Homework 4

Setup:

In [1]:
```
import numpy as np
import scipy as sp
import matplotlib.pyplot as plt
import newton_method_script as newton
import secant_method_script as secant
def driver(F='Nan',A='Nan',B='Nan',tol='Nan'):
    # use routines
    f = lambda x: (x-1)*x**2
    a = 0.5
    b = 2

    if F =='Nan':
        F=f
    if A =='Nan':
        A=a
    if B =='Nan':
        B=b

    # f = lambda x: np.sin(x)
    # a = 0.1
    # b = np.pi+0.1
    if tol == 'Nan':

        tol = 1e-5
    [astar,ier,count] = bisection(F,A,B,tol)
    print('the approximate root is',astar)
    print('the error message reads:',ier)
    print("The number of iterations is:",count)
    print('f(astar) =', F(astar))
# define routines
def bisection(f,a,b,tol):
# Inputs:
# f,a,b - function and endpoints of initial interval
# tol - bisection stops when interval length < tol
# Returns:
# astar - approximation of root
# ier - error message
# - ier = 1 => Failed
# - ier = 0 == success
# first verify there is a root we can find in the interval
    fa = f(a)
    fb = f(b)
    if (fa*fb>0):
        ier = 1
        astar = a
        return [astar, ier]
    # verify end points are not a root
    if (fa == 0):
        astar = a
```

```
        ier =0
        return [astar, ier]
    if (fb ==0):
        astar = b
        ier = 0
        return [astar, ier]
    count = 0
    d = 0.5*(a+b)
    while (abs(d-a)> tol):
        fd = f(d)
        if (fd ==0):
            astar = d
            ier = 0
            return [astar, ier]
        if (fa*fd<0):
            b = d
        else:
            a = d
            fa = fd
        d = 0.5*(a+b)
        count = count +1
        # print('abs(d-a) = ', abs(d-a))
    astar = d
    ier = 0
    return [astar, ier]
```

Problem 1

a)

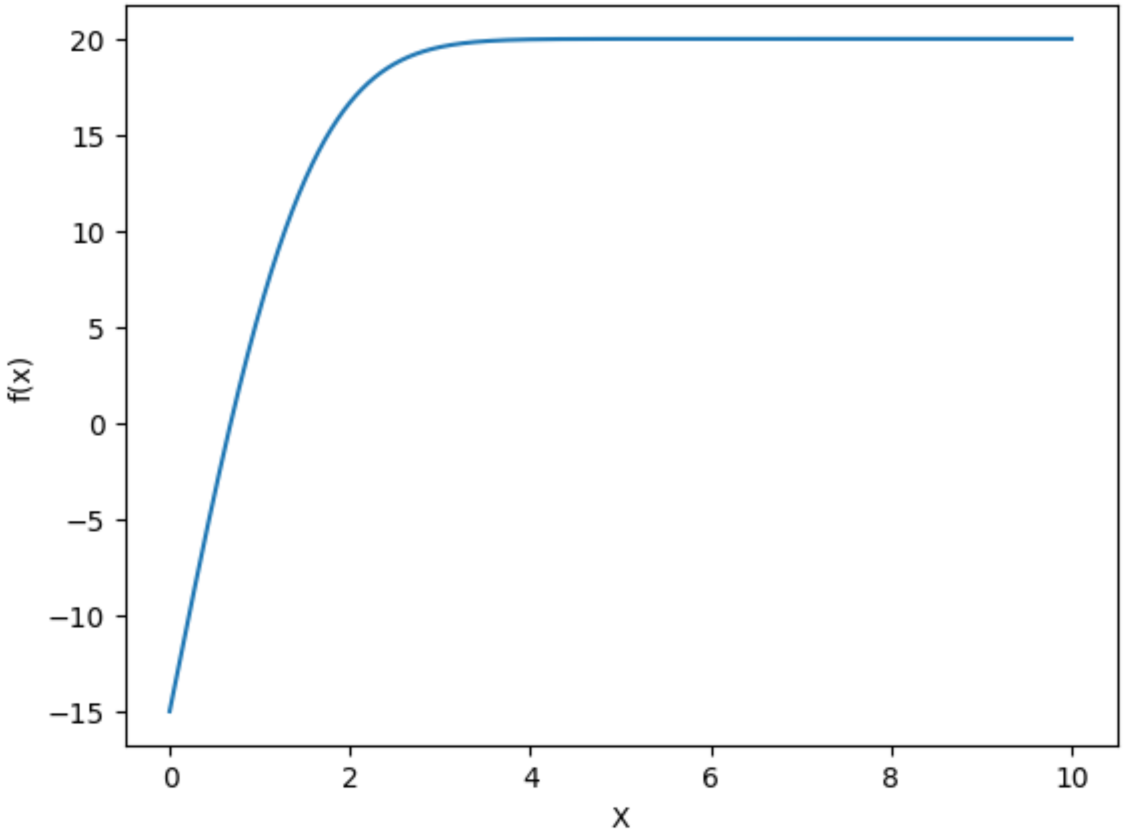
The function to solve is $f(x) = 35erf(x/(2\sqrt{\alpha t})) - 15$

$f'(x) = \frac{35}{2\sqrt{\alpha t}} e^{-x^2/(4\alpha t)}$

In [2]:
```
x11=np.linspace(0,10,10000)
alpha11 = 0.138e-6
t11 = 5184000
f11= lambda x: 35*sp.special.erf(x/(2*np.sqrt(alpha11*t11)))-15
df11 = lambda x: (35/(2*np.sqrt(alpha11*t11)))*np.exp(-x**2/(4*alpha11*t11))
```

In [3]:
```
plt.plot(x11,f11(x11))
plt.xlabel('X')
plt.ylabel("f(x)")
```

Out[3]:
```
Text(0, 0.5, 'f(x)')
```



b)

In [4]:
```
depth, error = bisection(f11,0,10,1e-13)
print('depth = ', depth)
```

```
depth =  0.6769618544819167
```

This makes sense, as our graph is also roughly showing that point as a zero.

c)

In [5]:
```
newton.newton_method(f11,df11,0.01,1e-13,100)
```

```
Newton method converged succesfully, niter=15, nfun=32, f(r)=3.4e-13
```

Out[5]:
```
(0.6769618544819533,
 array([0.01      , 0.72371999, 0.67034388, 0.67779986, 0.67685408,
        0.67697569, 0.67696008, 0.67696208, 0.67696183, 0.67696186,
        0.67696185, 0.67696185, 0.67696185, 0.67696185, 0.67696185,
        0.67696185]),
 32)
```

Converges succesfully.

In [6]:
```
print(df11(10))
```

```
1.3770983942531758e-14
```

In [7]:
```
newton.newton_method(f11,df11,10,1e-13,100)
```

Out[7]: (10, array([10]), 2)

Here the function breaks as $\frac{df}{dx} < 1 * 10^{-13}$

Problem 2

a)

$f(x) = (x - \alpha)^m h(x)$

b)

b)

$$g(x) = x - \frac{f(x)}{f'(x)} = x_{n+1}$$

$$x_n + 1 = x_n - f(x_n)/f'(x_n) = x_n - \frac{h(x)}{m(x-\alpha)^{-1} h(x) + h'(x)}$$

$$g'(x) = 1 - \frac{(h(x) + x - \alpha)[m h(x) + \_\_\_] - ((x-\alpha)(h(x)))[\_\_\_\quad]}{[m h(x) + (x-\alpha) h'(x)]^2}$$

$$\lim_{x \to \alpha} g'(x) = 1 - 1/m \leftarrow \text{converges.}$$

c)

c) Newton's Thm, (lass notes 11:

$$g(x) = x - m\frac{f(x)}{f'(x)}, \quad g'(x) = 1 + m f(x)(f''(x))^{-2} - m$$

$$g''(x) = m f'(x)(f''(x))^{-2} - 2f(x)(f'''(x))^{-3}$$

$$g'(\alpha) = 1 - m \qquad g''(\alpha) = 0$$

so quatratically conv.

d)

This proves that newtons method converges quadratically for roots greater than 1.

Problem 3

3

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = const = A$$

$$|x_n-\alpha|^p = \frac{|x_{n+1}-\alpha|}{A}$$

$$p\log(|x_n-\alpha|) = \log(|x_{n+1}-\alpha|) - \log(A)$$

$$p = \frac{\log(|x_{n+1}-\alpha|)}{\log(|x_n-\alpha|)} - \frac{\log(A)}{\log(|x_n-\alpha|)}$$

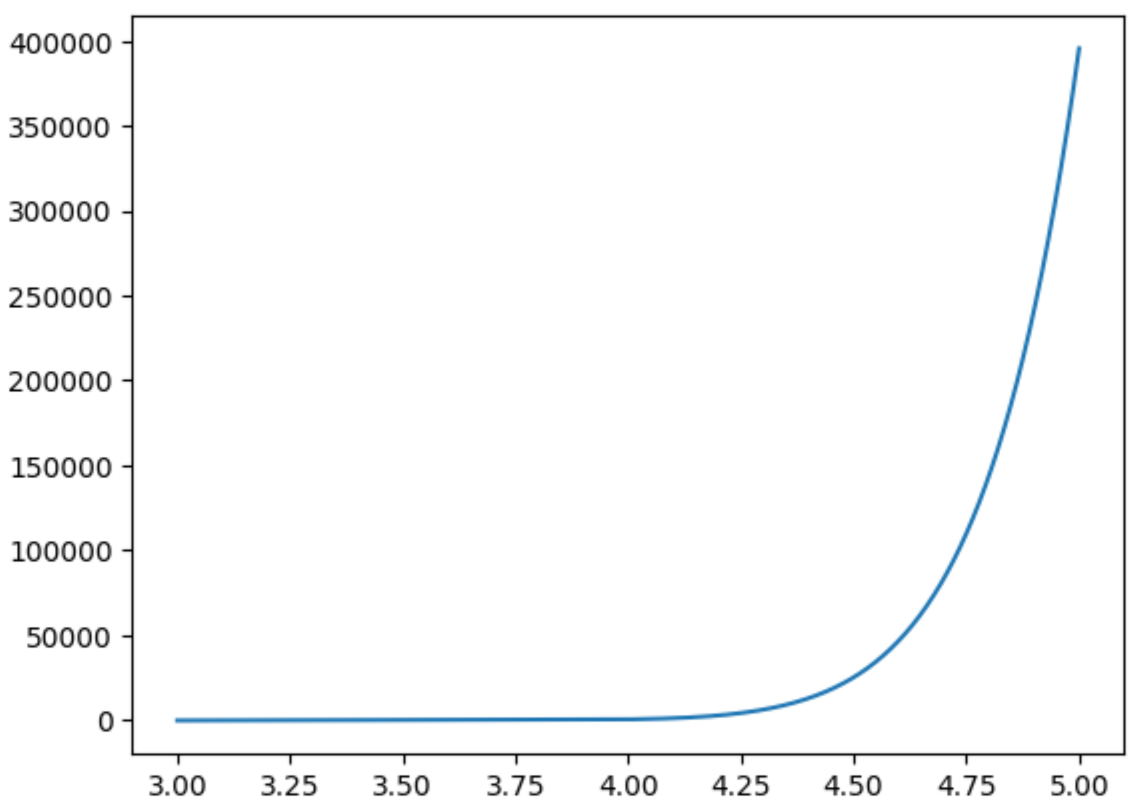
Problem 4

i)Newtons's

In [8]:
```
x4 = np.linspace(3,5,1000)
f4 = lambda x: np.exp(3*x)-27*x**6+27*x**4*np.exp(x)-9*x**2*np.exp(2*x)
df4 = lambda x: 3*np.exp(3*x)-27*6*x**5+27*4*x**3*np.exp(x)+27*x**4*np.exp(x)-18*x*
```

In [9]:
```
plt.plot(x4,f4(x4))
```

Out[9]:
```
[<matplotlib.lines.Line2D at 0x238929976d0>]
```



In [10]:
```
newton.newton_method(f4,df4,4,1e-13,1000)
```

```
Newton method converged succesfully, niter=50, nfun=102, f(r)=0.0e+00
```

Out[10]:
```
(3.733060006927688,
 array([4.        , 3.92812038, 3.87280282, 3.83145651, 3.80135832,
        3.77993501, 3.76495935, 3.75463482, 3.74758922, 3.74281624,
        3.73959936, 3.7374389 , 3.73599143, 3.73502323, 3.73437632,
        3.73394441, 3.73365618, 3.7334639 , 3.73333563, 3.73325007,
        3.73319315, 3.73315506, 3.73312959, 3.73311247, 3.7331018 ,
        3.73309412, 3.73308246, 3.73319502, 3.73315648, 3.73313039,
        3.73311279, 3.73310465, 3.7330905 , 3.73308042, 3.73239736,
        3.73262473, 3.73277622, 3.73287718, 3.73294441, 3.73298917,
        3.73301876, 3.73303886, 3.73305202, 3.73306656, 3.7330751 ,
        3.73316115, 3.733134  , 3.73311556, 3.73310463, 3.73309046,
        3.73306001]),
 102)
```

ii)

In [11]:
```
secant.secant_method(f4,3,5,1e-13,1000)
```

```
Secant method converged succesfully, niter=38, nfun=40, f(r)=0.0e+00
```

Out[11]:
```
(3.733066621214533,
 array([5.        , 3.00166963, 3.00333534, 4.08841815, 3.30462432,
        3.43336334, 3.56200094, 3.60936402, 3.64544772, 3.66884354,
        3.68582134, 3.69799039, 3.70692778, 3.71351725, 3.71841327,
        3.72206413, 3.72479567, 3.72684385, 3.72838227, 3.72953923,
        3.73041014, 3.73106617, 3.73156061, 3.7319334 , 3.73221455,
        3.73242665, 3.73258667, 3.73270742, 3.73279855, 3.73286728,
        3.73291928, 3.73295827, 3.73298798, 3.73301018, 3.73302793,
        3.73304001, 3.73304749, 3.73305945, 3.73306662]),
 40)
```

In [12]:
```
newton.mod_newton_method(f4,df4,4,1e-13,1000)
```

```
287.2543140879716
287.2543140879716
Newton method converged succesfully, niter=50, nfun=102, f(r)=0.0e+00
```

```
---------------------------------------------------------------------------
RecursionError                            Traceback (most recent call last)
Cell In[12], line 1
----> 1 newton.mod_newton_method(f4,df4,4,1e-13,1000)

File c:\Users\xman7\OneDrive - UCB-O365\Documents\APPM4600\Homework\Homework4\newton
_method_script.py:86, in mod_newton_method(f, df, x0, tol, nmax, verb)
     83 rnm = np.append(rnm,rn)
     84 nfunm = nfunm+nfun
---> 86 print(fnew(r))
     87 print(fnew(rn[-1]))
     89 while fnew(r)-fnew(rn[-1]) != 0 :

File c:\Users\xman7\OneDrive - UCB-O365\Documents\APPM4600\Homework\Homework4\newton
_method_script.py:81, in mod_newton_method.<locals>.<lambda>(x)
     79 print(f(rn[-1]))
     80 r,rn,nfun = newton_method(f,df,x0,tol,nmax,verb)
---> 81 fnew = lambda x: f(x)/(x-r)
     82 f = fnew
     83 rnm = np.append(rnm,rn)

File c:\Users\xman7\OneDrive - UCB-O365\Documents\APPM4600\Homework\Homework4\newton
_method_script.py:81, in mod_newton_method.<locals>.<lambda>(x)
     79 print(f(rn[-1]))
     80 r,rn,nfun = newton_method(f,df,x0,tol,nmax,verb)
---> 81 fnew = lambda x: f(x)/(x-r)
     82 f = fnew
     83 rnm = np.append(rnm,rn)

    [... skipping similar frames: mod_newton_method.<locals>.<lambda> at line 81 (29
69 times)]

File c:\Users\xman7\OneDrive - UCB-O365\Documents\APPM4600\Homework\Homework4\newton
_method_script.py:81, in mod_newton_method.<locals>.<lambda>(x)
     79 print(f(rn[-1]))
     80 r,rn,nfun = newton_method(f,df,x0,tol,nmax,verb)
---> 81 fnew = lambda x: f(x)/(x-r)
     82 f = fnew
     83 rnm = np.append(rnm,rn)

RecursionError: maximum recursion depth exceeded
```

Problem 5

a)

In [14]:
```
f5 = lambda x: x**6-x-1
df5 = lambda x: 6*x**5-1
```

Newton:

In [20]:
```
rn, rtn, countn = newton.newton_method(f5,df5,2,1e-13,1000)
print("root = ", rn)
```

```
Newton method converged succesfully, niter=8, nfun=18, f(r)=8.9e-16
root =  1.1347241384015194
```

In [25]:
```
for i,a in enumerate(rtn):
    print('|',i,"|",a-rn)
```

```
| 0 | 0.8652758615984806
| 1 | 0.5459041338497894
| 2 | 0.296014849837543
| 3 | 0.12024681770791701
| 4 | 0.026814294371793723
| 5 | 0.0016291357689859343
| 6 | 6.389942109663593e-06
| 7 | 9.870171346904044e-11
| 8 | 0.0
```

Secant:

In [21]:
```
rs, rts, counts = secant.secant_method(f5,2,1,1e-13,1000)
print("root = ", rs)
```

```
Secant method converged succesfully, niter=8, nfun=10, f(r)=1.6e-15
root =  1.1347241384015196
```

In [26]:
```
for i,a in enumerate(rts):
    print('|',i,"|",a-rs)
```

```
| 0 | -0.13472413840151964
| 1 | -0.11859510614345514
| 2 | 0.055853630275117805
| 3 | -0.017068307459968013
| 4 | -0.0021925881853865903
| 5 | 9.26696033332064e-05
| 6 | -4.92452814748745e-07
| 7 | -1.1030376612097825e-10
| 8 | 0.0
```

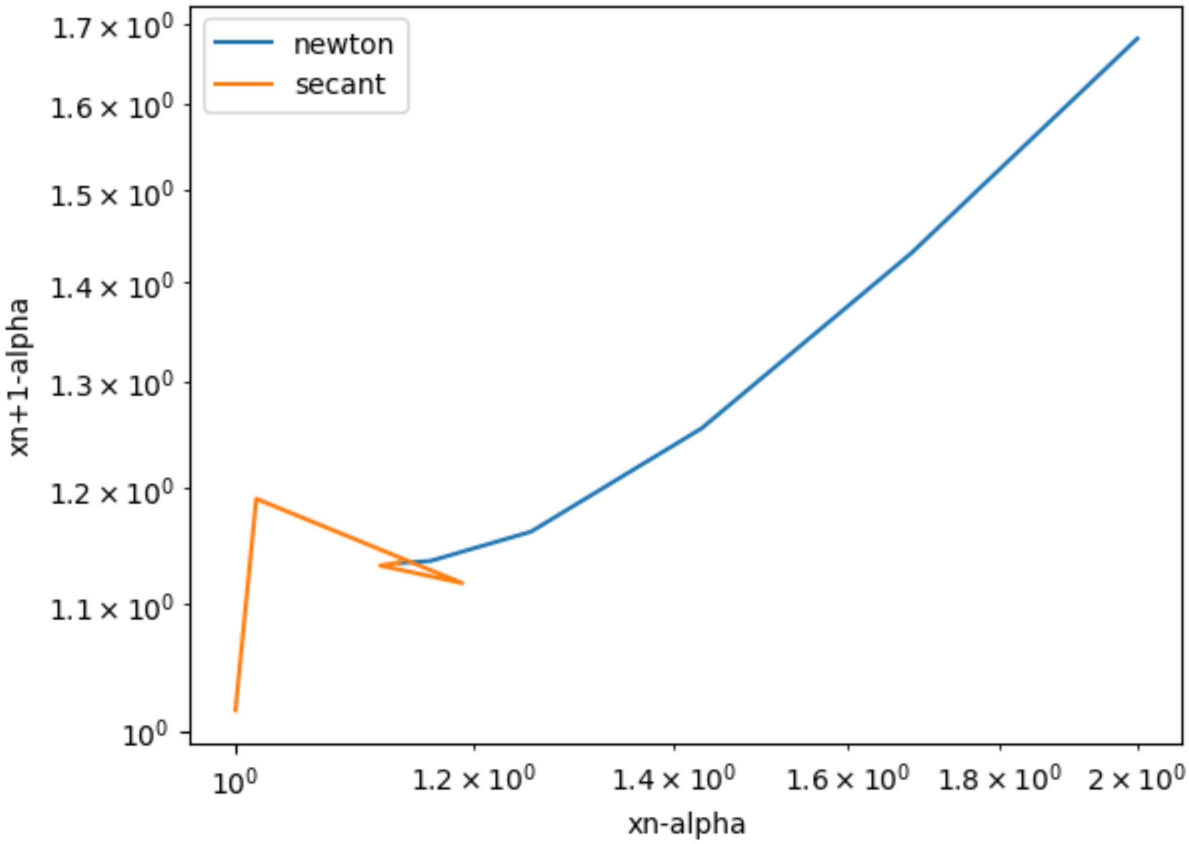
This does reduce as expected.

b)

Plots:

In [39]:
```
plt.loglog(rtn[0:7],rtn[1:8], label = 'newton')
plt.loglog(rts[0:7],rts[1:8], label = 'secant')
plt.xlabel('xn-alpha')
plt.ylabel("xn+1-alpha")
plt.legend()
```

Out[39]:
```
<matplotlib.legend.Legend at 0x2389669c7d0>
```



The slopes are proportional to the order of convergence