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Innovative Applications of O.R.

An unpunctual preventive maintenance policy under two-dimensional warranty☆

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ABSTRACT

In the business-to-consumer context, e.g., the automobile industry, preventive maintenance (PM) of warranted items usually relies upon customers to return their items to authorized maintenance centers according to prescribed schedules. However, item owners may be unpunctual, causing actual maintenance instants deviating from scheduled instants. This paper studies the impact of customer unpunctuality on the optimization of PM policy and the resultant warranty expenses. An unpunctual imperfect PM policy, which allows customers to advance or postpone scheduled PM activities in a tolerated range, is proposed for repairable items sold with a two-dimensional warranty. The expected total warranty costs of the unpunctual (and punctual) PM policies are derived under the assumption that customer unpunctuality is governed by a specific probability distribution. The optimization and comparison of the two policies are investigated in different scenarios regarding the product's failure rate function. The results for two possible unpunctuality distributions—uniform and triangular—are discussed and compared. Numerical studies show that the expected total warranty cost of the unpunctual policy could be either higher or lower than that of the punctual policy, depending on customer behaviors and the shape of failure rate function. Accordingly, manufacturers could induce customers to adjust their unpunctuality behaviors by modifying PM policies or introducing penalty/bonus mechanisms.

1. Introduction

1.1. Background and Motivation

In order to retain in today's highly competitive market, manufacturers are forced not only to provide high quality products on time but also to offer appealing after-sales services for their products. One such kind of after-sales services is product warranty. Nowadays, almost all products are sold with warranty contracts due to competition, industrial obligations, and/or customer requirements (Luo & Wu, 2018; Xie & Ye, 2016). Offering warranty is by no means free from the manufacturers' perspective. The warranty cost, resulting from the servicing of warranty claims, is a huge financial burden to manufacturers, and can account for as much as 15% of net sales (Murthy & Djamaludin, 2002). In reality, most manufacturers regard warranty as an overhead element, and they are always under pressure to keep warranty expenses staying within limits (Liu, Wu, & Xie, 2015; Zhao, He, & Xie, 2018). As a result, manufacturers are constantly seeking effective warranty cost reduction strategies. Incorporating appropriate preventive maintenance (PM) programs into warranty policies is one feasible way that offers potential solution to warranty cost reduction (Chien, Zhang, & Yin, 2019; Kim, Djamaludin, & Murthy, 2004).

Generally, PM activities are performed before an item breaks down and aim to reduce the item's degradation and its risk of failure (Shafiee & Chukova, 2013). In practice, the PM schedule of warranted items is usually prescribed by the manufacturer, while PM implementation relies upon either maintenance crews to execute the operations onsite (for installed systems, such as household heating systems and heavy machines) or item owners to return their items to authorized maintenance centers. This paper focuses particularly on the latter case, though our model can be easily applied to the former case. In this situation, one common yet challenging issue is maintenance unpunctuality that stems from the separate nature of PM specification and execution (He, Maillart, & Prokopyev, 2017). For instance, in the automobile warranty and maintenance context, it is not uncommon that some owners do not follow the manufacturer-prescribed PM schedule

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strictly. Possible reasons include (*i*) scheduled PM instants are at working days, (*ii*) exact PM instants may be difficult to record precisely, especially for heavy-usage customers, and (*iii*) some customers might simply miss certain PM instants.

In recent years, a Chinese automaker, SAIC-GM-Wuling Automobile, has adopted an unpunctual PM policy in its warranty and maintenance practice. This policy allows customers to return their vehicles slightly earlier or later than scheduled PM instants in a tolerated range, provided that the vehicles are still under warranty. From the manufacturer's perspective, customer unpunctuality may deteriorate maintenance effectiveness, which in turn increases warranty servicing cost. Therefore, it is necessary to explore the impacts of customer unpunctuality on the optimization of PM policy and the resultant warranty expenses. This paper aims to serve this necessity, with specific focus on automobiles protected by a two-dimensional (2-D) warranty.

1.2. Related literature

The literature on warranty-oriented PM optimization generally deals with optimal trade-off between the investment in a PM program and the resultant reduction in warranty servicing costs. This topic has received considerable attentions over the last decades; see Shafiee and Chukova (2013) for a comprehensive literature review. Since this paper is interested in optimal unpunctual PM policy under 2D warranty, we confine our attention to the interplay between 2D warranties and PM policies.

A 2D warranty simultaneously employs two variables, usually age and usage, to characterize the warranty policy (Xie, Shen, & Zhong, 2017; Ye & Murthy, 2016). A typical example of 2D warranties is automobile warranties, with usage being mileage travelled. In the 2D warranty context, the optimization of imperfect PM policies has been gaining in popularity. Huang and Yen (2009) and Huang, Chen, and Ho (2013) sought to determine optimal 2D warranty terms with consideration of periodic and non-periodic (reliability-based) PM strategies, respectively. Wang and Su (2016) and Su and Wang (2016) proposed a 2D PM policy for items sold with a 2D warranty, under which items should be preventively maintained every K units of age or every L units of usage, whichever comes first. Iskandar and Husniah (2017) studied a 2D lease contract for repairable equipment subject to a periodic imperfect PM policy. Wang, Zhou, and Peng (2017a) developed a game-theoretical decision model for a 2D warranty policy with a periodic PM policy, where customers are entitled to either accept or reject each PM during extended warranty period. Huang, Huang, and Ho (2017) studied a new 2D extended warranty policy in which customized PM schedules are offered to different customer categories. Wang, Liu, Liu, and Li (2017b) examined the worthiness of pre-sale upgrade and post-sale PM policies for second-hand products sold with a 2D warranty. Wang, Li, and Xie (2019) investigated optimal upgrade and PM strategies for industrial equipment under successive usage-based lease contracts with a 2D warranty period. Li, Liu, Wang, and Li (2019) studied optimal burn-in and PM strategies for repairable products sold with a 2D base warranty and an optional extended warranty. Recently, Wang and Xie (2018) reviewed the state of the art on integrated studies of 2D warranties and PM policies.

Nevertheless, the existing studies on warranty-oriented PM optimization *implicitly* assume that scheduled PM activities during the warranty period are punctually executed. As stated earlier, however, maintenance crews/customers may be unpunctual, i.e., the PM activities may be performed either earlier or later than intended. To the best of our knowledge, there are currently three reported publications related to the optimal planning of unpunctual PM/inspection policies. He et al. (2017) studied optimal age replacement policies with and without minimal repair in anticipation of maintenance unpunctuality. They assumed that the potential unpunctuality of maintenance crew causes actual replacement instants to deviate from planned instants in a probabilistic manner. Zhao, Al-Khalifa, Hamouda, and Nakagawa (2017) also discussed a random age replacement policy in which an item is replaced before failure at a random point of time. In a different problem setting, Scarf, Cavalcante, and Lopes (2019) investigated an inspection-replacement strategy for a critical system (whose failures can be immediately revealed) in which inspection opportunities arise at random.

Both He et al. (2017) and Zhao et al. (2017) focus on optimal preventive replacement problems over an infinite planning horizon, where optimal replacement thresholds are determined by minimizing long-run expected cost rates. Similarly, Scarf et al. (2019) deal with the long-run cost rate induced by inspections and subsequent replacements. The long-run cost rate is formulated as the ratio of expected maintenance cost per renewal cycle to expected renewal cycle length, according to the renewal reward theorem. In real applications, however, maintenance planning horizons are mostly *finite*, e.g., the limited warranty period for Ford vehicles is 3 years or 36000 miles, whichever comes first. In this scenario, the optimization problem is then to identify an optimal number of PM activities to minimize the expected total maintenance cost over the finite horizon of interest (e.g., warranty period). On the other hand, a PM activity is usually *imperfect* in the sense that the item's reliability status, after PM, is improved but not as good as new. Our work distinguishes itself from He et al. (2017), Zhao et al. (2017), and Scarf et al. (2019) by well incorporating the two aspects—finite planning horizon and imperfect PM effect.

1.3. The proposed unpunctual PM policy

This paper makes an early attempt to study an unpunctual imperfect PM policy for repairable items sold with a 2D warranty. In our setting, the potential unpunctuality of customers returning their items for PM is what causes actual PM instants differing from scheduled instants. The unpunctual PM policy is developed upon the 2D PM policy in Wang and Su (2016) and Su and Wang (2016). The unique characteristic of this policy is that it allows customers to slightly advance or postpone planned (originally periodical) PM interventions in a tolerated range. Specifically, under the unpunctual policy, the jth PM action should be performed at age $jK \pm \Delta K$ or at usage $jL \pm \Delta L$, whichever occurs first. In particular, K and L are scheduled age- and usage-based PM intervals, and $\pm \Delta K$ and $\pm \Delta L$ are tolerated unpunctual ranges in age and usage dimensions, respectively. The unpunctual policy gives customers a flexibility, to some extent, of planning PM activities according to their personal schedules, instead of following the manufacturer-prescribed PM schedule strictly.

Since the proposed unpunctual PM policy tolerates a degree of customer unpunctuality in PM execution, two natural questions arise: *What are the impacts of customer unpunctuality on the optimization of PM policy and the resulting warranty cost? How to design (or modify) a new (or an existing) PM policy in anticipation of customer unpunctuality?* The primary purpose of this paper is to answer these questions. To this end, we first identify the optimal PM policy (i.e., number and degree of PM activities) and the corresponding warranty cost that anticipate customer unpunctuality, and then compare them with those of the punctual policy. The remainder of the article is organized as follows. Section 2 develops two optimization models to determine optimal PM decisions for both punctual and unpunctual PM policies. Essentially, the unpunctual model extends its punctual counterpart by assuming that customer unpunctuality is governed by a specific probability distribution. Section 3 deals with the optimization and comparison of the two PM policies. Because it is difficult to carry out the analysis

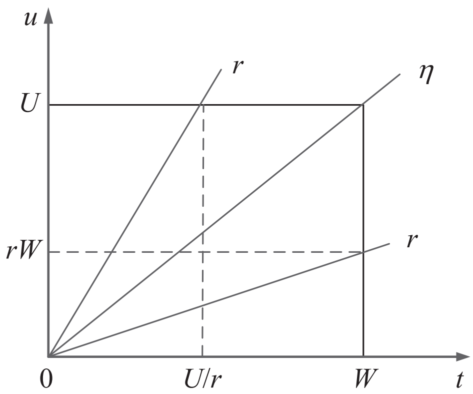

Fig. 1. Illustration of the two-dimensional warranty policy.

in a general setting, we first consider two special forms—linear and quadratic—of the failure rate function, and then extend the analysis to a general failure rate form. The corresponding results under two possible unpunctuality distributions—uniform and triangular—are also discussed and compared. Section 4 presents numerical studies to demonstrate the unpunctual PM policy and then answer the two questions raised above. Finally, Section 5 concludes this paper and suggests some future research topics.

2. Model formulation

2.1. Two-dimensional warranty and failure modeling

Consider that a repairable item is sold with a 2D free repair warranty policy which is characterized by a rectangular region $\Omega_R(W,U)=\{(t,u): t\in(0,W) \text{ and } u\in(0,U)\}$, as shown in Fig. 1. Under this policy, any eligible failures within the warranty period will be rectified by the manufacturer at no cost to the customer. The warranty contract terminates when either the item age reaches its limit W or the cumulative usage exceeds its limit U, whichever comes first. That is, if the usage rate is low ($r\le\eta=U/W$), then the warranty expires at age W; while if the usage rate is high ($r>\eta$), then the warranty ceases at age U/r when usage reaches U (see Fig. 1).

In this work, the repairable item of interest is treated as a black-box system, and the marginal approach is adopted to model item failures in terms of age and usage (Wang & Xie, 2018; Wu, 2014). The marginal approach reduces the 2D failure modeling problem to a one-dimensional problem, by treating the item's usage rate R as a covariate. An assumption adopted here is that each item has a constant usage rate during the warranty period, while the usage rates vary randomly across the customer population (Ye, Murthy, Xie, & Tang, 2013). Rationality of this assumption has been supported by real-life warranty data analysis (Gupta, De, & Chatterjee, 2014; Lawless, Crowder, & Lee, 2009) which reveals that the cumulative usage of a vehicle is approximately linear over its age. In this manner, the usage rate R can be modeled by a random variable with cumulative distribution function (cdf) $G(r)$ and probability density function (pdf) $g(r)$, $0\le r<\infty$.

The item is designed for some nominal usage rate r_0, conditional on which the distribution for the time to first failure T_0 is $F_0(t)$ and the corresponding failure rate is $\lambda_0(t)$. When the item's actual usage rate differs from the nominal value, its reliability may be affected. In this study, the effect of usage rate on the item reliability is modeled by the *accelerated failure time* (AFT) model. Let T_r represent the time to first failure under actual usage rate r. Conditional on $R=r$, T_r is linked to r through the following formulation (Jack, Iskandar, & Murthy, 2009):

$$\frac{T_r}{T_0}=\left(\frac{r_0}{r}\right)^{\gamma}, \tag{1}$$

where $\gamma>0$ is the acceleration factor. The AFT relationship (1) implies that a heavy usage increases the deterioration rate, and this, in turn, shortens the time to failure. Through this relationship, the cdf $F(t|r)$ and failure rate $\lambda(t|r)$ of T_r can be easily linked to $F_0(t)$ and $\lambda_0(t)$, respectively.

The AFT model has been widely adopted for failure modeling; see, e.g., Iskandar and Husniah (2017), Li et al. (2019), Tong, Song, and Liu (2017), Wang et al. (2017b), and Ye and Murthy (2016), among others. It is acknowledged that there are other models or approaches to characterize item failures for 2D warranty modeling, e.g., the bivariate approach (Murthy, Iskandar, & Wilson, 1995) and the alternative time scale model with random usage (Finkelstein, 2004; 2008). Interested readers are referred to Wang and Xie (2018) and Wu (2014) for more discussions.

The two-parameter Weibull distribution with scale parameter α and shape parameter β is adopted to model item failures under nominal usage rate r_0, due to its flexibility of describing increasing, constant, and decreasing failure rates. In this case, the failure rate under actual usage rate r is given by

$$\lambda(t|r)=\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}\left(\frac{r}{r_0}\right)^{\gamma\beta}. \tag{2}$$

We confine our attention to item rectification through minimal repair with negligible duration. This consideration is appropriate for complex systems, e.g., vehicles, where a system failure occurs due to a component breakdown and can be rectified by replacing the failed component with a new one (Murthy, 1991). As such, the item's failure intensity after repair is nearly the same as that immediately before failure. Under the minimal repair assumption, it is well known that item failures over time will occur according to a non-homogeneous Poisson process (NHPP) with an intensity function identical to the failure rate $\lambda(t|r)$.

2.2. Punctual PM policy

In this subsection, the expected warranty servicing cost, in the absence of maintenance unpunctuality, is formulated from the manufacturer's perspective. That is to say, the execution of PM activities strictly follows the planned schedule. This basic framework will be further extended in Section 2.3 to incorporate the impact of customer unpunctuality.

We consider the following (punctual) PM policy: during the warranty period, the items should be preventively maintained every $K=W/(n+1)$ units of time or every $L=U/(n+1)$ units of usage, whichever occurs first (Wang & Su, 2016). Notice that $L/K=U/W=\eta$. In this way, all customers will receive n periodical PM services, regardless of their usage rates. This policy gives customers a sense of fairness, to some extent.

In real applications, customers are required to return their items for PM according to the aforementioned schedule, with which some age reductions would occur between the actual and virtual ages. The *virtual age* concept is proposed by Kijima, Morimura, and Suzuki (1988) and Kijima (1989) and then widely adopted in imperfect maintenance modeling. In the automobile industry, a free PM program generally covers a set of moderate services, such as cleaning, oil and filter changes, adjustments, multipoint inspection, and/or replacement of certain worn components.[1]

[1] For a summary of free vehicle maintenance programs offered by major automakers, please refer to https://static.ed.edmunds-media.com/unversioned/img/pdf/free.vehicle.maintenance.programs/free.vehicle.maint.5.pdf.

As a result, it is reasonable to assume that an imperfect PM action only compensates the damage accumulated since the last action, which is in accordance with the Kijima Type I model (Kijima, 1989). The amount of age reduction depends on the PM effort m applied. Larger value of m corresponds to greater PM effort and thus more age reduction. The maintenance effort is constrained so that $0 \le m \le M$, with $m = 0$ implying no PM action and M representing the upper limit of the PM effort.

To analyze the punctual PM policy in detail, two cases are considered, i.e., $r \le \eta$ and $r > \eta$.

Case 1: $r \le \eta$. In this case, the 2D warranty would terminate at age W, and the age-based PM interval is then $K = W/(n+1)$. Thus, the time instant of performing the jth PM activity is $\tau_j = jK$, $j = 1, 2, \ldots, n$, with $\tau_0 = 0$ and $\tau_{n+1} = W$. Based on the discussion above, the virtual age v_j immediately after performing the jth PM activity is modeled as (Kim et al., 2004):

$$v_j = v_{j-1} + \delta(m)(\tau_j - \tau_{j-1}). \tag{3}$$

The age reduction factor $\delta(m) \in [0, 1]$ is a decreasing function of m with $\delta(0) = 1$ and $\delta(M) = 0$. If $m = M$, then the damage accumulated during the $(j-1)$th and the jth PM interventions is fully compensated, i.e., $v_j = v_{j-1}$; if $m = 0$, then no PM action is taken, i.e., $v_j = \tau_j$; while in a general case $m \in (0, M)$, a PM activity restores the condition of an item only partially. In this study, the level of PM effort m is assumed to remain unchanged throughout the warranty period. As such, Eq. (3) can be recursively derived as $v_j = \delta(m)\tau_j = j\delta(m)K$, $j = 1, 2, \ldots, n$.

With the assumption of minimal repair upon a failure, the failure process between two successive PM activities is also an NHPP. Then, the conditional expected number of item failures over the warranty period is given by

$$E[N(n, m)|r \le \eta] = \sum_{j=0}^{n} \int_{j\delta(m)K}^{j\delta(m)K+K} \lambda(t|r)\mathrm{d}t. \tag{4}$$

Case 2: $r > \eta$. In this situation, the 2D warranty would terminate at age U/r, and the actual age-based PM interval is $L/r = U/((n+1)r)$. The time instant of performing the jth PM activity is $\tau_j = jL/r$, $j = 1, 2, \ldots, n$, with $\tau_0 = 0$ and $\tau_{n+1} = U/r$. The virtual age right after the jth PM activity is $v_j = j\delta(m)L/r$, $j = 1, 2, \ldots, n$. Thus, the conditional expected number of item failures during the warranty period, i.e., $E[N(n, m)|r > \eta]$, is also given by Eq. (4), with L/r replacing W.

Therefore, by considering the random nature of usage rate R, the expected total warranty servicing cost of the punctual PM policy can be calculated by

$$\begin{aligned} E[\mathcal{C}(n, m)] = {} & c_f \int_0^{\eta} E[N(n, m)|r \le \eta] g(r)\mathrm{d}r \\ & + c_f \int_{\eta}^{\infty} E[N(n, m)|r > \eta] g(r)\mathrm{d}r + nC_p(m), \end{aligned} \tag{5}$$

where c_f is the average cost of a minimal repair under warranty and $C_p(m)$ denotes the cost for a PM activity with level m, and $C_p(m)$ increases as m increases.

For the punctual policy, the manufacturer's problem is to determine the optimal PM decision over the warranty period, i.e., n^* and m^*, so as to minimize the expected total warranty cost (5). That is,

$$\min E[\mathcal{C}(n, m)] \ s.t. \ 0 \le m \le M, n \in Z^+ \cup \{0\}.$$

Once the optimal number of PM actions n^* is determined, the optimal age- and usage-based PM intervals are set to $K^* = W/(n^* + 1)$ and $L^* = U/(n^* + 1)$, respectively.

2.3. Unpunctual PM policy

In this subsection, the PM policy studied in Section 2.2 is extended to incorporate customer unpunctuality. The unpunctual PM policy extends its punctual counterpart by allowing slight earliness or tardiness of customers executing each (originally periodical) PM activity, and the tolerated ranges are $\pm\Delta K$ and $\pm\Delta L$ in age and usage dimensions, respectively. In other words, the jth PM activity should be performed at age $jK \pm \Delta K$ or at usage $jL \pm \Delta L$, whichever occurs first. We suppose that ΔK and ΔL are predetermined constants and, for the sake of simplicity, $\Delta L/\Delta K = \eta$. Note that when $\Delta K = \Delta L = 0$, the unpunctual policy reduces to its punctual counterpart.

For illustrative purposes, Table 1 shows the detailed schedule of n unpunctual PM activities over the warranty period. Fig. 2 further shows the age-based tolerated range for the jth unpunctual PM activity. One can see that for a customer with usage rate $r \le \eta$, the jth PM activity should be carried out within age interval $[jK - \Delta K, jK + \Delta K]$; while for a customer with usage rate $r > \eta$, the jth PM activity should be performed during age interval $[(jL - \Delta L)/r, (jL + \Delta L)/r]$.

Under the unpunctual PM policy, one essential issue is how to characterize customer unpunctuality, as different customers may bear different unpunctual behaviors. Let Y_j, $j = 1, 2, \ldots, n$, represent the deviation between the actual instant of the jth PM action and the punctual instant. That is,

$$Y_j = \begin{cases} \tau_j - jK, & \text{if } r \le \eta, \\ \tau_j - jL/r, & \text{if } r > \eta. \end{cases} \tag{6}$$

It is clear that if $Y_j < 0$, then the jth PM is implemented earlier than planned and *vice versa* if $Y_j > 0$. Since customer behaviors are random, this study models Y_j's, $j = 1, 2, \ldots, n$, as independent and identically distributed (*i.i.d.*) random variables, defined in a finite support, with cdf $Z(y)$ and pdf $z(y)$. More specifically,

- for a customer with usage rate $r \le \eta$, Y_j's, $j = 1, 2, \ldots, n$, follow a specific distribution with cdf $Z(y_j|r \le \eta)$ and pdf $z(y_j|r \le \eta)$, $Y_j \in [-\Delta K, \Delta K]$.
- for a customer with usage rate $r > \eta$, Y_j's, $j = 1, 2, \ldots, n$, follow a specific distribution with cdf $Z(y_j|r > \eta)$ and pdf $z(y_j|r > \eta)$, $Y_j \in [-\Delta L/r, \Delta L/r]$.

An underlying assumption adopted here is that customers will respect the tolerated range, which guarantees that the support of Y_j will be $[-\Delta K, \Delta K]$ or $[-\Delta L/r, \Delta L/r]$. This assumption is reasonable as many automakers (e.g., Ford and SAIC-GM-Wuling) prescribe that *failure to perform scheduled maintenance as specified in the owner's manual will invalidate warranty coverage on parts affected by the lack of maintenance.*[2] This prescription will force customers to respect the tolerated range of the unpunctual PM policy. Another key assumption is that Y_j is independent of the item's failure time and the scheduled PM instants, i.e., Y_i is independent of Y_l if $i \ne l$. Further assume that the nth moment about random variable Y_j exists. It is also necessary to point out that $Y_j|_{r>\eta} = \eta Y_j|_{r\le\eta}/r$, according to the discussions above (recall that $\Delta L = \eta\Delta K$). This insight is essential in the derivation of Eqs. (12), (21), and (25) later on.

Now we are ready to analyze the proposed unpunctual PM policy in detail. As before, two cases—$r \le \eta$ and $r > \eta$—are considered.

Case 1: $r \le \eta$. In this case, Y_j is modeled by a random variable with conditional pdf $z(y_j|r \le \eta)$, $y_j \in [-\Delta K, \Delta K]$. The jth PM instant is thus $\tau_j = jK + y_j$, $j = 1, 2, \ldots, n$, with $\tau_0 = 0$ and $\tau_{n+1} = W$. Hence, the virtual age immediately after the jth PM activity is $v_j = \delta(m)\tau_j = \delta(m)(jK + y_j)$, with $v_0 = 0$.

Conditioning on $Y_j = y_j$, $j = 1, 2, \ldots, n$, the conditional expected number of item failures over the warranty period is

[2] 2019 Model Year Ford Warranty Guide: https://www.ford.com/cmslibs/content/dam/brand_ford/en_us/brand/resources/general/pdf/warranty/2019-Ford-Car-Truck-Warranty-version-1_frdwa_EN-US_04_2018.pdf.

Table 1
Illustration of the unpunctual PM schedule over the warranty period.

	1st PM	2nd PM	...	nth PM	Warranty limits
Age-based interval	$K \pm \Delta K$	$2K \pm \Delta K$	...	$nK \pm \Delta K$	$W = (n+1)K$
Usage-based interval	$L \pm \Delta L$	$2L \pm \Delta L$	...	$nL \pm \Delta L$	$U = (n+1)L$

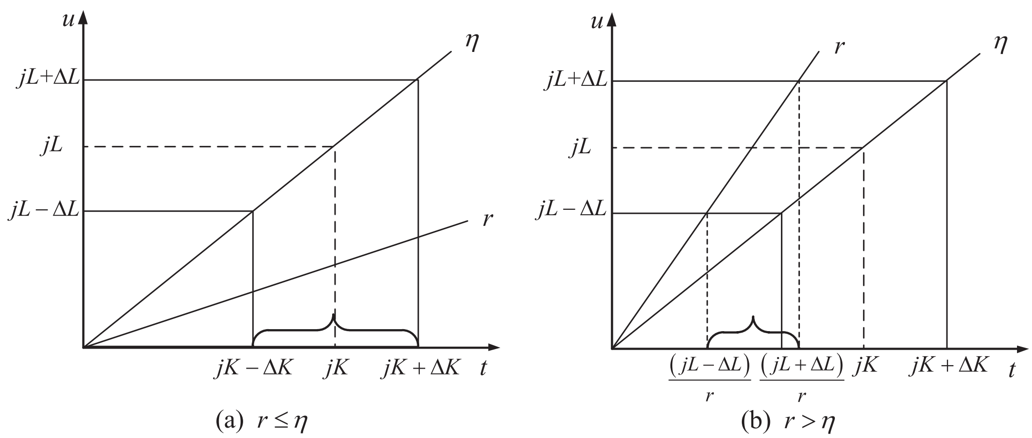

Fig. 2. Illustration of the tolerated range of the jth unpunctual PM activity.

$$
\begin{aligned}
&E[\widetilde{N}(n,m)|r \le \eta; y_1, \ldots, y_n] \\
&= \int_0^{K+y_1} \lambda(t|r)\mathrm{d}t + \sum_{j=1}^{n-1} \int_{\delta(m)(jK+y_j)}^{\delta(m)(jK+y_j)+K+y_{j+1}-y_j} \lambda(t|r)\mathrm{d}t \\
&+ \int_{\delta(m)(nK+y_n)}^{\delta(m)(nK+y_n)+K-y_n} \lambda(t|r)\mathrm{d}t.
\end{aligned} \tag{7}
$$

By removing the conditioning on Y_j, the conditional number of failures during the warranty period becomes

$$
\begin{aligned}
&E[\widetilde{N}(n,m)|r \le \eta] \\
&= \int_{-\Delta K}^{\Delta K} \cdots \int_{-\Delta K}^{\Delta K} E[\widetilde{N}(n,m)|r \le \eta; y_1, \ldots, y_n] \\
&\times \prod_{j=1}^{n} z(y_j|r \le \eta)\mathrm{d}y_1 \ldots \mathrm{d}y_n,
\end{aligned} \tag{8}
$$

where $E[\widetilde{N}(n,m)|r \le \eta; y_1, \ldots, y_n]$ is given by (7).

Case 2: $r > \eta$. In this situation, Y_j is a random variable with conditional pdf $z(y_j|r > \eta)$, $y_j \in [-\Delta L/r, \Delta L/r]$, $j = 1, 2, \ldots, n$. The virtual age after the jth PM activity is $v_j = \delta(m)(jL/r + y_j)$, with $v_0 = 0$. Then, conditioning on $Y_j = y_j$, $j =$ 1, 2,..., n, the conditional number of item failures over the warranty period, $E[\widetilde{N}(n,m)|r > \eta; y_1, \ldots, y_n]$, is given by Eq. (7), with L/r replacing K and U/r replacing W.

Similarly, by removing the conditioning on Y_j, the conditional expected number of item failures during the warranty period becomes

$$
\begin{aligned}
&E[\widetilde{N}(n,m)|r > \eta] \\
&= \int_{-\frac{\Delta L}{r}}^{\frac{\Delta L}{r}} \cdots \int_{-\frac{\Delta L}{r}}^{\frac{\Delta L}{r}} E[\widetilde{N}(n,m)|r > \eta; y_1, \ldots, y_n] \\
&\times \prod_{j=1}^{n} z(y_j|r > \eta)\mathrm{d}y_1 \ldots \mathrm{d}y_n.
\end{aligned} \tag{9}
$$

Finally, by considering the random nature of usage rate R, the expected total warranty servicing cost of the unpunctual PM policy can be calculated by

$$
\begin{aligned}
E[\widetilde{C}(n,m)] = c_f \int_0^{\eta} E[\widetilde{N}(n,m)|r \le \eta]g(r)\mathrm{d}r \\
+ c_f \int_{\eta}^{\infty} E[\widetilde{N}(n,m)|r > \eta]g(r)\mathrm{d}r + nC_p(m),
\end{aligned} \tag{10}
$$

where $E[\widetilde{N}(n,m)|r \le \eta]$ is given by (8) and $E[\widetilde{N}(n,m)|r > \eta]$ by (9).

For the unpunctual policy, the optimization problem is to identify the optimal PM decision, i.e., $\widetilde{n}^*$ and $\widetilde{m}^*$, over the warranty period to minimize the expected total warranty cost (10). Mathematically, it can be expressed as:

$$
\min \; E[\widetilde{C}(n,m)] \; s.t. \; 0 \le m \le M, n \in Z^+ \cup \{0\}.
$$

3. Model analysis

In this section, the effects of customer unpunctuality on the optimal PM policy and the total warranty cost are explored by comparing optimal punctual and unpunctual policies. It is well known that the Weibull failure rate in Eq. (2) bears different shapes according to the value of shape parameter β. For the sake of mathematical tractability, we first consider two special cases in which $\beta = 2$ and $\beta = 3$, respectively. The two cases represent linear and quadratic approximations of a general failure rate function. These simple failure rate forms are helpful for obtaining some analytical insights. Afterwards, a simple but efficient simulation algorithm is presented to examine the results for any general value of β.

3.1. Model analysis for $\beta = 2$

In this subsection, we analyze the punctual and unpunctual PM policies in the case of $\beta = 2$. The failure rate is given by $\lambda(t|r) = (r/r_0)^{2\gamma}(2/\alpha^2)t$, which is a linearly increasing function. We note that linear failure rate/intensity functions have been widely adopted in 2D warranty modeling and analysis (see, e.g., Huang et al., 2013; Huang & Yen, 2009; Su & Wang, 2016; Wang et al., 2017a).

3.1.1. Warranty cost models

To facilitate detailed analysis, we first derive the warranty cost models of the two PM policies. Recall that $K = W/(n+1)$ and $L = U/(n+1)$. Substituting $\lambda(t|r) = (r/r_0)^{2\gamma}(2/\alpha^2)t$ into Eq. (5) and after some algebraic operations, the expected total warranty servicing cost of the punctual PM policy is derived as

$$E[\mathcal{C}_1(n,m)] = c_f\big(\mathcal{A}W^2 + \mathcal{B}U^2\big)\frac{1+n\delta(m)}{n+1} + nC_p(m), \tag{11}$$

where $\mathcal{A} = \alpha^{-2}\int_0^\eta (\frac{r}{r_0})^{2\gamma} g(r)\mathrm{d}r$ and $\mathcal{B} = \alpha^{-2}\int_\eta^\infty \frac{r^{2\gamma-2}}{r_0^{2\gamma}} g(r)\mathrm{d}r$.

Similarly, for $n \ge 1$, the expected total warranty servicing cost of the unpunctual PM policy becomes

$$\begin{aligned} E[\widetilde{\mathcal{C}}_1(n,m)] = {} & c_f\big(\mathcal{A}W^2 + \mathcal{B}U^2\big)\frac{1+n\delta(m)}{n+1} + nC_p(m) \\ & + c_f(\mathcal{A}+\mathcal{B}\eta^2)(1-\delta(m))(E^2[Y_1|_{r\le\eta}] \\ & + nVar(Y_1|_{r\le\eta})), n \ge 1, \end{aligned} \tag{12}$$

where $E[Y_1|_{r\le\eta}]$ and $Var(Y_1|_{r\le\eta})$ are the mean and variance of $Y_1|_{r\le\eta}$ in the case of $r \le \eta$, respectively. The detailed derivation of (12) can be found in Appendix A. Note that when $n = 0$, $E[\widetilde{\mathcal{C}}_1(n = 0, m)] = E[\mathcal{C}_1(n = 0, m)] = c_f(\mathcal{A}W^2 + \mathcal{B}U^2)$.

It is worth mentioning that when $\beta = 2$, only the mean and variance of $Y_1|_{r\le\eta}$ (recall that $Y_j|_{r\le\eta}$, $j = 1, 2, \ldots, n$, are *i.i.d.* and $Y_j|_{r>\eta} = \eta Y_j|_{r\le\eta}/r$) are needed for obtaining the expected warranty cost of the unpunctual policy, and the exact distribution of $Y_1|_{r\le\eta}$ is not a necessity. This property makes it easy to evaluate the expected warranty cost, because the mean and variance of $Y_1|_{r\le\eta}$ can be empirically estimated from field survey data.

3.1.2. Optimization of the two PM policies

We then determine the optimal PM decisions for the two PM policies whose expected total warranty costs have been derived as in (11) and (12), respectively. The following two propositions demonstrate the existence and uniqueness of optimal number of PM actions for the two PM policies, i.e., n^* and $\tilde{n}^*$, respectively.

Proposition 1. *For any fixed value of $m \in (0, M]$, the optimal number of PM actions, n^*, for the punctual PM policy is unique and finite, and it is the solution of*

$$\mathcal{J}(n^*+1) \le \frac{C_p(m)}{c_f(1-\delta(m))(\mathcal{A}W^2+\mathcal{B}U^2)} \le \mathcal{J}(n^*), \tag{13}$$

where $\mathcal{J}(n) = \frac{1}{n(n+1)}$.

Proof. For a given PM level m, the expected total warranty cost, $E[\mathcal{C}_1(n,m)]$, becomes a discrete univariate function of n. To obtain the optimal number of PM actions that minimizes $E[\mathcal{C}_1(n,m)]$, the following two inequalities are formulated: $E[\mathcal{C}_1(n,m)] \le E[\mathcal{C}_1(n-1,m)]$ and $E[\mathcal{C}_1(n,m)] \le E[\mathcal{C}_1(n+1,m)]$, n = 1, 2,..., which lead to expression (13) above. We know that $\mathcal{J}(n)$ is a decreasing function of n, with $\lim_{n\to 0}\mathcal{J}(n) = +\infty$ and $\lim_{n\to\infty}\mathcal{J}(n) = 0$. Hence, there exists a finite and unique n^* that satisfies (13) for any $m > 0$. □

Proposition 2. *For any fixed value of $m \in (0, M]$, the optimal number of PM actions, $\tilde{n}^*$, for the unpunctual PM policy is unique and finite, and it is the solution of*

$$\mathcal{J}(\tilde{n}^*+1) \le \frac{C_p(m)}{c_f(1-\delta(m))(\mathcal{A}W^2+\mathcal{B}U^2)} + \frac{Var(Y_1|_{r\le\eta})}{W^2} \le \mathcal{J}(\tilde{n}^*). \tag{14}$$

Proof. The proof is similar to that of Proposition 1 and thus omitted. □

Based on Propositions 1 and 2, we can obtain the following corollary that compares the optimal number of PM actions under the two policies.

Corollary 1. *For any fixed value of $m \in (0, M]$, we have $\tilde{n}^* \le n^*$.*

Proof. By examining (13) and (14), it is straightforward to have $\mathcal{J}(\tilde{n}^*) \ge \mathcal{J}(n^*)$ and $\mathcal{J}(\tilde{n}^*+1) \ge \mathcal{J}(n^*+1)$, as $Var(Y_1|r \le \eta)$ is nonnegative. Since $\mathcal{J}(n)$ is a decreasing function of n, we have $\tilde{n}^* \le n^*$. □

Corollary 1 shows that incorporating customer unpunctuality into the warranty policy may lead to fewer number of PM actions over the warranty period. The reason is that under the unpunctual policy, implementing more PM actions within the warranty period would amplify the impact of maintenance unpunctuality on the warranty servicing cost, as indicated in Proposition 3 below. As a consequence, manufacturers should reduce the number of PM actions to mitigate the unpunctuality impact. It is important to note that as $\tilde{n}^*$ and n^* are integers in nature, there is a high possibility of $\tilde{n}^* = n^*$, especially when the term $Var(Y_1|_{r\le\eta})/W^2$ in (14) is small.

Unfortunately, it is difficult, if not impossible, to obtain closed-form solutions to the optimal PM levels m^* and $\tilde{m}^*$ as they are dependent on the formulations of $\delta(m)$ and $C_p(m)$. Nevertheless, both m^* and $\tilde{m}^*$ are integers, so simple numerical search methods are efficient to obtain the optimal solutions. We will examine m^* and $\tilde{m}^*$ through numerical examples in Section 4.

3.1.3. Comparison of the two PM policies

For policy comparison, our ideal goal is to compare the two PM policies under their optimal conditions, i.e., comparing $E[\widetilde{\mathcal{C}}_1(\tilde{n}^*, \tilde{m}^*)]$ and $E[\mathcal{C}_1(n^*, m^*)]$. However, as the closed-form solutions to m^* and $\tilde{m}^*$ are not available, we are unable to perform this analytical comparison. Alternatively, the punctual and unpunctual PM policies are compared when the same parameters n and m are applied to both of them. Numerical examples in Section 4 show that $\tilde{n}^* = n^*$ and $\tilde{m}^* = m^*$ in most cases, thus the comparison of $E[\widetilde{\mathcal{C}}_1(\tilde{n}^*, \tilde{m}^*)]$ and $E[\mathcal{C}_1(n^*, m^*)]$ is meaningful.

Nevertheless, by comparing (11) with (12), we obtain the following result.

Proposition 3. *When $\lambda(t|r) = (r/r_0)^{2\gamma}(2/\alpha^2)t$ and PM deviations Y_j's follow a known distribution, we have $E[\widetilde{\mathcal{C}}_1(n,m)] \ge E[\mathcal{C}_1(n,m)]$. For any $n \ge 1$, the cost difference is given by*

$$\begin{aligned} E[\widetilde{\mathcal{C}}_1(n,m)] - E[\mathcal{C}_1(n,m)] = {} & c_f(\mathcal{A}+\mathcal{B}\eta^2)(1-\delta(m))(E^2[Y_1|_{r\le\eta}] \\ & + nVar(Y_1|_{r\le\eta})) \ge 0. \end{aligned} \tag{15}$$

Proof. The cost difference can be easily obtained by subtracting $E[\mathcal{C}_1(n,m)]$ from $E[\widetilde{\mathcal{C}}_1(n,m)]$. As $\delta(m) \in [0, 1]$, $E^2[Y_1|_{r\le\eta}] \ge 0$, and $Var(Y_1|_{r\le\eta}) \ge 0$, the inequality holds. □

Proposition 3 tells us that when the failure rate function has a linear form, $E[\widetilde{\mathcal{C}}_1(n,m)]$ is always greater than or equal to $E[\mathcal{C}_1(n,m)]$. According to Eq. (15), the warranty cost difference between the two PM policies increases as n and/or m increases. This means that within a fixed warranty period, increasing the number of PM actions and/or enhancing the PM effort would amplify the impact of maintenance unpunctuality on the warranty cost. On the other hand, the warranty cost difference grows when the mean and/or variance of the deviation Y_1 increases. This implies that when the PM deviations are larger in magnitude and/or they are more dispersed across the customer population, the unpunctual PM policy would results in more warranty expenses, which is in line with our intuition.

3.1.4. Special unpunctuality distributions

The models and results in Sections 3.1.1–3.1.3 are obtained under a general distributional assumption for PM deviations. That is, the results can accommodate any eligible unpuncutality distributions. In what follows, we discuss two possible candidates of the unpunctuality distribution—uniform and triangular—and then analyze the results under the two distributions.

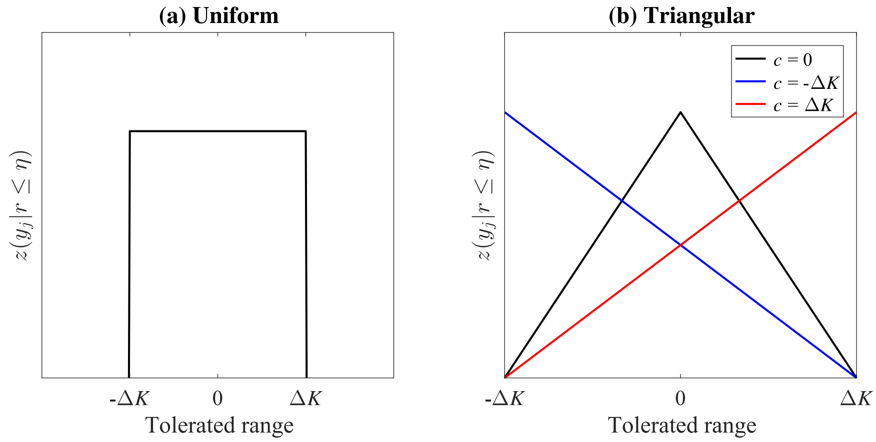

Fig. 3. Illustration of the uniform and triangular unpunctuality distributions.

(i) Uniform distribution. In general, the PM deviations Y_j's are difficult to observe and measure, thus it is a good starting point to assume that Y_j's follow an *uniform* distribution; see Fig. 3(a). The uniform distribution is often recommended for analytical modeling of an unobservable parameter.

In the case of $r \le \eta$, the uniform pdf of $Y_j|_{r\le\eta}$ is expressed as

$$z(y_j|r \le \eta) = \frac{1}{2\Delta K}, \text{ for } -\Delta K \le y_j \le \Delta K. \tag{16}$$

The uniform distribution of $Y_j|_{r>\eta}$ can be obtained in a similar way and is thus omitted. Indeed, as mentioned before, the mean and variance of $Y_1|_{r\le\eta}$ are the only quantities needed for obtaining (12). By substituting $E[Y_1|_{r\le\eta}] = 0$ and $Var(Y_1|_{r\le\eta}) = (\Delta K)^2/3$ into (12), we have

$$E[\widetilde{\mathcal{C}}_1^{\mathcal{U}}(n,m)] = E[\mathcal{C}_1(n,m)] + \frac{1}{3}c_f(\mathcal{A}+\mathcal{B}\eta^2)(1-\delta(m))n(\Delta K)^2. \tag{17}$$

The condition (14) for determining the optimal $\widetilde{n}^{\mathcal{U}*}$ thus becomes

$$\mathcal{J}(\widetilde{n}^{\mathcal{U}*}+1) \le \frac{C_p(m)}{c_f(1-\delta(m))(\mathcal{A}W^2+\mathcal{B}U^2)} + \frac{1}{3}\left(\frac{\Delta K}{W}\right)^2 \le \mathcal{J}(\widetilde{n}^{\mathcal{U}*}).$$

(ii) Triangular distribution. The second candidate of the unpunctuality distribution is the *triangular* distribution (He et al., 2017), which is able to describe left- and right-skewed scenarios; see Fig. 3(b). In the case of $r \le \eta$, the triangular pdf of $Y_j|_{r\le\eta}$ is given by

$$z(y_j|r \le \eta) = \begin{cases} \dfrac{y_j+\Delta K}{\Delta K(c+\Delta K)}, & \text{for } -\Delta K \le y_j \le c, \\ \dfrac{\Delta K - y_j}{\Delta K(\Delta K - c)}, & \text{for } c \le y_j \le \Delta K, \end{cases} \tag{18}$$

where $c \in [-\Delta K, \Delta K]$ is the mode. Notice that a positive (resp. negative) c corresponds to the case that most customers postpone (resp. advance) their PM activities.

The mean and variance of the triangular distribution above are $E[Y_1|_{r\le\eta}] = c/3$ and $Var(Y_1|_{r\le\eta}) = (\Delta K)^2/6 + c^2/18$, respectively, and the expected total warranty cost in (12) reduces to

$$E[\widetilde{\mathcal{C}}_1^{\mathcal{T}}(n,m)] = E[\mathcal{C}_1(n,m)] + \frac{1}{6}c_f(\mathcal{A}+\mathcal{B}\eta^2)(1-\delta(m)) \times \left(n + \frac{1}{3}(n+2)\left(\frac{c}{\Delta K}\right)^2\right)(\Delta K)^2, n \ge 1. \tag{19}$$

Hence, the condition (14) for determining the optimal $\widetilde{n}^{\mathcal{T}*}$ reduces to

$$\mathcal{J}(\widetilde{n}^{\mathcal{T}*}+1) \le \frac{C_p(m)}{c_f(1-\delta(m))(\mathcal{A}W^2+\mathcal{B}U^2)} + \frac{1}{6}\left(\frac{\Delta K}{W}\right)^2 + \frac{1}{18}\left(\frac{c}{W}\right)^2 \le \mathcal{J}(\widetilde{n}^{\mathcal{T}*}).$$

Corollary 2. *For any $n \ge 0$ and $m \in [0, M]$, $E[\widetilde{\mathcal{C}}_1^{\mathcal{T}}(n,m)|c=0] \le E[\widetilde{\mathcal{C}}_1^{\mathcal{T}}(n,m)|c=-\Delta K] = E[\widetilde{\mathcal{C}}_1^{\mathcal{T}}(n,m)|c=\Delta K]$.*

Proof. This result is trivial, thus the proof is omitted. □

Corollary 3. *For any $n \ge 0$ and $m \in [0, M]$, $E[\widetilde{\mathcal{C}}_1^{\mathcal{T}}(n,m)] \le E[\widetilde{\mathcal{C}}_1^{\mathcal{U}}(n,m)]$ always holds, regardless of the value of $c \in [-\Delta K, \Delta K]$.*

Proof. By subtracting $E[\widetilde{\mathcal{C}}_1^{\mathcal{T}}(n,m)]$ from $E[\widetilde{\mathcal{C}}_1^{\mathcal{U}}(n,m)]$, we obtain

$$E[\widetilde{\mathcal{C}}_1^{\mathcal{U}}(n,m)] - E[\widetilde{\mathcal{C}}_1^{\mathcal{T}}(n,m)] = \frac{1}{6}c_f(\mathcal{A}+\mathcal{B}\eta^2)(1-\delta(m))\left(n - \frac{1}{3}(n+2)\left(\frac{c}{\Delta K}\right)^2\right)(\Delta K)^2, n \ge 1.$$

Let $\xi(c^2) = n - \frac{1}{3}(n+2)\frac{c^2}{(\Delta K)^2}$. Then, we know that $\xi(c^2)$ is a decreasing function of $c^2 \in [0, (\Delta K)^2]$, and its minimum value is $\xi((\Delta K)^2) = 2(n-1)/3$. Therefore, for any $n \ge 1$, $\xi(c^2) \ge 0$, and thus $E[\widetilde{\mathcal{C}}_1^{\mathcal{U}}(n,m)] \ge E[\widetilde{\mathcal{C}}_1^{\mathcal{T}}(n,m)]$ for any value of $c \in [-\Delta K, \Delta K]$. Since $E[\widetilde{\mathcal{C}}_1^{\mathcal{U}}(n=0,m)] = E[\widetilde{\mathcal{C}}_1^{\mathcal{T}}(n=0,m)]$, the result holds for any $n \ge 0$. □

Corollary 2 means that when $\beta = 2$ and Y_j's obey the triangular distribution (18), the expected total warranty costs for $c = -\Delta K$ and $c = \Delta K$ are identical, but higher than that for $c = 0$. On the other hand, Corollary 3 implies that when $\beta = 2$, the expected total warranty cost for the triangular unpunctuality distribution is consistently lower than that for the uniform distribution.

3.2. Model analysis for $\beta = 3$

In this subsection, we analyze the punctual and unpunctual PM policies in the case of $\beta = 3$. The failure rate is given by $\lambda(t|r) = (r/r_0)^{3\gamma}(3/\alpha^3)t^2$, which is a quadratically (strictly convex) increasing function.

3.2.1. Warranty cost models

Similar to Section 3.1.1, we first derive the closed-form warranty cost models for the two PM policies. Substituting $\lambda(t|r) = (r/r_0)^{3\gamma}(3/\alpha^3)t^2$ into (5) and after some algebraic operations, the expected total warranty servicing cost of the punctual PM policy is derived as

$$E[\mathcal{C}_2(n,m)] = \frac{1}{2}c_f\left(\mathcal{D}W^3+\mathcal{E}U^3\right)\frac{2+n(2n+1)\delta(m)^2+3n\delta(m)}{(n+1)^2} + nC_p(m), \quad (20)$$

where $\mathcal{D}=\alpha^{-3}\int_0^\eta(\frac{r}{r_0})^{3\gamma}g(r)\mathrm{d}r$ and $\mathcal{E}=\alpha^{-3}\int_\eta^\infty\frac{r^{3\gamma-3}}{r_0^{3\gamma}}g(r)\mathrm{d}r$.

Likewise, for $n \geq 1$, the expected total warranty cost of the unpunctual PM policy is given by

$$\begin{aligned} &E[\widetilde{\mathcal{C}}_2(n,m)] \\ &= \frac{1}{2}c_f\left(\mathcal{D}W^3+\mathcal{E}U^3\right)\frac{2+n(2n+1)\delta(m)^2+3n\delta(m)}{(n+1)^2}+nC_p(m) \\ &+3c_f(\mathcal{D}+\mathcal{E}\eta^2)(1-\delta(m)) \\ &\times\left\{\begin{array}{l}\delta(m)\left(nE[(Y_1|_{r\leq\eta})^3]-(n-1)E[(Y_1|_{r\leq\eta})^2]E[Y_1|_{r\leq\eta}]\right) \\ +W\frac{n\delta(m)+2}{n+1}(E^2[Y_1|_{r\leq\eta}]+nVar(Y_1|_{r\leq\eta})) \\ -W^2\frac{n}{(n+1)^2}\delta(m)E[Y_1|_{r\leq\eta}]\end{array}\right\}. \end{aligned} \quad (21)$$

The detailed derivation of (21) is analogous to that of (12) and is omitted. Note that when no PM is performed (i.e, $n=0$), $E[\widetilde{\mathcal{C}}_2(n=0,m)]=E[\mathcal{C}_2(n=0,m)]=c_f(\mathcal{D}W^3+\mathcal{E}U^3)$.

When $\beta=3$, the complex structures of warranty cost models (20) and (21) hinder us from deriving the conditions for optimal number of PM actions as in Propositions 1 and 2, and comparing the warranty costs of the two PM policies as in Proposition 3. Numerical search methods can be employed to obtain the optimal number and level of PM actions. It is noteworthy that the optimal number of PM actions should be bounded by $n_{max}=\lfloor W/(2\Delta K)-1\rfloor$, where $\lfloor x\rfloor$ represents the largest integer that is smaller than x; otherwise, the PM interval would be shorter than the tolerated range, which is impractical.

3.2.2. Special unpunctuality distributions

(i) Uniform distribution. When customer unpunctual behaviors are governed by the uniform distribution (16), we have $E[Y_1|_{r\leq\eta}]=0$, $E[(Y_1|_{r\leq\eta})^2]=Var(Y_1|_{r\leq\eta})=(\Delta K)^2/3$, and $E[(Y_1|_{r\leq\eta})^3]=0$. Then, Eq. (21) is simplified to

$$E[\widetilde{\mathcal{C}}_2^{\mathcal{U}}(n,m)] = E[\mathcal{C}_2(n,m)]+c_f(\mathcal{D}+\mathcal{E}\eta^3)(1-\delta(m))(n\delta(m)+2) \times\frac{nW}{n+1}(\Delta K)^2, \quad (22)$$

where $E[\mathcal{C}_2(n,m)]$ is given by (20). It is clear that $E[\widetilde{\mathcal{C}}_2^{\mathcal{U}}(n,m)]\geq E[\mathcal{C}_2(n,m)]$ for any fixed n and m.

(ii) Triangular distribution. When the triangular distribution (18) is appropriate for characterizing customer unpunctuality, we have $E[Y_1|_{r\leq\eta}]=c/3$, $E[(Y_1|_{r\leq\eta})^2]=((\Delta K)^2+c^2)/6$, $Var(Y_1|_{r\leq\eta})=(\Delta K)^2/6+c^2/18$, and $E[(Y_1|_{r\leq\eta})^3]=((\Delta K)^2+c^2)c/10$, respectively. Thus, for $n\geq 1$, the expected total warranty cost in (21) reduces to

$$\begin{aligned} &E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)] \\ &= E[\mathcal{C}_2(n,m)]+c_f(\mathcal{D}+\mathcal{E}\eta^3)(1-\delta(m)) \\ &\times\left\{\begin{array}{l}\delta(m)\left((\Delta K)^2+c^2\right)c\left(\frac{2}{15}n+\frac{1}{6}\right)-\left(\frac{W}{n+1}\right)^2n\delta(m)c \\ +\frac{1}{2}(n\delta(m)+2)\frac{W}{n+1}(\Delta K)^2\left(n+\frac{1}{3}(n+2)\left(\frac{c}{\Delta K}\right)^2\right)\end{array}\right\}, n\geq 1. \end{aligned} \quad (23)$$

It is also clear that when $c=0$, $E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)|c=0]\geq E[\mathcal{C}_2(n,m)]$. After evaluating $E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)]$ at $c=-\Delta K$, 0, and ΔK, respectively, we have the following result.

Corollary 4. *For any $n\geq 0$ and $m\in[0, M]$, $E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)|c=-\Delta K]\geq E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)|c=\Delta K]$ and $E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)|c=-\Delta K]\geq E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)|c=0]$ always hold.*

Proof. When $n=0$, $E[\widetilde{\mathcal{C}}_2(0,m)|c=\Delta K]=E[\mathcal{C}_2(0,m)|c=-\Delta K]=c_f(\mathcal{D}W^3+\mathcal{E}U^3)$. While for $n\geq 1$, it is easy to obtain

$$\begin{aligned} &E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)|c=-\Delta K]-E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)|c=\Delta K] \\ &\quad= 2c_f(\mathcal{D}+\mathcal{E}\eta^3)(1-\delta(m))\delta(m)\Delta K \\ &\quad\times\left[n\left(\frac{W^2}{(n+1)^2}-\frac{4}{15}(\Delta K)^2\right)-\frac{1}{3}(\Delta K)^2\right], n\geq 1. \end{aligned}$$

Let $\zeta(n)=n(\frac{W^2}{(n+1)^2}-\frac{4}{15}(\Delta K)^2)-\frac{1}{3}(\Delta K)^2$. Recall that $n\leq n_{max}\leq W/(2\Delta K)-1$, we thus have $W/(n+1)\geq 2\Delta K$. Then, we know that $\zeta(n)\geq(\frac{56}{15}n-\frac{1}{3})(\Delta K)^2$, which is greater than zero for any $n\geq 1$. Therefore, for any $n\geq 1$, $\zeta(n)>0$, and thus $E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)|c=-\Delta K]>E[\widetilde{\mathcal{C}}_2^{\mathcal{T}}(n,m)|c=\Delta K]$. Considering all values of $n\in\{0,1,\dots,n_{max}\}$, the first result follows. Analogously, the second result can be proved easily. □

Corollary 4 delivers an interesting result: In the case of $\beta=3$ ($\lambda(t|r)$ is strictly convex and increasing in t), when most customers postpone their PM activities (e.g., $c=\Delta K$), the expected total warranty cost can be even lower than that in the case where most customers advance their PM (e.g., $c=-\Delta K$). This finding will be further interpreted in Section 4.4.

3.3. Model analysis for a general value of $\beta>1$

When the failure rate $\lambda(t|r)$ has a general form (with any real value of $\beta>1$ in Eq. (2)), it might exhibit concave ($1<\beta<2$) or convex ($\beta>2$) increasing patterns. It should be pointed out that for $\beta\leq 1$, the failure rate function is either decreasing and constant over time, therefore it is never beneficial to perform PM actions in these scenarios.

For a general value of β, the expected total warranty cost for the punctual PM policy can be explicitly expressed in a closed form, as follows:

$$E[\mathcal{C}(n,m)] = c_f(\mathcal{G}+\mathcal{H}\eta^\beta)\sum_{j=0}^{n}\left[(j\delta(m)K+K)^\beta-(j\delta(m)K)^\beta\right] + nC_p(m), \quad (24)$$

where $\mathcal{G}=\alpha^{-\beta}\int_0^\eta(\frac{r}{r_0})^{\gamma\beta}g(r)\mathrm{d}r$ and $\mathcal{H}=\alpha^{-\beta}\int_\eta^\infty\frac{r^{\gamma\beta-\beta}}{r_0^{\gamma\beta}}g(r)\mathrm{d}r$. Notice that when $\beta=2$, $\mathcal{G}$ and $\mathcal{H}$ reduce to $\mathcal{A}$ and $\mathcal{B}$, respectively; when $\beta=3$, they are simplified to $\mathcal{D}$ and $\mathcal{E}$, respectively.

However, in the general case, it is difficult to derive the closed-form expected total warranty cost for the unpunctual PM policy. Numerical integration and Monte Carlo simulation offer feasible solutions to warranty cost evaluation. The latter method is employed in this study, which is elaborated as follow.

In each simulation run (with fixed n and m), we first generate a series of deviations $y_1,\dots,y_n\sim Z(y|r\leq\eta)$, and then calculate the expected warranty cost by the following formula:

$$\begin{aligned} \widetilde{\mathcal{C}}(n,m) = c_f(\mathcal{G}+\mathcal{H}\eta^\beta)&\Bigg\{(K+y_1)^\beta+\sum_{j=1}^{n-1}\Big[\left(\delta(m)(jK+y_j)+K\right. \\ &\left.+y_{j+1}-y_j\right)^\beta-\left(\delta(m)(jK+y_j)\right)^\beta\Big]+(\delta(m)(nK+y_n) \\ &+K-y_n)^\beta-(\delta(m)(nK+y_n))^\beta\Bigg\}+nC_p(m). \end{aligned} \quad (25)$$

After a large number of simulation runs, the expected total warranty cost of the unpunctual policy can be well approximated by the average of $\widetilde{\mathcal{C}}(n,m)$, i.e., $\overline{\mathcal{C}}(n,m)=\sum_{i=1}^{Sim}\widetilde{\mathcal{C}}_i(n,m)/Sim$, where $\widetilde{\mathcal{C}}_i(n,m)$ is the total warranty cost in the ith simulation run and Sim is the number of simulation runs for each single case. The Monte Carlo simulation procedure is summarized in Algorithm 1.

Algorithm 1

Input: $G(r)$, $Z(y|r \le \eta)$, η, n, m, etc.
Output: $\overline{C}(n,m)$

1: **for** $i = 1$ to Sim **do**
2: Simulate $y_1, \ldots, y_n \sim Z(y|r \le \eta)$
3: Compute $\widetilde{C}_i(n,m)$ by Eq. (25)
4: **end for**
5: **return** $\overline{C}(n,m) = \sum_{i=1}^{Sim} \widetilde{C}_i(n,m)/Sim$

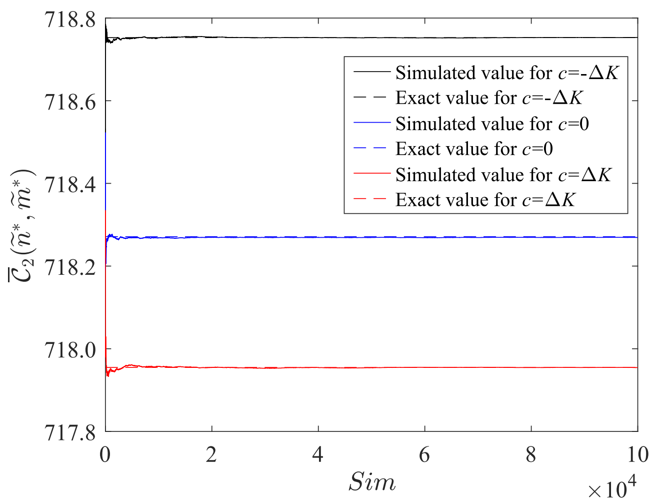

Fig. 4. Comparison of simulated and actual warranty costs for triangular unpunctuality distribution, with $\beta = 3$, $c_f = \$300$, and $\Delta K = 1$ weeks. Other parameter settings follow directly from Section 4.1.

In order to illustrate the accuracy and convergence of the simulation algorithm, Fig. 4 shows simulated and actual warranty costs for the triangular unpunctuality distribution, with the same parameters as in Section 4.1. It can be observed that the proposed simulation algorithm has a good performance in terms of both accuracy and convergence speed. In this study, the simulation runs are set to $Sim = 100000$ for each single case to guarantee a high accuracy, and the computation time for a single case is about 75 s using a personal computer. The computation time and simulation accuracy are acceptable for practical use.

Nevertheless, in the general case, the following proposition shows that under some conditions, the expected warranty servicing cost of the unpunctual PM policy is higher than that of the punctual policy, for identical n and m.

Proposition 4. *When the cumulative failure rate* $\Lambda(t|r) = \int_0^t \lambda(x|r)\mathrm{d}x$ *is convex and increasing in* t *and the mean of* Y_j*'s under the unpunctual PM policy is* $E[Y_j] = 0$, *we have* $E[\widetilde{C}(n,m)] \ge E[C(n,m)]$.

The proof of Proposition 4 can be found in Appendix B. The first condition in Proposition 4 is equivalent to $\Lambda'(t|r) = \lambda(t|r) \ge 0$ and $\Lambda''(t|r) = \lambda'(t|r) \ge 0$, which is quite common for aging items. In particular, the Weibull failure rate with $\beta \ge 2$ well satisfies this condition. The second condition (i.e., $E[Y_j] = 0$) is stricter than that in Proposition 3. It can be met when Y_j follows a symmetric distribution, e.g., uniform, symmetric triangular (i.e., $c = 0$), and truncated normal distributions, among others.

The result in Proposition 4 is well supported by the cases of $\beta = 2$ and $\beta = 3$ discussed above. Specifically, when $\beta = 2$, $E[\widetilde{C}_1^{\mathcal{U}}(n,m)] \ge E[C_1(n,m)]$ and $E[\widetilde{C}_1^{\mathcal{T}}(n,m)|c=0] \ge E[C_1(n,m)]$; when $\beta = 3$, $E[\widetilde{C}_2^{\mathcal{U}}(n,m)] \ge E[C_2(n,m)]$ and $E[\widetilde{C}_2^{\mathcal{T}}(n,m)|c=0] \ge E[C_2(n,m)]$. When β takes a general value and $E[Y_j] \ne 0$ (e.g., $c \ne 0$ for the triangular distribution), the expected total warranty costs in different scenarios will be examined via numerical studies in Section 4.

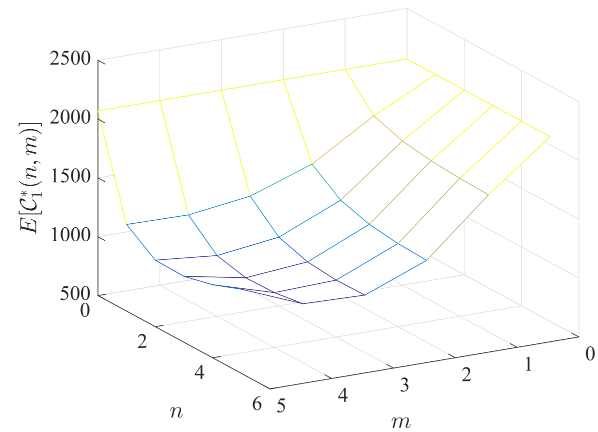

Fig. 5. Illustration of the existence of an optimal punctual PM decision ($\beta = 2$ and $c_f = 300$).

4. Numerical studies

In this section, numerical studies are presented to demonstrate the unpunctual PM policy. Section 4.1 describes parameter settings used in the studies. Sections 4.2 and 4.3 then present numerical results for the punctual and unpunctual PM policies, respectively. Finally, Section 4.4 discusses observations from the numerical studies and their managerial implications, and also answers the two questions raised earlier.

4.1. Parameter setting

We consider an automaker whose automobiles are sold with a 2D warranty of 3 years and 10×10^4 km (whichever occurs first). The automobile lifetime is Weibull distributed with scale parameter $\alpha = 3.2$ and shape parameter β. The nominal usage rate and AFT factor are set to $r_0 = 1 \times 10^4$ km/year and $\gamma = 0.8$, respectively. Besides, the automobile usage rates (in 10^4 km/year) obey a Gamma distribution $g(r) = \frac{1}{\phi^\rho \Gamma(\rho)} r^{\rho-1} e^{-r/\phi}$ with shape parameter $\rho = 5.88$ and scale parameter $\phi = 0.35$. That is, the average usage rate is $\bar{r} = \rho\phi = 2.058 \times 10^4$ km/year. These parameter values are arbitrarily selected for illustrative purposes, and they can be estimated from historical warranty data in real applications. Furthermore, the PM degree $\delta(m)$ is modeled as an exponentially decreasing function of m, namely, $\delta(m) = (1+m)e^{-m}$ (Kim et al., 2004), and the corresponding PM costs are $C_p(0) = \$0$, $C_p(1) = \$10$, $C_p(2) = \$30$, $C_p(3) = \$60$, $C_p(4) = \$100$, and $C_p(5) = \$160$. We further consider three pre-specified value of ΔK, i.e., 1 weeks (1/52 year), 2 weeks, and 4 weeks, for the unpunctual PM policy. This means that customers are entitled to have a 2-week, 4-week, or 8-week tolerated range (at most) of maintenance unpunctuality, respectively.

4.2. Optimal decisions for the punctual PM policy

We first look at optimal decisions for the punctual PM policy. Based on the parameter values above, Table 2 lists the optimal number and degree of PM actions and the expected total warranty cost for the punctual PM polity under various combinations of β and c_f. In particular, Fig. 5 illustrates the existence of an optimal PM decision with $\beta = 2$ and $c_f = \$300$, though the closed forms of n^* and m^* are difficult to obtain. It is worth noting that here we consider four specific values for β, i.e., $\beta = 1.5$, 2, 3 and 4.5, corresponding to concave, linear, and convex increasing failure rates,

Table 2
Numerical results for the punctual PM policy with different values of β and c_f.

c_f	$\beta = 1.5$			$\beta = 2$			$\beta = 3$			$\beta = 4.5$		
	n^*	m^*	$E[\mathcal{C}^*]$	n^*	m^*	$E[\mathcal{C}^*]$	n^*	m^*	$E[\mathcal{C}^*]$	n^*	m^*	$E[\mathcal{C}^*]$
50	1	1	182.98	1	3	237.27	2	3	275.49	3	3	286.39
100	1	2	343.75	2	3	395.60	3	3	405.26	4	3	384.65
150	1	3	494.62	2	3	533.40	4	3	511.99	3	4	452.48
200	2	3	634.27	3	3	652.27	3	4	592.33	3	4	503.31
250	2	3	762.83	3	3	770.34	4	4	665.09	4	4	542.93
300	2	3	891.40	3	4	865.37	4	4	718.11	4	4	571.52
350	3	3	1015.55	3	4	959.60	4	4	771.12	4	4	600.10
400	3	3	1134.91	4	4	1046.40	4	4	824.14	4	4	628.69
450	3	3	1254.28	4	4	1127.20	5	4	873.66	4	4	657.28
500	3	4	1363.14	4	4	1208.00	5	4	915.18	5	4	684.43

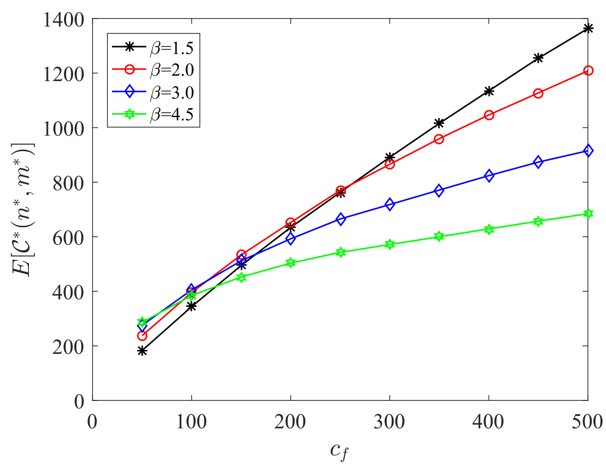

Fig. 6. Minimal warranty costs for the punctual PM policy under various β.

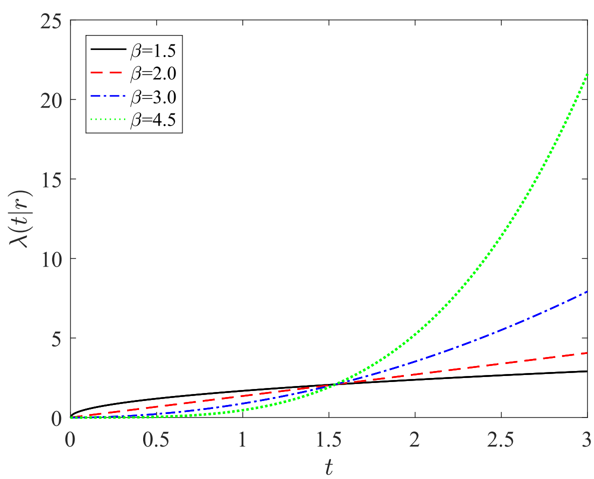

Fig. 7. Failure rate functions for different β with $r = 2.5$.

respectively. In this way, we attempt to examine all typical shapes of an increasing failure rate, which makes our investigation more thorough.

Table 2 shows that the optimal number of PM actions, the optimal PM degree and the corresponding total warranty cost tend to increase as the minimal repair cost c_f increases. This is because a large minimal repair cost will increase warranty servicing cost, thus the manufacturer should enhance PM investments to mitigate warranty cost growth. One thing particularly noteworthy is that when $\beta = 3$ and c_f increases from \$150 to \$200, the optimal PM level m^* grows from 3 to 4; however, it is unexpected that n^* jumps from 4 back to 3. A similar phenomenon can be observed in the case of $\beta = 4.5$, when c_f increases from \$100 to \$150. These interesting observations imply that when c_f increases, manufacturers could adopt a mixed strategy by enhancing (lowering) the PM effort and slightly decreasing (increasing) the number of PM actions at the same time.

Furthermore, when comparing the numerical results for different β, we find that the optimal number of PM actions and the optimal PM level exhibit upward trend as β increases. On the other hand, for small minimal repair cost c_f, a smaller β leads to lower warranty cost; while for medium to large c_f, a smaller β even results in higher warranty cost (see Fig. 6). This observation is due to the concavity/convexity of the failure rate function (Fig. 7). Therefore, it is essential for manufacturers to estimate the failure rate parameters as accurate as possible.

4.3. Optimal decisions for the unpunctual PM policy

Now we investigate optimal decisions for the unpunctual PM policy. Here we also consider the four values for β (i.e., 1.5, 2, 3, and 4.5), following Section 4.2. To facilitate the presentation of results, we define $\Psi = (E[\widetilde{\mathcal{C}}^*] - E[\mathcal{C}^*])/E[\mathcal{C}^*]$ as the relative growth of expected warranty cost of the unpunctual PM policy in comparison to its punctual counterpart. Clearly, a negative (resp. positive) Ψ means that the warranty cost of the unpunctual PM policy is lower (resp. higher) than that of the punctual policy.

Tables 3 and 4 summarize the numerical results for $\beta = 3$ with uniform and triangular unpunctuality distributions, respectively (Detailed results for β =1.5, 2, and 4.5 are omitted for space consideration). Figs. 8–11 show the relative warranty cost growths Ψ for uniform and triangular unpunctuality distributions with $\beta =$ 2, 3, 4.5, and 1.5, respectively. From these tables and figures, the following observations can be drawn:

(1) First of all, the optimal number of PM actions $\widetilde{n}^*$ and the optimal PM degree $\widetilde{m}^*$ of the unpunctual PM policy are identical to those of the punctual policy, respectively, *in most cases*. Specifically, the statement holds for all cases of β =1.5, 2, 4.5, and for most cases of $\beta = 3$ (This is why we omit the detailed results for β =1.5, 2, 4.5). One exception is that when $\beta = 3$ and c_f =\$250, the optimal number of PM actions for $\Delta K = 4$ weeks (i.e., $\widetilde{n}^{\mathcal{U}*} = 3$ and $\widetilde{n}^{\mathcal{T}*} = 3$, marked by bold in Tables 3 and 4) is lower than that of the punctual policy. This is somewhat consistent with the result in Corollary 1.

(2) According to the first observation, it is straightforward that the optimal number of PM actions, the optimal PM degree, and the expected total warranty cost of the unpunctual PM policy also tend to increase as the minimal repair cost c_f increases. This is in line with the case of the punctual policy.

(3) When $\beta = 2$, Fig. 8 shows that (a) the expected total warranty cost of the unpunctual PM policy is higher than that of the punctual policy (reflected by positive Ψ), regardless of the unpunctuality distributions (Proposition 3); (b) the expected warranty cost under the uniform unpunctuality

Table 3
Numerical results for uniform unpunctuality distribution with $\beta = 3$.

c_f	$\Delta K = 1$ weeks				$\Delta K = 2$ weeks				$\Delta K = 4$ weeks			
	$\tilde{n}^{\mathcal{U}*}$	$\tilde{m}^{\mathcal{U}*}$	$E[\tilde{C}_2^{\mathcal{U}*}]$	Ψ	$\tilde{n}^{\mathcal{U}*}$	$\tilde{m}^{\mathcal{U}*}$	$E[\tilde{C}_2^{\mathcal{U}*}]$	Ψ	$\tilde{n}^{\mathcal{U}*}$	$\tilde{m}^{\mathcal{U}*}$	$E[\tilde{C}_2^{\mathcal{U}*}]$	Ψ
50	2	3	275.53	0.01%	2	3	275.65	0.06%	2	3	276.14	0.24%
100	3	3	405.36	0.02%	3	3	405.66	0.10%	3	3	406.86	0.39%
150	4	3	512.16	0.03%	4	3	512.67	0.13%	4	3	514.74	0.54%
200	3	4	592.53	0.03%	3	4	593.12	0.13%	3	4	595.50	0.54%
250	4	4	665.36	0.04%	4	4	666.19	0.17%	**3**	4	669.38	0.65%
300	4	4	718.44	0.05%	4	4	719.43	0.18%	4	4	723.39	0.74%
350	4	4	771.51	0.05%	4	4	772.67	0.20%	4	4	777.29	0.80%
400	4	4	824.58	0.05%	4	4	825.90	0.21%	4	4	831.19	0.86%
450	5	4	874.20	0.06%	5	4	875.81	0.25%	5	4	882.24	0.98%
500	5	4	915.78	0.07%	5	4	917.57	0.26%	5	4	924.72	1.04%

Table 4
Numerical results for triangular unpunctuality distribution with $\beta = 3$.

	c_f	ΔK=1 week				ΔK=2 weeks				ΔK=4 weeks			
		$\tilde{n}^{\mathcal{T}*}$	$\tilde{m}^{\mathcal{T}*}$	$E[\tilde{C}_2^{\mathcal{T}*}]$	Ψ	$\tilde{n}^{\mathcal{T}*}$	$\tilde{m}^{\mathcal{T}*}$	$E[\tilde{C}_2^{\mathcal{T}*}]$	Ψ	$\tilde{n}^{\mathcal{T}*}$	$\tilde{m}^{\mathcal{T}*}$	$E[\tilde{C}_2^{\mathcal{T}*}]$	Ψ
$c = -\Delta K$	50	2	3	275.70	0.08%	2	3	275.98	0.18%	2	3	276.74	0.45%
	100	3	3	405.64	0.09%	3	3	406.17	0.22%	3	3	407.69	0.60%
	150	4	3	512.50	0.10%	4	3	513.27	0.25%	4	3	515.57	0.70%
	200	3	4	592.79	0.08%	3	4	593.57	0.21%	3	4	596.04	0.63%
	250	4	4	665.63	0.08%	4	4	666.58	0.22%	4	4	669.72	0.70%
	300	4	4	718.75	0.09%	4	4	719.89	0.25%	4	4	723.66	0.77%
	350	4	4	771.88	0.10%	4	4	773.21	0.27%	4	4	777.60	0.84%
	400	4	4	825.00	0.10%	4	4	826.53	0.29%	4	4	831.54	0.90%
	450	5	4	874.58	0.10%	5	4	876.27	0.30%	5	4	882.02	0.96%
	500	5	4	916.20	0.11%	5	4	918.08	0.32%	5	4	924.46	1.01%
$c = 0$	50	2	3	275.51	0.01%	2	3	275.57	0.03%	2	3	275.82	0.12%
	100	3	3	405.31	0.01%	3	3	405.46	0.05%	3	3	406.06	0.20%
	150	4	3	512.07	0.02%	4	3	512.33	0.07%	4	3	513.36	0.27%
	200	3	4	592.43	0.02%	3	4	592.72	0.07%	3	4	593.92	0.27%
	250	4	4	665.23	0.02%	4	4	665.64	0.08%	4	4	667.29	0.33%
	300	4	4	718.27	0.02%	4	4	718.77	0.09%	4	4	720.75	0.37%
	350	4	4	771.32	0.02%	4	4	771.89	0.10%	4	4	774.21	0.40%
	400	4	4	824.36	0.03%	4	4	825.02	0.11%	4	4	827.67	0.43%
	450	5	4	873.93	0.03%	5	4	874.74	0.12%	5	4	877.95	0.49%
	500	5	4	915.48	0.03%	5	4	916.37	0.13%	5	4	919.95	0.52%
$c = \Delta K$	50	2	3	275.34	−0.05%	2	3	275.27	−0.08%	2	3	275.33	−0.06%
	100	3	3	405.04	−0.05%	3	3	404.97	−0.07%	3	3	405.31	0.01%
	150	4	3	511.73	−0.05%	4	3	511.74	−0.05%	4	3	512.53	0.11%
	200	3	4	592.17	−0.03%	3	4	592.32	0.00%	3	4	593.56	0.21%
	250	4	4	664.96	−0.02%	4	4	665.25	0.02%	**3**	4	666.94	0.28%
	300	4	4	717.95	−0.02%	4	4	718.30	0.03%	4	4	720.48	0.33%
	350	4	4	770.95	−0.02%	4	4	771.35	0.03%	4	4	773.90	0.36%
	400	4	4	823.94	−0.02%	4	4	824.40	0.03%	4	4	827.31	0.38%
	450	5	4	873.54	−0.01%	5	4	874.20	0.06%	5	4	877.89	0.48%
	500	5	4	915.04	−0.02%	5	4	915.78	0.07%	5	4	919.88	0.51%

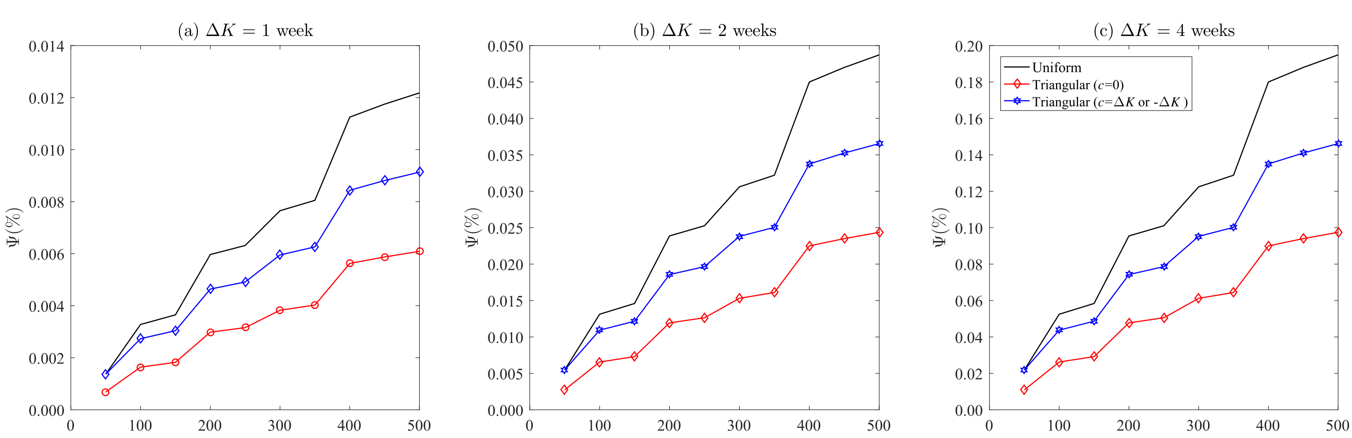

Fig. 8. Relative warranty cost growth Ψ for uniform and triangular unpunctuality distributions with $\beta = 2$.

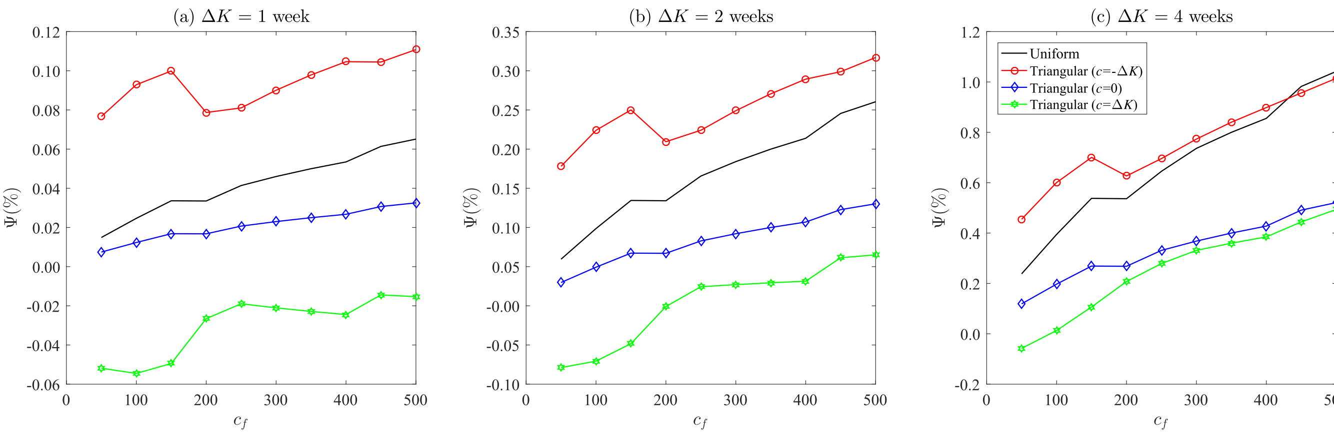

Fig. 9. Relative warranty cost growth Ψ for uniform and triangular unpunctuality distributions with $\beta = 3$.

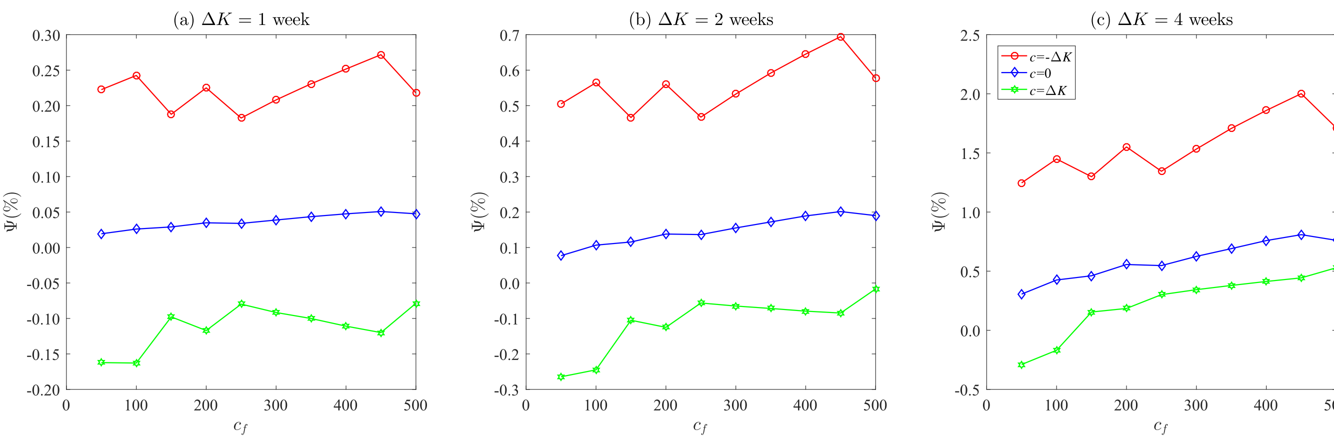

Fig. 10. Relative warranty cost growth Ψ for triangular unpunctuality distribution with $\beta = 4.5$.

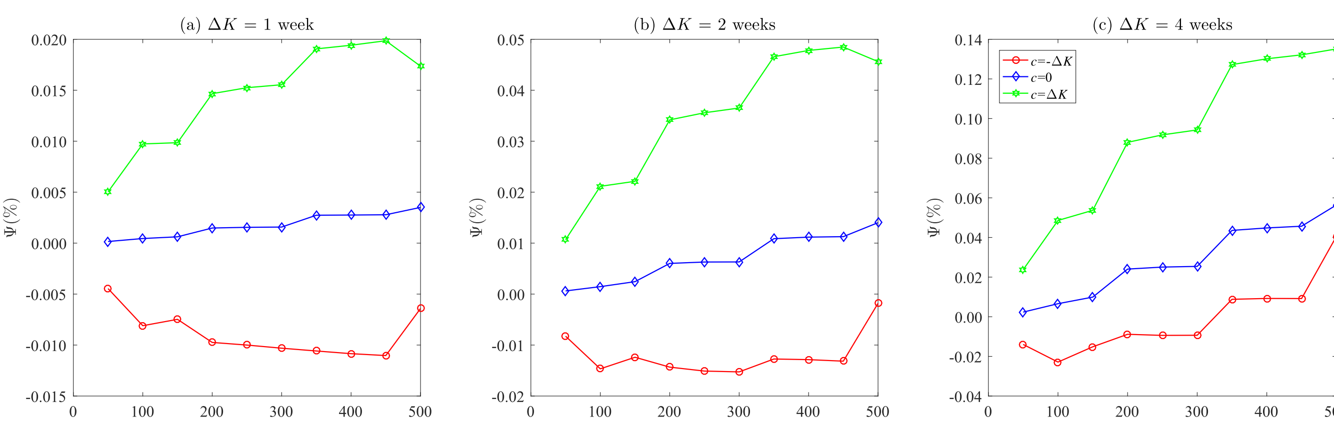

Fig. 11. Relative warranty cost growth Ψ for triangular unpunctuality distribution with $\beta = 1.5$.

distribution is higher than that under the triangular distribution (Corollary 3); (c) under the triangular unpunctuality distribution, the expected warranty cost with $c = 0$ is lower than that with $c = -\Delta K$ or ΔK (Corollary 2); and (d) the expected total warranty cost for the unpunctual PM policy increases as ΔK increases (reflected by increasing Ψ). Nevertheless, when $\beta = 2$, the warranty cost growth of the unpunctual policy is small (less than 1%) in all cases.

(4) When $\beta = 3$ and 4.5, Figs. 9 and 10 show that the relative warranty cost growth Ψ exhibits increasing trend as ΔK and/or c_f increases, as to be expected. More interestingly, under the triangular unpunctuality distribution, the expected warranty cost with $c = -\Delta K$ is higher than those with $c = 0$ and $c = \Delta K$ (Corollary 4); while the expected warranty cost with $c = \Delta K$ can be even lower than that of the punctual policy (reflected by negative Ψ), especially for small ΔK. This implies that when the failure rate is a convex increasing function, most customers postponing their PM interventions could even result in lower warranty servicing costs, if ΔK is relatively small.

(5) Fig. 11 shows that for $\beta = 1.5$, i.e., $\lambda(t|r)$ is a concave increasing function, the expected warranty cost for $c = \Delta K$ is

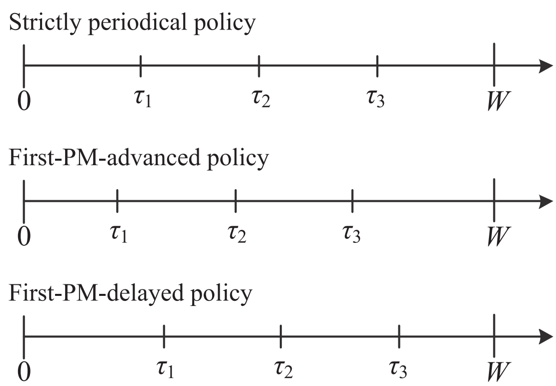

Fig. 12. Illustration of the strictly periodical, first-PM-advanced, and first-PM-delayed PM policies in a 1D characterization.

higher than those of $c = 0$ and $c = -\Delta K$; while the warranty cost growth for $c = -\Delta K$ is the lowest and less than zero in most cases. This means that when the failure rate has a concave increasing trend, most customers advancing their PM schedules is favorable to manufacturers, especially for relatively small ΔK.

4.4. Discussions: answering the two questions above

Based on the numerical results, we are able to answer the first question raised before: Customer unpunctuality has *little* impact on the optimal PM policy in the sense that the unpunctual policy has the same optimal parameters as its punctual counterpart in most cases. This is because the proposed unpunctual policy only allows a small degree of maintenance unpunctuality around exact instants (ΔK and ΔL are relatively small in real applications), and such small departures can hardly affect the decision variables (n and m) that are discrete in nature. On the other hand, the impact of customer unpunctuality on the total warranty servicing cost is quite sensitive to customer behaviors and the shape of failure rate function. More specifically,

- When the failure rate is a concave function ($1 < \beta < 2$ for Weibull distribution), most customers advancing their PM activities is favorable to manufacturers. That is, manufacturers would prefer customers to return their items for PM activities slightly earlier than the strict schedule.
- When the failure rate is a linear function ($\beta = 2$), the unpunctual PM policy always incurs more warranty servicing cost to manufacturers, irrespective of customers' unpunctuality behaviors. In this case, manufacturers would prefer customers to be as punctual as possible. Nevertheless, the warranty cost growth of the unpunctual policy in comparison to its punctual counterpart is quite small (less than 1%).
- When the failure rate has a convex shape ($\beta > 2$), most customers postponing their PM activities is favorable to manufacturers. That is, manufacturers would prefer customers to perform their PM activities slightly later than the strict schedule.

Basically, these interesting findings originate from the concavity/convexity of the failure rate function (see Fig. 7). To explain these observations, taking $r \le \eta$ as an example, let us consider two extreme cases in which (*i*) all customers carry out the jth PM activity at instant $jK - \Delta K$, while (*ii*) all customers perform the jth PM activity at $jK + \Delta K$ (Fig. 12). When the failure rate is convex, it is relatively low at the early stage of an item's useful life, thus postponing the first PM action ($\tau_1 > K$) could avoid maintenance excess; while the failure rate becomes high in the late stage, thus shortening the last period ($W - \tau_n < K$) could avoid insufficient maintenance. In this situation, the so-called "*first-PM-delayed*" policy (Cheng, Zhao, Chen, & Sun, 2014; Wu, Xie, & Ng, 2011) has advantages over the "*first-PM-advanced*" policy. However, when the failure rate is concave, the situation is reversed, i.e., the failure rate is relatively high in the early stage, while relatively low in the late stage. In this case, performing the first PM earlier ($\tau_1 < K$) and extending the last period ($W - \tau_n > K$) is a better choice. However, when $\beta = 2$, $\lambda(t|r)$ is a linear function, thus the customer unpunctuality only results in a slight warranty cost growth.

Now we are ready to answer the second question. Numerical studies above show that in some scenarios, the expected warranty costs of the unpunctual policy are even lower than those of the punctual policy. This is because the PM schedule under the punctual (strictly periodical) policy is inherently sub-optimal. Accordingly, manufacturers could induce customers to adjust their unpunctuality behaviors by modifying PM policies or introducing penalty/bonus mechanisms. For instance, if the failure rate exhibits a convex increasing pattern, then the jth PM instants in the age and usage dimensions can be revised to $[jK, jK + 2\Delta K]$ and $[jL, jL + 2\Delta L]$, respectively, so as to force customers to slightly postpone their PM activities.

A further remark to the second question is that manufacturers should obtain as much information on the product failure rate and customer behaviors as possible, in view of their importance. On the one hand, the parameters of the failure rate function are relatively easy to estimate, using either field failure data or reliability testing data (Wang & Xie, 2018; Wu, 2013). However, historical data about customers' unpunctual behaviors are difficult to obtain, especially for a new product. A possible solution to this issue is as follows: The manufacturer could first offer this unpunctual PM policy to a small group of (randomly selected) customers. After a period of time, the data about these customers' PM deviations are recorded and the manufacturer will have a better understanding of customer behaviors by analyzing these data. After that, a comprehensive warranty cost analysis should be performed so that the manufacturer is able to determine reasonable ΔK and ΔL to balance warranty cost growth and customer satisfaction. The unpunctual PM policy can then be applied to a broader scale.

5. Concluding remarks

This paper investigates an unpunctual PM policy for repairable items sold with a 2D warranty, under which the jth PM activity should be performed at age $jK \pm \Delta K$ or at usage $jL \pm \Delta L$, whichever occurs first. This policy offers customers a large degree of flexibility and convenience by tolerating their unpunctual PM behaviors to some extent. The expected total warranty costs for the unpunctual PM policy and its punctual counterpart are formulated under the assumption that customer unpunctuality is governed by a specific distribution. The optimization and comparison of the two policies are studied in different scenarios regarding the failure rate function and unpunctuality distribution. On this basis, the effects of customer unpunctuality on the optimal PM decision and the resultant warranty cost are investigated by comparing the two policies.

Numerical studies show that customer unpunctuality has little impact on the optimal PM policy; while the expected total warranty servicing cost for the unpunctual PM policy is quite sensitive to customer behaviors and the shape of failure rate function. In some scenarios, the expected warranty costs of the unpunctual policy are even lower than those of the punctual policy. In other words, the unpunctual PM policy, if well designed, has potential to attain both customer satisfaction growth and warranty cost reduction against its punctual counterpart. This gives rise to managerial guidelines when it is (most) (dis)advantageous for manufacturers to induce their customers to advance or postpone their PM schedules slightly. It is therefore of interest to manufacturers, like SAIC-GM-Wuling Automobile, to adopt this policy in their warranty and maintenance practice.

In summary, the main contributions of this paper to the warranty management literature include (*i*) it raises the attention to customer unpunctuality in PM execution; (*ii*) it provides a new warranty cost model in anticipation of customer unpunctuality; and (*iii*) it reveals several interesting insights that shed light on how a manufacturer should design/adjust a PM policy when customers' unpunctual maintenance executions are tolerated. Nevertheless, this work can be further extended in the following aspects:

- In this paper, the 2-D warranty policy of interest is a free minimal repair policy and the warranty region is rectangular. It is also valuable to examine the effect of customer unpunctuality on other warranty policies, such as pro-rata and lump-sum, and other warranty regions, e.g., L-shaped and triangular regions (Wang & Xie, 2018).
- The failure modeling approach adopted in this work is the well-known AFT model, while considering other approaches to 2D warranty modeling (e.g., bivariate model and alternative time scale model) could be another important direction for future work.
- Finally, motivated by the first-PM-delayed and first-PM-advanced phenomena, it is also interesting to extend the proposed unpunctual PM policy by treating the first PM instant as an additional decision variable.

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Appendix A. Detailed derivation of Eq. (12)

Substituting $\lambda(t|r) = (r/r_0)^{2\gamma}(2/\alpha^2)t$ into (10) and after some algebraic operations, for $n \geq 1$, the expected total warranty servicing cost of the unpunctual PM policy becomes

$$\begin{aligned}
&E[\widetilde{\mathcal{C}}_1(n,m)]\\
&= c_f\left(\mathcal{A}W^2 + \mathcal{B}U^2\right)\frac{1+n\delta(m)}{n+1} + nC_p(m)\\
&\quad + c_f\mathcal{A}(1-\delta(m))\int_{-\Delta K}^{\Delta K}\cdots\int_{-\Delta K}^{\Delta K}\left(\sum_{j=1}^{n}{y_j}^2 - \sum_{j=1}^{n-1}y_{j+1}y_j\right)\\
&\quad \times\prod_{j=1}^{n}z(y_j|r\leq\eta)\mathrm{d}y_1\ldots\mathrm{d}y_n + c_f(1-\delta(m))\\
&\quad \times\int_{\eta}^{\infty}\left\{\alpha^{-2}\left(\frac{r}{r_0}\right)^{2\gamma}\int_{-\frac{\Delta L}{r}}^{\frac{\Delta L}{r}}\cdots\int_{-\frac{\Delta L}{r}}^{\frac{\Delta L}{r}}\left(\sum_{j=1}^{n}{y_j}^2 - \sum_{j=1}^{n-1}y_{j+1}y_j\right)\right.\\
&\quad \left.\times\prod_{j=1}^{n}z(y_j|r>\eta)\mathrm{d}y_1\ldots\mathrm{d}y_n\right\}\mathrm{d}G(r),
\end{aligned}\tag{A.1}$$

where $\mathcal{A} = \alpha^{-2}\int_0^{\eta}(\frac{r}{r_0})^{2\gamma}g(r)\mathrm{d}r$.

According to the assumption that Y_j's, $j = 1, 2, \ldots, n$, are *i.i.d.* random variables, we have

$$\begin{aligned}
&\int_{-\Delta K}^{\Delta K}\cdots\int_{-\Delta K}^{\Delta K}\left(\sum_{j=1}^{n}{y_j}^2 - \sum_{j=1}^{n-1}y_{j+1}y_j\right)\prod_{j=1}^{n}z(y_j|r\leq\eta)\mathrm{d}y_1\ldots\mathrm{d}y_n\\
&= nE[(Y_1|_{r\leq\eta})^2] - (n-1)E^2[Y_1|_{r\leq\eta}]\\
&= n\left(E^2[Y_1|_{r\leq\eta}] + Var(Y_1|_{r\leq\eta})\right) - (n-1)E^2[Y_1|_{r\leq\eta}]\\
&= E^2[Y_1|_{r\leq\eta}] + nVar(Y_1|_{r\leq\eta}),\ n\geq 1,
\end{aligned}\tag{A.2}$$

where $E[Y_1|_{r\leq\eta}]$ and $Var(Y_1|_{r\leq\eta})$ are the mean and variance of $Y_1|_{r\leq\eta}$ in the case of $r\leq\eta$, respectively.

Similarly, for the last term in (A.1), we have

$$\begin{aligned}
&\int_{\eta}^{\infty}\left\{\alpha^{-2}\left(\frac{r}{r_0}\right)^{2\gamma}\int_{-\frac{\Delta L}{r}}^{\frac{\Delta L}{r}}\cdots\int_{-\frac{\Delta L}{r}}^{\frac{\Delta L}{r}}\left(\sum_{j=1}^{n}{y_j}^2 - \sum_{j=1}^{n-1}y_{j+1}y_j\right)\right.\\
&\quad \left.\times\prod_{j=1}^{n}z(y_j|r>\eta)\mathrm{d}y_1\ldots\mathrm{d}y_n\right\}\mathrm{d}G(r)\\
&= \int_{\eta}^{\infty}\alpha^{-2}\left(\frac{r}{r_0}\right)^{2\gamma}\left(nE[(Y_1|_{r>\eta})^2] - (n-1)E^2[Y_1|_{r>\eta}]\right)\mathrm{d}G(r)\\
&= \int_{\eta}^{\infty}\alpha^{-2}\left(\frac{r}{r_0}\right)^{2\gamma}\left(E^2[Y_1|_{r>\eta}] + nVar(Y_1|_{r>\eta})\right)\mathrm{d}G(r)\\
&= \int_{\eta}^{\infty}\alpha^{-2}\left(\frac{r}{r_0}\right)^{2\gamma}\frac{\eta^2}{r^2}\left(E^2[Y_1|_{r\leq\eta}] + nVar(Y_1|_{r\leq\eta})\right)\mathrm{d}G(r)\\
&= \mathcal{B}\eta^2\left(E^2[Y_1|_{r\leq\eta}] + nVar(Y_1|_{r\leq\eta})\right),\ n\geq 1,
\end{aligned}\tag{A.3}$$

where $\mathcal{B} = \alpha^{-2}\int_{\eta}^{\infty}\frac{r^{2\gamma-2}}{r_0^{2\gamma}}g(r)\mathrm{d}r$. The third equality is due to the fact that $Y_1|_{r>\eta} = \eta Y_1|_{r\leq\eta}/r$, and thus $E[Y_1|_{r>\eta}] = \eta E[Y_1|_{r\leq\eta}]/r$ and $Var(Y_1|_{r>\eta}) = \eta^2 Var(Y_1|_{r\leq\eta})/r^2$. The fourth equality is due to the fact that $E[Y_1|_{r\leq\eta}]$ and $Var(Y_1|_{r\leq\eta})$ are independent of usage rate r.

Finally, by substituting (A.2) and (A.3) into (A.1), it is easy to obtain Eq. (12).

Appendix B. Proof of Proposition 4

We first consider the case of $r\leq\eta$. When the mean of Y_j is $E[Y_j|r\leq\eta] = \int_{-\Delta K}^{\Delta K}y_j z(y_j|r\leq\eta)\mathrm{d}r = 0$. Applying Jensen's Inequality iteratively, the convexity of $\Lambda(t|r)$ leads to:

$$\begin{aligned}
&E[\widetilde{\mathcal{C}}(n,m)|r\leq\eta]\\
&= c_f\int_{-\Delta K}^{\Delta K}\cdots\int_{-\Delta K}^{\Delta K}\left[\sum_{j=0}^{n}\int_{\delta(m)\tau_j}^{\delta(m)\tau_j+\tau_{j+1}-\tau_j}\lambda(t|r)\mathrm{d}t\right]\\
&\quad \times\prod_{j=1}^{n}z\left(y_j|r\leq\eta\right)\mathrm{d}y_1\ldots\mathrm{d}y_n\\
&\geq c_f\int_{-\Delta K}^{\Delta K}\cdots\int_{-\Delta K}^{\Delta K}\left[\begin{array}{l}\sum_{j=0}^{n-2}\int_{\delta(m)\tau_j}^{\delta(m)\tau_j+\tau_{j+1}-\tau_j}\lambda(t|r)\mathrm{d}t\\ +\int_{\delta(m)\tau_{n-1}}^{\delta(m)\tau_{n-1}+nK-\tau_{n-1}}\lambda(t|r)\mathrm{d}t\\ +\int_{\delta(m)nK}^{\delta(m)nK+W-nK}\lambda(t|r)\mathrm{d}t\end{array}\right]\\
&\quad \times\prod_{j=1}^{n-1}z(y_j|r\leq\eta)\mathrm{d}y_1\ldots\mathrm{d}y_{n-1}\\
&\geq\cdots\\
&\geq c_f\int_{-\Delta K}^{\Delta K}\left[\int_0^{\tau_1}\lambda(t|r)\mathrm{d}t + \int_{\delta(m)\tau_1}^{\delta(m)\tau_1+2K-\tau_1}\lambda(t|r)\mathrm{d}t\right.\\
&\quad \left.+\sum_{j=2}^{n}\int_{\delta(m)jK}^{\delta(m)jK+K}\lambda(t|r)\mathrm{d}t\right]z(y_1|r\leq\eta)\mathrm{d}y_1\\
&\geq c_f\sum_{j=0}^{n}\int_{\delta(m)jK}^{\delta(m)jK+K}\lambda(t|r)\mathrm{d}t = E[\mathcal{C}(n,m)|r\leq\eta].
\end{aligned}$$

In a similar manner, when $r > \eta$, it is easy to obtain $E[\widetilde{\mathcal{C}}(n,m)|r > \eta] \geq E[\mathcal{C}(n,m)|r > \eta]$.

By removing the conditioning on R, we have

$$\begin{aligned} & E[\widetilde{\mathcal{C}}(n,m)] \\ & = \int_0^\eta E[\widetilde{\mathcal{C}}(n,m)|r \leq \eta]g(r)\mathrm{d}r + \int_\eta^\infty E[\widetilde{\mathcal{C}}(n,m)|r > \eta]g(r)\mathrm{d}r + nC_p(m) \\ & \geq \int_0^\eta E[\mathcal{C}(n,m)|r \leq \eta]g(r)\mathrm{d}r + \int_\eta^\infty E[\mathcal{C}(n,m)|r > \eta]g(r)\mathrm{d}r + nC_p(m) \\ & = E[\mathcal{C}(n,m)]. \end{aligned}$$

Therefore, the result follows directly.

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