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Decision Support

Design and pricing of extended warranty menus based on the multinomial logit choice model

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ABSTRACT

This paper studies the design and pricing of an extended warranty menu, which offers multiple options with differentiated lengths and prices. The power law process is used to model product failures and evaluate warranty costs. The multinomial logit model is adopted to describe customer choice behaviors. From a warrantor's perspective, the design and pricing problem is to determine which candidate options to offer and the associated prices so as to maximize the expected warranty profit. We show that the optimal strategy is to offer all candidate options associated with a *cost-plus-margin* pricing policy, with the same profit margins for all options. If only a limited number of options can be offered, then the options with the highest valuation margins should be selected. In addition, we present three extended models by incorporating heterogeneous warranty breadths, free preventive maintenance programs, and customers with heterogeneous perceptions of product failure probability. Major findings in the extended models include: *(i)* a free preventive maintenance program should be attached to a warranty option only when it is *feasible* for that option; and *(ii)* when the customer population is heterogeneous, the equal-margin pricing policy is no longer optimal. Overall, this work will equip practitioners with a quantitative tool to design and price extended warranty menus in various practical scenarios.

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1. Introduction

Nowadays, most consumer durables are sold with a manufacturer's warranty (also called base warranty) which protects customers from product failures in certain years after purchase. In addition to base warranties, manufacturers, retailers, and even third-party insurers (hereafter referred to as *warrantors)* are selling extended warranty (EW) contracts in the market. Unlike a base warranty that is free-of-change and bundled with the product, an EW is an optional contract offering customers an additional protection, usually after the expiration of base warranty, and has to be bought separately with an extra premium (Murthy & Jack, 2014). In recent years, the market for EWs is flourishing, attributed largely to providers' pursuit for high profitability and buyers' desire for peace of mind. It is widely recognized that EWs are highly profitable, which is especially important in the context that profit margins of products themselves are decreasing (Gallego, Wang, Ward, Hu, & Beltran, 2014). According to the Business Week,[1] profit margins on EWs of Best Buy and Circuit City were around 50–60% in 2003, which was almost 18 times the margins on their products. On the other hand, although the economic value of EWs is often questioned by consumer magazines and experts, there are still many customers willing to buy EWs for peace of mind (Huysentruyt & Read, 2010). Generally, customers who suspect their products are vulnerable to breakdowns are more likely to purchase an EW (Abito & Salant, 2019). The attach rates of EWs (i.e., the fraction of product buyers who also buy an EW) vary across product categories—from around 20% on computer-related items to 50% on major appliances, and even higher.[2]

A common practice of selling EWs is that when customers purchase a new product either online or at stores, they are recommended to buy an optional EW as well. For example, Fig. 1 shows a screenshot from tmall.com, where EW information is posted in the "Confirm Order Information" page. In this manner, customers can choose to buy any EW option or not immediately after they

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[1] "The Warranty Windfall" *Business Week*, December 20, 2004.
[2] "Service Contract Market Research", *Warranty Week*, March 9, 2017.

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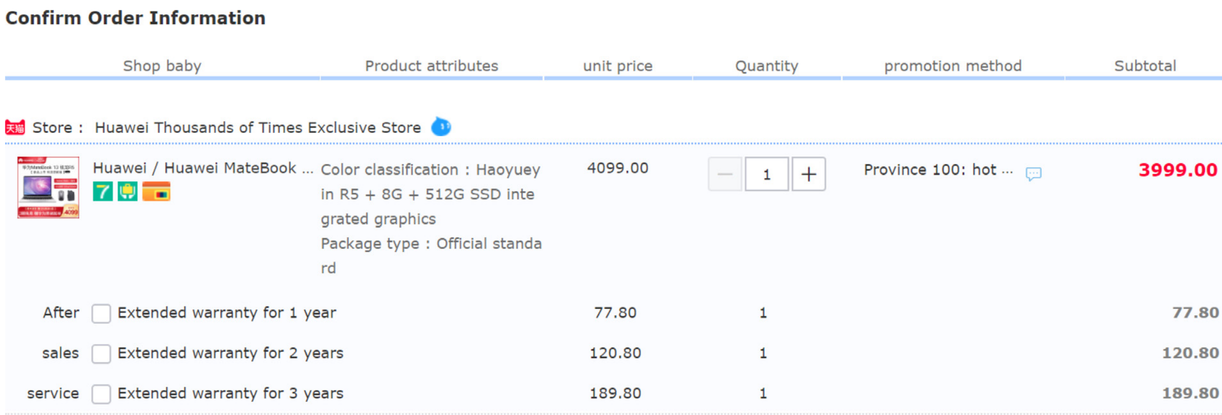

Fig. 1. Huawei MateBook's EW menu offered by an online store on tmall.com (translated version).

decide to purchase the product. More often than not, customers have diverse preferences on warranty features such as length and price. Recently, many warrantors have launched flexible EW programs in which differentiated options are designed to segment customers and attain competitive advantages. In general, the flexibility of an EW program can be reflected in its length, price, breadth (which components and services are covered), starting and ending points, renewal/cancellation policy, and maintenance strategy (Jack & Murthy, 2007). For instance, the three warranty options in Fig. 1 differ in two key features—length and price. By offering such an EW menu, the warrantor can adjust warranty lengths and prices to attract customers who are willing to pay more for longer protection, without losing those who have a "lesser taste" for protection.

The aim of this paper is to study the design and pricing of such kind of EW menus. We refer to individual contracts in a warranty menu as different *options* of the generic contract. Warranty options might differ from each other in the aforementioned features such as price and length. This creates a major hurdle in the warranty-menu design and pricing problem: different options might serve as substitutes so that the demand of a specific option depends not only on its length, price, or other features, but also on those of the other options. This research intends to solve this problem by adopting the multinomial logit (MNL) choice model which is widely used in assortment planning and multi-product pricing problems. The MNL model can be derived from a random utility maximization theory where each customer chooses the option with the highest utility out of the options available (Ben-Akiva & Lerman, 1985). Based on this model, we first investigate the following problem from a warrantor's perspective: given a list of candidate contract lengths, determine which options to offer and the associated prices so as to maximize the expected warranty profit per unit sold. We find that the optimal strategy is to offer all candidate options associated with a *cost-plus-margin* pricing policy, with a uniform profit margin for all options. More importantly, if only a limited number of options can be offered—which is a common industrial practice, then the options with the highest valuation margins should be selected.

In addition, we present three extended models to generalize the original model. The first one deals with the design and pricing of an EW menu with heterogeneous warranty breadths. We show that the cost-plus-margin pricing policy with an identical profit margin is still optimal in this case. The second one studies the problem of bundling a free preventive maintenance (PM) program with a subset of warranty options. We find that it is worthwhile to bundle a free PM program with a specific option only when it is *feasible* for that option. As a result, the warranty attach rate and expected warranty profit would become higher if a feasible PM program is bundled. Finally, the third one considers customers with heterogeneous perceptions of product failure probability via the mixed MNL (MMNL) model. An interesting finding is that the equal-margin pricing policy is no longer optimal in this scenario. Overall, the three extensions not only enrich the degree of flexibility in a warranty menu but also enhance scientific contributions of this work.

The rest of the paper is organized as follows. Section 2 reviews the relevant literature. Sections 3 and 4 formulate and then analyze the EW-menu design and pricing problem, respectively. Then, Section 5 explores the three extended models. Numerical examples are also given to demonstrate the models and results. Finally, Section 6 concludes the paper. All technical proofs are presented in the Appendix.

2. Literature review

The topic of this work falls in the design and pricing of EW contracts. In broad terms, the existing research on this topic can be classified into two types: The first type focuses mainly on the EW design and pricing in supply chain contexts (see, e.g., Bian, Xie, Archibald, & Sun, 2019; Desai & Padmanabhan, 2004; He, Huang, & He, 2018; Heese, 2012; Jiang & Zhang, 2011; Li, Mallik, & Chhajed, 2012; Ma, Ai, Yang, & Pan, 2019; Mai, Liu, Morris, & Sun, 2017), whereas the second stream does not consider supply chain structures (see, e.g., Huang, Huang, & Ho, 2017; Musakwa, 2015; Su & Shen, 2012; Su & Wang, 2016b; Tong, Liu, Men, & Cao, 2014). In the latter case, EW contracts are designed and/or priced predominately from the warrantor's perspective, through warranty cost analysis; while most studies in the former case adopt linear demand functions—either directly assumed or derived from a utility-based formulation of heterogeneous risk preferences—and then make optimal EW design decisions by examining both customer's and warrantor's problems. Our work distinguishes itself from theirs in the sense that (*i*) we aim at designing an EW menu that contains multiple substitutable options, rather than a single EW contract; and (*ii*) the demands of EW options are captured by the MNL model which is a nonlinear function of their lengths and prices.

A concept that is similar, sometimes equivalent, to the warranty menu is flexible warranties. In this regard, a number of studies focus on flexible two-dimensional warranty contracts, by deriving multiple combinations of age and usage limits for a given warranty cost (see, e.g., Manna, Pal, & Sinha, 2006; Shahanaghi, Noorossana, Jalali-Naini, & Heydari, 2013; Su & Wang, 2016a; Ye & Murthy, 2016). The main deficiency of these studies is that the flexible contracts are designed for a fixed warranty cost, without considering customer choices and the profit-maximization objective. Moreover, flexible EW contracts have aroused many interests and different aspects have been investigated, e.g., starting point (Jack & Murthy, 2007; Tong et al., 2014), deferral and renewal options (Hartman & Laksana, 2009; Lam & Lam, 2001), month-by-month commitment (Gallego et al., 2014), as well as flexible PM strategies (Huang et al., 2017; Su & Wang, 2016b; Wang, Zhou, & Peng, 2017). In addition, Padmanabhan and Rao (1993), Padmanabhan (1995), and Lutz and Padmanabhan (1998) study the design of warranty menus with self-selection constrains—that is, customers are able to self-select into the coverage level specifically designed for them. In this work, however, the EW menu contains multiple substitutable options with distinct lengths, prices, or other features; more importantly, without self-selection constraints, the MNL choice model adopted here is able to account for the *substitution* among available options.

The MNL model is a widely used utility-based choice model where each customer chooses the option with the highest utility from the options available (Ben-Akiva & Lerman, 1985). Talluri and Van Ryzin (2004) make the first attempt to apply the standard MNL model in revenue management. They show that the *revenue-ordered assortment* without constraints on the offer sets is optimal. Hopp and Xu (2005) study strategic impacts of modular design on the optimal length and price of a differentiated product line. Li and Huh (2011) study a product-line pricing problem with the MNL and Nested Logit models. A salient finding in Hopp and Xu (2005) and Li and Huh (2011) is that the profit margin, defined as price minus cost, at optimality is constant across all products. Rusmevichientong, Shmoys, Tong, and Topaloglu (2014) and Li, Webster, Mason, and Kempf (2019) further investigate the assortment planning and product-line pricing problems, respectively, under the MMNL model. They show that the revenue-ordered assortment and the equal-margin pricing policy are generally no longer optimal under the MMNL model. Recent applications of MNL in assortment planning and product-line pricing problems can be found in, e.g., Wang and Cui (2017), Strauss, Klein, and Steinhardt (2018), Wang (2018), Lee and Eun (2020), and Maclean and Ødegaard (2020). To the best of our knowledge, this work represents the first attempt to apply MNL in the design and pricing of *after-sales service contracts*, instead of physical products. In essence, EW is a typical kind of after-sales service that protects customers from product failures, and its value resolves only when the warranted item fails in the protection period. In this work, the cost and utility functions in MNL are specifically tailored to the warranty context by incorporating product failure probability and customers' perception of this probability.

We note that the MNL model has already been adopted in several studies to characterize warranty demands. Among them, Chu and Chintagunta (2009) and Guajardo, Cohen, and Netessine (2016) deal with base warranties, while Jindal (2015) and Abito and Salant (2019) look at EWs. Our work differs from theirs in the sense that they focus primarily on *empirical* investigations of the values and demand drivers of warranties for different types of products, whereas we deal with the design and pricing of EW menus from an *analytical* perspective. Nevertheless, the successful applications of MNL in empirical studies well support the adoption of this model in our work.

Overall, this research aims to contribute to the warranty literature by studying a new warranty-menu design and pricing problem. The MNL-based customer choice model tailored here provides a basis that can be built upon in order to incorporate more features in the warranty menu, such as heterogeneous warranty breadths, free PM programs, and customers with heterogeneous perceptions of product failure probability.

3. The model

Consider a single product (e.g., laptop, home appliance) whose design life is L. Each unit of the product is sold with a "free" base warranty of length w_b whose price is merged into the product selling price. At the point of sale, an EW menu is also recommended to customers, immediately after they finalize their decision to buy the warranted product (Abito & Salant, 2019). The EW menu is flexible in the sense that each customer can choose any option out of the options available. Of course, customer might decide not to buy any option at all. Suppose that n warranty options are available in the menu, denoted by $\mathcal{N} = \{1, 2, \dots, n\}$, and each option i comes with distinct length w_i and price p_i, $i \in \mathcal{N}$. Then, the menu can be represented by $\{\boldsymbol{w}, \boldsymbol{p}\} = \{(w_i, p_i)\}_{i\in\mathcal{N}}$, where $\boldsymbol{w} := \{w_i\}_{i\in\mathcal{N}}$ and $\boldsymbol{p} := \{p_i\}_{i\in\mathcal{N}}$. An EW contract takes effect after the expiry of base warranty. So, if option i is bought by a customer, then the total warranty length is $w_b + w_i$; otherwise, the total warranty length is simply w_b. One thing noteworthy is that by considering the product's design life, we confine ourselves to $w_i \in [0, L - w_b]$.

3.1. Warranty cost model

Offering warranty[3] contracts is not free from the warrantor's perspective, because servicing warranty claims in the field incurs expenses. For warranty cost modeling purposes, we assume that any item failures during the base warranty and EW periods will be immediately rectified by minimal repairs, as is typically done in the literature (Huang et al., 2017; Su & Shen, 2012; Ye & Murthy, 2016). A minimal repair restores a failed item to its functioning state just prior to failure. Under this assumption, item failures will occur according to a non-homogeneous Poisson process with an intensity function $\lambda(t)$ having the same form as the hazard rate of the time to first failure (Murthy & Jack, 2014). Thus, the expected servicing cost $c(w_i)$ of warranty option $i \in \mathcal{N}$ can be expressed as

$$c(w_i) = c_r \int_{w_b}^{w_b+w_i} \lambda(t)\mathrm{d}t = c_r(\Lambda(w_b + w_i) - \Lambda(w_b)), \tag{1}$$

where $\Lambda(t) = \int_0^t \lambda(x)\mathrm{d}x$ and c_r is the average minimal repair cost borne by the warrantor.

We further assume that the product of interest has a power law intensity function with shape parameter β and scale parameter α, i.e., $\lambda(t) = \beta/\alpha(t/\alpha)^{\beta-1}$ and $\Lambda(t) = (t/\alpha)^{\beta}$. This form has been widely employed in the reliability literature due to its flexibility in describing different kinds of failure characteristics (Wu, 2019). Specifically, $\beta < 1$, $\beta = 1$, and $\beta > 1$ correspond to infant mortality, random failures, and aging/wear-out failures, respectively, which exhibit decreasing, constant, and increasing failure intensities. In this situation, Eq. (1) can be rewritten as

$$c(w_i) = c_r\left(\left(\frac{w_b + w_i}{\alpha}\right)^{\beta} - \left(\frac{w_b}{\alpha}\right)^{\beta}\right). \tag{2}$$

In this work, only $\beta \geq 1$ is considered as infant mortality shall be eliminated by screening or burn-in tests before product release. This implies that the expected warranty cost $c(w_i)$ is non-negative and convex increasing in w_i. In the warranty-menu design and pricing problem, we assume that an identical cost model $c(\cdot)$ is applied to all warranty options, because its parameters (e.g., α and β in Eq. (2)) depend solely on the product's reliability characteristics and the warranty options' breadths. In particular, the servicing cost of a specific warranty option would increase (resp. decrease) if the product reliability is deteriorated (resp. improved) and/or

[3] Henceforth, we will use the terms "warranty" and "extended warranty" interchangeably, if not mentioned otherwise. Any reference to base warranty will be explicitly stated.

more (resp. less) components/services are covered. As we are dealing with a single product and its warranty options have the same breadth, the form of warranty cost $c(\cdot)$ should be the same for all options. One thing noteworthy is that the warranty cost model in Eq. (1) is general and can accommodate to any forms of intensity function $\lambda(t)$.

3.2. Customer choice model

We adopt the well-known MNL model to characterize discrete customer choice behaviors. As mentioned earlier, MNL assumes that customers act as utility maximizers. The random customer utility for warranty option $i \in \mathcal{N}$ can be defined as

$$u_i = v(w_i) - p_i + \varepsilon_i, \tag{3}$$

where $v(w_i)$ is the average customer-perceived valuation (measured in money units) of option i and ε_i measures the randomness of the utility function that is affected by unobservable characteristics.

The customer-perceived valuation $v(w_i)$—also called reservation price in Aydin and Ryan (2000)—of a warranty option is the maximal price a customer is willing to pay for that option. For each warranty option $i \in \mathcal{N}$, $v(w_i)$ can be further modeled as

$$v(w_i) = c_m \delta(r_i), \tag{4}$$

where c_m is the customer-paid repair cost for an out-of-warranty failure (generally, c_m is much larger than c_r, and known for customers), r_i is the product failure probability during the protection period of option i, and $\delta(\cdot)$ is the weight customers assign to failure probability r_i. Note that r_i is an increasing function of w_i, although the function argument is omitted for clarity. With the minimal repair assumption above, r_i can be derived as $r_i = r_i(w_i) = 1 - \exp(\Lambda(w_b) - \Lambda(w_b + w_i))$.

The detailed explanation of utility function (3) is as follows. From a customer's perspective, the value of warranty option i resolves only when the warranted unit fails in the time interval $[w_b, w_b + w_i]$ and the associated rectification cost is waived. To be specific, if a unit covered by option i fails, then the customer would save c_m units of money for failure repair. As a result, conditional on product failing under option i, with a perceived probability $\delta(r_i)$, the customer's average utility of buying this option is given by $c_m - p_i$; whereas conditional on no failure, the average utility becomes $-p_i$. In this vein, customers make their purchase decision regarding a warranty option by assessing the perceived repair expenses, in the event that the option is not purchased, against the premium to be paid for buying the option. This is consistent with the observation in UK Competition Commission (2003).

Furthermore, it is evident that customers tend to distort actual failure probability due to the lack of product reliability information (Abito & Salant, 2019). Such probability distortion behavior can be characterized by $\delta(\cdot)$. Commonly adopted formulations of this function include

$$\delta(r_i) = \exp(-(-\log(r_i))^{\lambda}), \tag{5}$$

used in Abito and Salant (2019) and

$$\delta(r_i) = r_i^{\lambda}/(r_i^{\lambda} + (1 - r_i)^{\lambda})^{1/\lambda}, \tag{6}$$

applied in Jindal (2015), where $\lambda \in (0, 1)$ is the model parameter, and a larger value of λ reflects a smaller distortion. Notice that when $\lambda \to 1$, we have $\delta(r_i) = r_i$ for both (5) and (6), corresponding to an ideal case that customers can precisely estimate the true failure probability. Fig. 2 illustrates the two probability distortion functions. One can observe that the failure probability tends to be overestimated when it is small, whereas underestimated when it is medium or large. This shows a commonly observed *inverse S shape*; see, e.g., Abito and Salant (2019) and Jindal (2015).

Again, we assume that all warranty options share the same form of valuation function $v(\cdot)$. This is appropriate when an identical breadth is applied to all options and the customer population is homogeneous in the perception of product failure probability. This assumption is made mainly for the sake of simplicity and tractability. It will be relaxed in Section 5 to accommodate more realistic scenarios.

According to the utility maximization theory, a rational customer will select the warranty option that maximizes her utility. Note that $u_0 \equiv \varepsilon_0$ is the utility of the outside option, corresponding to the case that a customer does not buy any warranty option. An underlying assumption of MNL is that the random terms ε_i's are independent and identically distributed Gumbel random variables, i.e., $\Pr(\varepsilon_i \le x) = \exp(-\exp(-(x/\mu + \chi)))$, where $\chi \approx 0.5772$ is Euler's constant and $\mu > 0$ is a scale parameter. It is a standard result for MNL (see Ben-Akiva and Lerman, 1985, for a derivation) that for given $\boldsymbol{w}$ and $\boldsymbol{p}$, a customer will choose warranty option $i \in \mathcal{N}$ with probability

$$q_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N}) = \Pr\left(u_i = \max_{j \in \mathcal{N}} u_j\right) = \frac{\exp\left(\frac{v(w_i) - p_i}{\mu}\right)}{1 + \sum_{j \in \mathcal{N}} \exp\left(\frac{v(w_j) - p_j}{\mu}\right)}, \tag{7}$$

and turn to the outside option with probability $q_0(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N}) = 1 - \sum_{i \in \mathcal{N}} q_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N})$.

One property noteworthy is that if $\delta(r_i)$ is an increasing function of r_i, which is well satisfied by (5) and (6), then $q_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N})$ would increase as w_i increases and/or p_i decreases. Recall that r_i is increasing in w_i. If $\delta(r_i)$ is an increasing function of r_i, then $\delta(r_i)$, as well as $v(w_i)$, are increasing in w_i (according to the chain rule). Due to the fact that $\partial q_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N})/\partial w_i > 0$ and $\partial q_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N})/\partial p_i < 0$ for all $i \in \mathcal{N}$, the property above is verified. Another property to note is that the scale parameter μ in (7) affects the MNL's behavior (Strauss et al., 2018): As $\mu \to 0$, MNL becomes purely deterministic (i.e., $q_i = 1$ if $u_i = \max_{j \in \mathcal{N}} u_j$; zero, otherwise); while when $\mu \to \infty$, the utility of each option becomes irrelevant and the choice probability is constant across all available options and the outside option (i.e., $q_i = 1/(1 + n)$, $\forall i \in \mathcal{N}$). This shows the flexibility and power of MNL as a customer choice model.

3.3. Expected profit per unit sold

We consider the purchase of warranty contracts occurs only at the point of product sale. This is not far away from the reality as most warranty contracts are bought at this point.[4] Moreover, we focus below on customers' warranty purchase decision conditional on they already buying the main product. This is because warranty is largely an afterthought in the buying process. In this manner, the potential market size for warranty is the total number of customers who have bought the product, which is considered exogenous for the warranty-menu design and pricing problem; the choice probability $q_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N})$ then represents the proportion of product buyers who purchase warranty option i as well. As a result, the total probability of warranty purchase (i.e., the warranty attach rate) is given by $Q = \sum_{i \in \mathcal{N}} q_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N})$.

As the total market size of the product is supposed to be fixed, we can simply deal with the expected warranty profit per unit sold, as follows:

$$\Omega(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N}) = \sum_{i \in \mathcal{N}} (p_i - c(w_i)) q_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{N}), \tag{8}$$

[4] For example, UK Competition Commission (2003) finds that more than 80% of customers in UK obtain their warranties at the point of product purchase. Also, Warranty Week suggests that the best time and place to sell an EW is at the time and place where the attached product is purchased; see the newsletter of July 25, 2007.

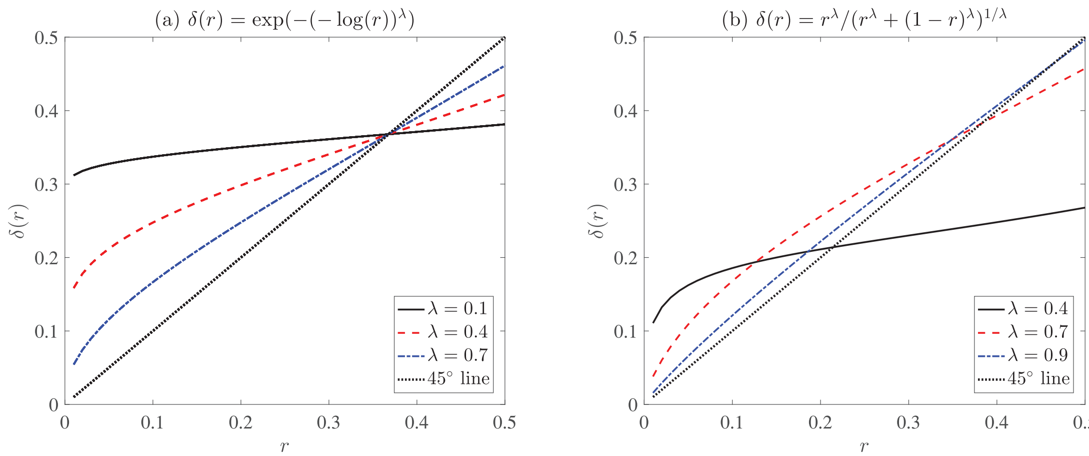

Fig. 2. Illustration of the probability distortion functions. One can see that $\delta(r) \geq r$ for small r, whereas $\delta(r) < r$ for medium or large r. In sub-figure (a), all curves intersect at point (e^{-1}, e^{-1}); while in sub-figure (b), the curves intersect with the 45-degree straight line in an ascending order of λ.

where $p_i - c(w_i)$ is the expected profit margin (markup) of warranty option i.

4. Design and pricing of an EW menu

In this section, we discuss the optimal design and pricing of a warranty menu. We consider a scenario in which the contract lengths of n warranty options are pre-specified. In the real world, the length of a warranty contract is usually a positive integer, which may be 6 months, 12 months, or 18 months, for example. It is thus a common practice for warrantors to pre-specify a list of warranty options with distinct lengths and then decide on which options to offer (i.e., offer set $\mathcal{S} \subseteq \mathcal{N}$) and the optimal prices for the offered options.

With the objective of profit maximization, the warranty-menu design and pricing problem can be formulated as

$$\max_{\mathcal{S} \subseteq \mathcal{N}, \boldsymbol{p}} \Omega(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S}). \tag{9}$$

Generally, it is not easy to solve this problem using standard optimization techniques, since the objective function is not well behaved. Following Wang (2018) and Wang and Cui (2017), we reformulate the problem as follows: Let $\Omega(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S}) = \pi$ for any prices $\boldsymbol{p}$. Rearranging terms and after some algebraic manipulations, we obtain

$$\sum_{i \in \mathcal{S}} (p_i - c(w_i) - \pi) \exp\left(\frac{v(w_i) - p_i}{\mu}\right) = \pi. \tag{10}$$

The derivation of this formula can be found in the proof of Theorem 1 (see the Appendix). Notice that the left-hand side (LHS) is strictly decreasing in π for any specific $\boldsymbol{p}$ while the right-hand side (RHS) is a 45-degree straight line. Therefore, the optimization problem (9) is equivalent to finding a unique intersection between a strictly decreasing function and a 45-degree line. After reformulation, the problem is much more tractable.

The optimal warranty-menu design and pricing decisions are presented in the following theorem.

Theorem 1.

(i) *The optimal offer set is* $\mathcal{S}^* = \mathcal{N}$.
(ii) *The optimal price of each option i is given by*

$$p_i^* = c(w_i) + \pi^* + \mu, \tag{11}$$

where π^* *is the unique solution to*

$$\mu \sum_{i \in \mathcal{N}} \exp\left(\frac{v(w_i) - c(w_i) - \pi - \mu}{\mu}\right) = \pi. \tag{12}$$

(iii) *The resulting warranty attach rate is given by*

$$Q^* = \sum_{i \in \mathcal{N}} q_i(\boldsymbol{w}, \boldsymbol{p}^*; \mathcal{N}) = \frac{\pi^*}{\mu + \pi^*}. \tag{13}$$

A key finding here is that all candidate options should be offered (i.e., $\mathcal{S}^* = \mathcal{N}$) when the lengths of warranty options are pre-specified and the warrantor determines the offer set and associated prices. Given any $\boldsymbol{w}$, the optimal price of each option i is given by $p_i^* = c(w_i) + \pi^* + \mu,\ i \in \mathcal{N}$. As a consequence, the profit margins of all options, at optimality, are identical and given by $p_i^* - c(w_i) = \pi^* + \mu,\ i \in \mathcal{N}$. In other words, the optimal warranty prices for given $\boldsymbol{w}$ should be specified in a way that the profit margins are the same for all options. This finding is consistent with the existing literature (see, e.g., Aydin & Ryan, 2000; Hopp & Xu, 2005; Li & Huh, 2011; Wang & Cui, 2017). This is because in standard logit-type choice models, cross-price elasticities are identical for all pairs of products (Hopp & Xu, 2005). Furthermore, it is interesting to find that when the optimal pricing policy above is applied, the resultant warranty attach rate Q^* can be explicitly expressed as a function of the expected profit π^*.

Corollary 1. *The price-length ratio* p_i^*/w_i *is either decreasing or first decreasing and then increasing in* $w_i \in [0, L - w_b],\ i \in \mathcal{N}$.

The price-length ratio is an essential indicator that customers may refer to when choosing from multiple warranty options. Corollary 1 presents an interesting property about the monotonicity of this ratio. There are two possible scenarios: (*a*) p_i^*/w_i is decreasing in w_i; or (*b*) it is first decreasing and then increasing in w_i, resulting in a "U-shaped" curve. This property is somewhat supported by the real case in Fig. 1, where the price-length ratios of the 1-year, 2-year, and 3-year options are 77.80, 60.40, and 63.27, respectively; in this case, the price-length ratio first decreases and then increases with the (discrete) growth of w_i. In essence, this property is attributed to the cost-plus-margin pricing policy, where the warranty cost $c(w_i)$ is convex increasing in w_i while the profit margin $\pi^* + \mu$ is constant across all options. The price-length ratio p_i^*/w_i is thus a combination of an increasing term and a decreasing term. As a result, it might exhibit a decreasing or U-shaped trend, depending on the relative magnitude of the two terms. In practice, one would expect that customers are more likely to choose one option if its price-length ratio is lower. Such behavior can be captured by the *contextual* reference effect (see, e.g., Wang, 2018). We leave this topic for future research.

Define $\eta_i := v(w_i) - c(w_i)$ as the *valuation margin* of option i, i.e., the difference between average customer valuation $v(w_i)$ and warranty cost $c(w_i)$ of that option. Recall that the difference between price p_i and warranty cost $c(w_i)$ is the *profit margin* of option *i*. Then we have the following corollary which is an intuitive follow-up result of Theorem 1.

Corollary 2. *The optimal profit* π^* *is increasing in* η_i*, for all* $i \in \mathcal{N}$*; and the optimal profit margin* $p_i^* - c(w_i)$ *for each option i is increasing in* η_j*, for all* $i, j \in \mathcal{N}$*.*

The first part implies that the warrantor can make more profits by selling warranty options with higher valuation margins. The customer valuation $v(w_i)$ reflects the average willing-to-pay for option *i*. Thus, a larger difference between customer valuation and warranty cost offers the warrantor more potentials to extract profit. The second part follows the first part but delivers a more interesting finding: If the warrantor replaces one of the existing options with one having a higher valuation margin, then the new menu will result in a higher profit, which in turn translates to an increase in the warranty prices for *all* options. This is due to the substitution and cannibalization effect and the fact that all profit margins are identical at optimality. More specifically, if the valuation margin of a specific option *i* becomes higher, then the warrantor would like more customers to buy it for profit maximization; since the total market size is fixed, the warrantor has to increase the prices of all other options to induce customers to purchase option *i*.

In reality, however, the warrantor may include only a limited number of options, say, m ($m < n$), in a warranty menu even though all of the n options are profitable. For example, only three warranty options are offered in Fig. 1. In this situation, the warrantor needs to determine which options to include in the menu with limited advertising space to maximize the warranty profit. Based on the results in Corollary 2, the following corollary describes the optimal warranty-option selection strategy when considering a limited menu size.

Corollary 3. *If the warrantor is limited to include only m out of the n warranty options in a menu, then the m options with the highest valuation margins should be selected.*

This result is quite natural. It implies that if we re-order the warranty options so that $\eta_1 > \eta_2 > \cdots > \eta_n$, then the warrantor should choose options 1 to m—the m options that have the highest valuation margins (Aydin & Ryan, 2000). This is because the valuation margin η_i is the "largest" possible profit margin that customers are willing to pay for option *i*. Therefore, the warrantor should give higher priorities to warranty options with higher valuation margins.

Example 1. Consider a hypothetical electronic appliance whose design life is $L = 10$ years. The product is covered by a free 1-year base warranty, i.e., $w_b = 1$ year. The warrantor intends to design a warranty menu for this product. Assume that the product has a power law intensity with scale parameter $\alpha = 6.06$ and shape parameter $\beta = 1.82$, and the average repair cost borne by the warrantor is $c_r = 200$ CNY. However, if a customer does not buy any warranty option, then she needs to pay, on average, $c_m = 450$ CNY for an out-of-warranty repair. Further assume that the distortion of product failure probability can be captured by Eq. (5), i.e., $\delta(r_i) = \exp(-(-\log(r_i))^\lambda)$, and the associated parameter is $\lambda = 0.69$. The scale parameter of MNL is set to $\mu = 12.5$. The optimally designed warranty menu with five options and the related metrics are shown in Table 1 and Fig. 3. Note that the units for time and money are years and CNY, respectively.

The optimal warranty menu is $\{\boldsymbol{w}, \boldsymbol{p}\} = \{(1, 87.02), (2, 116.06), (3, 154.33), (4, 201.37), (5, 256.84)\}$. The corresponding profit per unit sold is $\pi^* = 55.46$ CNY, and the

Table 1
The optimally designed warranty menu and related metrics.

w_i	r_i	$v(w_i)$	$c(w_i)$	η_i	p_i^*	q_i (%)	p_i^*/w_i
1	0.09	72.30	19.06	53.24	87.02	5.66	87.02
2	0.21	116.79	48.10	68.70	116.06	19.51	58.03
3	0.35	160.21	86.37	73.84	154.33	29.44	51.44
4	0.49	202.78	133.41	69.37	201.37	20.58	50.34
5	0.61	243.67	188.88	54.79	256.84	6.41	51.37

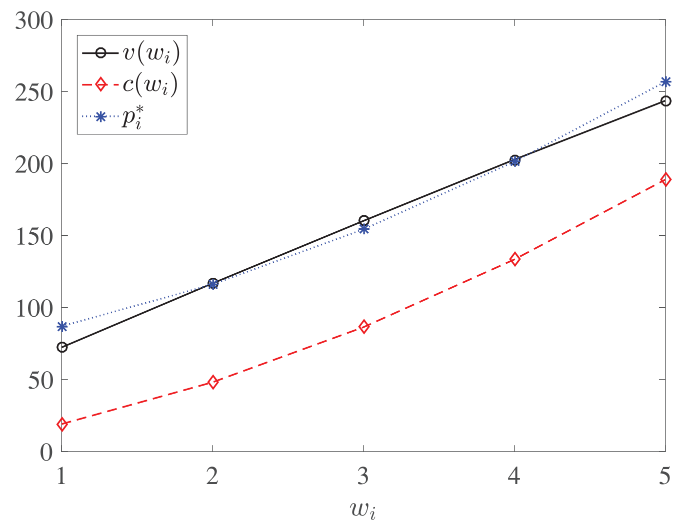

Fig. 3. Customer valuations, warranty costs, and optimal prices for the five options.

warranty attach rate is $Q^* = 81.61\%$. As can be seen from Table 1, the choice probabilities of the second, third, and fourth options are significantly larger than others. This is because the average customer utilities $v(w_i) - p_i$ for these options are positive (see Fig. 3), which in turn results in large choice probabilities. On the other hand, the first and fifth options still have small choice probabilities, even though the corresponding average customer utilities are negative. This phenomenon is caused by the influence of random term ε_i in utility model (3), as well as the fact that exponent terms $\exp(\cdot)$ in MNL always generate positive choice probabilities. As the valuation function $v(w_i)$ (resp. warranty cost $c(w_i)$) is concave (resp. convex) increasing in w_i, the valuation margins—defined as average valuation minus warranty cost—of the second, third, and fourth options are the largest (see Table 1). As a result, if the warrantor is limited to include only m ($m \geq 3$) out of the five options in a menu, then the second, third, and fourth options ought to be chosen for profit maximization purposes (refer to Corollary 3).

Moreover, the price-length ratio p_i^*/w_i, at optimality, is decreasing in w_i for $w_i \in \{1, 2, 3, 4\}$, and becomes slightly increasing thereafter (see Table 1). To illustrate a different pattern, Fig. 4 shows the price-length ratios for three different values of λ, i.e., 0.2, 0.5, and 0.8. One can observe that when $\lambda = 0.2$, the price-length ratio is consistently decreasing in w_i; whereas the ratios for $\lambda = 0.5$ and $\lambda = 0.8$ exhibit a decreasing-then-increasing pattern when w_i is increasing in a discrete manner. This validates the analytical result in Corollary 1.

Furthermore, if we consider nested warranty menus with increasing number of options, i.e., $\{(w_1 = 1, p_1)\}$, $\{(w_1 = 1, p_1), (w_2 = 2, p_2)\}$, …, then Fig. 5 shows the expected profit per unit sold versus the number of options under different values of λ. We find that as the number of available options increases, the expected warranty profit also grows, until reaching a specific limit. The warranty attach rate Q^* shall exhibit the same pattern, as it equals to $\pi^*/(\mu + \pi^*)$. This is due to the fact that

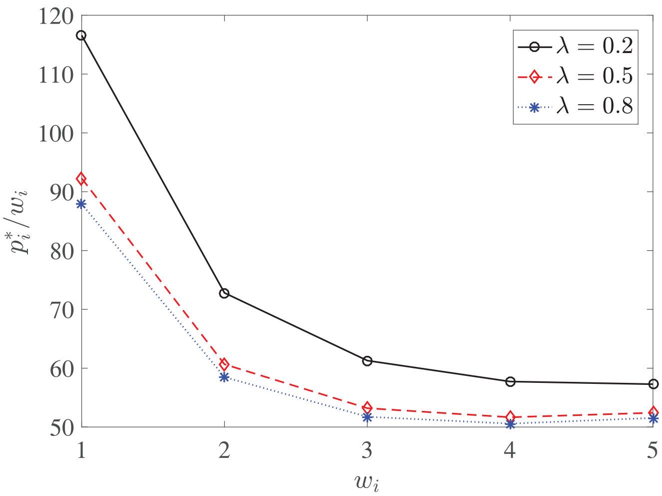

Fig. 4. Trends of the price-length ratios under different values of λ.

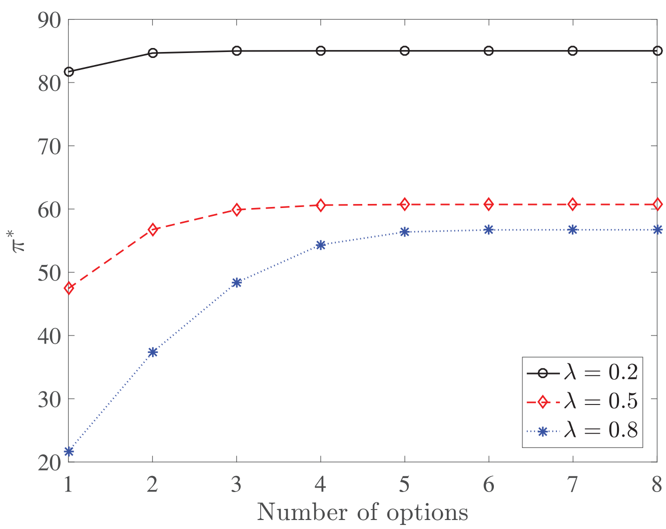

Fig. 5. Expected warranty profit versus the number of options under different values of λ.

warranty options with long protection periods will have negative customer utilities, thus contribute little to the warranty profit and attach rate. In addition, the profit curve approaches to its limiting value faster when λ is smaller. This observation stems from the concavity of valuation function. Specifically, when λ becomes smaller, the concavity of $v(w_i)$ is severer (see Fig. 2), and thus the customer utility would become negative even for smaller w_i. Consequently, the profit curve will approach to its limiting value faster as w_i increases. Another observation to highlight is that the warranty profit is smaller when λ becomes larger. This is because a larger λ reflects a smaller distortion of failure probability, implying that customers are able to estimate the true failure probability in a more accurate manner. Therefore, the warrantor's expected profit per unit sold will become smaller due to the loss of asymmetric advantage in product reliability information.

5. Extensions

The warranty-menu design and pricing problem studied above can serve as a workhorse model on top of which many additional features can be incorporated. In this section, three extended models are developed by incorporating heterogeneous warranty breadths, free PM programs, and customers with heterogeneous perceptions of product failure probability, respectively.

5.1. Heterogeneous warranty breadths

In the previous sections, we presumed that warranty options in a menu differ only in price and length. In reality, however, warranty options might have different types of breadth, namely, they might cover different sets of components and/or add-on services. For example, Ford's extended service plans include PowertrainCare, BaseCare, ExtraCare, and PremiumCare, under which the number of components covered are 29, 84, 113, and 1000+, respectively.[5] Another example is the protection plans of HP Spectre $\times$360 laptop, which differ not only in length and price but also in the inclusion/exclusion of accidental damages such as spills, surges, and drops (see Fig. 6). In this subsection, we extend our original model to incorporate heterogeneous breadths into the warranty-menu design and pricing problem. As a side note, a specific type of warranty breadth resembles a *bundle* of components or add-on services (refer to Fuerderer, Herrmann, and Wuebker, 1999, for bundle selling). That is to say, a warranty menu with heterogeneous breadths is equivalent to a menu of *nested* bundles along with different protection lengths. To our knowledge, quite limited studies on the design and pricing of bundle menus can be found in the literature. Ferrer, Mora, and Olivares (2010) and Cataldo and Ferrer (2017) investigate the optimal composition and pricing of multiple bundles, i.e., which components or services to include in a set of bundles, and the associated optimal price; whereas the aim of our work is to determine which bundles (with given compositions) to include in the menu and the associated optimal prices. Nevertheless, the proposed model below provides a new perspective on the design and pricing of bundle menus.

Suppose that there are K types of breadths across all warranty groups,[6] each covering a fixed number of components and/or add-on services. The warranty menu can then be denoted by $\{\boldsymbol{w}, \boldsymbol{p_1}, \boldsymbol{p_2}, \ldots, \boldsymbol{p_K}\} = \{(w_i, p_{i,1}, p_{i,2}, \ldots, p_{i,K})\}_{i\in\mathcal{N}}$, where $\boldsymbol{p_k} := \{p_{i,k}\}_{i\in\mathcal{N}}$, and $p_{i,k}$ is the price of warranty option with length w_i and breadth type k. Notice that warranty options in a specific group i, with any types of breadths, are of an identical length w_i, $i \in \mathcal{N}$. In other words, all warranty options in a specific group differ only in breadth type and price. Take the protection plans in Fig. 6 as an example. There are two types of warranty breadths: $k = 1$ (resp. $k = 2$) implies that the protection plans do not (resp. do) cover accidental damages. The warranty menu can then be represented as $\{(w_1 = 2, p_{1,1} = 231.99, p_{1,2} = 251.99), (w_2 = 3, p_{2,1} = 251.99, p_{2,2} = 275.99)\}$.

Although all options in a specific group have an identical warranty length, their servicing costs and customer-perceived valuations are indeed different due to the heterogeneity in breadth type. In general, when more components and/or add-on services are covered by a warranty option, the associated servicing cost and customer-perceived valuation become larger. Let $c_k(w_i)$ and $v_k(w_i)$ represent the expected warranty cost and customer valuation of type-k breadth in group i, respectively. Then, the random customer utility of type-k breadth in group i is given by $u_{i,k} = v_k(w_i) - p_{i,k} + \varepsilon_{i,k}$, where $\varepsilon_{i,k}$ is a Gumbel random variable. According to the setting above, the customer choice probability of type-k breadth in group $i \in \mathcal{N}$ can be derived as

$$q_{i,k}(\boldsymbol{w}, \boldsymbol{p_1}, \ldots, \boldsymbol{p_K}; \mathcal{N}) = \frac{\exp\left(\frac{v_k(w_i)-p_{i,k}}{\mu}\right)}{1 + \sum_{l=1}^{K}\sum_{j\in\mathcal{N}}\exp\left(\frac{v_l(w_j)-p_{j,l}}{\mu}\right)}, \tag{14}$$

[5] Refer to http://www.lombardfordwarrantys.com.

[6] For clear presentation, in this subsection we refer to all options with the same length w_i as in group i.

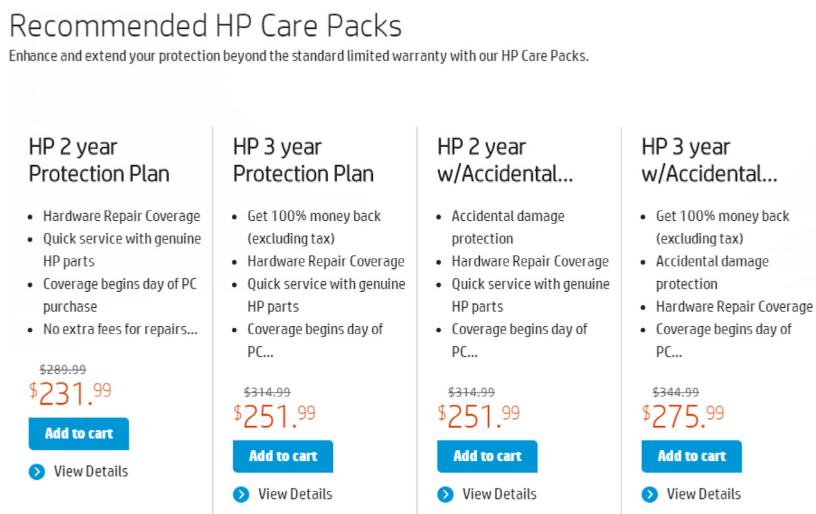

Fig. 6. HP Spectre $\times$360 laptop's EW menu (with two types of breadths) from store.hp.com.

and the expected warranty profit per unit sold is given by

$$\Omega(\boldsymbol{w}, \boldsymbol{p_1}, \ldots, \boldsymbol{p_K}; \mathcal{N}) = \sum_{k=1}^{K} \sum_{i \in \mathcal{N}} (p_{i,k} - c_k(w_i)) \times q_{i,k}(\boldsymbol{w}, \boldsymbol{p_1}, \ldots, \boldsymbol{p_K}; \mathcal{N}). \tag{15}$$

As before, the warrantor's problem is to determine which groups to offer and the associated prices for given $\boldsymbol{w}$, that is,

$$\max_{\mathcal{S} \subseteq \mathcal{N}; \boldsymbol{p_1}, \ldots, \boldsymbol{p_K}} \Omega(\boldsymbol{w}, \boldsymbol{p_1}, \ldots, \boldsymbol{p_K}; \mathcal{N}). \tag{16}$$

The optimization problem can be easily solved following the analysis in Section 4, and the optimal warranty-menu design and pricing decisions are presented as follows.

Theorem 2.

(i) *The optimal strategy is to offer all groups and all breadth types.*
(ii) *The optimal price of type-k breadth in group i is given by*

$$p^{\dagger}_{i,k} = c_k(w_i) + \pi^{\dagger} + \mu, \tag{17}$$

where $\pi^{\dagger}$ *is the unique solution to*

$$\mu \sum_{k=1}^{K} \sum_{i \in \mathcal{N}} \exp\left(\frac{v_k(w_i) - c_k(w_i) - \pi - \mu}{\mu}\right) = \pi. \tag{18}$$

(iii) *The associated warranty attach rate is given by*

$$Q^{\dagger} = \sum_{k=1}^{K} \sum_{i \in \mathcal{N}} q_{i,k}(\boldsymbol{w}, \boldsymbol{p_1}^{\dagger}, \ldots, \boldsymbol{p_K}^{\dagger}; \mathcal{N}) = \frac{\pi^{\dagger}}{\mu + \pi^{\dagger}}. \tag{19}$$

Basically, the optimal warranty-menu design and pricing strategy with heterogeneous breadth types is consistent with that in Theorem 1. The optimal strategy is to offer all groups and all breadth types, and then determine the associated prices according to the cost-plus-margin pricing policy. The profit margins of any breadth types in any groups, at optimality, are identical and given by $\pi^{\dagger} + \mu$. The corresponding attach rate is still a function of the expected warranty profit and is given by $Q^{\dagger} = \pi^{\dagger}/(\mu + \pi^{\dagger})$.

Similar to Corollary 1, we can easily obtain the following result on the monotonicity of $p^{\dagger}_{i,k}/w_i$ for any breadth type k.

Corollary 4. *For any breadth type* $k = 1, 2, \ldots, K$, *the price-length ratio* $p^{\dagger}_{i,k}/w_i$ *is either decreasing or first decreasing and then increasing in* $w_i \in [0, L - w_b], i \in \mathcal{N}$.

The property follows directly from Corollary 1, and it shows that the decreasing or U-shaped trend of the price-length ratio well preserves for each breadth type. This property also stems from the cost-plus-margin pricing policy which is optimal for this problem.

Table 2
Parameter setting used in Example 2.

	α_l	β_l	$c_{r,l}$	$c_{m,l}$
Cluster 1 ($l = 1$)	6.06	1.82	200	450
Cluster 2 ($l = 2$)	7.12	2.55	150	300
Cluster 3 ($l = 3$)	6.88	1.00	180	230

Example 2. Consider that the warrantor now plans to design five warranty groups of lengths 1–5 years for the electronic appliance, and each of them has three breadth types. Suppose that there are three mutually exclusive clusters of components and/or add-on services for the product, which have distinct reliability characteristics and servicing costs. In particular, breadth type 1 only covers the first cluster, breadth type 2 covers the first and second clusters, and breadth type 3 covers all the three clusters. This setting is consistent with those in the Ford and HP cases mentioned above. In this manner, the warranty cost $c_k(w_i)$ and customer valuation $v_k(w_i)$ can be expressed as $c_k(w_i) = \sum_{l=1}^{k} c_{r,l}[(w_b + w_i)^{\beta_l} - (w_b)^{\beta_l}]/\alpha_l^{\beta_l}$ and $v_k(w_i) = \sum_{l=1}^{k} c_{m,l}\delta(r_{i,l})$, where $r_{i,l} = 1 - \exp([(w_b)^{\beta_l} - (w_b + w_i)^{\beta_l}]/\alpha_l^{\beta_l})$, and α_l, β_l, $c_{r,l}$, and $c_{m,l}$ are the model parameters associated with cluster $l = 1, 2, 3$. The values of λ and μ are still set to $\lambda = 0.69$ and $\mu = 12.5$, and the setting of other cluster-specific parameters is summarized in Table 2. Notice that the parameter values for the first cluster is the same as those in Example 1.

Table 3 shows the optimally designed warranty menu and related metrics. One can see that only four options have sufficiently large (say, over 10%) choice probabilities, i.e., $q_{3,2} = 15.79\%$, $q_{4,2} = 12.06\%$, $q_{2,3} = 18.52\%$, and $q_{3,3} = 21.44\%$. This is well reflected in Fig. 7 that only these four options have positive average customer utilities. Likewise, other options still have small choice probabilities, although the associated average customer utilities are negative. In addition, Fig. 8 shows that the price-length ratio for breadth type 1 exhibits a continuously decreasing trend in w_i; while the price-length ratios for breadth types 2 and 3 first decrease and then slightly increase when w_i increases in a discrete manner. This observation agrees with Corollary 4.

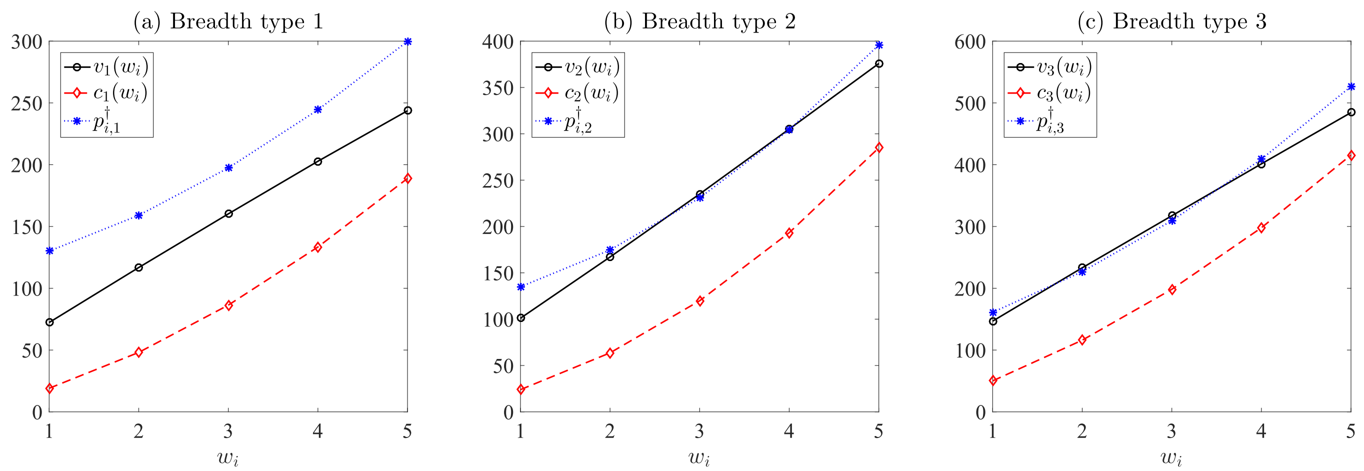

Fig. 7. Customer valuations, warranty costs, and optimal prices for the three breadth types.

Table 3
The optimally designed warranty menu and related metrics with three breadth types.

Breadth	w_i	$v_k(w_i)$	$c_k(w_i)$	$\eta_{i,k}$	$p^{\dagger}_{i,k}$	$q_{i,k}$ (%)	$p^{\dagger}_{i,k}/w_i$
$k=1$	1	72.30	19.06	53.24	130.03	0.11	130.03
	2	116.79	48.10	68.70	159.06	0.38	79.53
	3	160.21	86.37	73.84	197.34	0.58	65.78
	4	202.78	133.41	69.37	244.38	0.40	61.10
	5	243.67	188.88	54.79	299.85	0.13	59.97
$k=2$	1	101.02	23.94	77.08	134.91	0.75	134.91
	2	167.07	63.64	103.43	174.61	6.16	87.31
	3	235.03	119.84	115.19	230.81	15.79	76.94
	4	305.13	193.31	111.82	304.28	12.06	76.07
	5	375.83	284.82	91.01	395.79	2.28	79.16
$k=3$	1	146.84	50.11	96.73	161.08	3.61	161.08
	2	233.15	115.97	117.18	226.94	18.52	113.47
	3	317.34	198.33	119.01	309.30	21.44	103.10
	4	401.37	297.96	103.41	408.93	6.15	102.23
	5	484.34	415.64	68.71	526.61	0.38	105.32

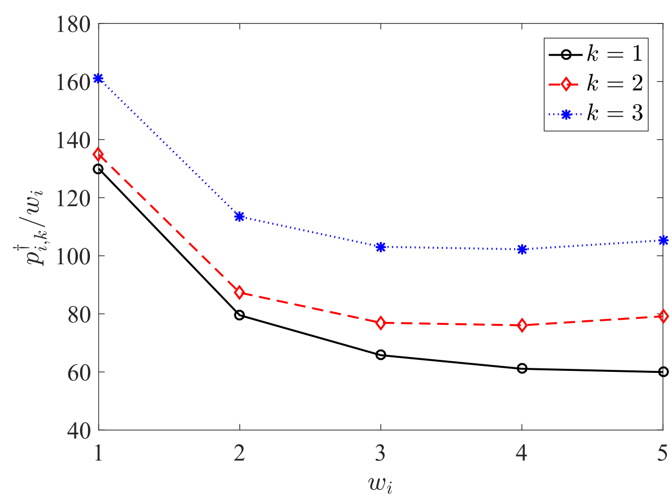

Fig. 8. Trends of the price-length ratios for different breadth types.

Under the optimal menu design, the expected warranty profit per unit sold is 98.47 CNY and the corresponding warranty attach rate is 88.74%. The two figures are remarkably larger than those with only a single breadth type (in Example 1). More interestingly, when two more breadth types are offered, customers shift their choices from options 2–4 in Example 1 (with breadth type 1) to options 3 and 4 with breadth type 2 and options 2 and 3 with breadth type 3 in this example. This demonstrates that offering warranty options with heterogeneous breadth types, if well designed, has a great potential to attract more customers by satisfying their diverse demands in protection breadth, which in turn leads to a higher warranty profit and attach rate.

5.2. Free preventive maintenance program

For capital-intensive products, such as vehicles and high-priced electronic equipment, regular PM activities are necessary to keep them in a good working condition. PM is particularly important in the context of product warranty, as it can improve product reliability and thus reduce warranty servicing cost. For instance, vehicles under warranty (base and/or extended) are usually required to be preventively maintained in a regular manner (Huang et al., 2017; Su & Wang, 2016b; Wang, Li, & Xie, 2020). A real example is that Nissan launched the so-called 6615 program in Taiwan offering members an EW contract for six years or 150,000 kilometers on six key systems, subject to PM every 6 months or 10,000 kilometers by authorized maintenance centers (Huang et al., 2017). It is thus of interest to incorporate an appropriate PM program when designing a warranty menu. In this subsection, we study an optimization problem of bundling a free PM program with a subset of warranty options.

Suppose that a warranty menu $\{\boldsymbol{w}, \boldsymbol{p}\}$ is recommended to customers who have bought the product, in which a free PM program is bundled with selected options. Let $\mathcal{M} \subseteq \mathcal{N}$ represent a set of warranty options that the warrantor decides to attach free PM services. We consider a periodical PM program with a constant maintenance interval Δ. In this situation, the number of PM services covered by option i is $n_i = \lfloor w_i/\Delta \rfloor$, where $\lfloor x \rfloor$ is the maximal integer that is smaller than or equal to x. Denote the expected *total* warranty servicing cost of option i with PM by $c^{pm}(w_i) = c'(w_i) + n_i c_{pm}$, where $c'(w_i)$ is the expected repair cost and $n_i c_{pm}$ is the cost of PM actions. Note that the expected repair cost with PM should be smaller than that without PM, i.e., $c'(w_i) \le c(w_i)$. On the other hand, regular PM services bring some benefits to customers: *(i)* PM services are usually costly so that customers could save money if they are covered by warranty; and *(ii)* the warranted items will exhibit a better performance after PM. The two kinds of PM benefits are on top of the valuation incurred by the warranty itself. We thus assume that the additional customer utility from a single PM service is u^{pm}, and the total utility of the free PM program for option i is $n_i u^{pm}$. Then, the random customer utility for warranty option $i \in \mathcal{M}$ can be expressed as

$$u_i^{pm} = v(w_i) + n_i u^{pm} - p_i + \varepsilon_i. \tag{20}$$

According to the MNL model, the choice probability of each warranty option and the expected profit per unit sold, respectively, are given by

$$\widehat{q}_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{M}) = \begin{cases} \dfrac{\exp\left(\frac{v(w_i)+n_i u^{pm}-p_i}{\mu}\right)}{1+\sum_{j\in\mathcal{M}}\exp\left(\frac{v(w_j)+n_j u^{pm}-p_j}{\mu}\right)+\sum_{j\notin\mathcal{M}}\exp\left(\frac{v(w_j)-p_j}{\mu}\right)}, & i \in \mathcal{M}; \\ \dfrac{\exp\left(\frac{v(w_i)-p_i}{\mu}\right)}{1+\sum_{j\in\mathcal{M}}\exp\left(\frac{v(w_j)+n_j u^{pm}-p_j}{\mu}\right)+\sum_{j\notin\mathcal{M}}\exp\left(\frac{v(w_j)-p_j}{\mu}\right)}, & i \notin \mathcal{M}, \end{cases} \tag{21}$$

and

$$\widehat{\Omega}(\boldsymbol{w}, \boldsymbol{p}; \mathcal{M}) = \sum_{i\in\mathcal{M}}(p_i - c^{pm}(w_i))\widehat{q}_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{M}) + \sum_{i\notin\mathcal{M}}(p_i - c(w_i))\widehat{q}_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{M}). \tag{22}$$

The PM bundling problem is, for given $\boldsymbol{w}$ and $\mathcal{N}$, to determine which options to bundle the free PM program and the optimal prices for all options, that is,

$$\max_{\mathcal{M}\subseteq\mathcal{N}, \boldsymbol{p}} \widehat{\Omega}(\boldsymbol{w}, \boldsymbol{p}; \mathcal{M}). \tag{23}$$

Definition 1. A free PM program with maintenance interval Δ is feasible for warranty option i if $n_i u^{pm} - c^{pm}(w_i) + c(w_i) \geq 0$.

The feasibility condition of a free PM program for option i can be rewritten as $n_i u^{pm} + c(w_i) - c'(w_i) \geq n_i c_{pm}$, where $n_i u^{pm} + c(w_i) - c'(w_i)$ is the total benefits of the PM program (consisting of an extra valuation, $n_i u^{pm}$, and a reduction in warranty cost, $c(w_i) - c'(w_i)$) and $n_i c_{pm}$ is the total PM expenses. That is to say, a free PM program is feasible for option i only if its benefits are higher than or at least equal to its extra expenses. We note that this feasibility condition can be well satisfied when an optimal PM decision with respect to maintenance interval Δ (and maintenance degree, for imperfect PM) is implemented, even in the case of $u^{pm} = 0$; refer to Huang et al. (2017), Shahanaghi et al. (2013), Su and Wang (2016b), and Wang et al. (2020), for example. Modeling and analysis of (imperfect) PM policies are not the main focus of the paper and thus briefly discussed in Example 3.

Theorem 3.

(i) *It is optimal to bundle a free PM program with a specific warranty option if the program is feasible for that option, i.e.,* $\mathcal{M}^\ddagger = \{i \in \mathcal{N} \mid n_i u^{pm} - c^{pm}(w_i) + c(w_i) \geq 0\}$.

(ii) *The optimal price for each option i is given by*

$$p_i^\ddagger = \begin{cases} c^{pm}(w_i) + \pi^\ddagger + \mu, & i \in \mathcal{M}^\ddagger; \\ c(w_i) + \pi^\ddagger + \mu, & i \notin \mathcal{M}^\ddagger, \end{cases} \tag{24}$$

where $\pi\ddagger$ is the unique solution to

$$\mu \sum_{i\in\mathcal{M}^\ddagger} \exp\left(\frac{v(w_i) + n_i u^{pm} - c^{pm}(w_i) - \pi - \mu}{\mu}\right) + \mu \sum_{i\notin\mathcal{M}^\ddagger} \exp\left(\frac{v(w_i) - c(w_i) - \pi - \mu}{\mu}\right) = \pi. \tag{25}$$

(iii) *The corresponding warranty attach rate is given by*

$$Q^\ddagger = \sum_{i\in\mathcal{N}} \widehat{q}_i(\boldsymbol{w}, \boldsymbol{p}^\ddagger; \mathcal{M}^\ddagger) = \frac{\pi^\ddagger}{\mu + \pi^\ddagger}. \tag{26}$$

Theorem 3 indicates that not all warranty options should be bundled with the free PM program. The PM program should be attached to a specific option only when it is feasible for that option. This bundling condition is a general result that is applicable to traditional product-line pricing problems in which free ancillary services are offered for selected products. Moreover, the optimal pricing policy in this problem is still the cost-plus-margin policy. However, the optimal prices for warranty options with PM might be lower or higher than those without PM, depending on the relative magnitudes of $c^{pm}(w_i)$ and $c(w_i)$. Furthermore, the profit margins of all options are still the same and the warranty attach rate is again a function of $\pi^\ddagger$, as before.

Corollary 5. *$\pi^\ddagger \geq \pi^*$ and $Q^\ddagger \geq Q^*$.*

Corollary 5 implies that the warranty profit and attach rate in the PM bundling problem, at optimality, are higher than or at least equal to those without considering free PM programs (in Section 4), respectively. In other words, among the customers who have bought the product, more of them would like to buy warranties if a free and feasible PM program is bundled. As a result, the expected warranty profit becomes higher. This result is driven by the net benefits (i.e., extra valuation plus warranty cost reduction) brought by the free PM program, and it shows the necessity of attaching a feasible PM program to selected warranty options.

Example 3. This example follows directly from Example 1. Suppose that the warrantor is considering to bundle a free imperfect PM program (having a constant maintenance interval Δ) with a subset of warranty options. To this end, the warrantor needs to evaluate the benefit and cost of the PM program to facilitate the decision-making process. We first model the expected repair cost with imperfect PM bundled. In the literature, numerous models have been developed to describe the effect of imperfect maintenance actions; refer to Wu (2019) for a recent summary. In this example, we adopt the age reduction model (see, e.g., Su & Wang, 2016b; Wang et al., 2020) for an illustration of imperfect PM modeling. This model assumes that a PM action can restore the item to a specific working state that corresponds to an age reduction between its actual and virtual ages. Let $\tau_k = w_b + k\Delta$, $k = 1, 2, \ldots, n_i$, denote the item's actual age at the kth PM action, with $\tau_0 = w_b$. Then, the item's virtual age immediately after the kth PM action is given by $a_k = a_{k-1} + (1-\delta)(\tau_k - \tau_{k-1})$, $k = 1, 2, \ldots, n_i$, where $a_0 = w_b$ and $\delta \in [0, 1]$ is the age reduction factor. If $\delta = 0$, then no PM action is taken, i.e., $a_k = \tau_k$; if $\delta = 1$, then the damage accumulated during the $(k-1)$th and the kth PM actions is fully compensated, i.e., $a_k = a_{k-1}$; while if $\delta \in (0, 1)$, then the PM effect is imperfect.

It is easy to verify that a_k can be recursively derived as $a_k = w_b + k(1-\delta)\Delta, k = 1, 2, \ldots, n_i$. According to the non-homogeneous Poisson process, the expected repair cost for warranty option i with PM is given by

$$c'(w_i) = c_r \sum_{k=0}^{n_i-1} \int_{w_b+k(1-\delta)\Delta}^{w_b+k(1-\delta)\Delta+\Delta} \lambda(t)\mathrm{d}t + c_r \int_{w_b+n_i(1-\delta)\Delta}^{w_b+n_i(1-\delta)\Delta+w_i-n_i\Delta} \lambda(t)\mathrm{d}t, \tag{27}$$

where the last term represents the expected repair cost incurred between the n_ith PM action and the end of the warranty period.

Usually, the maintenance interval Δ and age reduction factor δ should be determined by minimizing the total warranty cost $c^{pm}(w_i)$; see, e.g., Huang et al. (2017), Su and Wang (2016b), and Wang et al. (2020). In this example, the main focus is not on PM optimization, but on showcasing the warranty-menu design and pricing model. For this purpose, the PM-related parameters are arbitrarily set to $\Delta = 0.5$, $\delta = 0.5$, $u^{pm} = 25$, $c_{pm} = 25$, and the values of other parameters are the same as those in Example 1.

Table 4
The optimal warranty menu and related metrics in the PM bundling problem.

w_i	PM	$v(w_i)+n_iu^{pm}$	$c^{pm}(w_i)$	η_i	p_i^*	q_i (%)	p_i^*/w_i
1		72.30	19.06	53.24	103.48	1.22	103.48
2	✓	191.79	119.58	72.21	204.00	5.58	102.00
3	✓	285.21	199.87	85.34	284.28	15.95	94.76
4	✓	377.78	284.79	92.99	369.20	29.41	92.30
5	✓	468.67	374.22	94.44	458.64	33.03	91.73

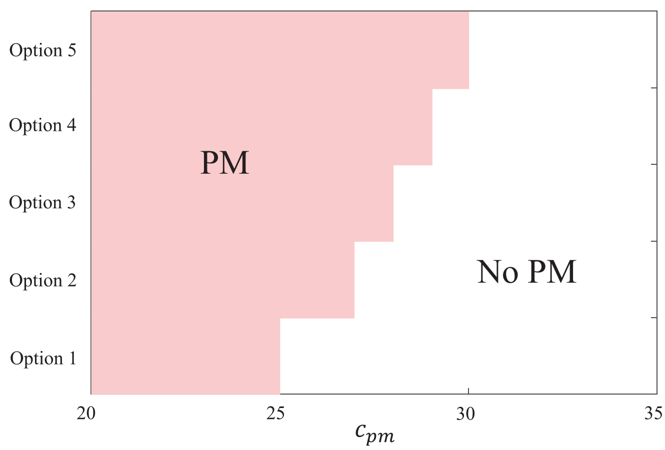

Fig. 9. Optimal PM bundling strategy for various PM costs.

Based on the parameter setting, Table 4 shows the optimally designed warranty menu and associated metrics. One can see that the free PM program should be attached to all warranty options except the first one, according to the feasibility condition. Due to the implementation of a free PM program, the customer valuations, warranty costs, and optimal warranty prices become larger than those in Example 1. In this example, the third, fourth, and fifth options have sufficiently large choice probabilities, implying that the free PM program helps improve the attractiveness of long-duration warranty options. Under the optimal menu design, the warranty profit per unit sold in the PM bundling problem is 71.91 CNY, and the associated attach rate is 85.19%. These figures are indeed larger than those in Example 1, which is consistent with Corollary 5. This demonstrates that attaching a free PM program to selected warranty options can increase warranty profit and attach rate.

Furthermore, we investigate the optimal PM bundling strategy for various PM costs (see Fig. 9). One can observe that as the PM cost increases, it is less attractive to attach the free PM program to warranty options. This is because the benefits of PM will gradually be overtaken by the maintenance expenses incurred, when the PM cost becomes higher.

5.3. Heterogeneous customer population

In the foregoing (sub)sections, we implicitly assumed that the customer population is homogeneous. In reality, however, customers might be heterogeneous (Lee & Eun, 2020) in the sense that they have diverse perceptions of failure probability for the same product. As a consequence, different customers would value the same warranty option differently. Such customer heterogeneity can be captured by the MNL model with random parameters, i.e., the MMNL model. In this subsection, we make an attempt to extend the original model by considering heterogeneous customer population.

Suppose that there are K segments of customers. The actual failure probability r_i and warranty cost $c(w_i)$ of option i is identical across all customer segments, whereas customers in different segments have different probability distortion behaviors. This way, each customer segment k is characterized by a unique distortion parameter λ_k, corresponding to a unique valuation function $v_k(w_i)=c_m\delta_k(r_i)$, where $\delta_k(\cdot)$ is given by (5) or (6) with λ_k replacing λ. The segment of a specific customer is unknown to the warrantor when she chooses a warranty option. In this manner, the parameters λ_k's are modeled as random variables drawn from a discrete mixing distribution. This leads to the so-called discrete MMNL model (Li et al., 2019), in which the customer population is split into a finite number of distinct segments and warranty demand in each segment is governed by the MNL model.

The random utility of warranty option i for a customer in segment k is expressed as $u_{i,k}=v_k(w_i)-p_i+\varepsilon_i$. According to the MMNL model, the conditional choice probability of option $i\in\mathcal{N}$ in customer segment $k=1,2,\ldots,K$, is given by

$$\breve{q}_{i,k}(\boldsymbol{w},\boldsymbol{p};\mathcal{N})=\frac{\exp\left(\frac{v_k(w_i)-p_i}{\mu}\right)}{1+\sum_{j\in\mathcal{N}}\exp\left(\frac{v_k(w_j)-p_j}{\mu}\right)}. \tag{28}$$

Denote by d_k the probability that a randomly selected customer belongs to segment k, with $\sum_{k=1}^{K}d_k=1$. Then, the choice probabilities of option $i\in\mathcal{N}$ and the outside option, respectively, across all customer segments are given by

$$\breve{q}_i(\boldsymbol{w},\boldsymbol{p};\mathcal{N})=\sum_{k=1}^{K}d_k\breve{q}_{i,k}(\boldsymbol{w},\boldsymbol{p};\mathcal{N}), \tag{29}$$

and

$$\breve{q}_0(\boldsymbol{w},\boldsymbol{p};\mathcal{N})=\sum_{k=1}^{K}d_k\breve{q}_{0,k}(\boldsymbol{w},\boldsymbol{p};\mathcal{N})=1-\sum_{i\in\mathcal{N}}\breve{q}_i(\boldsymbol{w},\boldsymbol{p};\mathcal{N}). \tag{30}$$

For given $\boldsymbol{w}$ and $\boldsymbol{p}$, the expected warranty profit per unit sold under the MMNL model is given by

$$\begin{aligned}\breve{\Omega}(\boldsymbol{w},\boldsymbol{p};\mathcal{N}) &= \sum_{i\in\mathcal{N}}(p_i-c(w_i))\breve{q}_i(\boldsymbol{w},\boldsymbol{p};\mathcal{N})\\ &= \sum_{i\in\mathcal{N}}(p_i-c(w_i))\sum_{k=1}^{K}d_k\breve{q}_{i,k}(\boldsymbol{w},\boldsymbol{p};\mathcal{N})\\ &= \sum_{k=1}^{K}d_k\sum_{i\in\mathcal{N}}(p_i-c(w_i))\breve{q}_{i,k}(\boldsymbol{w},\boldsymbol{p};\mathcal{N})\\ &= \sum_{k=1}^{K}d_kR_k(\boldsymbol{w},\boldsymbol{p};\mathcal{N}),\end{aligned} \tag{31}$$

where $R_k(\boldsymbol{w},\boldsymbol{p};\mathcal{N})=\sum_{i\in\mathcal{N}}(p_i-c(w_i))\breve{q}_{i,k}(\boldsymbol{w},\boldsymbol{p};\mathcal{N})$ is the expected warranty profit contributed by customer segment k.

Likewise, the warranty-menu design and pricing problem taking into account customer heterogeneity can be formulated as

$$\max_{\mathcal{S}\subseteq\mathcal{N},\boldsymbol{p}}\breve{\Omega}(\boldsymbol{w},\boldsymbol{p};\mathcal{S}). \tag{32}$$

We highlight that the problem is much more difficult to solve than those before because the MMNL model is highly ill-behaved. In particular, the above-used reformulation technique fails in this problem. Recently, Li et al. (2019) developed efficient optimization algorithms for product-line pricing problem under the discrete MMNL model, which is quite similar to the warranty-menu design and pricing problem. However, our problem is indeed more complicated than theirs, as it involves the selection of candidate warranty options. In what follows, Li et al.'s (2019)) methodology will be followed to discuss the warranty-menu design and pricing problem.

We first take the first derivatives of $\breve{\Omega}(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S})$ with respect to p_i:

$$\frac{\partial \breve{\Omega}(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S})}{\partial p_i} = \breve{q}_i + \sum_{j \in \mathcal{S}} (p_j - c(w_j)) \frac{1}{\mu} \sum_{k=1}^{K} d_k \breve{q}_{i,k} \breve{q}_{j,k} - (p_i - c(w_i)) \frac{1}{\mu} \sum_{k=1}^{K} d_k \breve{q}_{i,k}, i \in \mathcal{S}. \tag{33}$$

Note that in the equation above, $\breve{q}_i(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S})$ and $\breve{q}_{i,k}(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S})$ are simplified as $\breve{q}_i$ and $\breve{q}_{i,k}$, respectively, for notation brevity.

By setting $\partial \breve{\Omega}(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S}) / \partial p_i$ to zero, we obtain the following iterative expression for the optimal profit margin:

$$p_i - c(w_i) = \mu + \sum_{k=1}^{K} \frac{d_k \breve{q}_{i,k}}{\breve{q}_i} R_k(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S}). \tag{34}$$

Following Lemma 2 in Li et al. (2019), we have the following result regarding the profit margins at optimality.

Proposition 1. *For any $\mathcal{S} \subseteq \mathcal{N}$, let $p_i^\star$ represent the optimal price of option $i \in \mathcal{S}$. Then, in general, $p_i^\star - c(w_i) \neq p_j^\star - c(w_j)$ for $i \neq j$.*

Proposition 1 reveals an essential finding: The equal-margin pricing policy in Theorems 1–3 no longer holds when taking into account heterogeneous customer population. An intuitive explanation is that under the MMNL model, customers in different segments generate an identical servicing cost for the same warranty option but have heterogeneous valuations for that option. As a result, optimal warranty prices and thus profit margins should be, more or less, also relevant to customer segments. Even when $d_1 = d_2 = \cdots = d_K$, i.e., the probability that an arriving customer is from each given segment is equal, it is clear from Eq. (34) that the profit margins $p_i^\star - c(w_i), i \in \mathcal{S}$, are still not equal. This suggests that the heterogeneity in customer population justifies a *nonequal-margin pricing* policy.

In contrast to the MNL-based warranty design and pricing problems studied above, the profit function $\breve{\Omega}(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S})$ under the MMNL model is far less well-behaved. We thus leave this challenging problem for future research. While, before ending the discussion, we briefly introduce the main findings in Li et al. (2019) to provide some clues for solving our problem. In particular, Li et al. (2019) have shown, in their product-line pricing problem, that the equal-margin pricing property and the profit concavity property with respect to the choice probability vector break down under the discrete MMNL model, even with strong symmetric assumption for price sensitivities. More importantly, by taking advantage of the profit concavity with respect to the choice probability vector in the MNL model, they reformulate the MMNL-based profit function as the summation of a set of quasi-concave functions. Three efficient algorithms are then developed to solve the product-line pricing problem in various scenarios, which can be potentially employed in the warranty-menu design and pricing problem.

6. Conclusions and further research

A recent trend in the EW market is that an increasing number of warrantors are offering flexible EW menus in which differentiated options are designed to segment customers and attain competitive advantages. In this paper, we investigate the design and pricing of an EW menu—offering warranty options with different lengths and prices—based on the MNL choice model. The warranty cost model is developed upon the well-established reliability and maintenance theory; while the customer utility model in MNL is built upon the empirical studies of EW from the economic/marketing perspective. We first consider a simple warranty-menu design and pricing problem from a warrantor's perspective: given a list of candidate warranty lengths, determine which candidate options to offer and the associated prices so as to maximize the warranty profit per unit sold. It is found that the optimal strategy is to offer all candidate options associated with a *cost-plus-margin* pricing policy, with a uniform profit margin for all options. However, if the menu size is limited due to practical constraints, then the options with the highest valuation margins should be selected.

In addition, we develop three extended models to generalize the original model. The first one deals with the design and pricing of a warranty menu with heterogeneous breadth types. Mathematically, this model provides a new perspective on the design and pricing of bundle menus. The second one focuses on a PM bundling problem in which a free PM program is bundled with selected warranty options. We find that the free PM program should be attached to a specific option only when it is feasible for that option. In particular, this PM bundling condition is a general result that is applicable to traditional product-line pricing problems in which free ancillary services are offered for selected products. Finally, the third one further considers customers with heterogeneous perceptions of product failure probability via the MMNL model. A finding particularly noteworthy is that the equal-margin pricing policy is no longer optimal in the presence of customer heterogeneity, justifying a nonequal-margin pricing policy.

This work makes an early effort to design and price EW menus based on the MNL model, and thus opens up many opportunities for future research. First, the warranty-menu design and pricing problem considering customer heterogeneity deserves in-depth investigations. Second, more features can be involved in a warranty menu, e.g., the starting and ending points, the renewal/cancellation policy, to name a few. Third, this study did not explicitly model the impact of product value on customers' warranty purchase decision. It is evident that customers are more likely to purchase an EW if the warranted product has a higher value (UK Competition Commission, 2003). We leave this interesting problem for future research. Finally, empirical studies on real data sets are still needed to demonstrate the goodness of fit and prediction accuracy of MNL in the warranty context.

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Appendix A. Technical proofs

Proof of Theorem 1. Let $\Omega(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S}) = \pi$ for any prices $\boldsymbol{p}$. Rearranging terms and after some algebraic manipulations, we have

$$\Omega(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S}) = \frac{\sum_{i \in \mathcal{N}} (p_i - c(w_i)) \exp\left(\frac{v(w_i) - p_i}{\mu}\right)}{1 + \sum_{i \in \mathcal{N}} \exp\left(\frac{v(w_i) - p_i}{\mu}\right)} = \pi$$
$$\Leftrightarrow \sum_{i \in \mathcal{S}} (p_i - c(w_i) - \pi) \exp\left(\frac{v(w_i) - p_i}{\mu}\right) = \pi.$$

One can see that the optimization problem (9) is equivalent to identifying a unique intersection between a strictly decreasing function and a 45-degree line, i.e.,

$$\pi = G(\pi) := \sum_{i \in \mathcal{S}} g_i(\pi), \tag{A.1}$$

where $g_i(\pi) = \max_{p_i} g_i(\pi, p_i)$ and

$$g_i(\pi, p_i) = (p_i - c(w_i) - \pi)\exp\left(\frac{\upsilon(w_i) - p_i}{\mu}\right).$$

By taking the first derivative of $g_i(\pi, p_i)$ with respect to p_i, we have

$$\frac{\partial g_i(\pi, p_i)}{\partial p_i} = \frac{1}{\mu}\exp\left(\frac{\upsilon(w_i) - p_i}{\mu}\right)(\mu - p_i + c(w_i) + \pi).$$

It is clear that $g_i(\pi, p_i)$ is increasing in p_i for $p_i \le c(w_i) + \pi + \mu$, and becomes decreasing thereafter. Hence, $g_i(\pi, p_i)$ attains its maximum at $p_i = c(w_i) + \pi + \mu$, and thus $g_i(\pi)$ becomes

$$g_i(\pi) = \mu \exp\left(\frac{\upsilon(w_i) - c(w_i) - \pi^* - \mu}{\mu}\right) > 0.$$

As $g_i(\pi)$ is always positive, it is beneficial to include all of the n warranty options in the menu. Also, there exists a unique π^* such that Eq. (A.1) (or equivalently, Eq. (12)) is satisfied. For a warranty option of length w_i, its optimal price is thus $p_i^* = c(w_i) + \pi^* + \mu$.

When the optimal pricing policy is applied, the warranty attach rate is given by

$$Q^* = \sum_{i\in\mathcal{N}} q_i(\boldsymbol{w}, \boldsymbol{p}^*; \mathcal{N}) = \frac{\sum_{i\in\mathcal{N}} \exp\left(\frac{\upsilon(w_i)-p_i^*}{\mu}\right)}{1 + \sum_{i\in\mathcal{N}} \exp\left(\frac{\upsilon(w_i)-p_i^*}{\mu}\right)}$$
$$= \frac{\pi^*/\mu}{1+\pi^*/\mu} = \frac{\pi^*}{\mu+\pi^*}.$$

The third equality follows directly from (12). This completes the proof. □

Proof of Corollary 1. By taking the first derivative of p_i^*/w_i with respect to w_i, we obtain

$$\frac{\partial(p_i^*/w_i)}{\partial w_i} = \frac{wc'(w_i) - c(w_i) - \pi^* - \mu}{w_i^2}.$$

Let $\zeta(w_i) := w_i c'(w_i) - c(w_i) - \pi^* - \mu$. Then, we have $\zeta'(w_i) = w_i c''(w_i) \ge 0$ because $c''(w_i) \ge 0$. Thus, $\zeta(w_i)$ is increasing in w_i. Recall that the feasible range of w_i is $[0, L - w_b]$. It is clear that $\zeta(0) = -\pi^* - \mu < 0$. However, there are two possible cases about the value of $\zeta(L - w_b)$ and thus the monotonicity of the price-length ratio:

(a) If $\zeta(L - w_b) \le 0$, then $\partial(p_i^*/w_i)/\partial w_i$ is negative for any $w_i \in [0, L - w_b]$. Thus, the price-length ratio p_i^*/w_i is decreasing in $w_i \in [0, L - w_b]$.
(b) If $\zeta(L - w_b) > 0$, then there exists a unique w° such that $\partial(p_i^*/w_i)/\partial w_i = 0$. In this case, $\partial(p_i^*/w_i)/\partial w_i$ is negative for $w_i \in [0, w^\circ)$ and becomes positive for $w_i \in (w^\circ, L - w_b]$. Hence, the price-length ratio p_i^*/w_i will be decreasing in $w_i \in [0, w^\circ)$ and then increasing in $w_i \in (w^\circ, L - w_b]$.

This completes the proof. □

Proof of Corollary 2. From Theorem 1, we know that π^* is the solution to (12). If $\eta_i = \upsilon(w_i) - c(w_i)$ increases, then the LHS of (12) also increases. As a result, π^* has to become larger in order to make the equality valid.

Theorem 1 also tells us that the profit margin of option i is $p_i^* - c(w_i) = \pi^* + \mu$. Then, it is straightforward to know that $p_i^* - c(w_i)$ is increasing in η_j, $\forall i, j \in \mathcal{N}$. This completes the proof. □

Proof of Corollary 3. This result can be proved by contradiction. Suppose the optimal offer set does not contain options 1 to m. Then, the warrantor can replace an existing option with the lowest valuation margin by a higher valuation-margin option that is currently not included in the offer set. According to Corollary 2, this will increase the expected warranty profit, and thus the original offer set is not optimal. This completes the proof. □

Proof of Theorem 2. The proof is quite similar to that of Theorem 1 and thus omitted. □

Proof of Corollary 4. The proof is quite similar to that of Corollary 1 and thus omitted. □

Proof of Theorem 3. Let $\widehat{\Omega}(\boldsymbol{w}, \boldsymbol{p}; \mathcal{M}) = \pi$. As before, the optimization problem (23) can be reformulated as finding an unique solution to

$$\pi = \max_{\mathcal{M}\subseteq\mathcal{N}, \boldsymbol{p}} \left\{ \sum_{i\in\mathcal{M}} g_i^{pm}(\pi, p_i) + \sum_{i\notin\mathcal{M}} g_i(\pi, p_i) \right\}, \tag{A.2}$$

where $g_i^{pm}(\pi, p_i) = (p_i - c^{pm}(w_i) - \pi)\exp(\frac{\upsilon(w_i)+n_i u^{pm} - p_i}{\mu})$ and $g_i(\pi, p_i) = (p_i - c(w_i) - \pi)\exp(\frac{\upsilon(w_i)-p_i}{\mu})$.

It is straightforward to know that for any given $\mathcal{M}$, the optimal pricing policy is

$$p_i = \begin{cases} c^{pm}(w_i) + \pi + \mu, & i \in \mathcal{M}; \\ c(w_i) + \pi + \mu, & i \notin \mathcal{M}. \end{cases}$$

By substituting p_i into (A.2), we have

$$\pi = \max_{\mathcal{M}\subseteq\mathcal{N}} \left\{ \sum_{i\in\mathcal{M}} g_i^{pm}(\pi) + \sum_{i\notin\mathcal{M}} g_i(\pi) \right\}, \tag{A.3}$$

where $g_i^{pm}(\pi) = \mu\exp(\frac{\upsilon(w_i)+n_i u^{pm} - c^{pm}(w_i) - \pi - \mu}{\mu})$ and $g_i(\pi) = \mu\exp(\frac{\upsilon(w_i)-c(w_i)-\pi-\mu}{\mu})$.

Then, for a given π, we need to choose an optimal $\mathcal{M}$ to solve (A.3) above. Mathematically, we need to compare $g_i^{pm}(\pi)$ and $g_i(\pi)$ for any given π. It is clear that $g_i^{pm}(\pi) \ge g_i(\pi)$ when $n_i u^{pm} - c^{pm}(w_i) + c(w_i) \ge 0$. Therefore, it is optimal to bundle the free PM program with option i if the program is feasible for that option.

Furthermore, when the optimal PM bundling and warranty pricing policies are applied, the warranty attach rate can be obtained by

$$Q^\ddagger = \sum_{i\in\mathcal{N}} \widehat{q}_i(\boldsymbol{w}, \boldsymbol{p}^\ddagger; \mathcal{M}^\ddagger)$$
$$= \frac{\sum_{i\in\mathcal{M}^\ddagger} \exp\left(\frac{\upsilon(w_i)+n_i u^{pm} - p_i^\ddagger}{\mu}\right) + \sum_{i\notin\mathcal{M}^\ddagger} \exp\left(\frac{\upsilon(w_i)-p_i^\ddagger}{\mu}\right)}{1 + \sum_{i\in\mathcal{M}^\ddagger} \exp\left(\frac{\upsilon(w_i)+n_i u^{pm} - p_i^\ddagger}{\mu}\right) + \sum_{i\notin\mathcal{M}^\ddagger} \exp\left(\frac{\upsilon(w_i)-p_i^\ddagger}{\mu}\right)}$$
$$= \frac{\pi^\ddagger}{\mu + \pi^\ddagger}.$$

The last equality follows directly from (25). This completes the proof. □

Proof of Corollary 5. By comparing (25) with (12), we find that if a feasible PM program is bundled with selected warranty options, then the LHS of (25) is higher than or at least equal to that of (12). Thus, it is straightforward to verify that $\pi\ddagger \ge \pi^*$.

By comparing $Q\ddagger$ with Q^*, we have

$$Q^\ddagger = \frac{\pi^\ddagger}{\mu+\pi^\ddagger} = \frac{1}{1+\mu/\pi^\ddagger} \ge \frac{1}{1+\mu/\pi^*} = Q^*.$$

The inequality is due to the fact that $\pi\ddagger \ge \pi^*$. This completes the proof. □

Proof of Proposition 1. First of all, notice that $\sum_{k=1}^{K} \frac{d_k \breve{q}_{i,k}}{\breve{q}_i} R_k(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S})$ is a weighted average of $R_k(\boldsymbol{w}, \boldsymbol{p}; \mathcal{S})$ with weights $d_k \breve{q}_{i,k}/\breve{q}_i$. It is clear that the weights are dependent on the option index i.

Assume for contradiction that $p_i^\star - c(w_i)$ is the same for all options $i \in \mathcal{S}$. Without loss of generality, denote $p_i^\star - c(w_i) = \vartheta$ for all i, where ϑ is a constant. Then,

$$\begin{aligned} R_k(\boldsymbol{w}, \boldsymbol{p}^\star; \mathcal{S}) &= \sum_{i \in \mathcal{S}} (p_i^\star - c(w_i)) \breve{q}_{i,k}(\boldsymbol{w}, \boldsymbol{p}^\star; \mathcal{S}) \\ &= \vartheta \sum_{i \in \mathcal{S}} \breve{q}_{i,k}(\boldsymbol{w}, \boldsymbol{p}^\star; \mathcal{S}) = \vartheta \left(1 - \breve{q}_{0,k}(\boldsymbol{w}, \boldsymbol{p}^\star; \mathcal{S})\right) \\ &= \vartheta \left(1 - \frac{1}{1 + \sum_{i \in \mathcal{S}} \exp\left(\frac{v_k(w_i) - p_i^\star}{\mu}\right)} \right) \\ &= \vartheta \left(1 - \frac{1}{1 + \sum_{i \in \mathcal{S}} \exp\left(\frac{v_k(w_i) - c(w_i) - \vartheta}{\mu}\right)} \right). \end{aligned}$$

As can be seen, the value of $R_k(\boldsymbol{w}, \boldsymbol{p}^\star; \mathcal{S})$ depends on the segment index k, thus in general $R_k(\boldsymbol{w}, \boldsymbol{p}^\star; \mathcal{S})$ is not equal across customer segments. As a result, the RHS of Eq. (34) is a weighted average of the vector $(R_1, R_2, \ldots, R_K)$ with non-equal elements and the weights are associated with the warranty option index i. Therefore, the weighted average must be a value that is dependent on index i. This results in a contradiction with our initial assumption that $p_i^\star - c(w_i) = \vartheta$ for all i. $\square$

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