Probability And Mathmatical Statistics
Homework 10

151220131 谢旻晖

3

矩估计

$$
\begin{aligned}
EX &= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} dx \\
令t &= \frac{lnx-\mu}{\sqrt{2}\sigma} \\
EX &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2} \sqrt{2}\sigma e^{\sqrt{2}\sigma t+\mu} dt \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2+\sqrt{2}\sigma t+\mu} dt \\
&= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-\frac{\sqrt{2}}{2}\sigma)^2} dt \\
令u &= t-\frac{\sqrt{2}}{2}\sigma \\
EX &= \frac{\sqrt{\pi}e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-u^2} du \\
&= e^{\mu+\frac{\sigma^2}{2}} \qquad\qquad (右边为N(0,\frac{1}{\sqrt{2}})的概率积分) \\
EX^2 &= \int_0^{+\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} dx \\
令t &= \frac{lnx-\mu}{\sqrt{2}\sigma} \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{2\sqrt{2}\sigma t+2\mu-t^2} dt \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-(t-\sqrt{2}\sigma)^2} e^{2\sigma^2+2\mu} dt \\
&= \frac{e^{2\sigma^2+2\mu}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-\sqrt{2}\sigma)^2} dt \\
&= e^{2\sigma^2+2\mu}
\end{aligned}
$$

使用样本矩代替标准矩，得

$$
\begin{cases}
\bar{X} & = e^{\mu+\frac{\sigma^2}{2}} \\
\frac{1}{n}\sum_{i=1}^n X_i^2 & = e^{2\mu+2\sigma^2}
\end{cases}
$$

解得

$$
\begin{aligned}
\hat{\mu} &= \frac{4ln\bar{X} - ln\frac{\sum_{i=1}^n X_i^2}{n}}{2} \\
\hat{\sigma^2} &= ln\frac{\sum_{i=1}^n X_i^2}{n} - 2ln\bar{X}
\end{aligned}
$$

极大似然估计

似然函数为

$$
\begin{aligned}
L(\mu,\sigma^2) &= \Pi_{i=1}^n (2\pi\sigma^2)^{-\frac{1}{2}} X_i^{-1} e^{-\frac{(lnX_i-\mu)^2}{2\sigma^2}} \\
&= (2\pi\sigma^2)^{-\frac{n}{2}} \frac{1}{\Pi_{i=1}^n X_i} e^{-\frac{\sum_{i=1}^n (lnX_i-\mu)^2}{2\sigma^2}} \\
lnL(\mu,\sigma^2) &= -\frac{n}{2}ln(2\pi\sigma^2) + ln\frac{1}{\Pi_{i=1}^n X_i} - \frac{\sum_{i=1}^n (lnX_i-\mu)^2}{2\sigma^2}
\end{aligned}
$$

$$
\begin{cases}
\frac{\partial lnL}{\partial \sigma^2} & = -\frac{n}{2}\frac{2\pi}{2\pi\sigma^2} + \frac{\sum_{i=1}^n (lnX_i-\mu)^2}{2(\sigma^2)^2} = 0 \qquad (1) \\
\frac{\partial lnL}{\partial \mu} & = -\frac{\sum_{i=1}^n 2(\mu-lnX_i)}{2\sigma^2} = 0 \qquad (2)
\end{cases}
$$

解得

$$
\begin{aligned}
\hat{\mu} &= \frac{\sum_{i=1}^n lnX_i}{n} \\
\hat{\sigma^2} &= \frac{\sum_{i=1}^n \left[lnX_i - \sum_{j=1}^n \frac{lnX_j}{n}\right]}{n}
\end{aligned}
$$

4

矩估计

极大似然估计

11

无偏估计意味着 $E[.]=\sigma^2$.

$$
\begin{aligned}
E[.] &= c\sum_{i=1}^{n-1} E(X_{i+1}-X_i)^2 \\
&= c\sum_{i=1}^{n-1}\left[D(X_{i+1}-X_i)+(E(X_{i+1}-X_i))^2\right] \\
&= c(n-1)2\sigma^2 \\
&= \sigma^2 \\
\therefore c &= \frac{1}{2(n-1)}
\end{aligned}
$$

12

$$
\begin{aligned}
L(\sigma^2) &= \Pi_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(X_i-\mu)^2}{2\sigma^2}} \\
lnL(\sigma^2) &= \sum_{i=1}^{n} -\frac{(X_i-\mu)^2}{2\sigma^2} + ln\frac{1}{\sqrt{2\pi\sigma^2}} \\
\frac{dlnL(\sigma^2)}{d\sigma^2} &= \sum_{i=1}^{n}\frac{(x-\mu)^2}{2\sigma^4} + \sqrt{2\pi\sigma^2}(-\frac{1}{2\pi\sigma^2})\frac{\sqrt{2\pi}}{2\sqrt{\sigma^2}} \\
&= \sum_{i=1}^{n}\frac{(x-\mu)^2}{2\sigma^4} - \frac{1}{2\sigma^2} \\
&\text{令}\frac{dlnL(\sigma^2)}{d\sigma^2} = 0 \\
&\frac{\sum_{i=1}^{n}(X_i-\mu)}{2\sigma^2} = \frac{n}{2\sigma^2} \\
\hat{\sigma^2} &= \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2
\end{aligned}
$$

13

14

18