Probability And Mathmatical Statistics
Homework 5

151220131 谢旻晖

习题四 4

1.

(U,V)	(1,1)	(2,1)	(2,2)
P	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

2.

$$E[U] = \frac{14}{9}$$

$$E[V] = \frac{10}{9}$$

3.

UV	1	2	4
P	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

$$\begin{aligned}
&Cov(U,V)\\
&= E[UV] - E[U]E[V]\\
&= \frac{16}{9} - \frac{140}{81}\\
&= \frac{4}{81}
\end{aligned}$$

习题四 19

两个泊松分布参数分别为 1 和 2，由方差的公式

$$\begin{aligned}
&E[X+Y]^2\\
&= Var[X+Y] + (E[X+Y])^2\\
&= Var[X] + Var[Y] + (E[X]+E[Y])^2\\
&= 1+2+9\\
&= 12
\end{aligned}$$

习题四 23

$$P(|X| \geq x) = P(f(|X|) \geq f(x)) \leq \frac{E[|f(X)|]}{f(x)}$$

2

令 $X_i = \begin{cases} 1 & \text{第 i 个数为不动点} \\ 0 & \text{第 i 个数非不动点} \end{cases}$ 令 X 为不动点的个数, 则 $X = \sum_{i=1}^{n} X_i$.

$$\begin{aligned}
E[X] &= E[\sum_{i=1}^{n} X_i]\\
&= \sum_{i=1}^{n} E[X_i]\\
&= \sum_{i=1}^{n} P(X_i = 1)\\
&= \sum_{i=1}^{n} \frac{1}{n}\\
&= 1
\end{aligned}$$

$$
\begin{aligned}
E[X^2] &= E[(\sum_{i=1}^{n} X_i)^2] \\
&= E[\sum_{i,j=1}^{n} X_i X_j] \\
&= \sum_{i,j=1}^{n} E[X_i X_j] \\
&= \sum_{i=1}^{n} E[X_i^2] + \sum_{i\neq j} E[X_i X_j] \\
&= 1 + \sum_{i\neq j} P(X_i X_j = 1) \\
&= 1 + \sum_{i\neq j} P(X_i = 1, X_j = 1) \\
&= 1 + \sum_{i\neq j} \frac{1}{n(n-1)} \\
&= 1 + 1 = 2
\end{aligned}
$$

$$Var(X) = E[X^2] - (E[X])^2 = 2 - 1 = 1$$

3

设随机变量 X_i 为第 i 天股票的价格. 由全期望公式

$$
\begin{aligned}
E[X_i] &= pE[X_i|\text{第 } i-1 \text{ 天涨}] + qE[X_i|\text{第 } i-1 \text{ 天跌}] \\
&= pE[X_{i-1} * r] + qE[X_{i-1} * \frac{1}{r}] \\
&= (pr + \frac{q}{r})E[X_{i-1}] \qquad \text{(期望的线性性质)}
\end{aligned}
$$

$$\because E[X_1] = 1 \therefore E[X_d] = (pr + \frac{q}{r})^{d-1}$$

$$
\begin{aligned}
E[X_i^2] &= pE[X_i^2|\text{第 } i-1 \text{ 天涨}] + qE[X_i^2|\text{第 } i-1 \text{ 天跌}] \\
&= pE[X_{i-1}^2 * r^2] + qE[X_{i-1}^2 * \frac{1}{r^2}] \\
&= (pr^2 + \frac{q}{r^2})E[X_{i-1}^2]
\end{aligned}
$$

$$\because E[X_1^2] = 1 \therefore E[X_d^2] = (pr^2 + \frac{q}{r^2})^{d-1}$$

$$
\begin{aligned}
Var[X_d] &= E[X_d^2] - (E[X_d])^2 \\
&= (pr^2 + \frac{q}{r^2})^{d-1} - (pr + \frac{q}{r})^{2d-2}
\end{aligned}
$$

where $q = 1 - p$

4

a)

Y_i	0	1
P	$\frac{1}{2}$	$\frac{1}{2}$

b)

取特定的 $Y_1 = Y_2 ... Y_{\binom{n}{2}} = 1$. 显然有

$$P(Y_1 = 1)P(Y_2 = 1)...P(Y_{\binom{2}{1}} = 1) = \frac{1}{2^{\binom{n}{2}}}$$

而

$$P(Y_1 = 1 \wedge Y_2 = 1 ... \wedge Y_n = 1) = 0$$

说明如下：考虑 n 个随机比特中某 3 个 a, b, c，若 $a \oplus b = 1$ 且 $a \oplus c = 1$，那么 $b \oplus c = b \oplus a \oplus a \oplus c = 0$. 所以不存在所有异或生成的比特均为 1 的情况。

因此,$P(Y_1 = 1)P(Y_2 = 1)...P(Y_{\binom{2}{1}} = 1) \neq P(Y_1 = 1 \wedge Y_2 = 1 ... \wedge Y_n = 1)$

$Y_i, i = 1, 2 ... \binom{n}{2}$ 并不相互独立.

c)

$$
\begin{aligned}
E[Y_i Y_j] &= 0 * P(Y_i Y_j = 0) + 1 * P(Y_i Y_j = 1) \\
&= P(Y_i Y_j = 1) \\
&= P(Y_i = 1 \wedge Y_j = 1) \\
&\overset{*}{=} P(Y_i = 1)P(Y_j = 1) \\
&= E[Y_i]E[Y_j]
\end{aligned}
$$

其中带 * 号的等号作进一步说明：分两种情况.

若 Y_i 和 Y_j 是由四个比特生成，则显然他们是独立的, 有 $P(Y_i = 1 \wedge Y_j = 1) = P(Y_i = 1)P(Y_j = 1)$。

若 Y_i 和 Y_j 是由三个比特生成，即他们之间使用了公共的比特，不妨设 $Y_i = a \oplus b$,$Y_j = b \oplus c$. 样本空间为 (a, b, c) 三元组个数，共 8 个，其中 $Y_i = 1 \wedge Y_j = 1$ 有 (0,1,0),(1,0,1) 两种，$P(Y_i = 1 \wedge Y_j = 1) = \frac{1}{4}$. 而 $Y_i = 1$ 和 $Y_j = 1$ 各 4 种，$P(Y_i = 1) = \frac{1}{2}$,$P(Y_j = 1) = \frac{1}{2}$, 有 $P(Y_i = 1 \wedge Y_j = 1) = P(Y_i = 1)P(Y_j = 1)$。

d)

$$
\begin{aligned}
Var[Y] = Var[\sum Y_i] &= \sum Var[Y_i] + 2 \sum_{1 \le i < j \le \binom{n}{2}} Cov[Y_i, Y_j] \\
&= \sum Var[Y_i] \\
&= \sum E[Y_i^2] - (E[Y_i])^2 \\
&= \sum \frac{1}{4} = \frac{\binom{n}{2}}{4}
\end{aligned}
$$

e) 使用切比雪夫不等式

$$P(|Y - E(Y)| \geq n) \leq \frac{Var[Y]}{n^2} = \frac{\binom{n}{2}}{4n^2} = \frac{n-1}{8n}$$