Problem 3: Stability analysis of Collins toggle switch.

f-1) $\frac{dR_i^*}{dt} = k_f L R_i - k_r R_i^*$ (1)

let (1) reach to 0.

$k_f L R_i - k_r R_i^* = 0$; $\Rightarrow k_f L R_1 - k_r R_1^* = 0 \Rightarrow R_1^* = \frac{k_f L R_1}{k_r}$ (A)

$k_f L R_2 - k_r R_2^* = 0 \Rightarrow R_2^* = \frac{k_f L R_2}{k_r}$

$\frac{dN_i^*}{dt} = k_f^{ND} N_i D_j - k_r^{ND} N_i^*$ (2)

let (2) reach to 0

$k_f^{ND} N_i D_j - k_r^{ND} N_i^* = 0 \Rightarrow k_f^{ND} N_1 D_2 - k_r^{ND} N_1^* \Rightarrow N_1^* = \frac{k_f^{ND} N_1 D_2}{k_r^{ND}}$ (B)

$k_f^{ND} N_2 D_1 - k_r^{ND} N_2^* \qquad N_2^* = \frac{k_f^{ND} N_2 D_1}{k_r^{ND}}$

$\frac{dD_i}{dt} = k_D R_i^* - \gamma_D D_i$ (3)

let (3) reach to 0

$k_D R_i^* - \gamma_D D_i = 0 \Rightarrow k_D R_1^* - \gamma_D D_1 = 0 \Rightarrow D_1 = \frac{k_D R_1^*}{\gamma_D}$ (C)

$k_D R_2^* - \gamma_D D_2 = 0 \qquad D_2 = \frac{k_D R_2^*}{\gamma_D}$

$\frac{dR_i}{dt} = \frac{\beta^n}{k^n + N_i^{*n}} - \gamma_R R_i$ (4)

(1) - (3) fast eqn.

$k_f L R_i - k_r R_i^* = 0$ ; $k_f^{ND} N_i D_j - k_r^{ND} N_i^* = 0$ ; $k_D R_i - \gamma_D D_i = 0$

Plug (A) into (C).

$D_1 = \frac{k_D}{\gamma_D}\left(\frac{k_f L R_1}{k_r}\right)$ ; $D_2 = \frac{k_D}{\gamma_D}\left(\frac{k_f L R_2}{k_r}\right)$

then plug the above eqn. to (B)

$$N_1^* = \frac{k_f^{ND} N_1}{k_r^{ND}} \frac{k_D}{\gamma_D}\left(\frac{k_f L R_2}{k_r}\right)$$

$$N_2^* = \frac{k_f^{ND} N_2}{k_r^{ND}} \frac{k_D}{\gamma_D}\left(\frac{k_f L R_1}{k_r}\right)$$

$$\therefore \boxed{\begin{aligned} \frac{dR_1}{dt} &= \frac{\beta^n}{k^n + \left(\frac{k_f^{ND} N_1}{k_r^{ND}} \frac{k_D}{\gamma_D}\left(\frac{k_f L R_2}{k_r}\right)\right)^n} - \gamma_R R_1 \\ \frac{dR_2}{dt} &= \frac{\beta^n}{k^n + \left(\frac{k_f^{ND} N_2}{k_r^{ND}} \frac{k_D}{\gamma_D}\left(\frac{k_f L R_1}{k_r}\right)\right)^n} - \gamma_R R_2 \end{aligned}}$$

f-2). let $u = \frac{R_1}{k}$, $v = \frac{R_2}{k}$, $\tau = \gamma R t$; then $R_1 = uk$ $\quad dR_1 = du k$

$R_2 = vk \quad dR_2 = dv k$

$t = \frac{\tau}{\gamma_R} \quad dt = \frac{d\tau}{\gamma_R}$

$$\frac{dR_1}{dt} = \frac{\beta^n}{k^n + \left(\frac{k_f^{ND} N_1}{k_r^{ND}} \frac{k_D}{\gamma_D}\left(\frac{k_f L R_2}{k_r}\right)\right)^n} - \gamma_R R_1$$

$$\Rightarrow \frac{du k \gamma_R}{d\tau} = \frac{\beta_n}{k^n + \left(\frac{k_f^{ND} N_1}{k_r^{ND}} \frac{k_D}{\gamma_D} \frac{k_f L v k}{k_r}\right)^n} - \gamma_R R_1$$

$$= \frac{\beta_n}{k^n ( 1 + \left(\frac{k_f^{ND} N_1 k_D k_f L v}{k_r^{ND} \gamma_D k_r}\right)^n} - \gamma_R R_1$$

$$\frac{du}{d\tau} = -u + \frac{\beta_n}{k^{n-1}\gamma_R \left(1 + \left(\frac{k_f^{ND} N_1 k_D k_f L v}{k_r \gamma_D k_r^{ND}}\right)^n\right)}$$

$\rightarrow$ has the form $\frac{du}{dt} = \frac{\alpha}{1 + (A_1 v)^n}$

where $\alpha = \frac{\beta_n}{k^{n-1}\gamma_R}$ $\quad A_1 = \frac{k_f^{ND} N_1 k_D k_f L}{k_r \gamma_D k_r^{ND}}$

$$\frac{dR_2}{dt} = \frac{\beta^n}{k^n + \left(\frac{k_f^{ND} N_2}{k_r^{ND}} \frac{k_D}{\gamma_D}\left(\frac{k_f L R_1}{k_r}\right)\right)^n} - \gamma_R R_2$$

$$\Rightarrow \frac{dv k \gamma_R}{d\tau} = \frac{\beta_n}{k^n + \left(\frac{k_f^{ND} N_2}{k_r^{ND}} \frac{k_D}{\gamma_D} \frac{k_f L u k}{k_r}\right)^n} - \gamma_R R_2$$

$$= \frac{\beta_n}{k^n ( 1 + \left(\frac{k_f^{ND} N_2 k_D k_f L u}{k_r^{ND} \gamma_D k_r}\right)^n} - \gamma_R R_2$$

$$\frac{dN}{d\tau} = -V + \frac{\beta_n}{k^{n-1}\gamma_R\left(1+\left(\frac{k_f^{ND} N_2 k_D k_f L U}{k_r \gamma_D k_r^{ND}}\right)^n\right)}$$

→ has the form $\dfrac{dN}{dt} = \dfrac{\alpha}{1+(A_2V)^n}$

where $\alpha = \dfrac{\beta_n}{k^{n-1}\gamma_R}$ $\quad A_2 = \dfrac{k_f^{ND} N_2 k_D k_f L}{k_r \gamma_D k_r^{ND}}$