Probability And Mathmatical Statistics
Homework 10

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矩估计

$$EX = \int_0^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} dx$$

$$令t = \frac{lnx - \mu}{\sqrt{2}\sigma}$$

$$\begin{aligned} EX &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2} \sqrt{2}\sigma e^{\sqrt{2}\sigma t+\mu} dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2+\sqrt{2}\sigma t+\mu} dt \\ &= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-\frac{\sqrt{2}}{2}\sigma)^2} dt \end{aligned}$$

$$令u = t - \frac{\sqrt{2}}{2}\sigma$$

$$\begin{aligned} EX &= \frac{\sqrt{\pi}e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-u^2} du \\ &= e^{\mu+\frac{\sigma^2}{2}} \qquad (右边为N(0, \frac{1}{\sqrt{2}})的概率积分) \end{aligned}$$

$$EX^2 = \int_0^{+\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} dx$$

$$令t = \frac{lnx - \mu}{\sqrt{2}\sigma}$$

$$\begin{aligned} &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{2\sqrt{2}\sigma t+2\mu-t^2} dt \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-(t-\sqrt{2}\sigma)^2} e^{2\sigma^2+2\mu} dt \\ &= \frac{e^{2\sigma^2+2\mu}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-\sqrt{2}\sigma)^2} dt \\ &= e^{2\sigma^2+2\mu} \end{aligned}$$

使用样本矩代替标准矩，得

$$\begin{cases} \bar{X} &= e^{\mu+\frac{\sigma^2}{2}} \\ \frac{1}{n}\sum_{i=1}^n X_i^2 &= e^{2\mu+2\sigma^2} \end{cases}$$

解得

$$\hat{\mu} = \frac{4ln\bar{X} - ln\frac{\sum_{i=1}^n X_i^2}{n}}{2}$$

$$\hat{\sigma^2} = ln\frac{\sum_{i=1}^n X_i^2}{n} - 2ln\bar{X}$$

极大似然估计

似然函数为

$$\begin{aligned} L(\mu, \sigma^2) &= \Pi_{i=1}^n (2\pi\sigma^2)^{-\frac{1}{2}} X_i^{-1} e^{-\frac{(lnX_i-\mu)^2}{2\sigma^2}} \\ &= (2\pi\sigma^2)^{-\frac{n}{2}} \frac{1}{\Pi_{i=1}^n X_i} e^{-\frac{\sum_{i=1}^n (lnX_i-\mu)^2}{2\sigma^2}} \\ lnL(\mu, \sigma^2) &= -\frac{n}{2} ln(2\pi\sigma^2) + ln\frac{1}{\Pi_{i=1}^n X_i} \\ &\quad - \frac{\sum_{i=1}^n (lnX_i - \mu)^2}{2\sigma^2} \end{aligned}$$

$$\begin{cases} \frac{\partial lnL}{\partial \sigma^2} &= -\frac{n}{2}\frac{2\pi}{2\pi\sigma^2} + \frac{\sum_{i=1}^n (lnX_i-\mu)^2}{2(\sigma^2)^2} = 0 \\ \frac{\partial lnL}{\partial \mu} &= -\frac{\sum_{i=1}^n 2(\mu - lnX_i)}{2\sigma^2} = 0 \end{cases}$$

解得

$$\hat{\mu} = \frac{\sum_{i=1}^n lnX_i}{n}$$

$$\hat{\sigma^2} = \frac{\sum_{i=1}^n \left[lnX_i - \sum_{j=1}^n \frac{lnX_j}{n}\right]}{n}$$

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矩估计

令 $Y = X - \mu$.

$$p(y; \mu, \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{y}{\theta}}, y \geq 0 \\ 0 \qquad O.W. \end{cases}$$

则 $Y \sim E(\theta)$,$EY=\theta$,$EY^2=2\theta^2$.

$$EX = E(Y+\mu) = \theta+\mu$$
$$EX^2 = E(Y+\mu)^2 = 2\theta^2+2\mu\theta+\mu^2$$
$$\therefore$$
$$\hat{\mu} = \bar{X} - S_n$$
$$\hat{\theta} = \sqrt{EX^2-(EX)^2} = \sqrt{\frac{1}{n}\sum X_i^2 - \bar{X}^2} = S_n$$

极大似然估计

$$L(\mu,\theta) = \Pi_{i=1}^n \frac{1}{\theta} e^{-\frac{X_i-\mu}{\theta}} = \frac{1}{\theta^n} e^{\sum_{i=1}^n -\frac{X_i-\mu}{\theta}}$$
$$lnL(\mu,\theta) = ln\frac{1}{\theta^n} + \sum_{i=1}^n -\frac{X_i-\mu}{\theta}$$
$$\frac{\partial lnL(\mu,\theta)}{\partial\mu} = \frac{n}{\theta} > 0$$
$$\frac{\partial lnL(\mu,\theta)}{\partial\theta} = -\frac{n}{\theta} + \sum_{i=1}^n \frac{X_i-\mu}{\theta^2} = -\frac{n}{\theta} + \frac{\sum_{i=1}^n X_i - n\mu}{\theta^2} \therefore$$

由于似然函数随 μ 递增而递增，μ 最大为 $\min_i X_i$. 所以

$$\hat{\mu} = \min_i X_i$$

此时

$$\hat{\theta} = \frac{\sum_{i=1}^n X_i - n\hat{\mu}}{n} = \bar{X} - \min_i X_i$$

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无偏估计意味着 $E[.] = \sigma^2$.

$$E[.] = c\sum_{i=1}^{n-1} E(X_{i+1}-X_i)^2 = c\sum_{i=1}^{n-1}\left[D(X_{i+1}-X_i) + (E(X_{i+1}-X_i))^2\right] = c(n-1)2\sigma^2 = \sigma^2$$
$$\therefore c = \frac{1}{2(n-1)}$$

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$$L(\sigma^2) = \Pi_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_i-\mu)^2}{2\sigma^2}}$$
$$lnL(\sigma^2) = \sum_{i=1}^n -\frac{(X_i-\mu)^2}{2\sigma^2} + ln\frac{1}{\sqrt{2\pi\sigma^2}}$$
$$\frac{dlnL(\sigma^2)}{d\sigma^2} = \sum_{i=1}^n \frac{(x-\mu)^2}{2\sigma^4} + \sqrt{2\pi\sigma^2}(-\frac{1}{2\pi\sigma^2})\frac{\sqrt{2\pi}}{2\sqrt{\sigma^2}} = \sum_{i=1}^n \frac{(x-\mu)^2}{2\sigma^4} - \frac{1}{2\sigma^2}$$
$$令 \frac{dlnL(\sigma^2)}{d\sigma^2} = 0$$
$$\frac{\sum_{i=1}^n (X_i-\mu)}{2\sigma^2} = \frac{n}{2\sigma^2}$$
$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (X_i-\mu)^2$$

所以极大似然估计为 $\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (X_i-\mu)^2$.
标准正态分布的 k 阶原点矩为

$$\begin{cases} (k-1)!! & \text{当 k 为偶} \\ 0 & \text{当 k 为奇数} \end{cases}$$

$$E\hat{\sigma^2} = E(X_i-\mu)^2 = \sigma^2 E(\frac{X_i-\mu}{\sigma})^2 = \sigma^2$$

$$M(\hat{\sigma^2},\sigma^2) = D(\hat{\sigma^2}) + (E\hat{\sigma^2}-\sigma^2)^2 = \frac{1}{n^2}\sum_{i=1}^n D(X_i-\mu)^2 = \frac{1}{n}\left[E(X_i-\mu)^4 - (E(X_i-\mu)^2)^2\right] = \frac{1}{n}\sigma^4\left[E(\frac{X_i-\mu}{\sigma})^4 - (E(\frac{X_i-\mu}{\sigma})^2)^2\right] = \frac{2\sigma^4}{n}$$

$$
\begin{aligned}
&ES^2 \\
&= \frac{1}{n-1}\sum_{i=1}^{n} E(X_i - \bar{X})^2 \\
&= \frac{1}{n-1}\sum_{i=1}^{n} D(X_i - \bar{X}) \\
&= \frac{1}{n-1}\sum_{i=1}^{n} D(-\frac{1}{n}X_1 ...... + \frac{n-1}{n}X_i - \frac{1}{n}X_{i+1} - ... - \frac{1}{n}X_n) \\
&= \frac{1}{n-1}\sum_{i=1}^{n} \frac{n-1}{n}\sigma^2 \\
&= \sigma^2
\end{aligned}
$$

$$
\begin{aligned}
&M(S^2, \sigma^2) \\
&= E(S^2 - \sigma^2)^2 \\
&= (ES^2 - \sigma^2)^2 + DS^2 \\
&= DS^2 \\
&\because \frac{(n-1)S^2}{\sigma^2} \ \sim \ \chi^2(n-1) \\
&\therefore \frac{(n-1)^2}{\sigma^4} D(S^2) = D(\chi^2(n-1)) = 2n-2 \\
&\therefore DS^2 = \frac{2\sigma^4}{n-1} \\
&\therefore M(S^2, \sigma^2) = \frac{2\sigma^4}{n-1}
\end{aligned}
$$

$$
\begin{aligned}
&\because M(\hat{\sigma^2}, \sigma^2) = \frac{2\sigma^4}{n} \\
&\quad M(S^2, \sigma^2) = \frac{2\sigma^4}{n-1} \\
&\therefore M(S^2, \sigma^2) > M(\hat{\sigma^2}, \sigma^2)
\end{aligned}
$$

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$$
\begin{aligned}
\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} &\sim N(0,1) \\
P(Y_1 \le Y \le Y_2) &= 1 - \alpha \\
P(0 \le Y \le Y_2) &= \frac{1-\alpha}{2} \\
Y_2 &= u_{\frac{\alpha}{2}} \\
-\frac{u_{\frac{\alpha}{2}}\sigma}{\sqrt{n}} &\le \bar{X} - \mu \le \frac{u_{\frac{\alpha}{2}}\sigma}{\sqrt{n}} \\
\bar{X} - \frac{u_{\frac{\alpha}{2}}\sigma}{\sqrt{n}} &\le \mu \le \bar{X} + \frac{u_{\frac{\alpha}{2}}\sigma}{\sqrt{n}} \\
\frac{2u_{\frac{\alpha}{2}}\sigma}{\sqrt{n}} &\le L \\
n &\ge \frac{4u_{\frac{\alpha}{2}}\sigma}{\sqrt{n}}
\end{aligned}
$$

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$$
\begin{aligned}
&n = 9 \\
&\frac{\bar{X} - \mu}{S/\sqrt{n}} \ \sim \ t(n-1) \\
&P\{-\lambda \le \frac{\bar{X} - \mu}{S/\sqrt{n}} \le \lambda\} = 95\% \\
&\lambda = t_{0.025} = 2.3060 \\
&S^2 = 0.33 \\
&S = 0.5745 \\
&\bar{X} = 6 \\
&-2.3060 \le \frac{6 - \mu}{0.5745/3} \le 2.3060 \\
&5.5584 \le \mu \le 6.4416
\end{aligned}
$$

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大样本法，当 n 足够大时，根据中心极限定理，有

$$
\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \ \to \ N(0,1)
$$

$$\frac{n\bar{X} - n\lambda}{\sqrt{n}\sqrt{\lambda}} \ \rightarrow \ N(0,1)$$

$$\bar{X} \ \rightarrow \ \lambda \ \text{ when } n \ \rightarrow \infty$$

$$\frac{\bar{X} - \lambda}{\sqrt{\frac{\bar{X}}{n}}} \ \rightarrow \ N(0,1)$$

$$P(X_1 \leq \frac{\bar{X} - \lambda}{\sqrt{\frac{\bar{X}}{n}}} \leq X_2) = 1 - \alpha$$

$$-u_{\frac{\alpha}{2}} \leq \frac{\bar{X} - \lambda}{\sqrt{\frac{\bar{X}}{n}}} \leq u_{\frac{\alpha}{2}}$$

$$\bar{X} - u_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}} \leq \lambda \leq \bar{X} + u_{\frac{\alpha}{2}}\sqrt{\frac{\bar{X}}{n}}$$