

Why the Bidirectional Breadth First Search Algorithm Finds a Shortest Path

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Suppose you have an unweighted undirected graph, given for every node x of the graph the set E_x of nodes on edges meeting x (i.e. E_x is the edgelist of x), and want a shortest path between two nodes a and b .

A **unidirectional breadth first search** starting from a will find one. Let $A_0 = \{a\}$, and for $k > 0$, $A_k = (\cup_{x \in A_{k-1}} E_x) \setminus \cup_{0 \leq j < k} A_j$ (all the nodes in all the edgelists of all the nodes in A_{k-1} , but removing nodes in all previous A_j). Define $d(x, a)$ as the distance from x to a , the length of the shortest path from x to a . This is a metric, in particular for any x, y, z , $d(x, y) = 0 \Leftrightarrow x = y$ and $d(x, z) \leq d(x, y) + d(y, z)$. The latter is true because you can make a path from x to z by concatenating shortest paths from x to y and y to z , though this concatenation may not be the shortest path from x to z .

Claim: $A_k = \{x | d(x, a) = k\}$. If $d(a, x) = k$, then $x \in A_k$ because there is a path of length k from a , and no shorter path (which would cause $x \in A_j$ for some $j < k$).

On the other hand, if $x \in A_k$, there is a path of length k from a to x , so $d(a, x) \leq k$. It's not possible that $d(a, x) = j < k$, because if it were, then $x \in A_j$ and thus not in A_k . Thus $d(a, x) = k$.

A breadth first search for b starting from a , implemented as in `bfs1.py`, in effect creates the sets A_j for $j < k = d(a, b)$, and some part of A_k until it encounters b in that set. This is because nodes are added to the queue from edgelists of nodes popped off the queue (which is initialized with a and is FIFO), but only if they have not been previously encountered. The code implicitly shifts from A_j to A_{j+1} when all the edgelists of nodes belonging to A_{j-1} have been processed.

For every node in A_j added to the queue, the code keeps track of which node in A_{j-1} owned the edgelist on which it was first encountered. A shortest path from a to b is then built by reversing what is obtained by following these parent relationships backwards.

A **bidirectional breadth first search** can be faster, due to the potentially exponential fan-out of breadth first search. Define the A_k as before, and the B_k similarly starting from $B_0 = \{b\}$. Supposing that we start from the one end with A_0 , from the other with B_0 , that some choice is made at each iteration whether to move from A_r to A_{r+1} or B_s to B_{s+1} , and that we stop when $A_r \cap B_s \neq \emptyset$

for the first time. From the two paths from the ends we get a path between the endpoints a and b of length $r + s$. In fact, there is no shorter path. To see this, use the fact that if the equations below for A_0 through A_k hold, then the logical equivalence for A_{k+1} holds.

$$\begin{array}{c}
\text{If} \\
A_0 \cap B_0 = \emptyset \\
A_1 \cap B_0 = \emptyset \\
\ldots \\
A_k \cap B_0 = \emptyset \\
\text{then} \\
A_{k+1} \cap B_0 = \emptyset \Leftrightarrow A_r \cap B_s = \emptyset,
\end{array}$$

for any nonnegative r, s with $r + s = k + 1$.

This can be seen as follows. If $A_{k+1} \cap B_0 \neq \emptyset$, then recalling the properties of unidirectional breadth first search from a , there is a shortest path of length $k + 1$ from a to b . This can be split into two pieces of lengths r and s , reversing one of them, that are shortest paths of those lengths from a to some x and from b to x , so that $x \in A_r \cap B_s$, which is thus nonempty. Conversely, if there is an $x \in A_r \cap B_s$, by concatenating paths, one of them reversed, we get a path of length $r + s = k + 1$ from a to b , so that $d(a, b) \leq k + 1$. But $d(a, b) \leq k$ is false because of all the intersections empty by hypothesis. Thus $d(a, b) = k + 1$, and thus $A_{k+1} \cap B_0 \neq \emptyset$.

Then, to see that the path we construct that meets at a point in the first nonempty $A_r \cap B_s$ is a shortest path from a to b , if $A_0 \cap B_0 \neq \emptyset$, then $a = b$ and the path is of length 0. Otherwise, use the above fact repeatedly, noting that we have some sequence of $A_{r_i} \cap B_{s_i} = \emptyset$, $r_i + s_i = k_i = 1, 2, \dots, r + s - 1$, implying $A_{k_i} \cap B_0 = \emptyset$ until finally we arrive at $A_r \cap B_s \neq \emptyset \Rightarrow A_{r+s} \cap B_0 \neq \emptyset \Rightarrow d(a, b) = r + s$.

The code in `bfs2.py` works with the A_{r_i} and B_{s_i} fairly explicitly, using the lists `r_level_nodes` and `t_level_nodes` for them (`r` for root or a , `t` for target or b). It processes a whole level, or one of these sets, at a time, before going on to the next. It finds a nonempty intersection of the two by finding a newly encountered node in the one that it's building, that is in the other that it's already built, and stops when this first happens.

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