manner.

- 4) From the managerial perspective, this article provides some interesting insights that are helpful for practitioners to adjust their PM schedules when unpunctual executions are anticipated.

The rest of this article is organized as follows. Sections II and III discuss the modeling and optimization of the unpunctual imperfect PM policies for infinite and finite planning horizons, respectively. Numerical studies are also conducted in these sections to demonstrate the models and results. Finally, Section IV concludes this article.

II. MODEL A: INFINITE PLANNING-HORIZON SETTING

The first model of interest is an unpunctual PM policy for an infinite planning horizon (hereafter termed *Model A*). We generalize the periodic imperfect PM model in [10] and [19] by assuming that actual PM instants deviate from the scheduled instants in a probabilistic manner—namely, the PM deviations are governed by a specific probability distribution. In what follows, we first formulate the average long-run cost rates for the punctual and unpunctual execution scenarios, respectively, and then examine the optimization and policy comparison of the two scenarios. Numerical studies are also conducted to demonstrate the punctual and unpunctual PM policies.

A. The Optimization Models

The following assumptions are made to facilitate the modeling and analysis.

- 1) When there is no maintenance action, the hazard rate of the system $\lambda(t)$ is strictly increasing to ∞ , with $\lambda(0) = 0$.
- 2) The j th PM interval is $T + Y_j$ for $j = 1, 2, \dots, N$, and the system is preventively replaced at the N th PM instant. Essentially, T is the constant interval between two successive PM actions, and Y_j , $j = 1, 2, \dots, N$, is the probabilistic deviation between the constant and actual intervals of the j th PM ($Y_0 = 0$). We assume that Y_j 's are independent and identically distributed random variables, with a cumulative distribution function $G(y)$, $y \in [l, u]$, where l and u ($l \leq u$) are the smallest and largest values Y_j can take.
- 3) The effect of imperfect PM actions on system reliability is described by the hazard rate adjustment model [15]. The hazard rate becomes $\lambda_j(t) = a_j \lambda_{j-1}(t)$, $t > 0$, after the j th PM action if it was $\lambda_{j-1}(t)$ before the PM, where a_j is the associated hazard rate adjustment factor. As a result, the system has a hazard rate $\lambda_j(t) = A_j \lambda(t)$ for $t \in (0, T + Y_{j+1})$, where $1 = a_0 < a_1 \leq a_2 \leq \dots \leq a_{N-1}$, $A_j = \prod_{k=0}^j a_k$, and $1 = A_0 < A_1 < \dots < A_{N-1}$. In essence, this model assumes that each PM action can reduce the hazard rate to zero and then the hazard rate function grows more quickly than it did before.
- 4) A replacement action brings the system to be as good as new. The system undergoes minimal repair upon a failure between two adjacent PM actions. The hazard rate remains unchanged after a minimal repair.

- 5) The durations for imperfect PM, minimal repair, and replacement are very small in comparison with the mean time to failure, and thus negligible.
- 6) The costs of an imperfect PM, a minimal repair, and a replacement are c_p , c_m , and c_r , respectively, with $c_r > c_m$ and $c_r > c_p$.

Based on the assumptions above, if PM actions are always performed on time—that is, $Y_j = 0$ for $j = 1, 2, \dots, N$, then the long-run average cost rate is as follows:

$$\tilde{C}_A(T, N) = \frac{c_m \sum_{j=1}^N A_{j-1} \Lambda(T) + (N-1)c_p + c_r}{NT} \quad (1)$$

where $\Lambda(t) = \int_0^t \lambda(x) dx$ is the cumulative hazard rate function. The optimization problem is to determine the optimal number of PM actions \tilde{N}^* and optimal PM interval \tilde{T}^* such that $\tilde{C}_A(T, N)$ is minimized.

Remark 1: If we take $N = 1$, then model (1) reduces to the well-known periodic replacement with minimal repair model in [40], and the long-run average cost rate becomes $\tilde{C}_A(T, N = 1) = (c_m \Lambda(T) + c_r)/T$. On the other hand, if $a_1 = \dots = a_{N-1} = 1$ and $c_p = c_r$, i.e., all PM actions are perfect replacements, then model (1) is also equivalent to the periodic replacement with minimal repair model. In this article, we consider $a_j > 1$, $j = 1, 2, \dots, N-1$, for imperfect PM consideration.

On the other hand, if PM actions are unpunctually implemented, then the long-run average cost rate can be formulated as follows:

$$\begin{aligned} C_A(T, N) &= \frac{c_m \sum_{j=1}^N \int_l^u A_{j-1} \Lambda(T + y_j) dG(y_j) + (N-1)c_p + c_r}{\sum_{j=1}^N \int_l^u (T + y_j) dG(y_j)} \\ &= \frac{c_m \sum_{j=1}^N \int_l^u A_{j-1} \Lambda(T + y_j) dG(y_j) + (N-1)c_p + c_r}{N(T + \mu_y)} \end{aligned} \quad (2)$$

where $\mu_y = \int_l^u y_j dG(y_j)$ is the expected deviation of each PM. Likewise, the objective is to seek the optimal values of N^* and T^* to minimize $C_A(T, N)$, when the maintenance unpunctuality is anticipated. It should be noted that as Y_j can take negative values, this model requires the feasible domain of T to be $\{T \mid T > \max\{-l, 0\}\}$ so as to guarantee $T + Y_j > 0$ for $j = 1, 2, \dots, N$.

Remark 2: If we take $N = 1$, then model (2) is simplified to the so-called “unpunctual age replacement with minimal repair” model in [38], and the long-run average cost rate becomes

$$C_A(T, N = 1) = \frac{c_m \int_l^u \Lambda(T + y_j) dG(y_j) + c_r}{T + \mu_y}.$$

B. Optimal PM Decisions

In this section, we examine the optimal infinite-horizon PM decisions in the punctual and unpunctual scenarios, respectively, which form the foundation of further analyses in Section II-C.

We first investigate the optimal N^* and T^* of the imperfect PM policy with unpunctual executions. For any feasible PM interval $T > \max\{-l, 0\}$, the optimal N^* can be obtained by

solving two inequalities, namely, $C_A(T, N-1) > C_A(T, N)$ and $C_A(T, N+1) \geq C_A(T, N)$, which yields the following:

$$\mathcal{J}_A(T, N-1) < \frac{c_r - c_p}{c_m} \text{ and } \mathcal{J}_A(T, N) \geq \frac{c_r - c_p}{c_m} \quad (3)$$

where

$$\begin{aligned} \mathcal{J}_A(T, N) = & N \int_l^u A_N \Lambda(T + y_{N+1}) dG(y_{N+1}) \\ & - \sum_{j=1}^N \int_l^u A_{j-1} \Lambda(T + y_j) dG(y_j), \end{aligned}$$

for $N = 1, 2, \dots$, and $\mathcal{J}_A(T, 0) = 0$.

Proposition 1: For any feasible $T > \max\{-l, 0\}$, there exists a finite and unique solution N^* to (3), which minimizes the average cost rate $C_A(T, N)$.

Now, suppose that the number of PM actions $N \geq 1$ is fixed, we attempt to identify the optimal T^* that minimizes $C_A(T, N)$. To this end, we differentiate $C_A(T, N)$ with respect to T and set $\partial C_A(T, N)/\partial T = 0$. Then, we have

$$\sum_{j=1}^N \mathcal{L}_{A,i}(T) = \frac{(N-1)c_p + c_r}{c_m} \quad (4)$$

where

$$\begin{aligned} \mathcal{L}_{A,i}(T) = & (T + \mu_y) \int_l^u A_{j-1} \lambda(T + y_j) dG(y_j) \\ & - \int_l^u A_{j-1} \Lambda(T + y_j) dG(y_j). \end{aligned}$$

The following proposition can be obtained regarding the optimal T^* for any fixed $N \geq 1$.

Proposition 2: For any finite $N \geq 1$, if

$$\lim_{T \rightarrow \max\{-l, 0\}} \sum_{j=1}^N \mathcal{L}_{A,i}(T) < \frac{(N-1)c_p + c_r}{c_m} \quad (5)$$

then there exists a unique solution T^* to (4), and the resultant cost rate is

$$C_A(T^*, N) = \frac{c_m}{N} \sum_{j=1}^N \int_l^u A_{j-1} \lambda(T^* + y_j) dG(y_j). \quad (6)$$

Otherwise,

$$\inf C_A(T, N) = \lim_{T \rightarrow \max\{-l, 0\}} C_A(T, N). \quad (7)$$

Note that condition (5) guarantees the uniqueness of the solution T^* to (4). To satisfy this condition, $[(N-1)c_p + c_r]/c_m$ should be relatively large, which indicates that the imperfect PM cost and the replacement cost should be large in comparison with the minimal repair cost. In this article, we suppose that condition (5) will hold, at least for the optimal N^* , throughout the remainder of Section II.

According to the analysis above, we have to solve (3) and (4) simultaneously, when determining the optimal N^* and T^* for Model A with unpunctual executions. To facilitate this process, we develop an enumeration algorithm to search N^* and T^* that minimize $C_A(T, N)$; see Algorithm 1. For a fixed N (starting

Algorithm 1: Searching T^* and N^* for Model A.

Input: $\lambda(t)$, c_m , c_p , c_r , $\{a_j\}$, $G(y)$, l , u

Output: N^* , T^*

```

1: for  $N = 1$  to  $\infty$  do
2:   if condition (5) is satisfied then
3:     obtain  $T$  by solving (4)
4:     if condition (3) is met then
5:        $T^* \leftarrow T$ ,  $N^* \leftarrow N$ 
6:       compute  $C_A(T^*, N^*)$  by (6)
7:       break
8:     end if
9:   end if
10: end for
11: return  $N^*$ ,  $T^*$ ,  $C_A(T^*, N^*)$ 

```

from $N = 1$), if condition (5) is satisfied, then a unique optimal PM interval T can be obtained by solving (4). We then turn to check condition (3) for current N and T . If the inequalities well hold, then the current N and T are optimal; otherwise, set $N \leftarrow N + 1$ and repeat the procedure. The analytical results in Propositions 1 and 2 guarantee that the algorithm will stop at a finite N^* , with the associated optimal T^* obtained.

Next, we look at the optimal decisions \tilde{N}^* and \tilde{T}^* of the periodic PM policy with punctual executions. As the optimization procedure is quite similar to the unpunctual scenario, we directly present the following result without detailed proof.

Proposition 3: For the infinite-horizon imperfect PM policy (Model A) with punctual executions, the optimal \tilde{N}^* and \tilde{T}^* exist when the following conditions are satisfied, simultaneously:

$$\tilde{\mathcal{J}}_A(T, N-1) < \frac{c_r - c_p}{c_m} \text{ and } \tilde{\mathcal{J}}_A(T, N) \geq \frac{c_r - c_p}{c_m}, \quad (8)$$

and

$$\sum_{j=1}^N \tilde{\mathcal{L}}_{A,i}(T) = \frac{(N-1)c_p + c_r}{c_m} \quad (9)$$

and the resulting average cost rate is

$$\tilde{C}_A(\tilde{T}^*, \tilde{N}^*) = \frac{c_m}{\tilde{N}^*} \sum_{j=1}^{\tilde{N}^*} A_{j-1} \lambda(\tilde{T}^*) \quad (10)$$

where $\tilde{\mathcal{L}}_{A,i}(T) = A_{j-1}(T\lambda(T) - \Lambda(T))$, $\tilde{\mathcal{J}}_A(T, N) = (NA_N - \sum_{j=1}^N A_{j-1})\Lambda(T)$, $N = 1, 2, \dots$, and $\tilde{\mathcal{J}}_A(T, 0) = 0$.

The optimal decision variables \tilde{N}^* and \tilde{T}^* in the punctual execution scenario can also be searched via Algorithm 1, with a slight modification.

C. Impact of Unpunctual Executions on the Optimal PM Decisions

In this section, we investigate the impact of unpunctual policy executions on the optimal PM decisions. For this purpose, we compare $(\tilde{T}^*, \tilde{N}^*)$ with (T^*, N^*) under different conditions on the PM deviations. More precisely, two different scenarios of maintenance unpunctuality—that is, PM actions are never

performed earlier (or later) than intended—are examined; see Theorems 1 and 2, respectively. Note that only convex hazard rate $\lambda(t)$ is considered below, as it is commonly used to characterize time-to-failure distributions for aging systems. Since it is difficult to derive analytical results for the case in which $\lambda(t)$ is concave increasing, we resort to numerical experiments instead.

Theorem 1: If $\lambda(t)$ is convex and $0 \leq l < u$, then we have the following results.

- 1) If the two policies adopt the same $N \geq 1$, and

$$\lim_{T \rightarrow 0} \mathcal{L}_{A,i}(T) \geq 0 \quad (11)$$

then $T^* < \tilde{T}^*$.

- 2) If the two policies adopt the same $T > 0$, then $N^* \leq \tilde{N}^*$.

Theorem 1 implies that if a fixed number N ($N \geq 1$) of PM actions are scheduled and all of them are never performed earlier than intended, then the maintenance planner should schedule PM earlier than that in the punctual scenario; on the other hand, if a fixed PM interval T ($T > 0$) is set and all PM actions are never performed earlier than intended, then a fewer number of PM actions should be planned in comparison with the punctual scenario. Note that the inequality $N^* \leq \tilde{N}^*$ is not strict because N is an integer in nature; thus, there is a possibility that $N^* = \tilde{N}^*$, especially when the values of $\mathcal{J}_A(T, N)$ and $\tilde{\mathcal{J}}_A(T, N)$ are close.

Theorem 2: If $\lambda(t)$ is convex and $l < u \leq 0$, then we have the following results.

- 1) If the two policies adopt the same $N \geq 1$, then $T^* > \tilde{T}^*$.
- 2) If the two policies adopt the same $T > -l$, then $N^* \geq \tilde{N}^*$.

In contrast, Theorem 2 indicates that if a fixed number N ($N \geq 1$) of PM actions are planned and they are always performed earlier than intended, then the maintenance planner should schedule PM later than that in the punctual scenario; while if a fixed PM interval T ($T > 0$) is specified and all PM actions are always performed earlier than intended, then a larger number of PM actions should be planned in comparison with the punctual scenario. Comparing Theorem 2 with Theorem 1, we find that the scenario of unpunctual execution indeed has an impact on the optimal PM decisions (in terms of PM interval and number of PM actions).

Remark 3: For scenario 1) in Theorem 2 (i.e., the two policies adopt the same $N \geq 1$), there are the following two possible cases about the relationship of \tilde{T}^* and $-l$.

- 1) If $\sum_{j=1}^N \tilde{\mathcal{L}}_{A,i}(-l) \geq \frac{(N-1)c_p + c_r}{c_m}$, then $T^* > -l > \tilde{T}^*$.
- 2) If $\sum_{j=1}^N \tilde{\mathcal{L}}_{A,i}(-l) < \frac{(N-1)c_p + c_r}{c_m}$, then $T^* > \tilde{T}^* > -l$.

The arguments in Remark 3 can be referred to the proof of Theorem 2 in the Appendix. This result originates from the different feasible domains of PM interval T in punctual and unpunctual scenarios. Specifically, when $l < u \leq 0$, the feasible domain of PM interval T in the unpunctual scenario is $\{T \mid T > -l\}$, whereas that in the punctual scenario is still $\{T \mid T > 0\}$.

It is worth emphasizing that the results in Theorems 1 and 2 are either for fixed N or for fixed T . The ideal comparison we attempt to perform is comparing $(\tilde{T}^*, \tilde{N}^*)$ with (T^*, N^*) in their optimal values. However, such a comparison is difficult,

TABLE I
OPTIMAL PM DECISIONS UNDER PUNCTUAL EXECUTIONS
IN THE INFINITE HORIZON SETTING

c_p	$\beta = 1.5$			$\beta = 3.5$		
	\tilde{N}^*	\tilde{T}^*	\tilde{C}_A	\tilde{N}^*	\tilde{T}^*	\tilde{C}_A
0.1	4	74.53	0.305	15	8.38	0.350
0.5	4	76.48	0.309	13	9.93	0.390
1	4	78.89	0.314	12	11.02	0.434
2	4	83.60	0.323	10	13.21	0.509
5	3	117.12	0.341	8	16.63	0.684

if not impossible, to conduct, and, thus, we resort to numerical experiments again.

D. Numerical Studies

In this section, numerical studies are presented to demonstrate the unpunctual PM policy for an infinite planning horizon and to examine the impact of unpunctual executions on the optimal PM decisions.

Suppose that the system lifetime is Weibull distributed, i.e., $\lambda(t) = (\beta/\alpha)(t/\alpha)^{\beta-1}$ and $\Lambda(t) = (t/\alpha)^\beta$, with $\alpha = 15$. For the shape parameter, only $\beta > 1$ is considered, as it is well-known that carrying out PM activities is uneconomic for $\beta \leq 1$ (the hazard rate function is decreasing or constant). In this article, we examine two typical shapes of hazard rate—concave ($1 < \beta < 2$) and convex ($\beta > 2$). Specifically, $\beta = 1.5$ and $\beta = 3.5$ are considered for concave and convex hazard rates, respectively. The hazard rate adjustment factor is assumed to be $a_j = (5j + 1)/(4j + 1)$, i.e., a_j is strictly increasing in j , from $a_0 = 1$ to $a_\infty = 1.25$. Further assume that the minimal repair cost is $c_m = 1$ and the replacement cost is $c_r = 30$.

We first look at the imperfect PM policy in the punctual execution scenario. Table I summarizes the optimal PM decisions and corresponding average cost rates for different values of c_p . It can be seen that as the PM cost c_p increases, the optimal number of PM actions \tilde{N}^* decreases, whereas both the optimal PM interval \tilde{T}^* and the cost rate $\tilde{C}_A(\tilde{N}^*, \tilde{T}^*)$ increase. This observation applies to both concave and convex hazard rates.

We then study the imperfect PM policy in the unpunctual execution scenario. For illustrative purposes, we assume that the PM deviation Y_j has a uniform distribution $U(l, u)$, though our model accommodates to any feasible distribution of Y_j . To examine the impact of maintenance unpunctuality, we consider three scenarios of unpunctual executions—advanced ($l < u \leq 0$), delayed ($0 \leq l < u$), and hybrid ($l < 0 < u$), coupled with three unpunctuality magnitudes (measured by $u - l$). Specifically, the three unpunctuality magnitudes correspond to $u - l = 1$, $u - l = 2$, and $u - l = 5$, respectively. Tables II and III report the optimal PM decisions and corresponding average cost rates under unpunctual executions for $\beta = 1.5$ and $\beta = 3.5$, respectively. In these tables, we define $\Psi = (T^* - \tilde{T}^*)/\tilde{T}^*$ and $\Omega = (C_A(N^*, T^*) - \tilde{C}_A(\tilde{N}^*, \tilde{T}^*))/\tilde{C}_A(\tilde{N}^*, \tilde{T}^*)$ as the relative changes of PM interval and cost rate, respectively, when comparing unpunctual PM policy with its punctual counterpart. It is clear that a positive (respectively negative) Ψ corresponds

TABLE II
OPTIMAL PM DECISIONS UNDER UNPUNCTUAL EXECUTIONS IN THE INFINITE HORIZON SETTING ($\beta = 1.5$)

c_p	$l = 0, u = 1$					$l = 0, u = 2$					$l = 0, u = 5$				
	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$
0.1	4	74.03	-0.67	0.305	0.00	4	73.53	-1.34	0.305	0.00	4	72.05	-3.33	0.305	0.00
0.5	4	75.98	-0.65	0.309	0.00	4	75.49	-1.29	0.309	0.00	4	74.00	-3.24	0.309	0.00
1	4	78.39	-0.63	0.314	0.00	4	77.90	-1.25	0.314	0.00	4	76.41	-3.14	0.314	0.00
2	4	83.10	-0.60	0.323	0.00	4	82.61	-1.18	0.323	0.00	4	81.12	-2.97	0.323	0.00
5	3	116.62	-0.43	0.342	0.00	3	116.12	-0.85	0.342	0.00	3	114.63	-2.13	0.342	0.00
c_p	$l = -0.5, u = 0.5$					$l = -1, u = 1$					$l = -2.5, u = 2.5$				
	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$
0.1	4	74.53	0.00	0.305	0.00	4	74.53	0.00	0.305	0.00	4	74.55	0.03	0.305	0.00
0.5	4	76.48	0.00	0.309	0.00	4	76.49	0.01	0.309	0.00	4	76.50	0.03	0.309	0.00
1	4	78.89	0.00	0.314	0.00	4	78.90	0.01	0.314	0.00	4	78.91	0.03	0.314	0.00
2	4	83.60	0.00	0.323	0.00	4	83.61	0.01	0.323	0.00	4	83.62	0.02	0.323	0.00
5	3	117.12	0.00	0.342	0.00	3	117.12	0.00	0.342	0.00	3	117.13	0.01	0.342	0.00
c_p	$l = -1, u = 0$					$l = -2, u = 0$					$l = -5, u = 0$				
	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$
0.1	4	75.03	0.67	0.305	0.00	4	75.53	1.34	0.305	0.00	4	77.05	3.38	0.305	0.00
0.5	4	76.98	0.65	0.309	0.00	4	77.49	1.32	0.309	0.00	4	79.00	3.29	0.309	0.00
1	4	79.39	0.63	0.314	0.00	4	79.90	1.28	0.314	0.00	4	81.41	3.19	0.314	0.00
2	4	84.10	0.60	0.323	0.00	4	84.61	1.21	0.323	0.00	4	86.12	3.01	0.323	0.00
5	3	117.62	0.43	0.342	0.00	3	118.12	0.85	0.342	0.00	3	119.63	2.14	0.342	0.00

TABLE III
OPTIMAL PM DECISIONS UNDER UNPUNCTUAL EXECUTIONS IN THE INFINITE HORIZON SETTING ($\beta = 3.5$)

c_p	$l = 0, u = 1$					$l = 0, u = 2$					$l = 0, u = 5$				
	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$
0.1	15	7.88	-5.97	0.350	0.14	15	7.37	-12.05	0.352	0.60	14	6.38	-23.87	0.362	3.43
0.5	13	9.43	-5.04	0.391	0.10	13	8.92	-10.17	0.392	0.44	13	7.38	-25.68	0.401	2.66
1	12	10.51	-4.63	0.435	0.09	12	10.01	-9.17	0.435	0.35	11	9.16	-16.88	0.443	2.10
2	10	12.71	-3.79	0.509	0.06	10	12.20	-7.65	0.510	0.24	10	10.67	-19.23	0.516	1.49
5	8	16.12	-3.07	0.684	0.03	8	16.53	-0.60	0.685	0.13	7	15.01	-9.74	0.690	0.85
c_p	$l = -0.5, u = 0.5$					$l = -1, u = 1$					$l = -2.5, u = 2.5$				
	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$
0.1	15	8.38	0.00	0.350	0.14	15	8.37	-0.12	0.352	0.60	14	8.88	5.97	0.362	3.43
0.5	13	9.93	0.00	0.391	0.10	13	9.92	-0.10	0.392	0.44	13	9.88	-0.50	0.401	2.66
1	12	11.01	-0.09	0.435	0.09	12	11.01	-0.09	0.436	0.35	11	11.66	5.81	0.443	2.10
2	10	13.21	0.00	0.509	0.06	10	13.20	-0.08	0.510	0.24	10	13.17	-0.30	0.516	1.49
5	8	16.62	-0.06	0.684	0.03	8	17.53	5.41	0.685	0.13	7	17.51	5.29	0.690	0.85
c_p	$l = -1, u = 0$					$l = -2, u = 0$					$l = -5, u = 0$				
	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$	N^*	T^*	$\Psi(\%)$	C_A	$\Omega(\%)$
0.1	15	8.88	5.97	0.350	0.14	15	9.37	11.81	0.352	0.60	14	11.38	35.80	0.362	3.43
0.5	13	10.43	5.04	0.391	0.10	13	10.92	9.97	0.392	0.44	13	12.38	24.67	0.401	2.66
1	12	11.51	4.45	0.435	0.09	12	12.01	8.98	0.436	0.35	11	14.16	28.49	0.443	2.10
2	10	13.71	3.79	0.509	0.06	10	14.20	7.49	0.510	0.24	10	15.67	18.62	0.516	1.49
5	8	17.12	2.95	0.684	0.03	7	18.53	11.43	0.685	0.13	7	20.01	20.32	0.690	0.85

to the case that $T^* > \tilde{T}^*$ (respectively $T^* < \tilde{T}^*$); and a larger (respectively smaller) absolute value of Ψ implies a larger (respectively smaller) difference between T^* and \tilde{T}^* . The same argument applies to Ω as well. Figs. 1 and 2 further illustrate the values of Ψ for $\beta = 1.5$ and $\beta = 3.5$, respectively.

From these tables and figures, the following observations can be drawn.

- 1) In the unpunctual PM policy, as the PM cost c_p increases, the optimal number of PM actions N^* decreases, whereas both the optimal PM interval T^* and the average cost

rate $C_A(N^*, T^*)$ increase. This finding, similar to that for the punctual policy, applies to all numerical cases in Tables II and III, regardless of the unpunctuality scenarios and magnitudes.

- 2) The unpunctual PM policy requires the same number of PM actions as the punctual policy for $\beta = 1.5$ (see Table II); however, it needs slightly fewer number of PM actions for $\beta = 3.5$, especially when the magnitude of maintenance unpunctuality $u - l$ is large (see Table III). Moreover, Table III shows that the difference between N^*

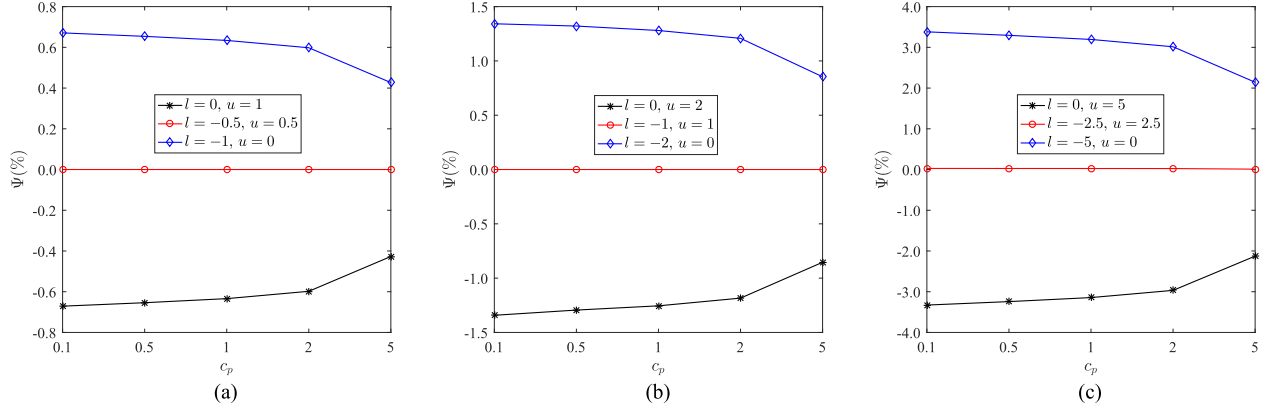


Fig. 1. Relative PM-interval change Ψ in the infinite horizon setting ($\beta = 1.5$). (a) $u - l = 1$. (b) $u - l = 2$. (c) $u - l = 5$.

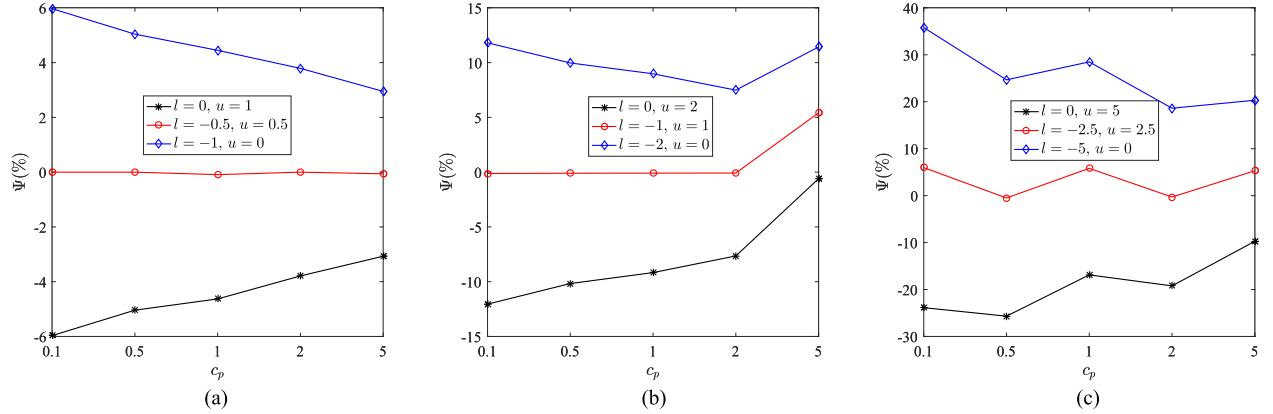


Fig. 2. Relative PM-interval change Ψ in the infinite horizon setting ($\beta = 3.5$). (a) $u - l = 1$. (b) $u - l = 2$. (c) $u - l = 5$.

and \tilde{N}^* , in the case of $\beta = 3.5$, depends more on the unpunctuality magnitude, rather than the unpunctuality scenario (advanced, delayed, or hybrid). This appears to be inconsistent with Theorems 1 and 2. Indeed, it is not the case because the optimal PM intervals T^* and \tilde{T}^* are not identical.

- 3) It is interesting to observe that the optimal PM interval of the unpunctual PM policy can be shorter or longer than that of the punctual policy, depending on the unpunctuality scenario and magnitude (see Figs. 1 and 2). More specifically, a) when PM actions are never performed earlier than intended ($0 \leq l < u$), $T^* < \tilde{T}^*$; b) when PM actions are never performed later than intended ($l < u \leq 0$), $T^* > \tilde{T}^*$; while c) when the unpunctuality scenario is hybrid ($l < 0 < u$), the values of T^* and \tilde{T}^* are close, especially when $u - l$ is not large. Observations a) and b) well support the results in Theorems 1 and 2, although we cannot analytically prove the result for concave hazard rate. Furthermore, the difference between T^* and \tilde{T}^* (measured by the absolute value of Ψ) tends to become larger when $u - l$ increases and/or c_p decreases.
- 4) In terms of average cost rates, we find that $C_A(N^*, T^*)$ and $\tilde{C}_A(\tilde{N}^*, \tilde{T}^*)$ are identical for $\beta = 1.5$; while $C_A(N^*, T^*)$

is slightly larger than $\tilde{C}_A(\tilde{N}^*, \tilde{T}^*)$ in the case of $\beta = 3.5$. One thing noteworthy is that when $\beta = 3.5$, the average cost rate $C_A(N^*, T^*)$ remains constant for the same value of $u - l$, irrespective of the unpunctuality scenarios (see Table III).

III. MODEL B: FINITE PLANNING-HORIZON SETTING

In this section, we investigate an unpunctual imperfect PM policy for a finite planning horizon (termed *Model B*). We generalize the periodic imperfect PM model in [16] by assuming that the PM deviations are governed by a specific probability distribution. Below, we first formulate the total expected maintenance costs for the punctual and unpunctual execution scenarios, respectively, and then discuss the optimization of the two scenarios. Numerical experiments are also performed to illustrate the proposed PM models.

A. The Optimization Models

For the imperfect PM policy in a finite horizon, the assumptions 1) and 3)–6) in Section II-A still hold; while the assumption 2) is revised as follows: The planning horizon is $[0, S]$, and the j th PM action is carried out at instant $jT + Y_j$, $j = 1, 2, \dots, N$,

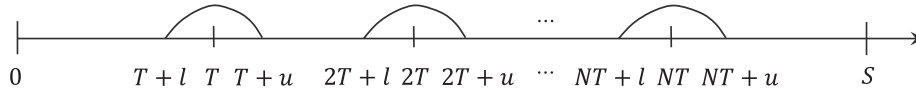


Fig. 3. Illustration of the PM schedule within a finite planning horizon ($l \leq 0 \leq u$).

with $S = (N + 1)T$; see Fig. 3. It should be noted that there is no imperfect PM or replacement at the end of the planning horizon. This setting is appropriate for a relatively short planning horizon (e.g., warranty period and lease period) in the sense that the maintainer is responsible for maintenance execution only in $[0, S]$ and the system will continue to operate after S .

Remark 4: Due to the infinite/finite nature of the planning horizons, the instants of performing the j th PM action are indeed different in the two settings. Specifically, the j th PM instant in the infinite horizon setting is $jT + \sum_{i=1}^j Y_i$, whereas that in the finite horizon setting is $jT + Y_j$. In the infinite horizon setting, the PM deviation Y_j is applied to the j th PM interval; thus, it will accumulate for subsequent PM instants. However, Y_j is applied to the j th PM instant directly in the finite horizon setting; so it has no impact on subsequent PM instants. This is because in the finite horizon setting, the planning horizon ends at S so that the N th PM instant shall be smaller than S ; whereas the infinite horizon setting does not need this constraint.

Based on the aforementioned assumptions, if all PM actions are always executed on time, then the total expected maintenance cost in $[0, S]$ is as follows:

$$\tilde{C}_B(N) = c_m \sum_{j=1}^{N+1} A_{j-1} \Lambda(T) + Nc_p. \quad (12)$$

The objective is to determine the optimal number of PM actions \tilde{N}^* such that $\tilde{C}_B(N)$ is minimized. Once the optimal number of PM actions \tilde{N}^* is identified, the optimal PM interval can be determined as $\tilde{T}^* = S/(\tilde{N}^* + 1)$.

On the other hand, in the unpunctual execution scenario (see Fig. 3), the j th PM instant is $jT + Y_j$, and the j th PM interval is thus $jT + Y_j - ((j-1)T + Y_{j-1}) = T + Y_j - Y_{j-1}$, $j = 2, 3, \dots, N$. Note that the first and the last PM intervals are $T + Y_1$ and $T - Y_N$, respectively. Then, the total expected maintenance cost in $[0, S]$ can be derived as follows:

$$\begin{aligned} C_B(N) = & c_m \int_l^u \Lambda(T + y_1) dG(y_1) + c_m \sum_{j=2}^N \int_l^u \int_l^u A_{j-1} \\ & \times \Lambda(T + y_j - y_{j-1}) dG(y_{j-1}) dG(y_j) \\ & + c_m \int_l^u A_N \Lambda(T - y_N) dG(y_N) + Nc_p. \end{aligned} \quad (13)$$

Likewise, the optimization problem is to seek the optimal N^* to minimize $C_B(N)$, when the maintenance unpunctuality is anticipated.

One thing noteworthy is that in order to avoid impractical PM plans, each PM interval should be larger than zero, i.e., the j th PM instant should be larger than the $(j-1)$ th instant

(see Fig. 3). Mathematically, the condition is equivalent to $T + Y_j - Y_{j-1} > 0$, $j = 2, 3, \dots, N$. Considering the range of Y_j and Y_{j-1} , i.e., $[l, u]$, we have $T > u - l$. This means that the PM interval T should be larger than the unpunctual range $u - l$. As a result, the optimal number of PM action N^* has an upper bound: $N^* \leq N_{\max} = \lfloor S/(u - l) - 1 \rfloor$, where $\lfloor x \rfloor$ represents the largest integer that is smaller than x .

Remark 5: When $N = 0$, i.e., no imperfect PM action is carried out within $[0, S]$, the total expected maintenance costs in (12) and (13) reduce to $\tilde{C}_B(N = 0) = C_B(N = 0) = c_m \Lambda(S)$.

B. Optimal PM Decisions

In this section, we examine the optimal finite-horizon PM decisions for the punctual and unpunctual scenarios, respectively.

We first look at the optimal number of PM actions N^* for the unpunctual execution scenario. By forming inequalities $C_B(N-1) > C_B(N)$ and $C_B(N) \leq C_B(N+1)$, we have

$$\mathcal{J}_B(N+1) \leq \frac{c_p}{c_m} \text{ and } \mathcal{J}_B(N) > \frac{c_p}{c_m} \quad (14)$$

where

$$\begin{aligned} \mathcal{J}_B(N) = & \int_l^u \Lambda\left(\frac{S}{N} + y_1\right) dG(y_1) + \sum_{j=2}^{N-1} \int_l^u \int_l^u A_{j-1} \\ & \times \Lambda\left(\frac{S}{N} + y_j - y_{j-1}\right) dG(y_{j-1}) dG(y_j) \\ & + \int_l^u A_{N-1} \Lambda\left(\frac{S}{N} - y_N\right) dG(y_N) \\ & - \int_l^u \Lambda\left(\frac{S}{N+1} + y_1\right) dG(y_1) - \sum_{j=2}^N \int_l^u \int_l^u \\ & \times A_{j-1} \Lambda\left(\frac{S}{N+1} + y_j - y_{j-1}\right) dG(y_{j-1}) dG(y_j) \\ & - \int_l^u A_N \Lambda\left(\frac{S}{N+1} - y_{N+1}\right) dG(y_{N+1}). \end{aligned}$$

Likewise, for the finite-horizon PM policy with punctual executions, the optimal \tilde{N}^* can be determined by

$$\tilde{\mathcal{J}}_B(N+1) \leq \frac{c_p}{c_m} \text{ and } \tilde{\mathcal{J}}_B(N) > \frac{c_p}{c_m} \quad (15)$$

where

$$\tilde{\mathcal{J}}_B(N) = \sum_{j=1}^N A_{j-1} \Lambda\left(\frac{S}{N}\right) - \sum_{j=1}^{N+1} A_{j-1} \Lambda\left(\frac{S}{N+1}\right).$$

Note that it is difficult, if not impossible, to show the uniqueness of optimal N^* and \tilde{N}^* for the unpunctual and punctual

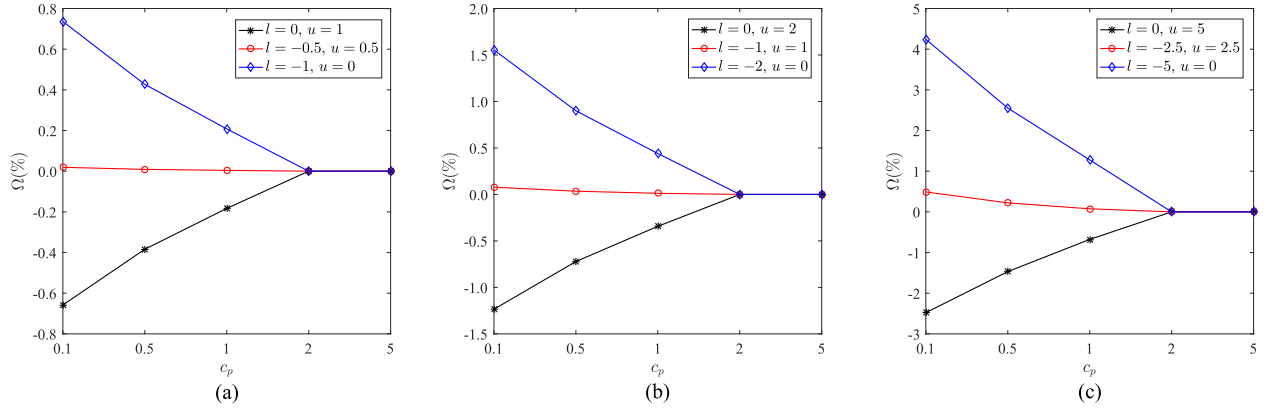


Fig. 4. Relative maintenance-cost change Ω in the finite horizon setting ($\beta = 1.5$). (a) $u - l = 1$. (b) $u - l = 2$. (c) $u - l = 5$.

TABLE IV
OPTIMAL PM DECISIONS UNDER PUNCTUAL EXECUTIONS IN THE FINITE HORIZON SETTING

c_p	$\beta = 1.5$		$\beta = 3.5$	
	\tilde{N}^*	\tilde{C}_B	\tilde{N}^*	\tilde{C}_B
0.1	3	5.772	10	2.106
0.5	2	6.645	6	4.959
1	1	7.223	4	7.527
2	0	8.000	3	11.472
5	0	8.000	2	20.036

execution scenarios. Fortunately, both N^* and \tilde{N}^* are integers; so simple search methods are efficient enough for obtaining them. Therefore, the impact of unpunctual policy executions on the optimal PM decisions in the finite horizon setting will be investigated via numerical experiments.

C. Numerical Studies

In this section, numerical experiments are conducted to illustrate the unpunctual PM policy for a finite planning horizon and to examine the impact of unpunctual executions on the optimal PM decisions. The parameter setting is the same as that in Section II-D. That is, we adopt the same setting with respect to hazard rate function and unpunctuality scenarios and magnitudes. In addition, the length of the finite planning horizon is set to $S = 60$.

We first examine the imperfect PM policy in the punctual execution scenario. Table IV lists the optimal number of PM actions and corresponding total maintenance cost for different values of c_p . It can be observed that as the PM cost c_p increases, the optimal number of PM actions \tilde{N}^* decreases, whereas the total maintenance cost $\tilde{C}_B(\tilde{N}^*)$ increases. This finding applies to both concave and convex hazard rates.

We then look at the imperfect PM policy in the unpunctual execution scenario. Tables V and VI summarize the optimal PM decisions and corresponding total maintenance costs under unpunctual executions for $\beta = 1.5$ and $\beta = 3.5$, respectively. Figs. 4 and 5 further show the values of relative maintenance

TABLE V
OPTIMAL PM DECISIONS UNDER UNPUNCTUAL EXECUTIONS IN THE FINITE HORIZON SETTING ($\beta = 1.5$)

c_p	$l = 0, u = 1$			$l = 0, u = 2$			$l = 0, u = 5$		
	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$
0.1	3	5.734	-0.66	3	5.701	-1.24	3	5.629	-2.48
0.5	2	6.620	-0.38	2	6.597	-0.72	2	6.548	-1.47
1	1	7.209	-0.18	1	7.198	-0.34	1	7.174	-0.68
2	0	8.000	0.00	0	8.000	0.00	0	8.000	0.00
5	0	8.000	0.00	0	8.000	0.00	0	8.000	0.00

c_p	$l = -0.5, u = 0.5$			$l = -1, u = 1$			$l = -2.5, u = 2.5$		
	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$
0.1	3	5.773	0.02	3	5.776	0.08	3	5.800	0.49
0.5	2	6.646	0.01	2	6.648	0.03	2	6.660	0.22
1	1	7.223	0.00	1	7.223	0.01	1	7.228	0.07
2	0	8.000	0.00	0	8.000	0.00	0	8.000	0.00
5	0	8.000	0.00	0	8.000	0.00	0	8.000	0.00

c_p	$l = -1, u = 0$			$l = -2, u = 0$			$l = -5, u = 0$		
	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$
0.1	3	5.814	0.74	3	5.862	1.55	2	6.017	4.24
0.5	2	6.674	0.43	2	6.705	0.90	1	6.815	2.55
1	1	7.238	0.21	1	7.254	0.44	1	7.315	1.28
2	0	8.000	0.00	0	8.000	0.00	0	8.000	0.00
5	0	8.000	0.00	0	8.000	0.00	0	8.000	0.00

cost changes Ω for $\beta = 1.5$ and $\beta = 3.5$, respectively. From these tables and figures, the following findings can be drawn.

- 1) Analogous to the observation in the punctual execution scenario, as the PM cost c_p increases, the optimal number of PM actions N^* decreases, whereas the total expected maintenance cost $C_B(N^*)$ increases. This finding applies to all numerical cases in Tables V and VI, irrespective of the unpunctuality scenarios and magnitudes.
- 2) Compared with the punctual execution scenario, the unpunctual PM policy requires the same, or slightly fewer (especially when the unpunctuality magnitude $u - l$ is large), number of PM actions for $\beta = 1.5$ and $\beta = 3.5$. It shows that the difference between N^* and \tilde{N}^* depends more on the unpunctuality magnitude, rather than the unpunctuality scenario.
- 3) Different from the infinite horizon setting, it is interesting to find that the total maintenance cost of the unpunctual

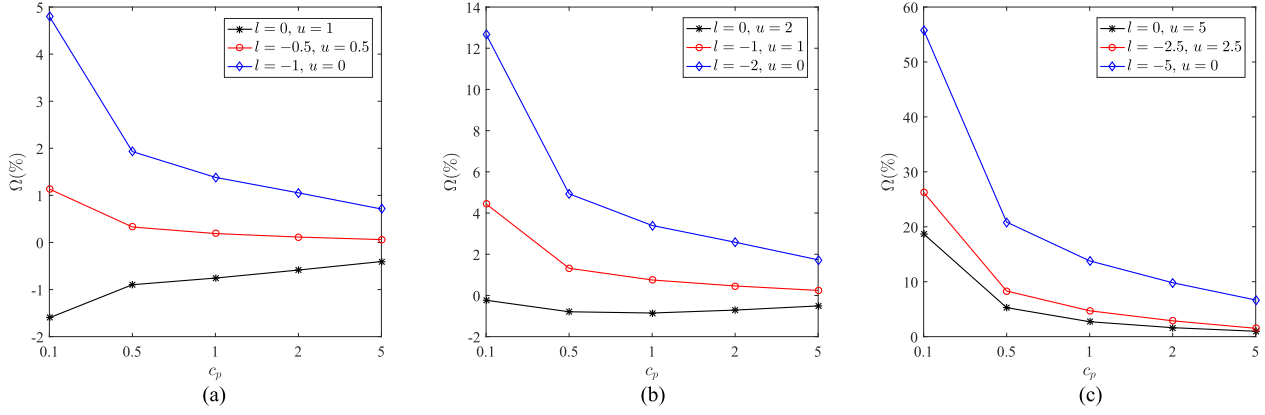


Fig. 5. Relative maintenance-cost change Ω in the finite horizon setting ($\beta = 3.5$). (a) $u - l = 1$. (b) $u - l = 2$. (c) $u - l = 5$.

TABLE VI
OPTIMAL PM DECISIONS UNDER UNPUNCTUAL EXECUTIONS IN THE FINITE HORIZON SETTING ($\beta = 3.5$)

c_p	$l = 0, u = 1$			$l = 0, u = 2$			$l = 0, u = 5$		
	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$
0.1	10	2.072	-1.60	9	2.101	-0.23	9	2.500	18.70
0.5	6	4.914	-0.90	6	4.919	-0.79	6	5.220	5.28
1	4	7.470	-0.76	4	7.463	-0.85	4	7.731	2.72
2	3	11.405	-0.59	3	11.390	-0.71	3	11.659	1.63
5	2	19.955	-0.41	2	19.934	-0.51	2	20.233	0.98

c_p	$l = -0.5, u = 0.5$			$l = -1, u = 1$			$l = -2.5, u = 2.5$		
	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$
0.1	10	2.130	1.13	9	2.200	4.44	9	2.659	26.26
0.5	6	4.975	0.33	6	5.024	1.32	6	5.370	8.30
1	4	7.541	0.19	4	7.583	0.75	4	7.881	4.71
2	3	11.485	0.12	3	11.525	0.46	3	11.802	2.88
5	2	20.048	0.06	2	20.085	0.24	2	20.340	1.52

c_p	$l = -1, u = 0$			$l = -2, u = 0$			$l = -5, u = 0$		
	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$	N^*	C_B	$\Omega(\%)$
0.1	10	2.207	4.80	9	2.373	12.68	8	3.282	55.83
0.5	6	5.054	1.93	6	5.203	4.93	6	5.992	20.84
1	4	7.631	1.38	4	7.783	3.40	4	8.565	13.79
2	3	11.593	1.05	3	11.768	2.58	3	12.597	9.81
5	2	20.178	0.71	2	20.383	1.73	2	21.372	6.67

PM policy can be either higher or lower than that of the punctual policy, depending on the unpunctuality scenario and magnitude (see Figs. 4 and 5). More specifically, a) when $0 \leq l < u$, $C_B(N^*) < \tilde{C}_B(\tilde{N}^*)$ in most cases ($C_B(N^*) > \tilde{C}_B(\tilde{N}^*)$ only when $\beta = 3.5$ and $u - l = 5$); b) when $l < 0$, $C_B(N^*) > \tilde{C}_B(\tilde{N}^*)$; while c) when $l < 0 < u$, $C_B(N^*)$ is slightly higher than $\tilde{C}_B(\tilde{N}^*)$, especially when $u - l$ is not large. Moreover, the difference between $C_B(N^*)$ and $\tilde{C}_B(\tilde{N}^*)$ (measured by the absolute value of Ω) tends to become larger when $u - l$ increases and/or c_p decreases.

IV. CONCLUSION

In this article, we studied two imperfect PM policies with unpunctual executions for infinite and finite planning horizons, respectively. The two PM models extend their punctual counterparts by taking into account the unpunctual executions

of scheduled PM activities. From the mathematical modeling perspective, the maintenance unpunctuality is assumed to be governed by a specific probability distribution. We discussed the modeling and optimization of the two unpunctual PM policies and then explored the impact of unpunctual executions on the optimal PM decisions and corresponding maintenance expenses in an analytical or numerical way.

Numerical experiments show that the impacts of unpunctual executions on the optimal PM decisions and maintenance costs (or cost rates), in the infinite and finite horizon settings, are indeed different. In the infinite horizon setting, the optimal PM interval in the unpunctual execution scenario can be longer or shorter than that in the punctual scenario, depending on the unpunctuality scenario and magnitude; however, the resultant cost rate is always higher than or equal to that in the punctual scenario. On the other hand, in the finite horizon setting, the total maintenance cost in the unpunctual scenario can be higher or lower than that in the punctual scenario, depending on the unpunctuality scenario and magnitude as well. These insights are helpful for maintenance planners to adjust their PM schedules according to the planning horizons of interest, when the maintenance unpunctuality is well-anticipated.

It is worth emphasizing that we have confined our analysis of the unpunctual PM policies to numerical experiments, as the complexity of the cost (rate) models makes it quite difficult to obtain analytical results. As a consequence, we were cautious about making *general* claims about the impacts of unpunctual executions. We recognized this as a limitation of the article. Deriving general analytical results on the comparison of punctual and unpunctual scenarios is thus valuable and needs in-depth investigations. Furthermore, a consideration of practical significance is that the unpunctual ranges might become wider or narrower when time elapsed, rather than remaining unchanged. A potential way of addressing this issue is to model the PM deviation as a stochastic process with an increasing or decreasing mean in N and T . In addition, future research could also extend the current framework to other imperfect PM models, e.g., age reduction model, hybrid model, and geometric process model, or more advanced maintenance paradigms, e.g., condition-based maintenance and predictive maintenance.

APPENDIX

A. Proof of Proposition 1

It is evident that $\mathcal{J}_A(T, N)$ is increasing in N and approaches to ∞ as $N \rightarrow \infty$ because

$$\begin{aligned} \mathcal{J}_A(T, N) - \mathcal{J}_A(T, N-1) \\ = N(A_N - A_{N-1}) \int_l^u \Lambda(T + y_N) dG(y_N) > 0, \end{aligned}$$

and

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathcal{J}_A(T, N) \\ = \lim_{N \rightarrow \infty} \left(\int_l^u \Lambda(T + y) dG(y) \right) \sum_{j=1}^N (A_N - A_{j-1}) = \infty. \end{aligned}$$

The last equality is due to $a_j > 1$, $j = 1, 2, \dots, N-1$, and thus $\lim_{N \rightarrow \infty} \sum_{j=1}^N (A_N - A_{j-1}) = \infty$. Therefore, there exists a finite and unique N^* that minimizes $C_A(T, N)$ for any feasible $T > \max\{-l, 0\}$.

B. Proof of Proposition 2

As the hazard rate $\lambda(t)$ is an increasing function of t , i.e., $\lambda'(t) > 0$, we have

$$\mathcal{L}'_{A,i}(T) = (T + \mu_y) \int_l^u A_{j-1} \lambda'(T + y_j) dG(y_j) > 0.$$

Following the same argument as in [40, p. 97] and [38], we show that $\mathcal{L}_{A,i}(T) \rightarrow \infty$ as $T \rightarrow \infty$. First, it is clear that $\mathcal{L}_{A,i}(T)$ is continuous on $[0, +\infty)$. Let $\{T_i\}$ be an infinite sequence such that $\max\{-\mu_y, 0\} < T_1 < T_2 < \dots$, with $\lambda(T_{i-1}) < \lambda(T_i)$ for $i = 2, 3, \dots$ and $\lim_{i \rightarrow \infty} \lambda(T_i) = \infty$. Such a sequence exists as $\lambda(t)$ is strictly increasing and unbounded [by Assumption 1)]. Then, it is evident that

$$\begin{aligned} \mathcal{L}_{A,i}(T_i) &= (T_i + \mu_y) \int_l^u A_{j-1} \lambda(T_i + y_j) dG(y_j) \\ &\quad - \int_l^u A_{j-1} \Lambda(T_i + y_j) dG(y_j) \\ &> (T_1 + \mu_y) \int_l^u A_{j-1} \lambda(T_i + y_j) dG(y_j) \\ &\quad - \int_l^u A_{j-1} \Lambda(T_1 + y_j) dG(y_j) \\ &\rightarrow \infty \text{ as } i \rightarrow \infty. \end{aligned}$$

In other words, $\mathcal{L}_{A,i}(T)$ is increasing in T and tends to ∞ as $T \rightarrow \infty$.

Therefore, the solution to (4) is unique and finite when condition (5) is met; otherwise, $\inf C_A(T, N) = \lim_{T \rightarrow \max\{-l, 0\}} C_A(T, N)$. Given the optimal solution T^* to (4), rearranging its terms yields (6).

C. Proof of Theorem 1

Part (i): Recall that both $\mathcal{L}_{A,i}(T)$ and $\tilde{\mathcal{L}}_{A,i}(T)$ are increasing functions of T . Also, $0 \leq l < u$ implies that $\mu_y > 0$ and both

problems have the same feasible domain $\{T \mid T > 0\}$. According to (4) and (9), if $\mathcal{L}_{A,i}(T) > \tilde{\mathcal{L}}_{A,i}(T)$ is true for all $T > 0$, then $T^* < \tilde{T}^*$ for a fixed N .

First, it is clear that $\lim_{T \rightarrow 0} \tilde{\mathcal{L}}_{A,i}(T) = 0$ as $\lambda(0) = 0$. Hence, if condition (11) is satisfied, then we have

$$\lim_{T \rightarrow 0} \mathcal{L}_{A,i}(T) \geq \lim_{T \rightarrow 0} \tilde{\mathcal{L}}_{A,i}(T).$$

Next, $\mathcal{L}_{A,i}(T)$ is increasing faster than $\tilde{\mathcal{L}}_{A,i}(T)$, since

$$\begin{aligned} \mathcal{L}'_{A,i}(T) &= (T + \mu_y) \int_l^u A_{j-1} \lambda'(T + y_j) dG(y_j) \\ &> T \int_l^u A_{j-1} \lambda'(T) dG(y_j) \\ &= T A_{j-1} \lambda'(T) = \tilde{\mathcal{L}}'_{A,i}(T). \end{aligned}$$

The inequality is due to $0 \leq l < u$ ($\mu_y > 0$) and the convexity of $\lambda(t)$. Hence, it is true that $\mathcal{L}_{A,i}(T) > \tilde{\mathcal{L}}_{A,i}(T)$ for all $T > 0$. If the two policies adopt the same $N \geq 1$, then $\sum_{j=1}^N \mathcal{L}_{A,i}(T) > \sum_{j=1}^N \tilde{\mathcal{L}}_{A,i}(T)$ for all $T > 0$, which implies $T^* < \tilde{T}^*$.

Part (ii): Analogously, note that both $\mathcal{J}_A(T, N)$ and $\tilde{\mathcal{J}}_A(T, N)$ are increasing functions of N . If we can show that $\mathcal{J}_A(T, N) > \tilde{\mathcal{J}}_A(T, N)$ for all $N \geq 1$, then $N^* \leq \tilde{N}^*$ follows directly, according to (3) and (8). The inequality is not strict because N should be integer; thus, there is a possibility that $N^* = \tilde{N}^*$.

Based on Jensen's inequality, we know that

$$\begin{aligned} \mathcal{J}_A(T, N) - \mathcal{J}_A(T, N-1) \\ = N(A_N - A_{N-1}) \int_l^u \Lambda(T + y_N) dG(y_N) \\ \geq N(A_N - A_{N-1}) \Lambda(T + \mu_y) \\ > N(A_N - A_{N-1}) \Lambda(T) \\ = \tilde{\mathcal{J}}_A(T, N) - \tilde{\mathcal{J}}_A(T, N-1). \end{aligned}$$

Thus, $\mathcal{J}_A(T, N)$ indeed increases faster than $\tilde{\mathcal{J}}_A(T, N)$. Combining with the fact $\mathcal{J}_A(T, 0) = \tilde{\mathcal{J}}_A(T, 0) = 0$, we conclude that $\mathcal{J}_A(T, N) > \tilde{\mathcal{J}}_A(T, N)$ for all $N \geq 1$. Therefore, if the two policies specify the same T , we have $N^* \leq \tilde{N}^*$.

D. Proof of Theorem 2

Part (i): In this case, $l < u \leq 0$ implies that $\mu_y < 0$ and the feasible domain of T in the unpunctual scenario is $\{T \mid T > -l\}$. Likewise, we first show $\lim_{T \rightarrow -l} \mathcal{L}_{A,i}(T) < \tilde{\mathcal{L}}_{A,i}(-l)$ and then show $\mathcal{L}_{A,i}(T)$ is increasing slower than $\tilde{\mathcal{L}}_{A,i}(T)$.

The expressions of $\lim_{T \rightarrow -l} \mathcal{L}_{A,i}(T)$ and $\tilde{\mathcal{L}}_{A,i}(-l)$ are given by

$$\begin{aligned} \lim_{T \rightarrow -l} \mathcal{L}_{A,i}(T) &= (-l + \mu_y) \int_l^u A_{j-1} \lambda(-l + y_j) dG(y_j) \\ &\quad - \int_l^u A_{j-1} \Lambda(-l + y_j) dG(y_j), \end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{L}}_{\mathcal{A},i}(-l) &= -lA_{j-1}\lambda(-l) - A_{j-1}\Lambda(-l) \\ &= (-l + \mu_y)A_{j-1}\lambda(-l) - A_{j-1}[\Lambda(-l) + \mu_y\lambda(-l)].\end{aligned}$$

Note that $\int_l^u \lambda(-l + y_j)dG(y_j) < \lambda(-l)$ for $l < u \leq 0$. According to the Jensen's inequality (and bear in mind that $\Lambda(t)$ is convex), we obtain

$$\begin{aligned}\int_l^u \Lambda(-l + y_j)dG(y_j) &\geq \Lambda(-l + \mu_y) \\ &= \Lambda(-l) - \int_{-l+\mu_y}^{-l} \lambda(t)dt.\end{aligned}$$

As $\lambda(t)$ is convex, we also have

$$-\int_{-l+\mu_y}^{-l} \lambda(t)dt \geq -\lambda(-l)(-\mu_y) = \lambda(-l)\mu_y.$$

The discussions above lead to the result: $\lim_{T \rightarrow -l} \mathcal{L}_{\mathcal{A},i}(T) < \tilde{\mathcal{L}}_{\mathcal{A},i}(-l)$.

We claim that $\mathcal{L}_{\mathcal{A},i}(T)$ is increasing slower than $\tilde{\mathcal{L}}_{\mathcal{A},i}(T)$ for $l < u \leq 0$, as follows:

$$\begin{aligned}\mathcal{L}'_{\mathcal{A},i}(T) &= (T + \mu_y) \int_l^u A_{j-1}\lambda'(T + y_j)dG(y_j) \\ &< T \int_l^u A_{j-1}\lambda'(T)dG(y_j) \\ &= TA_{j-1}\lambda'(T) = \tilde{\mathcal{L}}'_{\mathcal{A},i}(T).\end{aligned}$$

The inequality is due to $l < u \leq 0$ and $\mu_y < 0$. Therefore, if the two policies adopt the same $N \geq 1$, then $\sum_{j=1}^N \mathcal{L}_{\mathcal{A},i}(T) < \sum_{j=1}^N \tilde{\mathcal{L}}_{\mathcal{A},i}(T)$ for all $T > -l$.

Moreover, as the feasible domain of PM interval T in the unpunctual scenario is $T > -l$, there are following two possible cases.

- 1) If $\sum_{j=1}^N \tilde{\mathcal{L}}_{\mathcal{A},i}(-l) \geq \frac{(N-1)c_p + c_r}{c_m}$, then $\tilde{T}^* < -l$, which means that $\tilde{T}^* < -l < T^*$.
- 2) if $\sum_{j=1}^N \tilde{\mathcal{L}}_{\mathcal{A},i}(-l) < \frac{(N-1)c_p + c_r}{c_m}$, then both T^* and \tilde{T}^* are greater than $-l$. Because $\sum_{j=1}^N \mathcal{L}_{\mathcal{A},i}(T) < \sum_{j=1}^N \tilde{\mathcal{L}}_{\mathcal{A},i}(T)$ for all $T > -l$, it is clear that $-l < \tilde{T}^* < T^*$.

Part (ii): In order to prove $N^* \geq \tilde{N}^*$, what we need to show is that $\mathcal{J}_{\mathcal{A}}(T, N) < \tilde{\mathcal{J}}_{\mathcal{A}}(T, N)$ for all $N \geq 1$. First of all, note that

$$\begin{aligned}\mathcal{J}_{\mathcal{A}}(T, N) - \mathcal{J}_{\mathcal{A}}(T, N-1) &= N(A_N - A_{N-1}) \int_l^u \Lambda(T + y_N)dG(y_N) \\ &< N(A_N - A_{N-1})\Lambda(T) \\ &= \tilde{\mathcal{J}}_{\mathcal{A}}(T, N) - \tilde{\mathcal{J}}_{\mathcal{A}}(T, N-1).\end{aligned}$$

The inequality is due to the negativity of l and u . Hence, $\mathcal{J}_{\mathcal{A}}(T, N)$ increases slower than $\tilde{\mathcal{J}}_{\mathcal{A}}(T, N)$. Coupled with the fact $\mathcal{J}_{\mathcal{A}}(T, 0) = \tilde{\mathcal{J}}_{\mathcal{A}}(T, 0) = 0$, we conclude that $\mathcal{J}_{\mathcal{A}}(T, N) < \tilde{\mathcal{J}}_{\mathcal{A}}(T, N)$ for all $N \geq 1$. Therefore, if the two policies specify the same $T > -l$, we have $N^* \geq \tilde{N}^*$.

REFERENCES

- [1] H. Wang, "A survey of maintenance policies of deteriorating systems," *Eur. J. Oper. Res.*, vol. 139, no. 3, pp. 469–489, 2002.
- [2] S. Wu and M. J. Zuo, "Linear and nonlinear preventive maintenance models," *IEEE Trans. Rel.*, vol. 59, no. 1, pp. 242–249, Mar. 2010.
- [3] R. Barlow and L. Hunter, "Optimum preventive maintenance policies," *Operations Res.*, vol. 8, no. 1, pp. 90–100, 1960.
- [4] M. Finkelstein, M. Shafiee, and A. N. Kotchap, "Classical optimal replacement strategies revisited," *IEEE Trans. Rel.*, vol. 65, no. 2, pp. 540–546, Jun. 2016.
- [5] Z. Liang and A. K. Parlikad, "Predictive group maintenance for multi-system multi-component networks," *Rel. Eng. Syst. Saf.*, vol. 195, 2020, Art. no. 106704.
- [6] J. Shen, L. Cui, and Y. Ma, "Availability and optimal maintenance policy for systems degrading in dynamic environments," *Eur. J. Oper. Res.*, vol. 276, no. 1, pp. 133–143, 2019.
- [7] Q. Sun, Z.-S. Ye, and N. Chen, "Optimal inspection and replacement policies for multi-unit systems subject to degradation," *IEEE Trans. Rel.*, vol. 67, no. 1, pp. 401–413, Mar. 2018.
- [8] H. C. Vu, P. Do, M. Fouladirad, and A. Grall, "Dynamic opportunistic maintenance planning for multi-component redundant systems with various types of opportunities," *Rel. Eng. Syst. Saf.*, vol. 198, 2020, Art. no. 106854.
- [9] X. Zhu, X. Bei, N. Chatwattanasiri, and D. W. Coit, "Optimal system design and sequential preventive maintenance under uncertain aperiodic-changing stresses," *IEEE Trans. Rel.*, vol. 67, no. 3, pp. 907–919, Sep. 2018.
- [10] T. Nakagawa, "Periodic and sequential preventive maintenance policies," *J. Appl. Probability*, vol. 23, no. 2, pp. 536–542, 1986.
- [11] H. Pham and H. Wang, "Imperfect maintenance," *Eur. J. Oper. Res.*, vol. 94, no. 3, pp. 425–438, 1996.
- [12] L. Doyen and O. Gaudoin, "Classes of imperfect repair models based on reduction of failure intensity or virtual age," *Rel. Eng. Syst. Saf.*, vol. 84, no. 1, pp. 45–56, 2004.
- [13] G. J. Wang and R. C. Yam, "Generalized geometric process and its application in maintenance problems," *Appl. Math. Model.*, vol. 49, pp. 554–567, 2017.
- [14] G.-J. Wang, Y.-L. Zhang, and R. C. M. Yam, "Preventive maintenance models based on the generalized geometric process," *IEEE Trans. Rel.*, vol. 66, no. 4, pp. 1380–1388, Dec. 2017.
- [15] T. Nakagawa, "Sequential imperfect preventive maintenance policies," *IEEE Trans. Rel.*, vol. 37, no. 3, pp. 295–298, Aug. 1988.
- [16] T. Nakagawa and S. Mizutani, "A summary of maintenance policies for a finite interval," *Rel. Eng. Syst. Saf.*, vol. 94, no. 1, pp. 89–96, 2009.
- [17] M. Kijima, "Some results for repairable systems with general repair," *J. Appl. Probability*, vol. 26, no. 1, pp. 89–102, 1989.
- [18] D. Lin, M. J. Zuo, and R. C. M. Yam, "Sequential imperfect preventive maintenance models with two categories of failure modes," *Naval Res. Logistics*, vol. 48, no. 2, pp. 172–183, 2001.
- [19] S. Wu and D. Clements-Croome, "Preventive maintenance models with random maintenance quality," *Rel. Eng. Syst. Saf.*, vol. 90, no. 1, pp. 99–105, 2005.
- [20] R. I. Zequeira and C. Bérenguer, "Periodic imperfect preventive maintenance with two categories of competing failure modes," *Rel. Eng. Syst. Saf.*, vol. 91, no. 4, pp. 460–468, 2006.
- [21] S.-H. Sheu and C.-C. Chang, "An extended periodic imperfect preventive maintenance model with age-dependent failure type," *IEEE Trans. Rel.*, vol. 58, no. 2, pp. 397–405, Jun. 2009.
- [22] J. H. Cha and M. Finkelstein, "Optimal long-run imperfect maintenance with asymptotic virtual age," *IEEE Trans. Rel.*, vol. 65, no. 1, pp. 187–196, Mar. 2016.
- [23] Q. Sun, Z.-S. Ye, and W. Peng, "Scheduling preventive maintenance considering the saturation effect," *IEEE Trans. Rel.*, vol. 68, no. 2, pp. 741–752, Jun. 2019.
- [24] X. Zhao, O. Gaudoin, L. Doyen, and M. Xie, "Optimal inspection and replacement policy based on experimental degradation data with covariates," *IIE Trans.*, vol. 51, no. 3, pp. 322–336, 2019.
- [25] D. N. P. Murthy and N. Jack, *Extended Warranties, Maintenance Service and Lease Contracts: Modeling and Analysis for Decision-Making*. London, UK: Springer, 2014.

- [26] Y.-H. Chien, "The optimal preventive-maintenance policy for a NHPBP repairable system under free-repair warranty," *Rel. Eng. Syst. Saf.*, vol. 188, pp. 444–453, 2019.
- [27] L. Shang, S. Si, S. Sun, and T. Jin, "Optimal warranty design and post-warranty maintenance for products subject to stochastic degradation," *IIEE Trans.*, vol. 50, no. 10, pp. 913–927, 2018.
- [28] C. Su and X. Wang, "A two-stage preventive maintenance optimization model incorporating two-dimensional extended warranty," *Rel. Eng. Syst. Saf.*, vol. 155, pp. 169–178, 2016.
- [29] X. Wang, L. Li, and M. Xie, "Optimal preventive maintenance strategy for leased equipment under successive usage-based contracts," *Int. J. Prod. Res.*, vol. 57, no. 18, pp. 5705–5724, 2019.
- [30] Q. Qiu, L. Cui, and L. Yang, "Maintenance policies for energy systems subject to complex failure processes and power purchasing agreement," *Comput. Ind. Eng.*, vol. 119, pp. 193–203, 2018.
- [31] S. Petchrompo, H. Li, A. K. Parlikad, A. Erguido, and C. Riches, "A value-based approach to optimising long-term maintenance plans for a multi-asset k-out-of-n system," *Rel. Eng. Syst. Saf.*, vol. 200, 2020, Art. no. 106924.
- [32] B. de Jonge, A. S. Dijkstra, and W. Romeijnders, "Cost benefits of postponing time-based maintenance under lifetime distribution uncertainty," *Rel. Eng. Syst. Saf.*, vol. 140, pp. 15–21, 2015.
- [33] L. Yang, Z.-S. Ye, C.-G. Lee, S.-F. Yang, and R. Peng, "A two-phase preventive maintenance policy considering imperfect repair and postponed replacement," *Eur. J. Oper. Res.*, vol. 274, no. 3, pp. 966–977, 2019.
- [34] P. A. Scarf, C. A. V. Cavalcante, and R. S. Lopes, "Delay-time modelling of a critical system subject to random inspections," *Eur. J. Oper. Res.*, vol. 278, no. 3, pp. 772–782, 2019.
- [35] X. Zhao, T. Nakagawa, and M. J. Zuo, "Optimal replacement last with continuous and discrete policies," *IEEE Trans. Rel.*, vol. 63, no. 4, pp. 868–880, Dec. 2014.
- [36] X. Zhao, K. N. Al-Khalifa, A. M. Hamouda, and T. Nakagawa, "First and last triggering event approaches for replacement with minimal repairs," *IEEE Trans. Rel.*, vol. 65, no. 1, pp. 197–207, Mar. 2016.
- [37] Z. Zhu, Y. Xiang, M. Li, W. Zhu, and K. Schneider, "Preventive maintenance subject to equipment unavailability," *IEEE Trans. Rel.*, vol. 68, no. 3, pp. 1009–1020, Sep. 2019.
- [38] K. He, L. M. Maillart, and O. A. Prokopyev, "Optimal planning of unpunctual preventive maintenance," *IIEE Trans.*, vol. 49, no. 2, pp. 127–143, 2017.
- [39] X. Wang, L. Li, and M. Xie, "An unpunctual preventive maintenance policy under two-dimensional warranty," *Eur. J. Oper. Res.*, vol. 282, no. 1, pp. 304–318, 2020.
- [40] R. E. Barlow and F. Proschan, *Mathematical Theory of Reliability*. New York, USA: Wiley, 1965.

Xiaolin Wang (Student Member, IEEE) received the B.E. and M.S. degrees in industrial engineering from Southeast University, Nanjing, China, in 2013 and 2016, respectively. He is currently working toward the Ph.D. degree in industrial engineering at the Department of Systems Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong.

His research outcomes have appeared in journals such as *IIEE Transactions*, *European Journal of Operational Research*, *Reliability Engineering & System Safety*, *International Journal of Production Research*, *Computers & Industrial Engineering*, etc. His current research interests include warranty analytics, maintenance planning, degradation modeling, etc.

Mr. Wang was a recipient of the prestigious Hong Kong PhD Fellowship from the Hong Kong Research Grants Council in 2017.

Hang Zhou received the Ph.D. degree in engineering asset management from University of Leicester, Leicester, U.K., in 2018.

He is currently a Postdoctoral Researcher with the Distributed Information and Automation Laboratory (DIAL), Institute for Manufacturing, Department of Engineering, University of Cambridge, Cambridge, U.K. His research is sponsored by InnovateUK. His current research interests include asset management, risk analysis, aerospace engineering, etc.

Ajith Kumar Parlikad (Member, IEEE) received the Ph.D. degree in manufacturing management from Cambridge University, Cambridge, U.K., in 2006.

He is currently a Reader in Asset Management with the Department of Engineering, University of Cambridge. He is based at the Institute for Manufacturing, where he is the Head of the Asset Management Research Group. His research focuses on examining how data and digital technologies can be exploited to improve through-life asset performance.

Dr. Parlikad is a Member of the steering committees of the IFAC Working Group on Advanced Maintenance Engineering, Services, and Technology and the UK Digital Twin Hub.

Min Xie (Fellow, IEEE) received the Ph.D. degree in quality technology from Linköping University, Linköping, Sweden, in 1987.

He is currently a Chair Professor with Department of Systems Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong. He has authored or coauthored about 300 peer-reviewed journal papers and 8 books on quality and reliability engineering, including *Computing System Reliability* (Kluwer), *Stochastic Aging and Dependence for Reliability* (Springer), and *Weibull Models* (Wiley). More than 50 Ph.D. students have graduated under his supervision.

Dr. Xie has chaired many international conferences and given keynote speeches. He also serves as an Editor and Associate Editor and on the Editorial Board of many established international journals. He was the recipient of the prestigious Lee Kuan Yew (LKY) Research Fellowship in 1991.