Probability And Mathmatical Statistics
Homework 9

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3.

(1)

$$
\begin{aligned}
S^2 &= \frac{\sum_{i=1}^n X_i^2}{n-1} \\
&\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1} \\
&= \frac{\sum_{i=1}^n (X_i^2 - 2X_i\bar{X} + \bar{X}^2)}{n-1} \\
&= \frac{\sum_{i=1}^n X_i^2 - 2\bar{X}\sum_{i=1}^n X_i + n\bar{X}^2}{n-1} \\
&= \frac{\sum_{i=1}^n X_i^2 - 2\bar{X}n\bar{X} + n\bar{X}^2}{n-1} \\
&= \frac{1}{n-1}\left[\sum_{i=1}^n X_i^2 - n\bar{X}^2\right]
\end{aligned}
$$

(2)

$$
\begin{aligned}
E(S^2) &= E(\frac{1}{n-1}\left[\sum_{i=1}^n X_i^2 - n\bar{X}^2\right]) \\
&= \frac{1}{n-1}\left[\sum_{i=1}^n EX_i^2 - nE\bar{X}^2\right] \\
&= \frac{1}{n-1}\left[\sum_{i=1}^n (DX_i + (EX_i)^2) - n\left(D\bar{X} + \left(E\bar{X}\right)^2\right)\right] \\
&= \frac{1}{n-1}\left[n(\sigma^2+\mu^2) - n\left(\frac{n\sigma^2}{n^2} + \mu^2\right)\right] \\
&= \frac{1}{n-1}\left[(n-1)\sigma^2\right] = \sigma^2
\end{aligned}
$$

8.

$$
\begin{aligned}
&EY \\
&= \sum_{i=1}^n E\left(X_i + X_{n+i} - 2\bar{X}\right)^2 \\
&= \sum_{i=1}^n \left[D\left(X_i + X_{n+i} - 2\bar{X}\right) + \left(E\left(X_i + X_{n+i} - 2\bar{X}\right)\right)^2\right] \\
&= \sum_{i=1}^n \left[\left(2\sigma^2 + 4D\bar{X} - 4cov(\bar{X}, X_i) - 4cov(\bar{X}, X_{n+i})\right) + 0\right] \\
&= \sum_{i=1}^n \left[2\sigma^2 + 4\frac{\sigma^2}{2n} - 4\frac{DX_i}{2n} - 4\frac{DX_{n+i}}{2n}\right] \\
&= \sum_{i=1}^n \left[2\sigma^2 - \frac{2\sigma^2}{n}\right] = (2n-2)\sigma^2
\end{aligned}
$$

9.

$$
\begin{aligned}
&\frac{X_{n+1} - \bar{X}}{S}\sqrt{\frac{n}{n+1}} \\
&= \frac{\frac{X_{n+1} - \bar{X}}{\sigma\sqrt{\frac{1+n}{n}}}}{\sqrt{\frac{\frac{(n-1)S^2}{\sigma^2}}{n-1}}}
\end{aligned}
$$

$$
\begin{aligned}
\text{令}A &= \frac{X_{n+1} - \bar{X}}{\sigma\sqrt{\frac{1+n}{n}}} \\
B &= \frac{(n-1)S^2}{\sigma^2} \\
\text{则原式化为}&\frac{A}{\sqrt{\frac{B}{n-1}}}
\end{aligned}
$$

由定理 $B \sim \chi^2(n-1)$.A 是独立的正态分布随机变量之和，仍然是正态分布，且 $A \sim N(0,1)$.
由定理 S^2 与 $\bar{X}$ 独立，且显然有 S^2 与 X_{n+1} 独立.

所以有 A 与 B 独立。
由 t 分布定义，原式服从 $t(n-1)$

10.

(1)

$$\bar{X} \ \sim \ N(12, \frac{4}{5})$$

$$\begin{aligned}
&P(\bar{X} > 13) \\
&= P(\frac{\bar{X}-12}{\sqrt{\frac{4}{5}}} > \frac{\sqrt{5}}{2}) \\
&= 1 - \Phi(\frac{\sqrt{5}}{2}) \\
&= 1 - 0.8686 = 0.1314
\end{aligned}$$

(2)

$$\begin{aligned}
&P(\min_{1\le i\le 5} X_i < 10) \\
&= 1 - P(\min_{1\le i\le 5} X_i \ge 10) \quad = 1 - \Pi_{i=1}^{5} P(X_i \ge 10) \\
&= 1 - \Pi_{i=1}^{5} 1 - \Phi(-1) \\
&= 1 - \Pi_{i=1}^{5} \Phi(1) \\
&= 0.5785
\end{aligned}$$

(3)

$$\begin{aligned}
&P(\max_{1\le i\le 5} X_i > 15) \\
&= 1 - P(\max_{1\le i\le 5} X_i \le 15) \quad = 1 - \Pi_{i=1}^{5} P(X_i \le 15) \\
&= 1 - \Pi_{i=1}^{5} \Phi(1.5) \\
&= 0.2923
\end{aligned}$$

11.

从答案来看 S_1^2 和 S_2^2 应该为样本修正方差，所求的也为样本修正方差，题目有误
设合并后的样本为 Z_i, 样本均值和样本修正方差为 $\bar{Z}$ 和 S_Z^2

$$\begin{aligned}
\bar{Z} &= \frac{n_1\bar{X} + n_2\bar{Y}}{n_1+n_2} \\
S_Z^2 &= \frac{\sum_{i=1}^{n_1+n_2} Z_i^2 - (n_1+n_2)\bar{Z}^2}{n_1+n_2-1} \\
&= \frac{(n_1-1)S_1^2 + n_1\bar{X}^2 + (n_2-1)S_2^2 + n_2\bar{Y}^2 - (n_1+n_2)\bar{Z}^2}{n_1+n_2-1} \\
&= \frac{(n_1-1)S_1^2 + n_1\bar{X}^2 + (n_2-1)S_2^2 + n_2\bar{Y}^2 - \frac{\left(n_1\bar{X}+n_2\bar{Y}\right)^2}{n_1+n_2}}{n_1+n_2-1}
\end{aligned}$$

补充

$$\begin{aligned}
&\frac{(X_1+X_2)^2}{(X_1-X_2)^2} \\
&= \frac{\frac{(X_1+X_2)^2}{2\sigma^2}}{\frac{(X_1-X_2)^2}{2\sigma^2}}
\end{aligned}$$

$$\begin{aligned}
\because \frac{X_1+X_2}{\sqrt{2}\sigma} &\ \sim \ N(0,1) \\
\therefore \frac{(X_1+X_2)^2}{2\sigma^2} &\ \sim \ \chi^2(1) \\
\because \frac{X_1-X_2}{\sqrt{2}\sigma} &\ \sim \ N(0,1) \\
\therefore \frac{(X_1-X_2)^2}{2\sigma^2} &\ \sim \ \chi^2(1)
\end{aligned}$$

下面证明 $\frac{(X_1+X_2)^2}{2\sigma^2}$ 和 $\frac{(X_1-X_2)^2}{2\sigma^2}$ 独立，只要证明 X_1+X_2 和 X_1-X_2 独立, 又由于他们都是多元正态分布，独立与不相关等价，只要证明:$cov(X_1+X_2, X_1-X_2)=0$ 即可。

$$\begin{aligned}
cov(X_1+X_2, X_1-X_2) &= D(X_1) - cov(X_1,X_2) + cov(X_1,X_2) - \\
&= \sigma^2 - \sigma^2 = 0
\end{aligned}$$

由 F 分布定义，原式子服从 $F(1,1)$.