Probability And Mathmatical Statistics
Homework 4

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1.

a

$$
\begin{aligned}
&P(X=Y)\\
&=\sum_{k=1}^{+\infty} P(X=Y=k)\\
&=\sum_{k=1}^{+\infty} P(X=k)P(Y=k)\\
&=\sum_{k=1}^{+\infty} \left((1-p)\,(1-q)\right)^{k-1} pq\\
&=\frac{pq}{p+q-pq}
\end{aligned}
$$

b

$$
\begin{aligned}
&P(\min{(X,Y)}=k)\\
&=P(X=k,Y\geq k)+P(X>k,Y=k)\\
&=P(X=k)P(Y\geq k)+P(X>k)P(Y=k)\\
&=(1-p)^{k-1}p(1-q)^{k-1}+(1-p)^{k}(1-q)^{k-1}q\\
&=(1-p)^{k-1}(1-q)^{k-1}(p+q-pq)
\end{aligned}
$$

c

Solution1.

由定义，先求概率 $P(\max(X,Y))=k$.

$$
\begin{aligned}
&P(\max(X,Y)=k)\\
&=P(X=k,Y\leq k)+P(X<k,Y=k)\\
&=P(X=k)P(Y\leq k)+P(X<k)P(Y=k)\\
&=P(X=k)(1-P(Y>k))+(1-P(X\geq k))P(Y=k)\\
&=(1-p)^{k-1}p(1-(1-q)^{k})+(1-q)^{k-1}q(1-(1-p)^{k-1})\\
&=(1-p)^{k-1}p+(1-q)^{k-1}q-(p+q-pq)(1-p)^{k-1}(1-q)^{k-1}
\end{aligned}
$$

再求期望:

$$
\begin{aligned}
&E(\max(X,Y))\\
&=\sum_{k=1}^{+\infty} kP(\max(X,Y)=k)\\
&=\sum_{k=1}^{+\infty} k(1-p)^{k-1}p+\sum_{k=1}^{+\infty} k(1-q)^{k-1}q-\sum_{k=1}^{+\infty}(p+q-pq)(1-p)^{k}\\
&=\frac{1}{p}+\frac{1}{q}-\frac{1}{p+q-pq}
\end{aligned}
$$

Solution2.

在 b. 的推导中发现 $k=\min(X,Y)\sim G(p+q-pq)$,
因此有 $E[\min(X,Y)]=(p+q-pq)^{-1}$

$$
\begin{aligned}
E[\max(X,Y)]&=E[X]+E[Y]-E[\min(X,Y)]\\
&=\frac{1}{p}+\frac{1}{q}-\frac{1}{p+q-pq}
\end{aligned}
$$

d

$$
\begin{aligned}
&E[X|X\leq Y]\\
&=\sum_{x=0}^{+\infty} xP(X=x|X\leq Y)\\
&=\sum_{x=0}^{+\infty} x\frac{P(X=x,X\leq Y)}{P(X\leq Y)}
\end{aligned}
$$

其中

$$P(X=x, X\le Y)=(1-p)^{x-1}p(1-q)^{x-1}$$

$$\begin{aligned}&P(X\le Y)\\&=\sum_{k=0}^{+\infty}P(X=k,Y\ge k)\\&=\sum(1-p)^{k-1}p(1-q)^{k-1}\\&=p\sum\left((1-p)(1-q)\right)^{k-1}\\&=\frac{p}{p+q-pq}\end{aligned}$$

$\therefore$

$$\begin{aligned}&E[X|X\le Y]\\&=\sum_{x=0}^{+\infty}(p+q-pq)(1-p)^{x-1}(1-q)^{x-1}\\&=\frac{1}{p+q-pq}\end{aligned}$$

2.

定义随机变量 X 为抛到连续出现两个 6 所抛次数, 定义两个指示器随机变量 Y,Z 为

$$Y=\begin{cases}1 & \text{第一次抛掷结果为 6}\\0 & \text{第一次抛掷结果不为 6}\end{cases}$$

$$Z=\begin{cases}1 & \text{第二次抛掷结果为 6}\\0 & \text{第二次抛掷结果不为 6}\end{cases}$$

$$\begin{aligned}&E[X]\\&=P(Y=0)E[X|Y=0]+P(Y=1)E(X|Y=1)\\&=\frac{5}{6}(1+E[X])+\frac{1}{6}E(X|Y=1)\\&=\frac{5}{6}(1+E[X])+\frac{1}{6}\left(P(Z=0)E[X|Y=1,Z=0]+P(Z=1)E[X|Y=1,Z=1]\right)\\&=\frac{5}{6}(1+E[X])+\frac{1}{6}\left(\frac{5}{6}(2+E[X])+\frac{1}{6}*2\right)\end{aligned}$$

解得

$$E[X]=42$$

3.

连着抛掷两次，若结果为正反，则为 1；若结果为反正，则为 0；若结果为正正或反反，则重新再抛两次。

易见 $P(generate\ 1)=P(generate\ 0)=p(1-p)$. 将连续的两次绑定为整体，所需连续抛掷整体的次数服从参数为 $2p(1-p)$ 的几何分布,$E=\frac{1}{2p(1-p)}$, 再乘以 2 为 $E=\frac{1}{p(1-p)}$