Robust Boosting for Functional Regression

September 13, 2021

1 Methodology

We address here the problem to estimate the regression function $F : X \to Y$, where the explanatory variable $X \in \mathcal{H}$ ($\mathcal{H}$ is a Hilbert space) and the response $Y \in \mathbb{R}$. Define the target function

$$F = \underset{G}{\operatorname{argmin}}\, L(Y, G(x)),$$

over joint distribution of (X, Y) where L is a pre-specified loss function, we estimate F by adapting the robust gradient boosting procedure (`RRBoost`) proposed by Ju and Salibián-Barrera (2021).

The generalization of `RRBoost` to functional data requires using functional regressors as the base learners. A convenient choice is the type B tree adopted by our previously proposed `TFBoost`. We recall the two stage estimating procedure of `RRBoost` below and use type B trees as base learners to approximate the negative gradients in both stages:

- Stage 1:compute an S-type boosting estimator with high robustness but possibly low efficiency;
- Stage 2:compute an M-type boosting estimator initialized at the function and scale estimators obtained in Stage 1.

We call the resulted estimator `TFBoost(RR)` and outline the algorithm as follows:

```
Algorithm 1: TFBoost(RR) algorithm
  Input      : A data set $(\mathbf{x}_i, y_i), i \in \mathcal{I}_{\text{train}}$
               The number of stage 1 iterations $T_1$
               The number of stage 2 iterations $T_2$
               The maximum depth of type B trees $d$
               Breakdown point in the first stage $b$
               The number of random directions of type B trees $P$
               Basis $\boldsymbol{\psi} = \{\psi_1, ..., \psi_d\}$
               Shrinkage parameter $\nu$
  Initialize:  $\hat{F}_0(\mathbf{x})$
  Stage 1  :
        for $t = 1 : T_1$ do
            $\hat{\sigma}_n(\hat{F}_{t-1}) = \{\sigma : \frac{1}{|\mathcal{I}_{\text{train}}|} \sum_{i\in\mathcal{I}_{\text{train}}} \rho_0\left(\frac{y_i - \hat{F}_{t-1}(x_i)}{\sigma}\right) = \kappa\}$
            $C_t = \left[\sum_{i\in\mathcal{I}_{\text{train}}} \psi_0\left(\frac{y_i - \hat{F}_{t-1}(x_i)}{\hat{\sigma}_n(\hat{F}_{t-1})}\right)\left(\frac{y_i - \hat{F}_{t-1}(x_i)}{\hat{\sigma}_n(\hat{F}_{t-1})}\right)\right]^{-1}$
            $g_{t,\ell} = -C_t\psi_0\left(\frac{y_\ell - \hat{F}_{t-1}(x_\ell)}{\hat{\sigma}_n(\hat{F}_{t-1})}\right)$
            Sample $\mathcal{P} = \{\mathbf{p}_1, ..., \mathbf{p}_P\}$
            $h_t = \text{argmin}_{h\in\mathcal{H}} \sum_{i\in|\mathcal{I}_{\text{train}}|} \left(g_{t,i} + h\left(\mathbf{p}_1^T\langle x_i\boldsymbol{\psi}\rangle, ..., \mathbf{p}_P^T\langle x_i\boldsymbol{\psi}\rangle\right)\right)^2$
            $\alpha_t = \text{argmin}_\alpha\, \hat{\sigma}\left(\hat{F}_{t-1} + \alpha h_t\right)$
            $\hat{F}_t(x) = \hat{F}_{t-1}(x) + \alpha_t h_t(x)$
  end
        $\hat{\sigma} = \hat{\sigma}_n(\hat{F}_{T_1})$
  Stage 2  :
        for  $t = 1 : T_2$ do
            $g_{t,i} = -\frac{1}{\hat{\sigma}_n}\psi_1\left(\frac{y_i - \hat{F}_{T_1+t-1}(\mathbf{x}_i)}{\hat{\sigma}_n}\right)$
            Sample $\mathcal{P} = \{\mathbf{p}_1, ..., \mathbf{p}_P\}$
            $h_t = \text{argmin}_{h\in\mathcal{H}} \sum_{i\in|\mathcal{I}_{\text{train}}|} \left(g_{t,i} + h\left(\mathbf{p}_1^T\langle x_i\boldsymbol{\psi}\rangle, ..., \mathbf{p}_P^T\langle x_i\boldsymbol{\psi}\rangle\right)\right)^2$
            $\alpha_t = \text{argmin}_\alpha \sum_{i\in\mathcal{I}_{\text{train}}} \rho_1\left(\frac{y_i - \hat{F}_{T_1+t-1}(x_i) - \alpha h_t(x_i)}{\hat{\sigma}}\right)$
            $\hat{F}_{T_1+t}(x) = \hat{F}_{T_1+t-1}(x) + \alpha_t h_t(x)$
  end
  Output   : $\hat{F}_{T_1+T_2}(x)$
```

1.1 Initialization

Since the loss functions involved in `TFBoost(RR)` are typically non-convex, a reliable initialization step is required to avoid reaching a local optima with a large objective value. Especially when the breakdown point b is large (close to 0.5), some "good" initial points will have large residuals and be treated as outliers. To avoid this issue, we suggest using a functional multi-index tree computed with the L1 loss, which we call "multi-index LAD tree". This generalizes `LADTree` recommended for `RRBoost` from a multivariate setting to our functional setting. To avoid overfitting, we use a relatively shallow Type A tree which finds the optimal directions that minimizes the L1 prediction error. Given projection directions $\beta_1, ..., \beta_K$, we describe below the procedure to fit a Type A multi-index LAD Tree.

Let $i \in \mathcal{I}_o$ be all the observations at a node o and $R_1(j,s) = \{i | \langle x_i, \beta_j \rangle \leq s\}$ and $R_2(j,s) = \{i | \langle x_i, \beta_j \rangle > s\}$ be the left and right regions of the split made on the j-th index, we find j and s that solves

$$\min_{j \in \{1,...,P\}} \left\{ \min_{s} \left(\sum_{i \in \{\mathcal{I}_o \cap R_1(j,s)\}} |y_i - \text{med}(y_i)| + \sum_{i \in \{\mathcal{I}_o \cap R_2(j,s)\}} |y_i - \text{med}(y_i)| \right) \right\}.$$

This procedure is repeated until the tree reaches the maximum depth (d_{init}) or minimum number of observations per node (m_{init}). We denote the resulted tree $h_{\text{init}}(\langle x, \beta_1 \rangle, ..., \langle x, \beta_K \rangle)$.

To fit a Type A multi-index LAD tree, we find $\beta_1, ..., \beta_K$ that minimizes the L1 loss on the entire tree:

$$\operatorname*{argmin}_{\beta_1,...,\beta_K} \sum_{i=1}^{n} |y_i - h_{\text{init}}(\langle x_i, \beta_1 \rangle, ..., \langle x_i, \beta_K \rangle)|.$$

1.2 Early stopping

Same as what we did for `RRBoost`, we use an early stopping rule to determine T_1 and T_2 in Algorithm (1) in order to reduce overfitting. We monitor the validation loss:

$$L_{1,\text{val}}(t) = \hat{\sigma}_{\text{val}}(\hat{F}_t),$$

for stage 1 where $\hat{\sigma}_{\text{val}}$ satisfies

$$\frac{1}{|\mathcal{I}_{\text{val}}|} \sum_{i \in \mathcal{I}_{\text{val}}} \rho_0 \left(\frac{y_i - \hat{F}_{t-1}(x_i)}{\hat{\sigma}_{\text{val}}} \right) = \kappa,$$

and

$$L_{2,\text{val}}(t) = \sum_{i \in \mathcal{I}_{\text{val}}} \rho_1 \left(\frac{y_i - \hat{F}_{T_1+t}(x_i)}{\hat{\sigma}} \right)$$

for stage 2 where $\hat{\sigma} = \hat{\sigma}_n(\hat{F}_{T_1})$.

We let $T_{1,\max}$ and $T_{2,\max}$ be the maximum number of iterations allowed in each stage. The early stopping times for stage 1 and stage 2 are defined as

$$T_1 = \operatorname*{argmin}_{t=1,...,T_{1,\max}} L_{1,\text{val}}(t)$$

and

$$T_2 = \operatorname*{argmin}_{t=1,...,T_{2,\max}} L_{2,\text{val}}(t)$$

respectively.

2 Simulation

2.1 Data generation

- We generated data sets $D = \{(x_i, y_i), i = 1, ..., N\}$, consisting of a predictor $x_i \in \mathcal{L}_2$ and a scalar response y_i that follow the model:

$$y_i = r(x_i) + \rho\epsilon_i, \tag{1}$$

  where the errors ϵ_i are i.i.d, r is the regression function, and $\rho > 0$ is a constant that controls the signal-to-noise ratio (SNR):

$$\text{SNR} = \frac{\text{Var}(r(X))}{\text{Var}(\rho\epsilon)}.$$

- To sample the functional predictors x_i, we considered the model:

$$x_i(t) = \mu(t) + \sum_{p=1}^{4}\sqrt{\lambda_j}\xi_{ij}\phi_j(t), \tag{2}$$

  where $\mu(t) = 2\text{sin}(t\pi)\text{exp}(1-t)$, $\lambda_1 = 0.8, \lambda_2 = 0.3, \lambda_3 = 0.2$, and $\lambda_4 = 0.1$, $\xi_{ij} \sim N(0,1)$, and ϕ_j are the first four eigenfunctions of the "Mattern" covariance function $\gamma(s,t)$ with parameters $\rho = 3, \sigma = 1, \nu = 1/3$:

$$\gamma(s,t) = C\left(\frac{\sqrt{2\nu}|s-t|}{\rho}\right), \; C(u) = \frac{\sigma^2 2^{1-\nu}}{\Gamma(\nu)}u^{\nu}K_{\nu}(u),$$

  where $\Gamma(.)$ is the Gamma function and K_ν is the modified Bessel function of the second kind. For each subject i, we evaluate x_i on a dense and regular grid $t_1, ..., t_{100}$ equally spaced in $\mathcal{I} = [0,1]$.

- We considered five regression functions:
  - $r_1(X) = \int_{\mathcal{I}}\left(\sin\left(\frac{3}{2}\pi t\right) + \sin\left(\frac{1}{2}\pi t\right)\right)X(t)dt$,
  - $r_2(X) = (\xi_1 + \xi_2)^{1/3}$, where $\xi_1 = \int_{\mathcal{I}}(X(t) - \mu(t))\psi_1(t)dt$ and $\xi_2 = \int_{\mathcal{I}}(X(t) - \mu(t))\psi_2(t)dt$ are projections onto the first two FPCs (ψ_1 and ψ_2) of X with mean $\mu(t) = E(X(t))$,
  - $r_3(X) = 5\text{exp}\left(-\frac{1}{2}\left|\int_{\mathcal{I}} x(t)\log(|x(t)|)dt\right|\right)$,
  - $r_4(X) = 5\text{sigmoid}\left(\int_{\mathcal{I}} X(t)^2\text{sin}(2\pi t)dt\right)$, where $\text{sigmoid}(u) = 1/(1+\exp(-u))$, and
  - $r_5(X) = 5\left(\sqrt{\left|\int_{\mathcal{I}_1}\cos(2\pi t^2)X(t)dt\right|} + \sqrt{\left|\int_{\mathcal{I}_2}\sin(X(t))dt\right|}\right)$, where $\mathcal{I}_1 = [0, 0.5]$ and $\mathcal{I}_2 = (0.5, 1]$.

- For clean data (C_0), we generated ϵ_i in (1) from $N(0,1)$ and selected ρ that corresponds to SNR = 5.

  For contaminated data, we considered moderate and severe scenarios and respectively sampled 10% or 30% individuals as outliers. We let the indices of outliers be $\mathcal{O}$. For $j \in \mathcal{O}$, we considered seven types of outliers C_1 to C_7 that follow the model

$$y_i = r(x_i) + \eta_i,$$

where the predictors of each type of outliers are specified as follows:

- C_1 and C_7: *Shape outliers (symmetric and asymmetric)*

  In (2), $\xi_{j,2} \sim N(2, 0.25)$, $\mu(t) = 1 + 2\cos(2t\pi)$, and the other parameters are the same as the model for clean data,

- C_2: *Magnitude outliers (curve-type)*

  $x_j = 2\tilde{x}_j + 5$, where $\tilde{x}_j$ follows the model for clean data,

- C_3: *Magnitude outliers (point-type)*

  Randomly sample 10 points form $t_1, ..., t_{100}$ and denote them as $t_{j,o_1}, ..., t_{j,o_{10}}$. For $k = 1, ..., 10$,

  $$x_j(t_{j,o_k}) = \tilde{x}_j(t_{j,o_k}) + \eta_{j,o_k},$$

  where $\eta_{j,o_k} \sim 0.5N(10, 0.25) + 0.5N(-10, 0.25)$, where $\tilde{x}_j$ follows the model for clean data,

- C_4: *Magnitude outliers (interval-type)* Randomly sample one interval from intervals $[t_1, ..., t_{10}]$, ...,$[t_{91}, ..., t_{100}]$, and denote the interval as $t_{j,o}, ..., t_{j,o+9}$
  For $k = 0, ..., 9$,

  $$x_j(t_{j,o+k}) = \tilde{x}_j(t_{j,o+k}) + \eta_{j,o+k},$$

  where $\eta_{j,o+k} \sim N(10, 0.25)$, and $\tilde{x}_j$ follows the model for clean data

- C_5 and C_6: *vertical outliers (symmetric and asymmetric)*

  $$x_j = \tilde{x}_j,$$

  where $\tilde{x}_j$ follows the model for clean data.

For symmetrically contaminated settings C_1,..., C_5, we generate η_i from

$$0.5N(30, 0.25) + 0.5N(-30, 0.25).$$

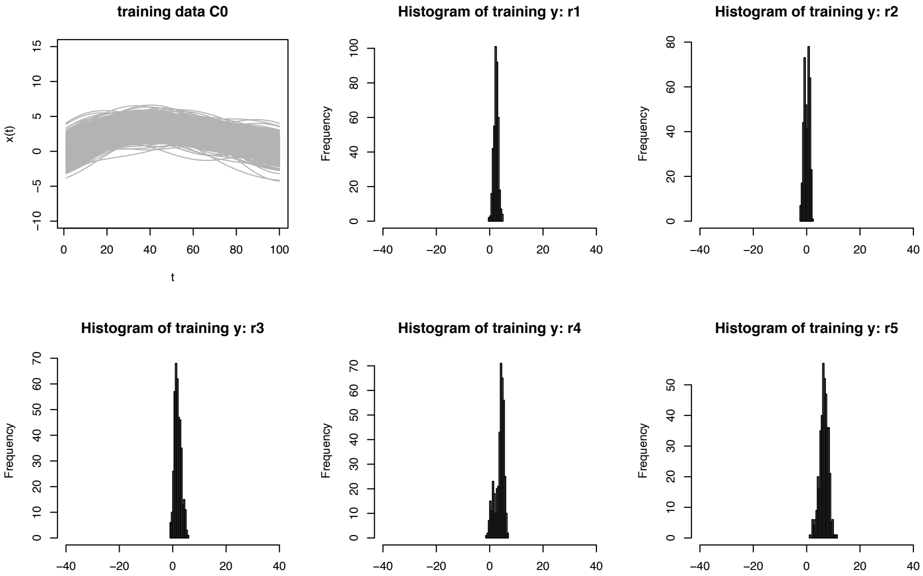
For asymmetrically contaminated settings C_6 and C_7, we generate η_i from

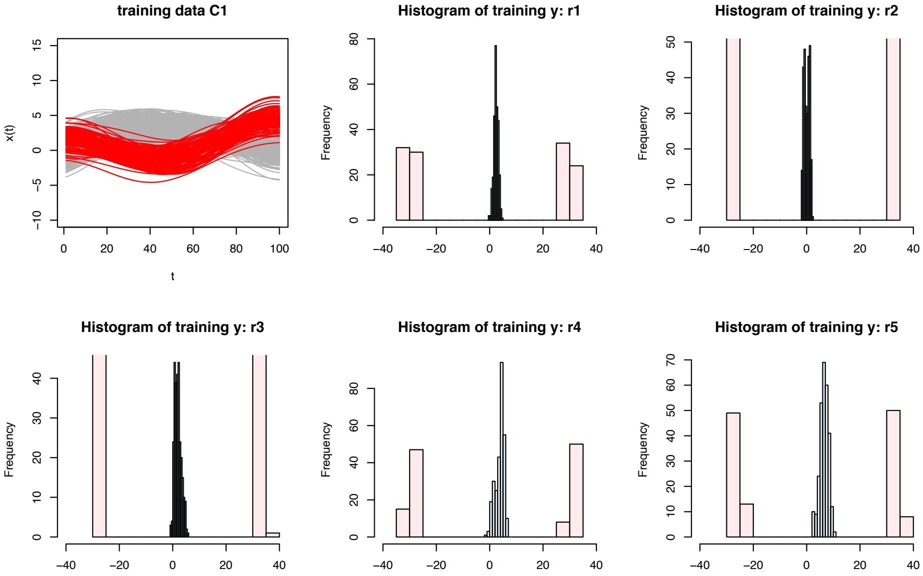
$$N(-30, 0.25).$$

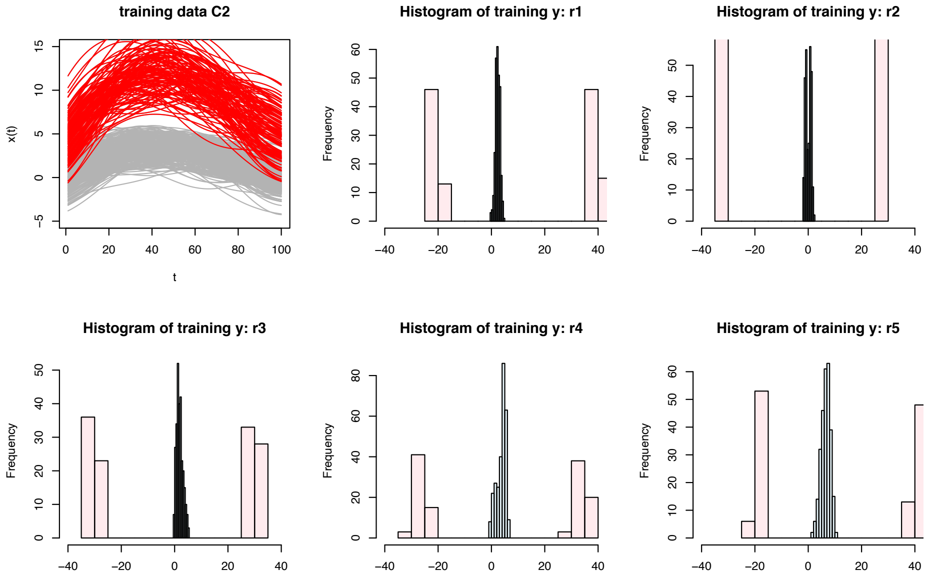
We generated data of size 1600, partitioned into a training set of size 400, validation set of size 200, and test set of size 1000. For C_1 to C_7, we contaminated the training set and validation set as described above. The test set always follow C_0.

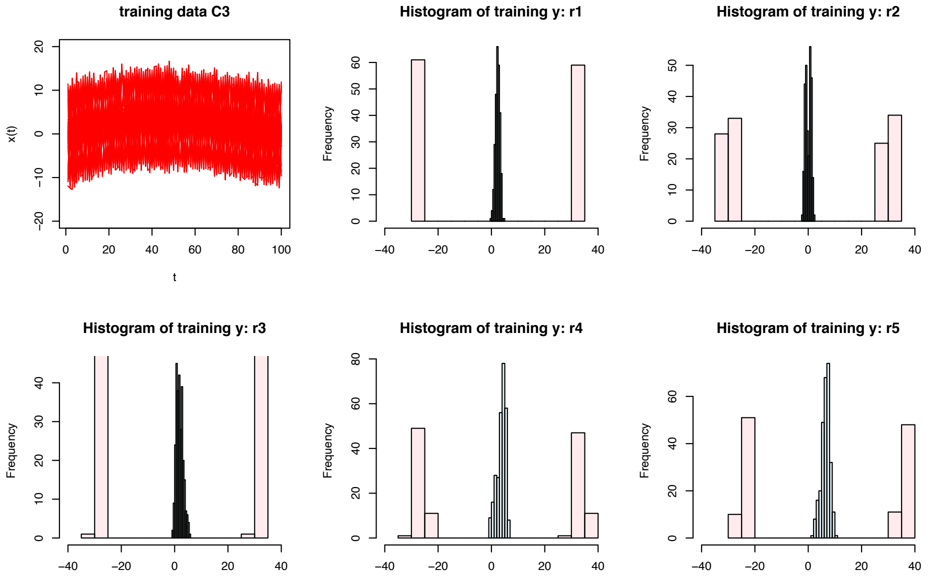
2.2 Visualization of the outliers

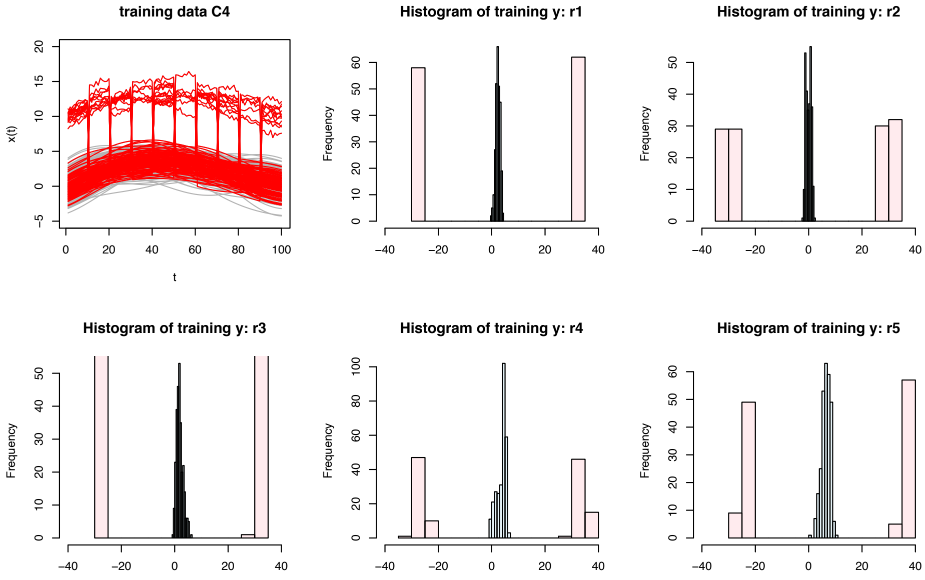
Below we plot the training data for each setting with red curves and pink histograms denoting the functional predictors and scalar response for outliers respectively. For C_1 to C_7, 10% of the training samples were selected as outliers.

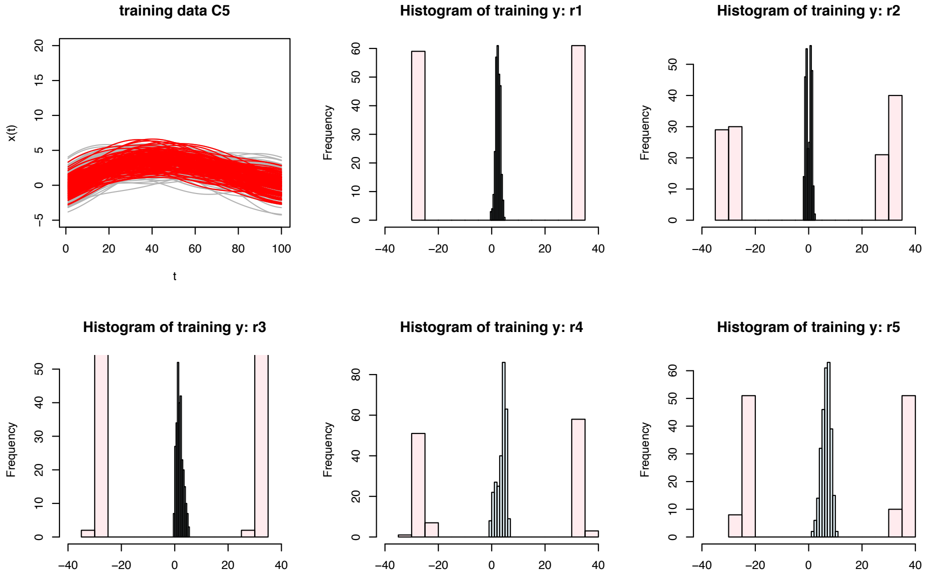

training data C0
x(t)
t
15
10
5
0
−5
−10
0
20
40
60
80
100
Histogram of training y: r1
Histogram of training y: r2
Histogram of training y: r3
Histogram of training y: r4
Histogram of training y: r5
Frequency
−40
−20
0
20
40

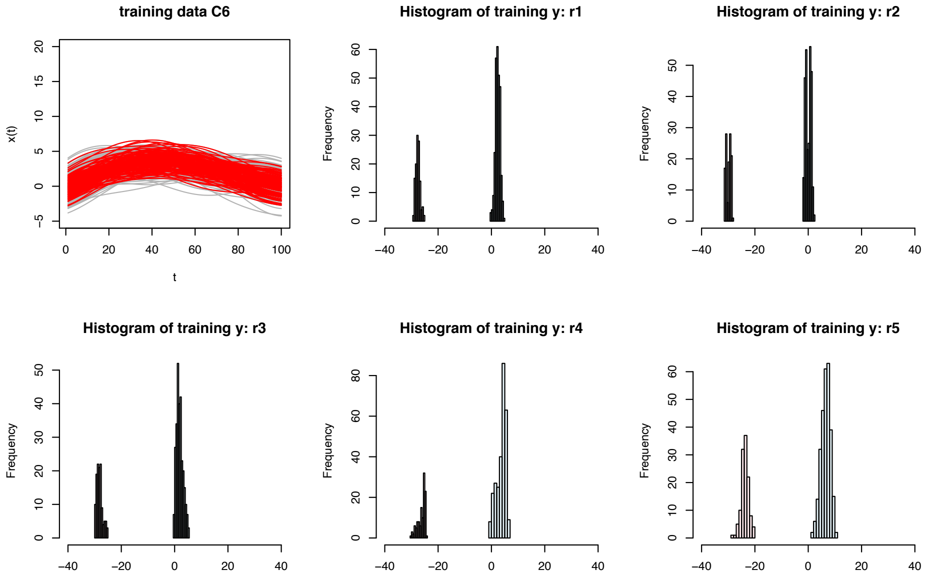

training data C1
x(t)
t
15
10
5
0
−5
−10
0
20
40
60
80
100
Histogram of training y: r1
Histogram of training y: r2
Histogram of training y: r3
Histogram of training y: r4
Histogram of training y: r5
Frequency
−40
−20
0
20
40

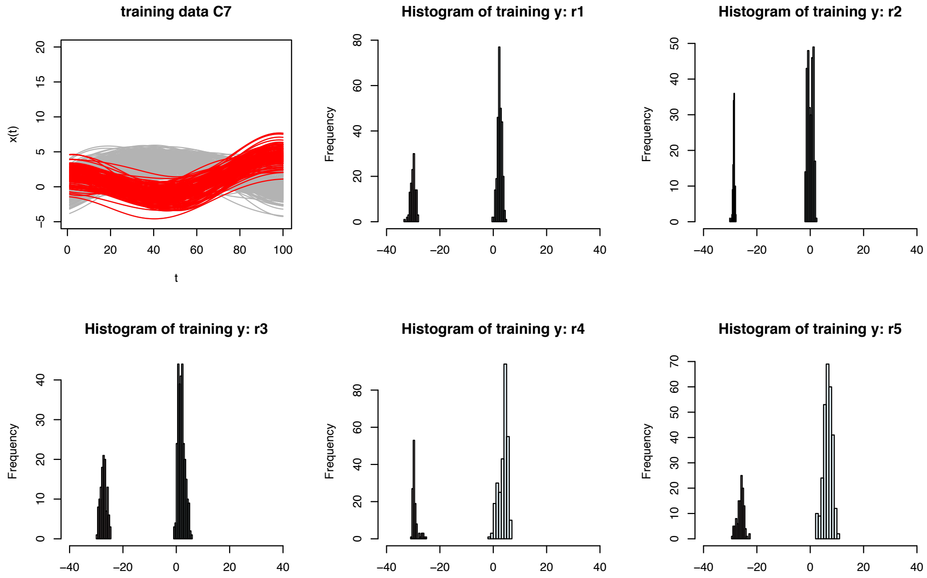

training data C2
x(t)
t
15
10
5
0
−5
0
20
40
60
80
100
Histogram of training y: r1
Histogram of training y: r2
Histogram of training y: r3
Histogram of training y: r4
Histogram of training y: r5
Frequency
−40
−20
0
20
40


training data C3
x(t)
t
20
10
0
−10
−20
0
20
40
60
80
100
Histogram of training y: r1
Histogram of training y: r2
Histogram of training y: r3
Histogram of training y: r4
Histogram of training y: r5
Frequency
−40
−20
0
20
40


training data C4
x(t)
t
0
20
40
60
80
100
−5
0
5
10
15
20
Histogram of training y: r1
Histogram of training y: r2
Histogram of training y: r3
Histogram of training y: r4
Histogram of training y: r5
Frequency
−40
−20
0
20
40
0
10
20
30
40
50
60
80
100


training data C5
x(t)
t
0
20
40
60
80
100
−5
0
5
10
15
20
Histogram of training y: r1
Histogram of training y: r2
Histogram of training y: r3
Histogram of training y: r4
Histogram of training y: r5
Frequency
−40
−20
0
20
40
0
10
20
30
40
50
60
80





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2.3 Model comparison

For each setting, we used 100 independently generated datasets and compared the performance of the following methods:

- `FPPR`: functional projection pursuit regression (Ferraty et al., 2013),
- `FGAM`: functional generalized additive models (McLean et al., 2014),
- `MFLM`: Sieve M-estimator for a semi-functional linear model (Huang et al., 2015),
- `RFSIR`: robust functional sliced inverse regression (Wang et al., 2017)
- `RFPLM`: robust estimation for semi-functional linear regression models (Boente et al., 2020),
- `TFBoost(L2)`: tree-based functional boosting with L2 loss,
- `TFBoost(LAD)`: tree-based functional boosting with LAD loss,
- `TFBoost(LAD-M)`: tree-based functional boosting initialized with the robust residual scale estimators and function estimator obtained from `TFBoost(LAD)` followed by the second stage of `TFBoost(RR)`,
- `TFBoost(RR.2)`: proposed robust tree-based functional boosting with $b = 0.2$,
- `TFBoost(RR.4)`: proposed robust tree-based functional boosting with $b = 0.4$, and
- `TFBoost(RR.5)`: proposed robust tree-based functional boosting with $b = 0.5$ and initialized with multi-index LAD Tree.

2.4 Implementation details

- For `FPPR`, we implemented the method based on the code to fit a functional single index model shared by the authors of Ferraty et al. (2013). We used cubic B-spline with 7 functions (3 evenly spaced interior knots) as the basis and selected the number of additive components between 1 to 15 to minimize the validation robust MSPE, which is defined as

  $$\mu_M(\{y_i - \hat{F}(x_i)\}|i \in \mathcal{I}_{\text{val}})^2 + \hat{\sigma}_M(\{y_i - \hat{F}(x_i)\}|i \in \mathcal{I}_{\text{val}}),$$

  where $\hat{\mu}_M$ and $\hat{\sigma}_M$ are M-location and M-scale estimators specified to achieve 95% asymptotic efficiency at the normal model.

- For `FGAM`, we adopted the implementation in `refund` package. We set basis to be bivariate cubic B-splines of dimension 15 by 15 and used REML to select penalization parameter.

- For `MFLM` and `RFPLM`, we adopted the implementation available from https://github.com/msalibian/RobustFPLM. We considered cubic B-spline basis of dimension 4 to 7 with evenly spaced interior knots, and selected the dimension that minimized the BIC.

- For `RFSIR`, we used the code shared by the author of Wang et al. (2017). We selected the number of leading functional principal components from 1 to 10 to minimize the validation robust MSPE.

- For `TFBoost` methods we used type B trees as base learners. With each training data, we selected the maximum depth of type B tree from 1 to 4 that achieves the lowest validation robust MSPE.

- Considering that the first stage of `TFBoost(RR.5)` uses $b = 0.5$ which tends to treat individuals with large residuals as outliers compared to `TFBoost(RR)` with smaller b values. To avoid getting many zero gradients in early iterations and getting trapped in suboptima with poor objective values, we initialize `TFBoost(RR.5)` with Type A multi-index LADTrees. We set the minimum number of observations per node to be 10, fixed the number of indices $K = 1$, and selected the maximum depth of these trees from 0,1,2,3.

  To select the initial tree from

  (a) median of the training responses (depth 0 tree),

  (b) Type A tree with depth 1 and index 1,

  (c) Type A tree with depth 2 and index 1, and

  (d) Type A tree with depth 3 and index 1,

  we follow the steps below.

  - Step 1: Fit `TFBoost(RR)` with (a) to obtain training and validation residuals

    $$r_{a,i} = \hat{F}_a(x_i) - y_i,$$

    and calculate M-scale of residuals (b = 0.5) for the training data and test data:

    $$M_{a,\text{train}} = \hat{\sigma}(r_{a,i}, i \in \mathcal{I}_{\text{train}})$$

    $$M_{a,\text{val}} = \hat{\sigma}(r_{a,i}, i \in \mathcal{I}_{\text{val}})$$

  - Step 2: Flag outliers in the training data: Let $\hat{\mu} = \text{median}(r_i, i \in \mathcal{I}_{\text{train}})$ and $\hat{\sigma} = \text{MAD}(r_i, i \in \mathcal{I}_{\text{train}})$, the outliers are defined as $\mathcal{O}_a = \{i, i \in \mathcal{I}_{\text{train}}, |r_i - \hat{\mu}| > 3\hat{\sigma}\}$.
  - Step 3: Repeat Step 2 for `TFBoost(RR)` initialized with (b), (c), and (d), and denote the flagged outliers as $\mathcal{O}_b$, $\mathcal{O}_c$, and $\mathcal{O}_d$ respectively. Note that $\hat{\mu}$ and $\hat{\sigma}$ are calculated using `TFBoost(RR)` initialized with (a).
  - Step 4: Among (b), (c), (d), we only consider the ones that "fix" some outliers in $\mathcal{O}_a$. For example, (b) will be considered if some points in $\mathcal{O}_a$ are not in $\mathcal{O}_b$.
  - Step 5: For the methods that passed the criterion in Step 4, we only consider the ones that have a lower validation residual scale compared to (a), and choose among them the one that achieves the lowest training residual scale.
  - If no method remains after Step 5, we choose (a).

2.5 Simulation results

Below we report summary statistics of test MSPEs, displayed in the form of mean (sd) with the lowest two average test MPSEs bold-faced for each setting.

10% outliers in training and validation sets

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	0.14 (0.01)	0.62 (0.39)	0.91 (0.33)	0.99 (0.37)	1.24 (0.26)	1.42 (0.84)	11.24 (2.00)	6.20 (1.87)
TFBoost(LAD)	0.15 (0.01)	0.17 (0.05)	0.19 (0.09)	0.15 (0.01)	0.19 (0.10)	0.15 (0.01)	0.16 (0.01)	0.23 (0.17)
TFBoost(LAD-M)	0.15 (0.01)	0.16 (0.04)	0.17 (0.05)	0.14 (0.01)	0.17 (0.05)	0.15 (0.01)	**0.15** (0.01)	0.22 (0.17)
TFBoost(RR.2)	0.15 (0.01)	0.15 (0.01)	**0.15** (0.01)	0.14 (0.01)	0.15 (0.01)	0.15 (0.01)	**0.15** (0.01)	0.15 (0.01)
TFBoost(RR.4)	0.15 (0.01)	0.15 (0.01)	0.16 (0.01)	0.14 (0.01)	0.15 (0.01)	0.15 (0.01)	**0.15** (0.01)	0.15 (0.01)
TFBoost(RR.5)	0.15 (0.01)	0.15 (0.01)	0.16 (0.01)	0.14 (0.01)	0.15 (0.01)	0.15 (0.01)	**0.15** (0.01)	0.15 (0.01)
FPPR	0.14 (0.01)	3.06 (3.05)	2.01 (2.84)	6.60 (4.91)	9.13 (5.03)	2.68 (2.52)	16.31 (4.85)	0.74 (1.19)
FGAM	**0.13** (0.01)	3.03 (7.95)	4.00 (7.02)	44.49 (103.16)	18.51 (49.94)	0.91 (0.52)	9.64 (0.60)	**0.13** (0.01)
RFPLM	**0.13** (0.01)	**0.13** (0.01)	**0.13** (0.01)	**0.12** (0.00)	**0.13** (0.00)	**0.13** (0.01)	**0.13** (0.01)	**0.13** (0.01)
MFLM	**0.13** (0.01)	**0.14** (0.01)	0.20 (0.16)	**0.13** (0.01)	0.17 (0.03)	**0.14** (0.01)	0.34 (0.06)	**0.13** (0.01)
RFSIR	0.14 (0.01)	0.94 (1.98)	1.74 (2.65)	0.14 (0.10)	**0.14** (0.01)	1.14 (1.20)	12.63 (4.23)	0.19 (0.23)

Table 1: Summary statistics of test errors for data generated from r_1; displayed in the form of mean (sd).

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	**0.18** (0.01)	0.72 (0.37)	1.02 (0.28)	1.18 (0.29)	1.49 (0.31)	1.90 (1.02)	11.56 (1.88)	7.12 (1.27)
TFBoost(LAD)	**0.18** (0.01)	0.20 (0.07)	0.22 (0.09)	0.18 (0.01)	0.24 (0.14)	**0.19** (0.01)	0.20 (0.02)	0.25 (0.09)
TFBoost(LAD-M)	**0.18** (0.01)	0.20 (0.04)	0.21 (0.07)	0.18 (0.02)	0.23 (0.11)	**0.19** (0.01)	**0.19** (0.01)	0.23 (0.08)
TFBoost(RR.2)	**0.18** (0.01)	**0.18** (0.01)	**0.19** (0.01)	**0.17** (0.01)	**0.18** (0.01)	**0.18** (0.01)	**0.18** (0.01)	**0.18** (0.01)
TFBoost(RR.4)	0.19 (0.01)	**0.19** (0.01)	**0.19** (0.01)	**0.17** (0.01)	**0.18** (0.01)	**0.19** (0.01)	**0.19** (0.01)	**0.19** (0.01)
TFBoost(RR.5)	0.19 (0.01)	**0.19** (0.01)	**0.19** (0.01)	**0.17** (0.01)	**0.18** (0.01)	**0.19** (0.01)	**0.19** (0.01)	**0.19** (0.01)
FPPR	**0.18** (0.01)	3.34 (3.35)	2.62 (2.68)	6.79 (5.35)	10.01 (4.48)	2.76 (2.38)	16.43 (4.73)	1.03 (1.11)
FGAM	0.22 (0.01)	3.13 (7.93)	4.74 (8.23)	42.13 (97.39)	21.86 (63.03)	1.04 (0.51)	9.79 (0.60)	0.23 (0.01)
RFPLM	0.29 (0.01)	0.28 (0.01)	0.29 (0.01)	0.28 (0.01)	0.29 (0.01)	0.29 (0.01)	0.29 (0.01)	0.29 (0.01)
MFLM	0.28 (0.01)	0.36 (0.15)	0.54 (0.26)	0.55 (0.16)	0.70 (0.06)	0.42 (0.17)	0.83 (0.11)	0.29 (0.01)
RFSIR	**0.18** (0.01)	1.37 (2.65)	2.05 (3.32)	**0.17** (0.01)	0.20 (0.07)	1.54 (1.80)	13.18 (5.36)	0.24 (0.23)

Table 2: Summary statistics of test errors for data generated from r_2; displayed in the form of mean (sd).

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	**0.31** (0.02)	0.88 (0.49)	1.52 (0.39)	1.75 (0.32)	2.07 (0.30)	2.14 (1.18)	11.73 (1.97)	6.90 (1.82)
TFBoost(LAD)	**0.31** (0.02)	0.35 (0.09)	0.35 (0.08)	0.30 (0.02)	0.38 (0.18)	**0.32** (0.02)	0.34 (0.04)	0.42 (0.14)
TFBoost(LAD-M)	**0.31** (0.02)	0.34 (0.06)	0.34 (0.08)	0.30 (0.01)	0.34 (0.07)	**0.32** (0.02)	**0.32** (0.02)	0.39 (0.12)
TFBoost(RR.2)	0.32 (0.03)	**0.32** (0.02)	**0.31** (0.02)	**0.29** (0.01)	**0.31** (0.01)	**0.31** (0.02)	**0.31** (0.02)	**0.32** (0.02)
TFBoost(RR.4)	0.32 (0.02)	**0.33** (0.04)	**0.32** (0.02)	0.30 (0.01)	**0.32** (0.02)	**0.32** (0.03)	**0.32** (0.02)	0.33 (0.03)
TFBoost(RR.5)	0.32 (0.02)	**0.33** (0.03)	**0.32** (0.02)	0.30 (0.02)	**0.32** (0.02)	**0.32** (0.02)	**0.32** (0.02)	0.33 (0.03)
FPPR	**0.30** (0.02)	3.22 (3.52)	2.62 (2.70)	6.45 (5.27)	10.38 (4.75)	2.60 (2.05)	17.13 (5.04)	1.26 (1.70)
FGAM	0.32 (0.01)	3.20 (7.73)	6.12 (15.06)	39.73 (87.88)	21.67 (58.30)	1.15 (0.51)	9.85 (0.58)	**0.32** (0.02)
RFPLM	0.38 (0.02)	0.38 (0.02)	0.38 (0.02)	0.36 (0.01)	0.38 (0.02)	0.38 (0.02)	0.38 (0.02)	0.38 (0.02)
MFLM	0.38 (0.02)	0.39 (0.03)	0.53 (0.43)	0.53 (0.47)	2.53 (0.22)	0.40 (0.03)	1.39 (0.99)	0.38 (0.02)
RFSIR	**0.31** (0.02)	1.73 (2.53)	2.17 (3.04)	**0.29** (0.01)	0.39 (0.64)	1.26 (0.88)	13.66 (4.77)	0.39 (0.27)

Table 3: Summary statistics of test errors for data generated from r_3; displayed in the form of mean (sd).

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	**0.52** (0.03)	1.04 (0.48)	2.20 (0.75)	2.62 (0.35)	2.96 (0.56)	2.85 (1.30)	12.35 (1.79)	0.82 (0.22)
TFBoost(LAD)	**0.53** (0.03)	0.60 (0.13)	0.61 (0.18)	0.52 (0.03)	0.68 (0.23)	0.56 (0.05)	0.55 (0.03)	0.71 (0.26)
TFBoost(LAD-M)	**0.53** (0.03)	0.58 (0.11)	0.57 (0.08)	**0.51** (0.03)	0.64 (0.18)	**0.54** (0.03)	**0.54** (0.03)	0.70 (0.26)
TFBoost(RR.2)	0.54 (0.05)	**0.55** (0.03)	**0.54** (0.03)	**0.50** (0.02)	**0.53** (0.03)	**0.53** (0.03)	**0.53** (0.03)	**0.54** (0.03)
TFBoost(RR.4)	0.54 (0.03)	**0.56** (0.04)	0.56 (0.05)	0.52 (0.03)	**0.54** (0.03)	0.55 (0.04)	0.55 (0.04)	**0.56** (0.04)
TFBoost(RR.5)	0.54 (0.03)	**0.56** (0.04)	**0.55** (0.04)	0.52 (0.04)	**0.54** (0.03)	0.55 (0.04)	0.55 (0.04)	**0.56** (0.04)
FPPR	0.60 (0.05)	4.30 (3.32)	4.81 (3.44)	9.05 (4.91)	10.52 (5.25)	4.24 (2.86)	16.78 (4.52)	1.49 (1.10)
FGAM	0.86 (0.04)	3.86 (7.24)	7.06 (11.76)	34.52 (64.76)	25.03 (67.45)	2.23 (0.52)	10.99 (0.71)	0.87 (0.05)
RFPLM	1.11 (0.06)	1.12 (0.08)	1.12 (0.07)	1.09 (0.06)	1.11 (0.05)	1.12 (0.06)	1.12 (0.06)	1.13 (0.10)
MFLM	1.10 (0.05)	1.22 (0.18)	1.34 (0.25)	1.46 (0.19)	1.72 (0.13)	1.33 (0.22)	1.94 (0.16)	1.12 (0.07)
RFSIR	0.56 (0.04)	2.17 (3.10)	2.87 (3.78)	0.54 (0.05)	0.59 (0.13)	2.38 (1.48)	14.00 (5.62)	0.61 (0.17)

Table 4: Summary statistics of test errors for data generated from r_4; displayed in the form of mean (sd).

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	**0.58** (0.04)	1.10 (0.49)	2.19 (0.56)	2.33 (0.27)	2.62 (0.38)	2.69 (1.05)	12.50 (2.12)	2.51 (2.26)
TFBoost(LAD)	0.61 (0.04)	0.67 (0.11)	0.69 (0.07)	0.61 (0.04)	0.68 (0.10)	0.65 (0.05)	0.65 (0.05)	0.81 (0.27)
TFBoost(LAD-M)	**0.60** (0.04)	**0.65** (0.10)	**0.66** (0.08)	**0.58** (0.04)	0.69 (0.19)	**0.62** (0.05)	**0.62** (0.04)	0.81 (0.25)
TFBoost(RR.2)	0.64 (0.06)	**0.61** (0.04)	**0.63** (0.04)	**0.58** (0.03)	**0.60** (0.03)	**0.61** (0.05)	**0.61** (0.04)	**0.60** (0.04)
TFBoost(RR.4)	0.66 (0.06)	0.66 (0.06)	0.69 (0.08)	0.62 (0.06)	**0.65** (0.06)	0.65 (0.07)	0.67 (0.07)	0.66 (0.07)
TFBoost(RR.5)	0.66 (0.07)	0.66 (0.06)	0.67 (0.07)	0.62 (0.06)	**0.65** (0.06)	0.66 (0.06)	0.66 (0.06)	0.65 (0.06)
FPPR	0.61 (0.06)	4.15 (3.25)	3.21 (2.61)	7.99 (5.43)	11.70 (5.34)	3.50 (2.42)	15.87 (4.70)	1.24 (0.86)
FGAM	**0.60** (0.04)	3.73 (8.41)	6.46 (10.95)	33.73 (67.78)	25.99 (82.71)	1.90 (0.53)	10.63 (0.73)	**0.60** (0.04)
RFPLM	0.88 (0.05)	0.88 (0.05)	0.88 (0.05)	0.85 (0.05)	0.89 (0.05)	0.88 (0.05)	0.88 (0.05)	0.88 (0.05)
MFLM	0.87 (0.05)	0.90 (0.05)	1.02 (0.15)	1.03 (0.43)	1.85 (0.59)	0.92 (0.07)	1.99 (1.09)	0.87 (0.04)
RFSIR	0.67 (0.06)	2.08 (1.99)	2.79 (3.05)	0.70 (0.66)	0.68 (0.20)	2.14 (1.28)	14.10 (4.84)	0.72 (0.20)

Table 5: Summary statistics of test errors for data generated from r_5; displayed in the form of mean (sd).

30% outliers in training and validation sets

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	0.14 (0.01)	0.96 (0.45)	1.41 (0.79)	1.10 (0.57)	0.91 (0.17)	2.78 (2.43)	82.75 (2.00)	69.67 (7.67)
TFBoost(LAD)	0.15 (0.01)	0.20 (0.11)	0.23 (0.15)	0.17 (0.02)	0.20 (0.16)	0.17 (0.03)	0.73 (2.33)	0.74 (0.54)
TFBoost(LAD-M)	0.15 (0.01)	0.18 (0.10)	0.21 (0.14)	0.16 (0.01)	0.17 (0.11)	0.16 (0.01)	0.41 (1.33)	0.62 (0.54)
TFBoost(RR.2)	0.15 (0.01)	0.60 (0.47)	0.30 (0.17)	5.39 (3.43)	1.19 (0.79)	1.77 (1.95)	45.94 (9.12)	0.60 (0.35)
TFBoost(RR.4)	0.15 (0.01)	**0.16** (0.01)	**0.16** (0.01)	**0.15** (0.01)	**0.15** (0.01)	**0.15** (0.01)	**0.15** (0.01)	0.16 (0.01)
TFBoost(RR.5)	0.15 (0.01)	**0.16** (0.01)	0.17 (0.02)	**0.15** (0.01)	**0.15** (0.01)	**0.15** (0.01)	**0.15** (0.01)	0.16 (0.01)
FPPR	0.14 (0.01)	2.67 (2.77)	0.96 (1.07)	16.57 (11.51)	11.73 (8.08)	6.86 (6.42)	109.00 (16.52)	0.76 (1.00)
FGAM	**0.13** (0.01)	1.21 (0.82)	1.17 (1.00)	5.16 (2.86)	2.32 (1.31)	2.44 (2.10)	82.17 (1.54)	**0.13** (0.01)
RFPLM	**0.13** (0.01)	**0.13** (0.01)	**0.13** (0.01)	**0.13** (0.00)	**0.13** (0.00)	**0.13** (0.01)	**0.13** (0.01)	**0.13** (0.01)
MFLM	**0.13** (0.01)	0.24 (0.16)	0.56 (0.48)	0.24 (0.13)	0.42 (0.23)	0.22 (0.12)	23.38 (2.91)	**0.13** (0.01)
RFSIR	0.14 (0.01)	1.27 (2.61)	1.14 (2.08)	0.16 (0.03)	0.27 (0.19)	2.03 (2.32)	87.49 (7.31)	0.29 (0.46)

Table 6: Summary statistics of test errors for data generated from r_1; displayed in the form of mean (sd).

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	**0.18** (0.01)	1.10 (0.55)	1.28 (0.48)	1.36 (0.62)	1.11 (0.13)	3.01 (2.18)	83.09 (2.18)	63.85 (7.28)
TFBoost(LAD)	**0.18** (0.01)	0.25 (0.18)	0.27 (0.15)	0.22 (0.07)	0.23 (0.24)	0.20 (0.01)	0.79 (2.27)	0.56 (0.42)
TFBoost(LAD-M)	**0.18** (0.01)	0.21 (0.06)	0.24 (0.11)	0.21 (0.07)	0.21 (0.19)	**0.19** (0.01)	0.73 (2.53)	0.35 (0.36)
TFBoost(RR.2)	**0.18** (0.01)	0.74 (0.66)	0.41 (0.27)	5.30 (3.50)	1.18 (0.74)	1.88 (1.85)	46.15 (8.02)	1.49 (0.27)
TFBoost(RR.4)	0.19 (0.01)	**0.19** (0.01)	**0.19** (0.01)	**0.19** (0.01)	**0.18** (0.01)	**0.18** (0.01)	**0.19** (0.01)	**0.21** (0.01)
TFBoost(RR.5)	0.19 (0.01)	**0.19** (0.01)	**0.19** (0.01)	**0.19** (0.01)	**0.18** (0.01)	**0.19** (0.01)	**0.19** (0.01)	**0.21** (0.02)
FPPR	**0.18** (0.01)	3.44 (3.39)	2.21 (3.04)	19.06 (12.71)	11.95 (6.73)	7.44 (6.88)	104.41 (15.89)	1.19 (1.49)
FGAM	0.22 (0.01)	1.33 (0.83)	1.35 (0.95)	5.26 (2.84)	2.27 (1.11)	2.60 (2.11)	82.34 (1.60)	0.23 (0.01)
RFPLM	0.29 (0.01)	0.29 (0.01)	0.33 (0.03)	0.29 (0.01)	0.29 (0.01)	0.29 (0.01)	0.29 (0.01)	0.31 (0.02)
MFLM	0.28 (0.01)	0.73 (0.22)	1.02 (0.51)	0.72 (0.27)	0.95 (0.24)	0.77 (0.17)	23.57 (2.90)	0.31 (0.01)
RFSIR	**0.18** (0.01)	1.63 (2.75)	1.51 (2.61)	0.22 (0.19)	0.47 (1.09)	2.09 (2.32)	86.15 (5.78)	0.29 (0.30)

Table 7: Summary statistics of test errors for data generated from r_2; displayed in the form of mean (sd).

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	**0.31** (0.02)	1.43 (0.60)	1.84 (0.51)	1.99 (0.62)	1.68 (0.15)	3.37 (2.43)	83.09 (2.10)	66.47 (7.74)
TFBoost(LAD)	**0.31** (0.02)	0.40 (0.12)	0.40 (0.11)	0.37 (0.04)	0.38 (0.17)	0.35 (0.04)	1.22 (2.96)	0.92 (0.51)
TFBoost(LAD-M)	**0.31** (0.02)	0.37 (0.10)	0.37 (0.13)	**0.33** (0.03)	0.35 (0.15)	**0.33** (0.02)	0.91 (2.82)	0.79 (0.48)
TFBoost(RR.2)	0.32 (0.03)	0.79 (0.45)	0.58 (0.29)	5.84 (3.62)	1.43 (0.83)	2.53 (2.48)	45.84 (8.25)	1.10 (0.67)
TFBoost(RR.4)	0.32 (0.02)	**0.34** (0.02)	**0.32** (0.02)	**0.32** (0.01)	**0.31** (0.01)	**0.32** (0.02)	**0.32** (0.02)	**0.34** (0.02)
TFBoost(RR.5)	0.32 (0.02)	**0.34** (0.02)	**0.33** (0.02)	**0.33** (0.02)	**0.31** (0.02)	**0.33** (0.02)	**0.33** (0.02)	0.35 (0.03)
FPPR	**0.30** (0.02)	3.97 (4.23)	2.90 (3.74)	18.64 (10.91)	11.70 (7.08)	6.78 (6.71)	108.99 (18.23)	1.28 (1.39)
FGAM	0.32 (0.01)	1.49 (0.88)	1.55 (0.89)	5.31 (2.86)	2.55 (1.24)	2.70 (2.10)	82.39 (1.68)	**0.33** (0.02)
RFPLM	0.38 (0.02)	0.38 (0.02)	0.43 (0.05)	0.39 (0.02)	0.37 (0.02)	0.38 (0.02)	0.38 (0.02)	0.40 (0.02)
MFLM	0.38 (0.02)	0.69 (0.66)	1.33 (1.13)	0.58 (0.19)	3.04 (0.41)	0.92 (0.95)	25.24 (2.92)	0.39 (0.02)
RFSIR	**0.31** (0.02)	1.36 (1.32)	1.60 (1.52)	0.36 (0.30)	0.44 (0.26)	2.49 (2.30)	86.70 (7.06)	0.47 (0.37)

Table 8: Summary statistics of test errors for data generated from r_3; displayed in the form of mean (sd).

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	**0.52** (0.03)	1.68 (0.65)	2.62 (0.51)	3.03 (0.80)	2.75 (0.22)	4.40 (2.66)	84.01 (2.36)	1.11 (0.36)
TFBoost(LAD)	**0.53** (0.03)	0.74 (0.23)	0.70 (0.19)	0.67 (0.10)	0.70 (0.21)	0.62 (0.12)	1.23 (2.40)	1.25 (0.60)
TFBoost(LAD-M)	**0.53** (0.03)	0.66 (0.14)	0.66 (0.19)	0.61 (0.09)	0.60 (0.20)	0.58 (0.04)	0.95 (2.22)	1.19 (0.55)
TFBoost(RR.2)	0.54 (0.05)	1.53 (0.90)	0.85 (0.37)	6.45 (2.98)	2.01 (1.04)	3.64 (2.21)	46.12 (9.48)	0.97 (0.32)
TFBoost(RR.4)	0.54 (0.03)	**0.61** (0.04)	**0.58** (0.04)	**0.56** (0.03)	**0.54** (0.04)	**0.55** (0.04)	**0.55** (0.03)	**0.58** (0.03)
TFBoost(RR.5)	0.54 (0.03)	**0.60** (0.04)	**0.59** (0.04)	**0.57** (0.03)	**0.54** (0.03)	**0.57** (0.04)	**0.56** (0.04)	**0.59** (0.03)
FPPR	0.60 (0.05)	4.11 (3.06)	3.87 (2.92)	17.87 (10.62)	14.34 (7.78)	8.36 (6.19)	105.62 (15.95)	2.07 (1.64)
FGAM	0.86 (0.04)	2.47 (0.86)	2.53 (0.89)	6.17 (3.01)	4.11 (1.65)	3.88 (2.18)	83.56 (2.01)	0.90 (0.05)
RFPLM	1.11 (0.06)	1.14 (0.06)	1.16 (0.07)	1.13 (0.06)	1.13 (0.07)	1.11 (0.06)	1.11 (0.06)	1.25 (0.12)
MFLM	1.10 (0.05)	1.69 (0.26)	1.97 (0.79)	1.73 (0.38)	2.11 (0.40)	1.76 (0.26)	24.63 (2.94)	1.22 (0.08)
RFSIR	0.56 (0.04)	1.70 (1.55)	2.70 (2.32)	0.86 (1.77)	0.91 (0.59)	3.68 (2.47)	87.10 (7.16)	0.65 (0.25)

Table 9: Summary statistics of test errors for data generated from r_4; displayed in the form of mean (sd).

	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7
TFBoost(L2)	**0.58** (0.04)	1.74 (0.60)	2.63 (0.86)	2.44 (0.53)	2.32 (0.20)	3.87 (1.79)	84.07 (2.27)	68.34 (7.96)
TFBoost(LAD)	0.61 (0.04)	0.72 (0.09)	0.82 (0.23)	0.77 (0.07)	0.79 (0.27)	0.75 (0.07)	1.41 (1.33)	1.45 (0.65)
TFBoost(LAD-M)	**0.60** (0.04)	**0.67** (0.07)	0.74 (0.21)	0.68 (0.10)	0.70 (0.25)	**0.67** (0.05)	1.27 (2.81)	1.32 (0.57)
TFBoost(RR.2)	0.64 (0.06)	1.24 (0.58)	0.91 (0.23)	6.02 (2.78)	1.75 (0.77)	3.15 (2.03)	45.84 (8.63)	1.04 (0.35)
TFBoost(RR.4)	0.66 (0.06)	**0.65** (0.05)	**0.70** (0.05)	**0.65** (0.05)	**0.64** (0.04)	**0.65** (0.04)	**0.64** (0.04)	**0.64** (0.04)
TFBoost(RR.5)	0.66 (0.07)	**0.67** (0.06)	**0.72** (0.05)	**0.67** (0.06)	**0.65** (0.05)	**0.67** (0.06)	**0.67** (0.06)	0.67 (0.06)
FPPR	0.61 (0.06)	4.00 (3.80)	2.59 (1.62)	17.34 (10.58)	12.88 (6.91)	7.94 (6.12)	106.34 (14.44)	1.58 (1.10)
FGAM	**0.60** (0.04)	2.17 (0.78)	2.11 (0.88)	5.76 (2.89)	3.35 (1.42)	3.48 (2.05)	83.32 (1.95)	**0.62** (0.04)
RFPLM	0.88 (0.05)	0.88 (0.04)	0.97 (0.06)	0.89 (0.05)	0.90 (0.05)	0.88 (0.05)	0.88 (0.05)	0.89 (0.04)
MFLM	0.87 (0.05)	1.22 (0.63)	1.70 (1.00)	1.15 (0.23)	3.84 (0.48)	1.47 (1.00)	26.11 (2.88)	0.91 (0.05)
RFSIR	0.67 (0.06)	1.92 (1.78)	2.41 (2.22)	0.73 (0.27)	1.09 (0.95)	3.23 (2.42)	86.80 (6.65)	0.84 (0.34)

Table 10: Summary statistics of test errors for data generated from r_5; displayed in the form of mean (sd).

3 Real example

We analyze the fruit fly data available at https://anson.ucdavis.edu/ mueller/data/data.html. The data set consists of number of eggs laid daily for each of 1000 medflies until time of death. We used a subset of the data including flies that lived at least 30 days, to predict the number of living days. The number of eggs laid in the first 30 days is treated as the functional predictor with values evaluated at each day.

We randomly portioned the data into 60% training set, 20% validation set, and 20% test set. The description and set-ups of all methods follows the details in Section 2.4. Since the test set may contain outliers, average absolute values (AAD) of the residuals on the test set are used for comparison. To avoid randomness introduced by a single partition, we considered 100 random partitions. The summary statistics of test robust MSPEs are shown in Figure 1. To be able to see the differences, the TFBoost methods are plotted separately from the other methods.

To further illustrate the performance of our method when data contain more extreme outliers. We considered adding symmetric and asymmetric outliers to the responses in the training and the validation sets. We randomly sample 20% of the individuals (denoted as $\mathcal{O}$), and let there responses be

$$\tilde{y}_i = y_i + \eta_i, i \in \mathcal{I}$$

where y_i is the original response, and $\eta_i \ = \text{round}(e_i)$, where $e_i \sim 0.5N(30,3) + 0.5N(-30,3)$ for symmetric outliers and $e_i \sim N(30,3)$ for asymmetric outliers. The robust MSPEs from 100 random partitions of the data are shown below.

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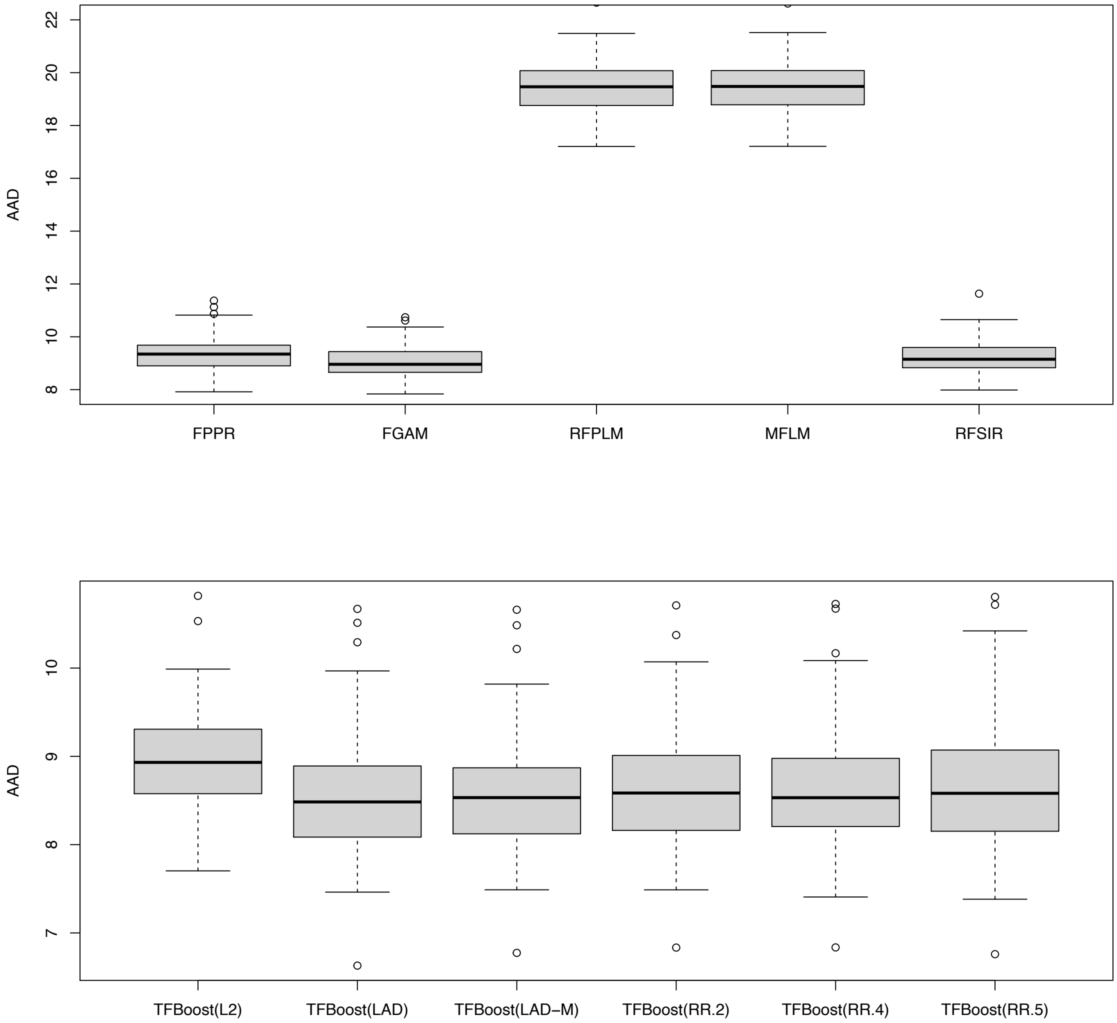

Figure 1: Test AADs from 100 random partitions

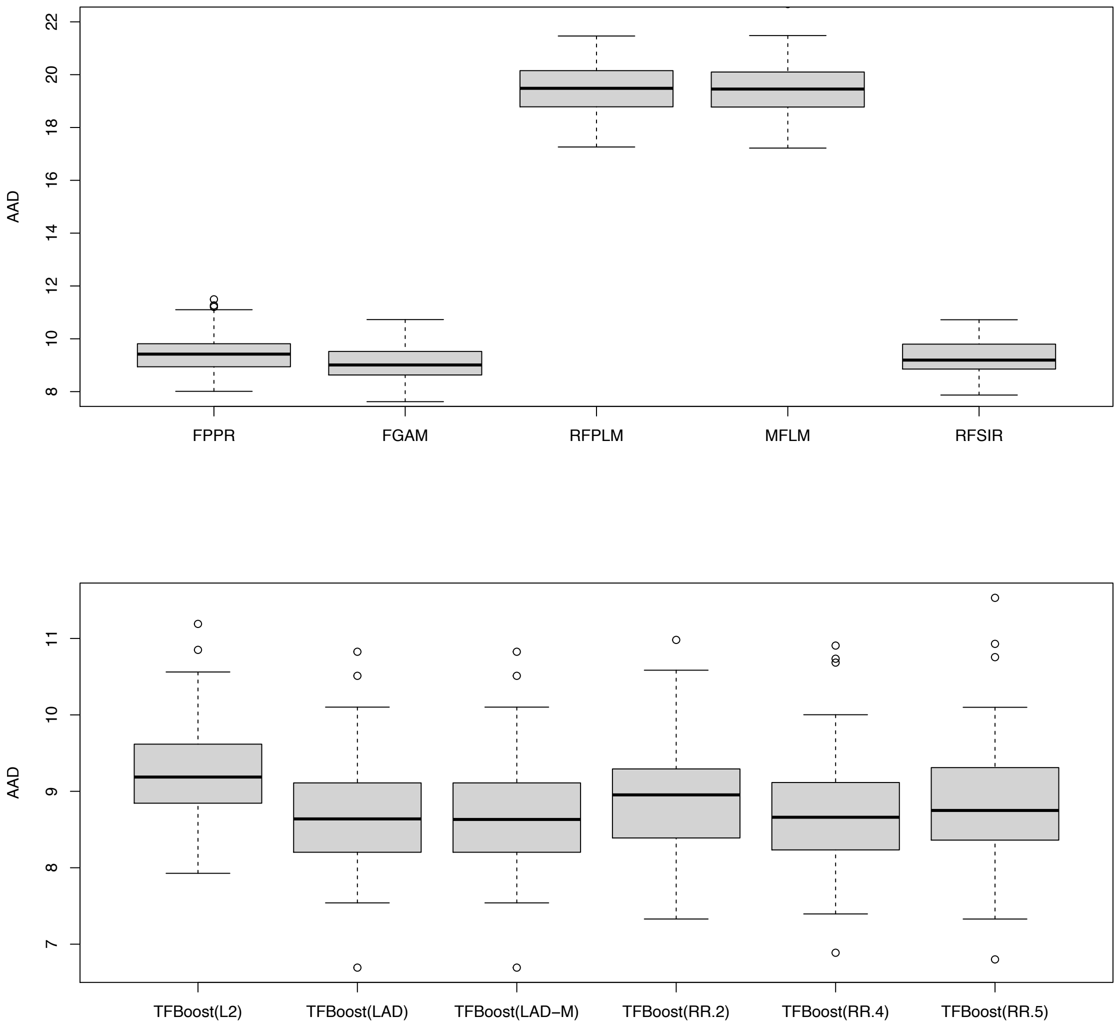

Figure 2: Test AADs from 100 random partitions for FruitFly data with symmetric outliers

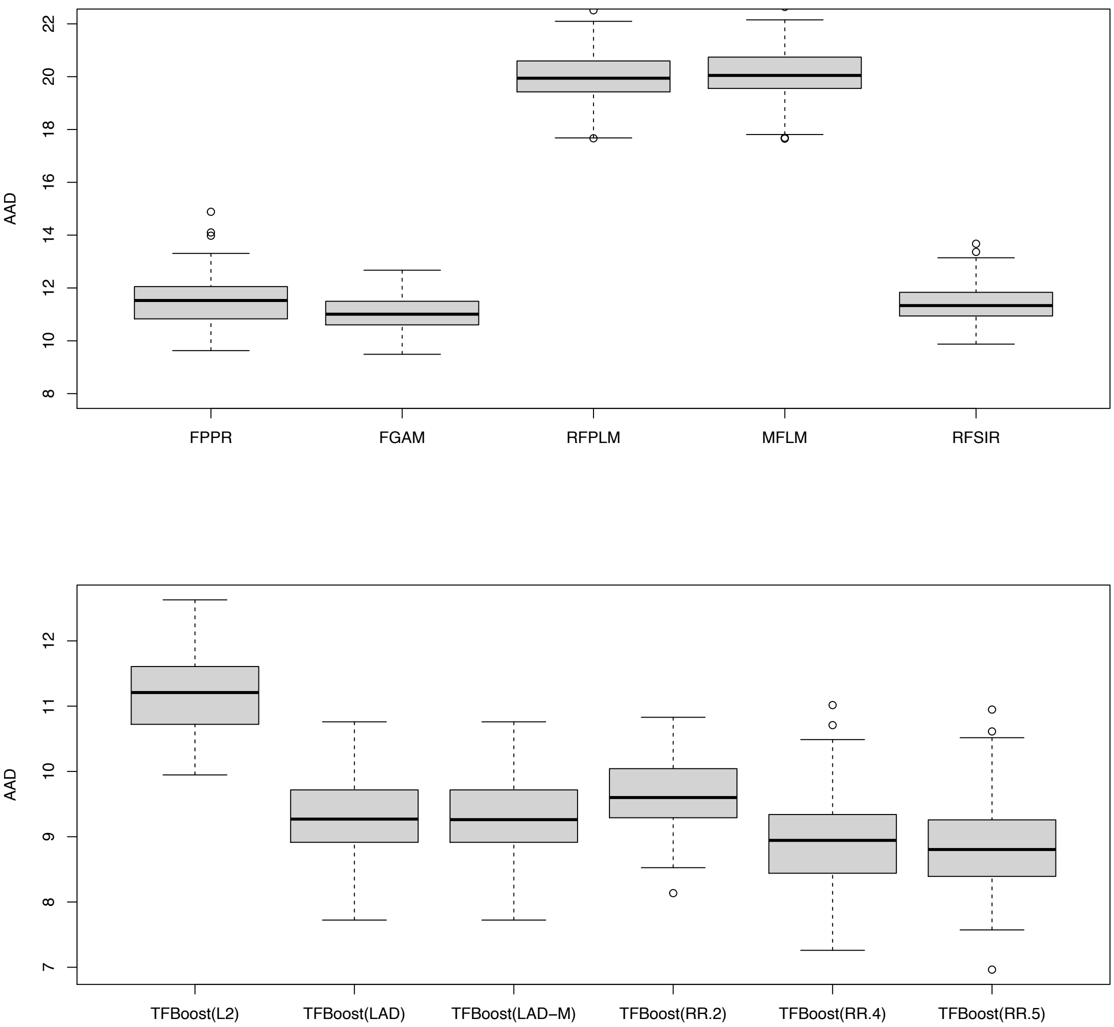

Figure 3: Test AADs from 100 random partitions for FruitFly data with asymmetric outliers