Model-Predictive Control for Path Tracking

Bicycle Kinematic Model

$$\begin{cases} \dot{x} = v\cos(\theta+\beta) \\ \dot{y} = v\sin(\theta+\beta) \\ \dot{\theta} = \frac{v}{l_r}\sin\beta \\ \dot{v} = a \\ \beta = \arctan(\frac{l_r}{l_f+l_r}\tan\delta) \end{cases}$$

Where

$$x : \text{x coordinate}$$
$$y : \text{y coordinate}$$
$$l_f : \text{distance from the center of the mass of the vehicle to the front axles}$$
$$l_r : \text{distance from the center of the mass of the vehicle to the rear axles}$$
$$\beta : \text{angle of the current velocity of the center of mass respected to x axis}$$
$$\theta : \text{heading}$$
$$a : \text{acceleration}$$
$$\delta : \text{steering angle}$$

Reference: [Kinematic and dynamic vehicle models for autonomous driving control design]

MPC formulation

State

$$\mathbf{z} = (\mathbf{x}, \mathbf{y}, \mathbf{v}, \theta)^{\mathbf{T}}$$
$$\mathbf{z} \in \mathbf{R}^{\mathbf{4x1}}$$

Control Input

$$\mathbf{u} = (\mathbf{a}, \delta)^{\mathbf{T}}$$
$$\mathbf{u} \in \mathbf{R}^{\mathbf{2x1}}$$

Linearization around equilibrium points

$$(\bar{\mathbf{z}}, \bar{\mathbf{u}})$$

$$
\dot{\mathbf{z}} = f(\mathbf{z}, \mathbf{u}) = \mathbf{A}^{'}\mathbf{z} + \mathbf{B}^{'}\mathbf{u}
$$
$$
\mathbf{A}^{'} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{z}}\right|_{\mathbf{z}=\bar{\mathbf{z}},\mathbf{u}=\bar{\mathbf{u}}}
$$
$$
\mathbf{B}^{'} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{z}=\bar{\mathbf{z}},\mathbf{u}=\bar{\mathbf{u}}}
$$

Therefore

$$
A^{'} = \left.\begin{pmatrix} \frac{\partial}{\partial x} v\cos(\theta+\beta) & \frac{\partial}{\partial y} v\cos(\theta+\beta) & \frac{\partial}{\partial v} v\cos(\theta+\beta) & \frac{\partial}{\partial \theta} v\cos(\theta+\beta) \\ \frac{\partial}{\partial x} v\sin(\theta+\beta) & \frac{\partial}{\partial y} v\sin(\theta+\beta) & \frac{\partial}{\partial v} v\sin(\theta+\beta) & \frac{\partial}{\partial \theta} v\sin(\theta+\beta) \\ \frac{\partial}{\partial x} a & \frac{\partial}{\partial y} a & \frac{\partial}{\partial v} a & \frac{\partial}{\partial \theta} a \\ \frac{\partial}{\partial x} \frac{v\sin\beta}{l_r} & \frac{\partial}{\partial y} \frac{v\sin\beta}{l_r} & \frac{\partial}{\partial v} \frac{v\sin\beta}{l_r} & \frac{\partial}{\partial \theta} \frac{v\sin\beta}{l_r} \end{pmatrix}\right|_{\mathbf{z}=\bar{\mathbf{z}},\mathbf{u}=\bar{\mathbf{u}}}
$$
$$
A^{'} = \begin{pmatrix} 0 & 0 & \cos(\bar{\theta}+\bar{\beta}) & -\bar{v}\sin(\bar{\theta}+\bar{\beta}) \\ 0 & 0 & \sin(\bar{\theta}+\bar{\beta}) & \bar{v}\cos(\bar{\theta}+\bar{\beta}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sin\bar{\beta}}{l_r} & 0 \end{pmatrix}
$$

And

$$
B^{'} = \left.\begin{pmatrix} \frac{\partial}{\partial a} v\cos(\theta+\beta) & \frac{\partial}{\partial \delta} v\cos(\theta+\beta) \\ \frac{\partial}{\partial a} v\sin(\theta+\beta) & \frac{\partial}{\partial \delta} v\sin(\theta+\beta) \\ \frac{\partial}{\partial a} a & \frac{\partial}{\partial \delta} a \\ \frac{\partial}{\partial a} \frac{v\sin\beta}{l_r} & \frac{\partial}{\partial \delta} \frac{v\sin\beta}{l_r} \end{pmatrix}\right|_{\mathbf{z}=\bar{\mathbf{z}},\mathbf{u}=\bar{\mathbf{u}}}
$$
$$
B^{'} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{\bar{v}\cos\bar{\beta}}{l_f+l_r} * \frac{1}{1+(\frac{l_r}{l_f+l_r}\tan\bar{\delta})^2} * \frac{1}{\cos^2\bar{\delta}} \end{pmatrix}
$$

Discrete-time mode with Forward Euler Discretization with sampling time dt

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$$
\mathbf{z_{k+1}} = \mathbf{z_k} + \mathbf{f}(\mathbf{z_k}, \mathbf{u_k})\mathbf{dt}
$$

Applying first degree Taylor expansion on f around equilibrium point, we get

$$
f(\mathbf{z_k}, \mathbf{u_k}) = \mathbf{f}(\bar{\mathbf{z}}, \bar{\mathbf{u}}) + \mathbf{A}^{'}(\mathbf{z_k} - \bar{\mathbf{z}}) + \mathbf{B}^{'}(\mathbf{u_k} - \bar{\mathbf{u}})
$$

So for

$$
\mathbf{z_{k+1}} = \mathbf{A}\mathbf{z_k} + \mathbf{B}\mathbf{u_k} + \mathbf{C}
$$

We have:

$$
A = (I + A^{'}dt) = \begin{pmatrix} 1 & 0 & \cos(\bar{\theta}+\bar{\beta})dt & -v\sin(\bar{\theta}+\bar{\beta})dt \\ 0 & 1 & \sin(\bar{\theta}+\bar{\beta}) & \bar{v}\cos(\bar{\theta}+\bar{\beta}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\sin\bar{\beta}}{l_r}dt & 1 \end{pmatrix}
$$

$$B = B^{'} dt = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ dt & 0 \\ 0 & \frac{\bar{v} \cos \bar{\beta}}{l_f + l_r} * \frac{1}{1 + (\frac{l_r}{l_f + l_r} \tan \bar{\delta})^2} * \frac{1}{\cos^2 \bar{\delta}} dt \end{pmatrix}$$

$$C = (f(\bar{z}, \bar{u}) - A^{'} \bar{z} - B^{'} \bar{u}) dt = dt(\begin{pmatrix} \bar{v} \cos (\bar{\theta} + \bar{\beta}) \\ \bar{v} \sin (\bar{\theta} + \bar{\beta}) \\ \bar{a} \\ \frac{\bar{v} \sin \bar{\beta}}{l_r} \end{pmatrix} - \begin{pmatrix} \bar{v} \cos (\bar{\theta} + \bar{\beta}) - \bar{v} \sin(\bar{\theta} + \bar{\beta}) \bar{\theta} \\ \bar{v} \sin (\bar{\theta} + \bar{\beta}) + \bar{v} \cos(\bar{\theta} + \bar{\beta}) \bar{\theta} \\ 0 \\ \frac{\bar{v} \sin \bar{\beta}}{l_r} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \bar{a} \\ \frac{\bar{v} \cos \bar{\beta}}{l_f + l_r} * \frac{1}{1 + (\frac{l_r}{l_f + l_r} \tan \bar{\delta})^2} * \frac{1}{\cos^2 \bar{\delta}} * \bar{\delta} \end{pmatrix}) = dt(\begin{pmatrix} \bar{v} \sin(\bar{\theta} + \bar{\beta}) \bar{\theta} \\ -\bar{v} \cos(\bar{\theta} + \bar{\beta}) \bar{\theta} \\ 0 \\ -\frac{\bar{v} \cos \bar{\beta}}{l_f + l_r} * \frac{1}{1 + (\frac{l_r}{l_f + l_r} \tan \bar{\delta})^2} * \frac{1}{\cos^2 \bar{\delta}} * \bar{\delta} \end{pmatrix})$$

Path Tracking Problem

$$\arg\min_{u_k} \sum_{k=0}^{N-1} ((z_k - z_{k,ref})^T Q (z_k - z_{k,ref}) + u_k^T R u_k) +$$
$$(z_N - z_{N,ref})^T Q_f (z_N - z_{N,ref}) +$$
$$\sum_{k=0}^{N-1} (u_{k+1} - u_k)^T R_d (u_{k+1} - u_k)$$

Subject to

$$\text{given initial state } z_0 \text{and}$$
$$z_{k+1} = A z_k + B u_k + C (k = 0, \ldots, N-1)$$
$$\frac{|u_{k+1} - u_k|}{dt} < u_{max}$$
$$u_{min} < u_k < u_{max}$$
$$v_{min} < v_k < v_{max}$$

Where

$$N : \text{horizon length}$$
$$z_{k,ref} : \text{reference state at k}$$
$$Q : \text{state cost matrix}$$
$$Qf : \text{state final matrix}$$
$$R : \text{input cost matrix}$$
$$Rd : \text{input difference cost matrix}$$
$$dt : \text{sampling time}$$