$\dot{m}_i = r_{X,i}u_i - (\mu + \theta_{m,i})m_i + \lambda_i \quad , \quad i = 1, 2, \cdots N$ (mRNA)

$\dot{P}_i = r_{L,i}w_i - (\mu + \theta_{p,i})p_i$ (protein)

- $r_{X,i}u_i$: production rate of mRNA i ; $r_{L,i}w_i$: production rate of protein
- λ_i : leak of gene g_i $\quad 0 \le u(\ldots) \le 1$, $0 \le w(\cdots) \le 1$
- $\mu + \theta_{m,i}$ or $\mu + \theta_{p,i}$: dilution and degradation term

Suppose we defined intracellular conc. as nmol (or μmol) per unit basis B, where B is an abstract volume basis. To derive the material balance governing the specific intracellular conc. of the j^{th} species x_j [*mol/B] we start from the general mole balance and the standard four terms (accumulation, in/out and generation):

$\dot{n}_{X,acc,j} = \dot{n}_{X,in,j} - \dot{n}_{X,out,j} + \dot{n}_{X,gen,j} \quad j = 1, 2, \ldots, \mathcal{M}$

- $\mathcal{M}$: # of intracellular species ; X : specific single cell from ppl of cell.

No connective transport into or from the cellular phase

$\dot{n}_{X,in,j} = \dot{n}_{X,out,j} = 0$

$\Rightarrow \dot{n}_{X,acc,j} = \dot{n}_{X,gen,j} \quad j = 1, 2 \ldots, \mathcal{M}$.

Assuming some abstract volume B, then:

$\frac{d}{dt}\int_B x_j \, dB = \int_B (\ldots) \, dB \quad , \quad j = 1, 2 \ldots, \mathcal{M}$.

first term is accumulation term, RHS is generation term.

After making the WMA, the integral balance eqn. reduce to a more recognizable form:

$\frac{d}{dt}(x_i B) = (\ldots)B \quad , \quad j = 1, 2, \cdots, \mathcal{M}$.

Expanding the derivative and simplifying gives:

$\dot{x}_j = (\ldots) - x_j B^{-1}\dot{B} \quad , \quad j = 1, 2, \ldots \mathcal{M}$.

- $B^{-1}\dot{B}$: intracellular dilution term

In unit system: $B = XV_R$, $\dot{B} = \dot{X}V_R + X\dot{V}_R$

$B^{-1}\dot{B} = X^{-1}\dot{X} + V_R^{-1}\dot{V}_R$, in a batch culture, V_R is constant, so $\dot{V}_R = 0$

$B^{-1}\dot{B} = X^{-1}\dot{X}$, where $\dot{X} = \mu X$, so $B^{-1}\dot{B} = X^{-1}\mu X = \mu = \frac{\dot{X}V_R}{XV_R} + \frac{X\dot{V}_R}{XV_R} = \frac{\dot{X}}{X} + \frac{\dot{V}_R}{V_R} \Rightarrow \boxed{\mu = 0}$

$$\boxed{\begin{aligned} &\text{Since no dilution in cell free: } \dot{m}_i = r_{X,i}u_i - \theta_{m,i}m_i + \lambda_i \\ &\qquad\qquad\qquad\qquad\qquad\qquad \dot{P}_i = r_{L,i}w_i - \theta_{p,i}P_i \end{aligned}}$$