d). $\frac{du}{dt} = \frac{\alpha}{1+v^n} - u \qquad \frac{dv}{dt} = \frac{\alpha}{1+u^n} - v$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \text{Jacobian matrix } \vec{J} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & \frac{\alpha n v^{n-1}}{(1+v^n)^2} \\ \frac{\alpha n u^{n-1}}{(1+u^n)^2} & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

At S.S. $\frac{du}{dt} = 0$ & $\frac{dv}{dt} = 0$

$$\frac{du}{dt} = \frac{\alpha}{1+v^n} - u = 0 \quad \Rightarrow \quad \frac{\alpha}{1+v_s^n} = u_s \quad (1)$$

$$\frac{dv}{dt} = \frac{\alpha}{1+u^n} - v = 0 \quad \Rightarrow \quad \frac{\alpha}{1+u_s^n} = v_s \quad (2)$$

plug (2) into (1), then:

$$\frac{\alpha}{1+\left(\frac{\alpha}{1+u_s^n}\right)^n} = u_s$$

$$\vec{J} = \begin{pmatrix} -1 & \frac{\alpha n v_s^{n-1}}{(1+v_s^n)^2} \\ \frac{\alpha n u_s^{n-1}}{(1+u_s^n)^2} & -1 \end{pmatrix}$$

Find eigenvalues:

$$(-1-\lambda)(-1-\lambda) - \frac{\alpha n v_s^{n-1}}{(1+v_s^n)^2} \cdot \frac{\alpha n u_s^{n-1}}{(1+u_s^n)^2} = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + \left(1 - \frac{\alpha^2 n^2 v_s^{n-1} u_s^{n-1}}{(1+v_s^n)^2 \cdot (1+u_s^n)^2}\right) = 0$$

$$\lambda_\pm = \frac{tr(\vec{J}) \pm \sqrt{tr(\vec{J})^2 - 4\det(J)}}{2}$$

$$tr(\vec{J}) = -2, \quad \det(J) = 1 - \frac{\alpha^2 n^2 v_s^{n-1} u_s^{n-1}}{(1+v_s^n)^2 (1+u_s^n)^2}$$

when $tr(\vec{J}) < 0$ and $\det(\vec{J}) > 0$, then system is stable

Following the criterion, $tr(\vec{J}) = -2 < 0$

$$\det(J) = 1 - \frac{\alpha^2 n^2 v_s^{n-1} u_s^{n-1}}{(1+v_s^n)^2 (1+u_s^n)^2} > 0$$

$(1+V_s^n)^2(1+U_s^n)^2 > 0$

$\Rightarrow (1+V_s^n)^2(1+U_s^n)^2 - \alpha^2 n^2 (V_s U_s)^{n-1} > 0$

$$\Rightarrow \frac{(1+V_s^n)^2(1-U_s^n)^2 - \alpha^2 n^2 (V_s U_s)^{n-1}}{(1+V_s^n)^2(1+U_s^n)^2} > 0$$

At center S.S. $V_s = U_s$

$\therefore (1+V_s^n)^4 - \alpha^2 n^2 V_s^{2(n-1)} > 0$

$\Rightarrow (1+V_s^n)^4 > (\alpha n V_s^{n-1})^2$

$\Rightarrow (1+V_s^n)^2 > \alpha n V_s^{n-1}$

For a stable system, the influence of degree of coorporativity and rate of synthesis follow the eqn. $(1+V_s^n)^2 > \alpha n V_s^{n-1}$

For example: $n=1$ $\alpha=1$ $\quad (1+V_s)^2 > 1$ $\quad \begin{cases} 1+V_s > 1 \Rightarrow V_s > 0 \\ 1+V_s < -1 \Rightarrow V_s < -2 \text{ (unrealistic)} \end{cases}$

e).

$$\lambda_{\pm} = \frac{tr(\vec{J}) \pm \sqrt{tr(\vec{J})^2 - 4\det(J)}}{2}$$

$$tr(\vec{J}) = -2, \quad \det(J) = 1 - \frac{\alpha^2 n^2 V_s^{n-1} U_s^{n-1}}{(1+V_s^n)^2(1+U_s^n)^2}$$

For center steady state, $\alpha=10$, $n=1$ $\quad U_s = V_s$

$$\det(J) = 1 - \frac{\alpha^2 n^2 V_s^{n-1} U_s^{n-1}}{(1+V_s^n)^2(1+U_s^n)^2} = 1 - \frac{100 \times 1 \times 1 \times 1}{(1+V_s)^2(1+U_s)^2}$$

$$= 1 - \frac{100}{(1+V_s)^2(1+U_s)^2}$$

read from figure.

From $n=1$ $\alpha=10$ figure, there's one point: $V_s = U_s \approx 2.5$

$$\det(J) = 1 - \frac{100}{(1+2.5)^2} = -7.16$$

$$\lambda_\pm = \frac{-2 \pm \sqrt{(-2)^2 + 4 \times 7.16}}{2} = \frac{-2 \pm 5.71}{2}$$

$\lambda_+ = 1.855 \qquad \lambda_- = -3.855$

Since both of them are negative, it's stable.

For center steady state. $\alpha = 10$ $n=2$ $u_s = v_s$

$$\det(J) = 1 - \frac{\alpha^2 n^2 v_s^{n-1} u_s^{n-1}}{(1+v_s^n)^2 (1+u_s^n)^2} = 1 - \frac{100 \times 4 \times v_s \times u_s}{(1+v_s^2)^2 (1+u_s^2)^2}$$

From $n=2$ $\alpha=10$ figure. I read three point $v_s \approx 10, 0, 2$

For $v_s \approx 10$

$$\det(J) = 1 - \frac{400 \times 10 \times 10}{(1+10^2)^2 (1+10^2)^2} = 1 - \frac{40000}{104060401} = 0.9996$$

$$\lambda_\pm = \frac{-2 \pm \sqrt{(-2)^2 - 4 \times 0.9996}}{2} = \frac{-2 \pm 0.04}{2}$$

$$\lambda_+ = \frac{-2+0.04}{2} = -0.98 \qquad \lambda_- = \frac{-2-0.04}{2} = -1.02$$

Since both of them are negative, it's stable.

For $v_s \approx 0$

$$\det(J) = 1 - \frac{0}{1^2+1^2} = 1$$

$$\lambda_\pm = \frac{-2 \pm \sqrt{(-2)^2 - 4 \times 0}}{2} = \frac{-2 \pm 2}{2}$$

$$\lambda_+ = \frac{-2+2}{2} = 0 \qquad \lambda_- = \frac{-2-2}{2} = -4$$

Since there's no positive λ, it's stable.

For $V_s \approx 2$

$$\det(J) = 1 - \frac{400 \times 2 \times 2}{(1+2^2)^2 (1+2^2)^2} = 1 - \frac{1600}{625} = -1.56$$

$$\lambda_\pm = \frac{-2 \pm \sqrt{(-2)^2 + 4 \times 1.56}}{2} = \frac{-2 \pm 3.2}{2}$$

$$\lambda_+ = \frac{-2+3.2}{2} = 0.6 \qquad \lambda_- = \frac{-2-3.2}{2} = -2.6$$

Since there's one positive λ, it's unstable