A Model of Holographic Dark Energy

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A model for holographic dark energy is proposed, following the idea that the short distance cut-off is related to the infrared cut-off. We assume that the infrared cut-off relevant to the dark energy is the size of the event horizon. With the input $\Omega_\Lambda = 0.73$, we predict the equation of state of the dark energy at the present time be characterized by $w = -0.90$. The cosmic coincidence problem can be resolved by inflation in our scenario, provided we assume the minimal number of e-foldings.

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The cosmological constant problem is a longstanding problem in theoretical physics, and has received even more serious considerations recently, due to the observational evidence for a non-vanishing value [1]. The direct evidence for the existence of the dark energy is further supported by other cosmological observations, in particular by the WMAP experiment [2]. For the first time in history, theorists are forced to explain not only why the cosmological constant is small, but also why it is comparable to the critical density (in this note we shall use terms the cosmological constant and the dark energy exchangeably.)

A. Cohen and collaborators suggested sometime ago [3], that in quantum field theory a short distance cut-off is related to a long distance cut-off due to the limit set by formation of a black hole, namely, if ρ_Λ is the quantum zero-point energy density caused by a short distance cut-off, the total energy in a region of size L should not exceed the mass of a black hole of the same size, thus $L^3 \rho_\Lambda \leq L M_p^2$. The largest L allowed is the one saturating this inequality, thus

$$\rho_\Lambda = 3c^2 M_p^2 L^{-2}. \tag{1}$$

For convenience, we introduced a numerical constant $3c^2$ in the above relation, and use M_p to denote the reduced Planck mass $M_p^{-2} = 8\pi G$. Taking L as the size of the current universe, for instance the Hubble scale, the resulting energy density is comparable to the present day dark energy. Related ideas were discussed in [4].

While the magnitude of the holographic energy of Cohen et al. matches the experimental data, S. Hsu recently pointed out that the equation of state does not [5]. Hsu's argument can be refined as follows. In the Friedman equation $3M_p^2 H^2 = \rho$, we put two terms ρ_m and ρ_Λ, the latter being given by (1), with $L = H^{-1}$. We find

$$\rho_m = 3(1 - c^2) M_p^2 H^2, \tag{2}$$

thus ρ_m behaves as H^2, the same as ρ_Λ. But ρ_m scales with the universe scale factor a as a^{-3}, so does ρ_Λ, thus the dark energy is pressureless, namely in the equation of state $p = w\rho$, $w = 0$. The accelerating universe certainly requires $w < -1/3$, and the most recent data indicate that $w < -0.76$ at the 95% confidence level [6].

To remedy the situation, we are forced to use a different scale other than the Hubble scale as the infrared cut-off. One possibility quickly comes to mind, the particle horizon used in the holographic cosmology of Fischler and Susskind [7]. The particle horizon is given by

$$R_H = a \int_0^t \frac{dt}{a} = a \int_0^a \frac{da}{Ha^2}. \tag{3}$$

Replacing L in (1) by R_H, we can solve the Friedmann equation exactly with another energy component (for instance matter). Unfortunately, this replacement does not work. To see this, we assume that the dark energy ρ_Λ dominates, thus the Friedmann equation simplifies to $HR_H = c$, or

$$\frac{1}{Ha^2} = c\frac{d}{da}\left(\frac{1}{Ha}\right). \tag{4}$$

We find $H^{-1} = \alpha a^{1+\frac{1}{c}}$ with a constant α. The "dark energy" assumes the form

$$\rho_\Lambda = 3\alpha^2 M_p^2 a^{-2(1+\frac{1}{c})}. \tag{5}$$

So $w = -\frac{1}{3} + \frac{2}{3c} > -\frac{1}{3}$.

In the relation $HR_H = c$, c is always positive, and in changing this integral equation into a differential equation (4), we find that the changing rate of $1/(Ha)$ with respect to a is always positive, namely, the Hubble scale $1/H$ as compared to the scale factor a always increases. To get an accelerating universe, we need a shrinking Hubble scale. To achieve this, we replace the particle horizon by the future event horizon

$$R_h = a\int_t^\infty \frac{dt}{a} = a\int_a^\infty \frac{da}{Ha^2}. \tag{6}$$

This horizon is the boundary of the volume a fixed observer may eventually observe. One is to formulate a theory regarding a fixed observer within this horizon.

Again, we assume that the dark energy dominates matter, solving equation

$$\int_a^\infty \frac{da}{Ha^2} = \frac{c}{Ha}, \tag{7}$$

we have

$$\rho_\Lambda = 3c^2 M_p^2 R_h^2 = 3\alpha^2 M_p^2 a^{-2(1-\frac{1}{c})}, \tag{8}$$

or

$$w = -\frac{1}{3} - \frac{2}{3c}. \tag{9}$$

Alas, we do obtain a component of energy behaving as dark energy. If we take $c = 1$, its behavior is similar to the cosmological constant. If $c < 1$, $w < -1$, a value achieved in the past only in the phantom model. A smaller c although makes the dark energy smaller for a fixed event horizon size, it also forces R_h to be smaller by the Friedmann equation $HR_h = c$, thus the changing rate of $1/(Ha)$ larger. This is the reason why a smaller c makes the universe accelerate faster.

Theoretically, we are more interested in the case $c = 1$. We can actually give an argument in favor of $c = 1$. Suppose the universe be spatially flat (as the observation suggests), the total energy within a sphere of radius R_h is $\frac{4\pi}{3} R_h^3 \rho_\Lambda$. On the other hand, the mass of a black hole of size R_h is $R_h/(2G)$. Equating these two quantities, we find

$$\rho_\Lambda = \frac{3}{8\pi G} R_h^{-2} = 3 M_p^2 R_h^{-2}, \tag{10}$$

it follows that $c = 1$.

Before we consider a more realistic cosmology, let us pause to discuss causality. Since the event horizon R_h, as defined in (6) depends on the future evolution of $a(t)$, it appears that our holographic dark energy grossly violates causality. Event horizon in the context of cosmology as well as in that of a black hole is always defined globally, as the casual structure of space-time is a global thing. The co-moving time is the intrinsic time of a co-moving observer, and in a time-dependent background it is not the best time to use in order to understand causality. Indeed, in the conformal time, the event horizon is no-longer as acausal as in the co-moving time, as we shall see shortly. The metric $ds^2 = -dt^2 + a^2(t)dx^2$ is rewritten in the conformal time

$$\eta = \int_0^t \frac{dt'}{a(t')}, \tag{11}$$

as

$$ds^2 = a^2(\eta)(-d\eta^2 + dr^2 + r^2 d\Omega_2^2). \tag{12}$$

Now, the range of the conformal time has a finite upper limit 0, for instance $\eta \in (-\infty, 0)$. Due to this finite upper limit, a light-ray starts from the origin at the time η can not reach arbitrarily far, thus there is a horizon at $r = -\eta$. (For a more detailed discussion on the global causal structure of such a universe, see [8].) A local quantum field theory for the observer sitting at the origin is to be defined within this finite box. We now imagine that a fundamental theory in this finite box will results in a zero-point energy which is just holographic dark energy. Now, the formula $R_h = a(\eta)|\eta|$ no longer appears acaual. Now, the puzzle transforms into the question how a fundamental theory can be formulated within a finite box, this is supposed to be a consequence of cosmological complementarity.

Still, it appears rather puzzling why holographic energy is given by the time-dependent horizon size, as its definition is global. We may pose a similar puzzle concerning the Gibbons-Hawking entropy. If the universe evolves adiabatically, then the potential total entropy of our universe at time η is given by $S(\eta) = \pi R_h^2/l_p^2$, it superficially violates

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causality as much as holographic dark energy does. If one eventually can understand the origin of this entropy, hopefully we may eventually understand the origin of holographic dark energy (for a discussion on the connection between entropy and dark energy, see the second reference of [4].)

Although we argued that $c = 1$ is preferred, in what follows we leave c as an arbitrary parameter. With an additional energy component, the Friedmann equation can always be solved exactly. For instance, with matter present, the Friedmann equation reads

$$3M_p^2 H^2 = \rho_0 a^{-3} + 3c^2 M_p^2 R_h^{-2}, \tag{13}$$

where ρ_0 is the value of ρ_m at the present time when $a = 1$. This equation can be rewritten as

$$\int_a^\circ \frac{da}{Ha^2} = c(H^2 a^2 - \Omega_m^0 H_0^2 a^{-1})^{-1/2}. \tag{14}$$

We may try to convert the above integral equation to a differential equation for the unknown function H.

However, it proves more convenient to use Ω_Λ as the unknown function. We have $\Omega_\Lambda = \rho_\Lambda/\rho_c$, where $\rho_c = 3M_p^2 H^2$. By definition, $R_h^2 = 3c^2 M_p^2/\rho_\Lambda = c^2/(\Omega_\Lambda H^2)$, or

$$\int_a^\circ \frac{da}{Ha^2} = \int_x^\circ \frac{dx}{Ha} = \frac{c}{\sqrt{\Omega_\Lambda}Ha}, \tag{15}$$

where $x = \ln a$. Next, we wish to express Ha in terms of Ω_Λ. To this end, we introduce the matter component $\rho_m = \rho_m^0 a^{-3}$. We set $a(t_0) = 1$, and ρ_m^0 is the present matter energy density. Now, the Friedmann equation is simply $1 - \Omega_\Lambda = \Omega_m = \Omega_m^0 H_0^2 H^{-2} a^{-3}$. This implies

$$\frac{1}{Ha} = \sqrt{a(1 - \Omega_\Lambda)} \frac{1}{H_0 \sqrt{\Omega_m^0}}. \tag{16}$$

Substituting this relation (as implied by the Friedmann equation) into (15)

$$\int_x^\circ \sqrt{a}\sqrt{1 - \Omega_\Lambda} dx = c\sqrt{a}\sqrt{\frac{1}{\Omega_\Lambda} - 1}. \tag{17}$$

Taking derivative with respect to x in the both sides of the above relation, and noting that the derivative of $\sqrt{a}$ is proportional to $\sqrt{a}$, we obtain

$$\frac{\Omega_\Lambda'}{\Omega_\Lambda^2} = (1 - \Omega_\Lambda)(\frac{1}{\Omega_\Lambda} + \frac{2}{c\sqrt{\Omega_\Lambda}}), \tag{18}$$

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where the prime denotes the derivative with respect to x. This equation can be solved exactly. Before solving the equation, we note that Ω_Λ' is always positive, namely the fraction of the dark energy increases in time, the correct behavior as we expect. Also, the expansion of the universe will never have a turning point so that the universe will not re-collapse, since Ω_Λ' never vanishes before Ω_Λ reaches its maximal value 1.

Let $y = 1/\sqrt{\Omega_\Lambda}$, the differential equation (18) is cast into the form

$$y^2 y' = (1 - y^2)(\frac{1}{c} + \frac{1}{2}y). \tag{19}$$

This equation can be solved exactly for arbitrary c, we write down the solution for $c = 1$ only for illustration purpose:

$$\ln \Omega_\Lambda - \frac{1}{3}\ln(1 - \sqrt{\Omega_\Lambda}) + \ln(1 + \sqrt{\Omega_\Lambda}) - \frac{8}{3}\ln(1 + 2\sqrt{\Omega_\Lambda}) = \ln a + x_0. \tag{20}$$

If we set $a_0 = 1$ at the present time, x_0 is equal to the L.H.S. of (20) with Ω_Λ replaced by Ω_Λ^0

As time draws by, Ω_Λ increases to 1, the most important term on the L.H.S. of (20) is the second term, we find, for large a

$$\sqrt{\Omega_\Lambda} = 1 - 3^{-8}2^3 e^{-3x_0} a^{-3}. \tag{21}$$

Since the universe is dominated by the dark energy for large a, we have

$$\rho_\Lambda \simeq \rho_c = \rho_m/(1 - \Omega_\Lambda) = \rho_m^0 a^{-3}/(1 - \Omega_\Lambda). \tag{22}$$

Thus, using (21) in the above relation

$$\rho_\Lambda = 3^8 2^{-4} e^{3x_0} \rho_m^0. \tag{23}$$

Namely, the final cosmological constant is related to ρ_m^0 through the above relation.

For very small a, matter dominated, and the most important term on the L.H.S. of (20) is the first term, we find

$$\Omega_\Lambda = e^{x_0} a, \tag{24}$$

thus

$$\rho_\Lambda = \Omega_\Lambda \rho_c \simeq \Omega_\Lambda \rho_m = e^{x_0} \rho_m^0 a^{-2}. \tag{25}$$

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So although the dark energy is larger for smaller a, it is still dominated over by matter, we do not have to worry about the possibility of ruining the standard big bang theory. A discussion of the dark energy behaving as a^{-2} in the early universe can be found in [9].

What we are interested in most is the prediction about the equation of state at the present time. Usually, in the cosmology literature such as [6], one measures w as in $\rho_\Lambda \sim a^{-3(1+w)}$. Expanding

$$\ln \rho_\Lambda = \ln \rho_\Lambda^0 + \frac{d \ln \rho_\Lambda}{d \ln a} \ln a + \frac{1}{2} \frac{d^2 \ln \rho_\Lambda}{d(\ln a)^2} (\ln a)^2 + \dots, \tag{26}$$

where the derivatives are taken at the present time $a_0 = 1$. The index w is then

$$w = -1 - \frac{1}{3} \left(\frac{d \ln \rho_\Lambda}{d \ln a} + \frac{1}{2} \frac{d^2 \ln \rho_\Lambda}{d(\ln a)^2} \ln a \right), \tag{27}$$

up to the second order. Since $\rho_\Lambda \sim \Omega_\Lambda H^2 \sim \Omega_\Lambda \frac{\rho_r}{\Omega_r} \sim \Omega_\Lambda/(1 - \Omega_\Lambda)a^{-3}$, the derivatives are easily computed using (18):

$$w = -\frac{1}{3} - \frac{2}{3c} \sqrt{\Omega_\Lambda^0} + \frac{1}{6c} \sqrt{\Omega_\Lambda^0}(1 - \Omega_\Lambda^0)(1 + \frac{2}{c} \sqrt{\Omega_\Lambda^0})z, \tag{28}$$

where we used $\ln a = -\ln(1 + z) \simeq -z$.

The above formula is valid for arbitrary c. Specifying to the case $c = 1$ when the holographic dark energy approaching to a constant in the far future, and plugging the optional value $\Omega_\Lambda^0 = 0.73$ into (28),

$$w = -0.903 + 0.104z. \tag{29}$$

Of course only the first two digits are effective. This result is in excellent agreement with new data [6]. At the one sigma level, the result of [6] is $w = -1.02^{+0.13}_{-0.19}$, with a slightly different value $\Omega_\Lambda^0 = 0.71$. If our holographic model for dark energy is viable, it is quite hopeful that this prediction will be verified by experiments in near future.

The choice $c < 1$ will leads to dark energy behaving as phantom, and in this case, the Gibbons-Hawking entropy will eventually decrease as the the event horizon will shrink, this violates the second law of thermodynamics. For $c > 1$, the second law of thermodynamics is not violated, while in a situation without any other component of energy, the space-time is not de Sitter, thus for symmetry reason we prefer to choose $c = 1$ and the result (29) in a sense is a prediction.

During the radiation dominated epoch, the dark energy also increases with time compared to the radiation energy, but it is still small enough not to ruin standard results such as nuclear genesis. We are also interested in whether our model will greatly affect the standard slow-roll inflation scenario. In this case, assume that the universe has only two energy components, the "dark energy" and the inflaton energy. If the latter is almost constant, we shall show that it is possible that the dark energy can be inflated away. Similar to (18), in this case we can derive an equation

$$\Omega'_\Lambda = 2\Omega_\Lambda(\Omega_\Lambda - 1)(1 - \sqrt{\Omega_\Lambda}). \tag{30}$$

Thus, Ω_Λ always decreases during inflation. The above equation can also be solved exactly. Instead of exhibiting the exact solution, we only show its behavior for small Ω_Λ:

$$\Omega_\Lambda \sim a^{-2}, \tag{31}$$

thus, if the initial value of Ω_Λ is reasonable, it will be red-shifted away quickly enough not to affect the standard inflation scenario.

This huge red-shift may be the resolution to the cosmic coincidence problem, since the coincidence problem becomes a problem of why the ratio between the dark energy density and the radiation density is a very tiny number at the onset of the radiation dominated epoch. A rough estimate shows that the ratio between ρ_Λ and ρ_r, the radiation density, is about 10^{-52}, if we choose the inflation energy scale be 10^{14}Gev. According to (31), this is to be equal to $\exp(-2N)$ where N is the number of e-folds, and we find $N = 60$, the minimal number of e-folds in the inflation scenario. Of course, we need to assume that all the dark energy is included in ρ_Λ in the end of inflation, namely, the inflaton energy completely decayed into radiation. Thus, inflation not only solves the traditional naturalness problems and helps to generate primordial perturbations, it also solves the cosmic coincidence problem! We may imagine that in another region of the universe, the number of e-folds is different, thus a different cosmological constant results.

This model requires, for a consistent solution to exist, that any other form of energy must eventually decay. Still, it is possible that there is an additional component of dark energy such as quintessence which will indeed decay in the far future. A couple of papers explored this question after the present paper appeared on the internet, so we shall not address this question here.

In conclusion, the holographic dark energy scenario is viable if we set the infrared cut-off by the event horizon. This is not only a viable model, it also makes a concrete prediction about the equation of state of the dark energy, thus falsifiable by the future experiments.

However, unlike expected earlier, we are not able to explain the cosmic coincidence along the line of [3], since the infrared cut-off is not the current Hubble scale. The eventual cosmological constant in the far future can be viewed as a boundary condition, or equivalently, the initial value of Ω_Λ can be viewed as a initial condition. This initial condition is affected by physics in very early universe, for instance physics of inflation. In this regard, it appears that inflation is able to explain the current value ρ_Λ if a proper number of e-folds is assumed, since the dark energy compared to the inflaton energy thus the radiation energy in the end of inflation is very small due to inflation. A detailed analysis will appear elsewhere.

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