Hamiltonicity of Cayley Graphs and Digraphs

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Abstract

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The existence of Hamiltonian paths and cycles has always been of interest, not necessarily just within graph theory. For example, the problem bears a strong relation with group theory: given a finite set of generators of a group, can one construct a finite sequence $s_1, s_2, \ldots s_r$ from these generators such that every word $s_1 s_2 \ldots s_i$ corresponds to a unique element of the group? Such a sequence of words would imply a Hamiltonian path in the Cayley graph of the group with the given generators forming the connecting set.

We begin by introducing fundamental elements of graph and group theory, and the notion of a Cayley graph. We then introduce a conjecture of Lovász, on the existence of Hamiltonian cycles in finite connected Cayley graphs. In this manner we reconcile the Hamiltonian problem in graph and group theory.

We then introduce a number of techniques, through which a number of classes of groups have been shown to have Hamiltonian Cayley graphs. We use these techniques to prove a result of Marušič (1983), that every Cayley graph for every finite Abelian group has a Hamiltonian cycle. We will also consider the pioneering work of Rankin (1948, 1966) on groups with a generating set of size 2.

We also consider Cayley digraphs and provide examples of infinite classes of such graphs which do not have a Hamiltonian cycle. Consequently, a variant of Lovász's conjecture for Cayley digraphs cannot be stated. However, we prove a result of Holsztyński and Strube (1978) that every Cayley digraph of an Abelian group has a Hamiltonian path, hence providing a holistic overview of the problem in the case of Abelian groups.

We conclude by proving a modern result, due to Pak and Radoičić (2009), showing that every group has a small generating set such that the corresponding Cayley graph has a Hamiltonian cycle. This is followed by a short survey of further results and open problems. In this manner, we hope to present the reader with a foundation for carrying out research in this area, along with evidence suggesting that Lovász's conjecture for Cayley graphs holds in the positive.

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Then you decide to take a walk by the old school
Nothing has changed it's still the same
I've got nothing to say but it's okay
Good morning, good morning

John Lennon,
Good Morning Good Morning

1 Introduction

1.1 | Foundations

We begin by introducing the necessary prerequisites, laying the foundations on which we can build on and pose simple (to state) yet deep questions, with the aim to not necessarily answer but to attain an understanding of them throughout this dissertation. To achieve this, we begin by introducing (di)graphs, groups, and the interplay between the two.

Before doing so, we must establish some notational conventions. The set difference of two sets A and B shall be denoted by $A - B$, while their intersection, union and Cartesian product shall be denoted (as usual) by $A \cap B$, $A \cup B$ and $A \times B$, respectively. To emphasise that A and B are disjoint, we shall sometimes denote their union by $A \dot{\cup} B$. The Cartesian product of the same set A for $n + 1$ times is recursively defined as $A^{n+1} := A \times A^n$. The cardinality of a set A shall be denoted by $|A|$, while the power–set of A is denoted by 2^A.

A *partition* of a set X is a collection $\mathcal{X} = \{X_1, \ldots, X_k\}$ of subsets of X, also known as *blocks*, such that the pairwise intersection of distinct blocks is empty and the disjoint union of all the blocks is exactly X.

We predominantly use the definitions, with notation adopted accordingly, of Cameron (1999); Diestel (2017); Lauri and Scapellato (2016).

1.1.1 | Graphs and digraphs

A graph $\mathsf{G} = (V(\mathsf{G}), E(\mathsf{G}))$ is a pair of sets $V = V(\mathsf{G})$, a non–empty set termed as the *vertex set*, and $E = E(\mathsf{G})$, a set of *unordered* pairs of distinct elements of V termed as the *edge set*. An element in E (called an edge) is written as a 2–element–set $\{u, v\}$. We may allow E to be a multi–set, in which case edges repeated more than once are termed as *multi–edges*. We may also allow the vertices in an edge to be the same, in which case such an edge is termed as a *loop*. However, unless otherwise stated, we shall assume our graphs to be without loops and multi–edges, and we shall assume that the vertex set is finite.

Two vertices in V belonging to the same edge in E are said to be *adjacent*. Similarly, if a vertex v in V belongs to an edge e in E, then e is said to be *incident* to v. The *degree* of a vertex v in V, denoted by $\deg(v)$, is the number of edges in E incident to v. A graph G is said to be *k–regular* if for every vertex v in V, $\deg(v) = k$. The minimum and maximum degree of a graph G are denoted by $\delta = \delta(\mathsf{G})$ and $\Delta = \Delta(\mathsf{G})$, respectively. Hence G is k–regular if and only if $\delta = \Delta = k$.

If U is a subset of the vertices V of a graph G, then $\mathsf{G} - U$ denotes the graph obtained by removing U from V and removing from E all the edges incident to some vertex in U. Similarly if F is a subset of the edges of a graph G, then $\mathsf{G} - F$ denotes the graph $(V, E - F)$. When $U = \{v\}$ or $F = \{e\}$ (which is to say, we remove a single vertex or edge), for brevity we shall simply write $\mathsf{G} - v$ and $\mathsf{G} - e$, respectively. If U is a subset of the vertices of a graph G, then $\mathsf{G}[U]$ denotes the *induced subgraph of* G *by* U, with vertex set U and with edge set containing all the edges join pairs of vertices of U in G.

We can generalise the notion of a graph to that of a *directed* graph (*digraph* for short), where a digraph $\mathsf{D} = (V(\mathsf{D}), A(\mathsf{D}))$ is a pair of sets $V = V(\mathsf{D})$, the vertex set as defined previously, and $A = A(\mathsf{D})$, a set of *ordered* pairs of distinct elements of V termed as the *arc set*. Elements in A (called arcs) are written as 2–tuples (u, v), the order of which determines the 'direction' of the arc (we say that (u, v) is an arc from u to v, and that it is incident *to v* and incident *from u*). It is worth noting that a graph can be considered as a digraph, by associating with each edge $\{u, v\}$ the pair of arcs (u, v) and (v, u).

Due to this notion of direction when discussing digraphs, we must adapt some of the terminology we use for (undirected) graphs. The *in–degree* $\deg_{\text{in}}(v)$ of a vertex v is the number of arcs incident to v, while the *out–degree* $\deg_{\text{out}}(v)$ of a vertex v is the number of arcs incident from v. We say that a digraph is k–regular if for every vertex v in V, $\deg_{\text{in}}(v) = \deg_{\text{out}}(v) = k$. A useful textbook result relating the number of arcs and the in/out degrees is the following.

Lemma 1.1.1 (Handshaking Lemma for Digraphs)**.** *Let* D *be a finite digraph. Then:*

$$\sum_{v \in V(\mathsf{D})} \deg_{\text{in}}(v) = |A(\mathsf{D})| = \sum_{v \in V(\mathsf{D})} \deg_{\text{out}}(v)$$

1.1.1.1 | Paths and cycles

A *path* is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ and edges $\{v_i, v_{i+1}\}$ for $1 \leq i < k$. If we include the additional edge $\{v_1, v_k\}$, we get a *cycle*. In the context of digraphs, we can similarly define *oriented* paths and cycles by considering the arcs (v_i, v_{i+1}) for $1 \leq i < k$, and including the additional arc (v_k, v_1) in the case of a cycle.

Paths and cycles are denoted by P and C, respectively, along with any sub or super–scripts as necessary. Whenever we write an oriented path P in a digraph D as a sequence of vertices $v_1 v_2 \ldots v_k$, we use the convention that the arcs are going *from left to right*, so (v_i, v_{i+1}) are the arcs in the path. In particular we refer to v_1 and v_k as the *source* and *sink* vertices of P, respectively. In the case of an undirected graph, there is no need for these conventions, and we simply term v_1 and v_k as *pendant* vertices of the path. A pendant vertex v of a graph G is a vertex such that $\deg(v) = 1$.

1.1.2 | Groups

A *group* is a pair $(\Gamma, *)$ where Γ is a non–empty set and $*$ is a binary operation with domain $\Gamma \times \Gamma$, satisfying the *group axioms*. Let $g, h, f \in \Gamma$:

G1 $g * h \in \Gamma$ (closure);

G2 $g * (h * f) = (g * h) * f$ (associativity);

G3 there exists $1 \in \Gamma$ such that for all g in Γ, $1 * g = g = g * 1$ (identity);

G4 there exists $g^{-1} \in \Gamma$ such that $g * g^{-1} = 1 = g^{-1} * g$ (inverse).

We say that a group is *Abelian* if it further satisfies the commutativity axiom,

G5 $g * h = h * g$ (commutativity).

For brevity, whenever there is no room for ambiguity, we may omit writing the binary operation $*$ and simply write gh instead of $g * h$. Unless otherwise stated, we shall always consider Γ to be finite.

Whenever the binary operation is not of concern and there is no room for ambiguity, we shall simply abuse the notation slightly and write Γ instead of $(\Gamma, *)$ whenever referring to a group.

We say that $g \in \Gamma$ is an *involution* if $g \neq 1$ and $g^2 = 1$. In other words g is its own inverse, meaning $g = g^{-1}$.

If Ψ is a subset of Γ and Ψ is a group, we say that Ψ is a *subgroup* of Γ. Moreover, for any element g in Γ, we say that $g\Psi = \{gh \colon h \in \Psi\}$ and $\Psi g = \{hg \colon h \in \Psi\}$ are *left* and *right–cosets* of Ψ in Γ, respectively. If for every $g \in \Gamma$ we have that $g\Psi = \Psi g$, we say that Ψ is a *normal subgroup* of Γ, which we denote by $\Psi \lhd \Gamma$.

A group Γ is said to be *simple* if its only normal subgroups are the trivial ones, namely $\{1\}$ and Γ itself.

The cosets of a subgroup Ψ partition the group Γ; in other words, cosets are either equal or disjoint. We denote the number of cosets of Ψ in Γ by $[\Gamma \colon \Psi]$. One of the most celebrated results in group theory is *Lagrange's Theorem*, which we state without proof.

Theorem 1.1.2 (Lagrange's Theorem). *Let Γ be a group and Ψ a subgroup of Γ. Then*

$$|\Gamma| = [\Gamma \colon \psi] \cdot |\Psi|$$

Given a set Y, let S_Y be the set of all permutations of the elements of Y[1]. The *symmetric group* on Y then is the pair $(S_Y, \circ)$, where $\circ$ denotes function composition. In the case that $Y = \{1, \ldots, n\}$, we use the alternative notation S_n for the symmetric group.

[1] Recall that a permutation of a set Y is a bijection $\phi \colon Y \to Y$.

A *group homomorphism* between two groups $(\Gamma, *)$ and $(\Psi, \cdot)$ is a function $\phi \colon \Gamma \to \Psi$ such that for every g_1 and g_2 in Γ, $\phi(g_1 * g_2) = \phi(g_1) \cdot \phi(g_2)$. If 1_Ψ denotes the identity of $(\Psi, \cdot)$, then the *kernel* of ϕ is the set of elements in Γ mapped to 1_Ψ under ϕ, $\ker(\phi) = \{g \in \Gamma \colon \phi(g) = 1_\Psi\}$. A *group isomorphism* is a bijective group homomorphism; indeed a group homomorphism ϕ is an isomorphism if and only if $\ker(\phi) = \{1_\Gamma\}$. In this case we say that the two groups are *isomorphic*, which we denote by $(\Gamma, *) \simeq (\Psi, \cdot)$.

1.1.2.1 | Group actions

If $(\Gamma, *)$ is a group and Y is a finite set, an *action* of $(\Gamma, *)$ on Y is a group homomorphism $\phi \colon \Gamma \to S_Y$ between $(\Gamma, *)$ and $(S_Y, \circ)$. More so, $(\phi(\Gamma), \circ)$ is a subgroup of $(S_Y, \circ)$. If ϕ is an isomorphism, then the action is said to be *faithful*.

Consider an action ϕ of the group $(\Gamma, *)$ on a finite set Y and let $y \in Y$. The *orbit* of y with respect to this action is the set

$$\Gamma(y) = \{x \in Y \colon \exists g \in \Gamma \text{ such that } (\phi(g))(x) = y\}$$

while the *stabiliser* of y with respect to this action is the set

$$\Gamma_y = \{g \in \Gamma \colon (\phi(g))(y) = y\}.$$

It is worth noting that orbits are in fact equivalence classes for the equivalence relation $\sim_\phi$, where for $x, y \in Y$ we have that $x \sim_\phi y$ if there exists $g \in \Gamma$ such that $(\phi(g))(x) = y$. Hence the orbits give a partition of Y.

1.1.2.2 | Group presentations

Let Γ be a group and let $S \subseteq \Gamma$. A *word* generated by S is a product of a *finite* number of terms, where each term is an element in S or some inverse of an element in S. We say that S is a *generating set* of Γ if every element in Γ can be written as a word generated by S. In this case, the elements of S are said to be *generators* of Γ.

Let w_1 and w_2 be two words generated by S. If $w_1 = w_2$ then this equality is known as a *relation in S*. Taking inverses, any such relation can be written as

$r = 1$ for some word r generated by S. If S generates Γ and every relation in S can be deduced from the relations $r_1 = 1, r_2 = 1, \dots$ then:

$$\Gamma = \langle S | r_1 = 1, r_2 = 1, \dots \rangle$$

is said to be a *presentation* of Γ.

The *cyclic group* of order n is denoted by Z_n. With some abuse of notation, given an element a of order n, $\langle a \rangle$ shall denote the group with presentation $\langle a | a^n = 1 \rangle$, which is isomorphic to Z_n. The *dihedral group* D_n of symmetries of a regular n–gon has order $2n$ and presentation $\left\langle a, b | a^n = b^2 = (ba)^2 = 1 \right\rangle$.

1.1.3 | Automorphism groups of graphs and digraphs

Let G_1 and G_2 be two graphs and $\sigma\colon V(\mathsf{G}_1) \to V(\mathsf{G}_2)$. We say that σ is a *graph isomorphism* if σ is a bijection such that $\{u, v\} \in E(\mathsf{G}_1)$ if, and only if, $\{\sigma(u), \sigma(v)\} \in E(\mathsf{G}_2)$. In this case we say that the two graphs are *isomorphic*, which we denote by $\mathsf{G}_1 \simeq \mathsf{G}_2$.

Similarly, let D_1 and D_2 be two digraphs and $\sigma\colon V(\mathsf{D}_1) \to V(\mathsf{D}_2)$. We say that σ is a *digraph isomorphism* if σ is a bijection such that $(u, v) \in A(\mathsf{D}_1)$ if, and only if, $(\sigma(u), \sigma(v)) \in A(\mathsf{D}_2)$. In this case we say that the two digraphs are *isomorphic*, which we denote by $\mathsf{D}_1 \simeq \mathsf{D}_2$.

Example 1.1.3. Consider Figure 1.1. Let $\mathbb{B}$ denote the set of all finite binary strings and let $b\colon \mathbb{N} \cup \{0\} \to \mathbb{B}$ be the bijection mapping every non–negative integer to its binary string representation. It is easy to verify that σ defined by $\sigma\colon v \mapsto b(v)$, for all $v \in V(\mathsf{G}_1)$, is an isomorphism and hence $\mathsf{G}_1 \simeq \mathsf{G}_2$.

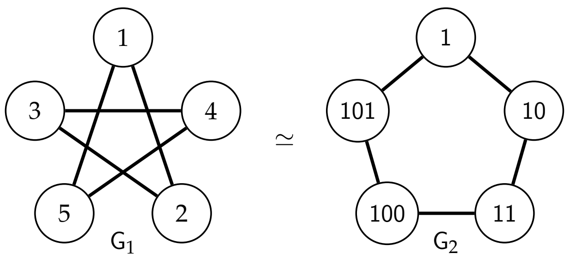


Figure 1.1: A pair of isomorphic graphs.

In the case that $V(\mathsf{G}_1) = V(\mathsf{G}_2)$ or $V(\mathsf{D}_1) = V(\mathsf{D}_2)$, the isomorphism is termed as an *automorphism*. A property of a (di)graph is said to be *invariant* if it is preserved under any automorphism of the (di)graph. The *automorphism group* of a (di)graph H, denoted by $\mathrm{Aut}(\mathsf{H})$, is the set of automorphisms of H under function composition. Note that $\mathrm{Aut}(\mathsf{H})$ is a subgroup of $\left(S_{V(\mathsf{H})}, \circ\right)$.

Let G be a graph, $v \in V$, and $\sigma \in \mathrm{Aut}(\mathsf{G})$. By definition of an isomorphism (and in particular an automorphism), if $\{u, v\}$ is an edge incident to v then $\{\sigma(u), \sigma(v)\}$ is an edge incident to $\sigma(v)$, and since σ is bijective the number of edges incident to v is equal to the number of edges incident to $\sigma(v)$. Hence $\deg(v) = \deg(\sigma(v))$. Similarly, considering a digraph D, it can be shown that $\deg_{\mathrm{in}}(v) = \deg_{\mathrm{in}}(\sigma(v))$ and $\deg_{\mathrm{out}}(v) = \deg_{\mathrm{out}}(\sigma(v))$.

1.2 | The Hamiltonian problem

In 1857, William R. Hamilton formulated the following puzzle while working on his *icosian calculus*.

Motivating Problem 1.2.1 (Hamilton's puzzle)**.** Given a regular dodecahedron, find a cycle along its edges which visits every vertex exactly once, starting and ending at the same vertex.

Definition 1.2.2. A path P in a graph G is said to be a *Hamiltonian path* if $V(\mathsf{P}) = V(\mathsf{G})$. Similarly, a cycle C in a graph G is said to be a *Hamiltonian cycle* if $V(\mathsf{C}) = V(\mathsf{G})$. In the context of digraphs, we prepend the term *oriented*.

1.2.1 | Historical perspectives

Icosian calculus as an algebraic structure was devised a year earlier in 1856, to study the symmetries of the icosahedron (from which the name is derived). One could observe many similarities between the ideas used in icosian calculus and modern day mathematics, for example the use of *generators* and *relations* to represent groups in order to study the symmetries of polyhedra.

Albeit spanning paths and cycles are named after Hamilton, earlier examples of problems similar to Hamilton's puzzle do exist. One such example is the exploration of chessboard *knight tours*, with some of the earliest known solutions (due to Pierre Remond de Montmort and Abraham de Moivre) dating to the late 17^{th} century. A knight's tour is a sequence of moves, on an $n \times n$ chessboard, by a knight chess–piece such that every square on the board is visited exactly once. While the work of de Montmort and de Moivre focused on the 'classical' 8×8 problem, it was Leonhard Euler (1778) who outlined a method of solution for the general $n \times n$ variant. Further historical perspectives are surveyed by Ball (1905); Wilson (1988). In the language of graph theory, *Hamilton's problem* can be stated more generally as follows.

Motivating Problem 1.2.3. What are the (necessary and) sufficient conditions for a (di)graph to have a Hamiltonian path (cycle)?

Necessary and sufficient conditions for a graph to be Hamiltonian are not as easy to find as, say, the criteria for a graph to be *Eulerian*. A graph G is said to be Eulerian if there exists a trail which visits every edge exactly once. It is well known that a graph is Eulerian if and only if every vertex has an even degree.

One of the most well-celebrated results in the area is Ore's Theorem, which establishes sufficient conditions for a graph to have a Hamiltonian cycle, namely that for any pair of distinct non–adjacent vertices, the sum of their degrees is at least the number of vertices n. We present here (without proof) a generalisation of Ore's result to digraphs, which is due to Woodall.

Theorem 1.2.4 (Woodall (1972))**.** *Let D be a digraph on n vertices. If for any two distinct vertices $u, v \in V$ such that $(u, v) \notin A$, we have that $\deg_{\text{out}}(u) + \deg_{\text{in}}(v) \geq n$, then D has an oriented Hamiltonian cycle.*

The area of Hamiltonicity of (di)graphs is broad in scope and still highly active, to the extent that a dissertation on the subject would barely scratch the tip of the iceberg. The rest of this chapter will henceforth be dedicated to gradually narrowing the scope of the classes of (di)graphs we consider – namely Cayley (di)graphs – and study Hamiltonicity within this narrower context.

1.2.2 | Probabilistic perspectives

While the primary domain of this dissertation is *algebraic* graph theory, we appeal briefly to the *probabilistic method*, pioneered by Paul Erdős and Alfréd Rényi. We take this opportunity to introduce two concepts: *random graphs* and the notion of properties that hold for *almost all* graphs.

When discussing random graphs, we generally start by considering some vertex set V with n vertices and associating a probability space over some family of graphs with the chosen vertex set V. Hence what we mean by a random graph depends on how we define our probability space. In this case, we shall consider the probability space[2] $\left({}^k\Omega_n, 2^{{}^k\Omega_n}, {}^k\mathbb{P}_n\right)$, where ${}^k\Omega_n$ is the set of all k–regular graphs on n vertices (up–to choice of V) and for all G in ${}^k\Omega_n$, ${}^k\mathbb{P}_n\left(\{\mathsf{G}\}\right) = \frac{1}{\left|{}^k\Omega_n\right|}$ (which is to say, we have a uniform probability distribution over all such graphs).

In the context of this probability space, we say that a property $\mathbf{P}$ holds for almost all k–regular graphs if,

$$\lim_{n\to\infty} {}^k\mathbb{P}_n\left(\left\{\mathsf{G} \in {}^k\Omega_n\colon\ \mathsf{G} \text{ has property } \mathbf{P}\right\}\right) = 1.$$

Similarly, the aforementioned can also be adapted accordingly for digraphs. Erdős and Rényi (1960) asked, as an open problem, for which classes of random graphs does almost every such graph have a Hamiltonian path. One of the biggest leaps made in this problem is due to Robinson and Wormald, concerning k–regular random graphs.

Theorem 1.2.5 (Robinson and Wormald (1992, 1994))**.** *For every fixed* $k \geq 3$*,*

$$\lim_{n\to\infty} {}^k\mathbb{P}_n\left(\left\{\mathsf{G} \in {}^k\Omega_n\colon\ \mathsf{G} \text{ has a Hamiltonian path}\right\}\right) = 1.$$

Even more strikingly, Cooper et al. (1994) extended this result to k–regular digraphs. Redefining ${}^k\Omega_n$ as the set of all k–regular digraphs, consider a probability space similar to the case for k–regular graphs.

[2]Recall that a probability space is a triple $(\Omega, \mathcal{S}, \mathbb{P})$ where Ω is the *sample space*, $\mathcal{S}$ is a σ–algebra on Ω (for example 2^Ω, for finite Ω) and $\mathbb{P}\colon\ \mathcal{S} \to [0,1]$ is a *probability measure*.

Theorem 1.2.6 (Cooper et al. (1994)).

$$\lim_{n\to\infty} {}^{k}\mathbb{P}_n\left(\left\{\mathsf{D} \in {}^{k}\Omega_n \colon \text{ D has an oriented Hamiltonian path}\right\}\right) = \begin{cases} 0 & k = 2 \\ 1 & k \geq 3 \end{cases}.$$

1.3 | Vertex–transitivity and Cayley (di)graphs

We say that a graph G is *vertex–transitive* if for any two vertices $u, v \in V$, there exists an automorphism $\sigma \in \mathrm{Aut}(\mathsf{G})$ such that $\sigma(u) = v$. Consequently $\mathrm{Aut}(\mathsf{G})$ acting on V results in the orbit partition $\Pi = \{V\}$. Indeed by the definition of an orbit, we have that G is vertex–transitive if and only if $\Pi = \{V\}$.

Recall that for an automorphism σ of a graph G, we have that $\deg(\sigma(u)) = \deg(u)$, and hence if G is vertex–transitive then it is regular. Similarly we can define vertex–transitive digraphs. In this case for any automorphism σ of a digraph D, we have that $\deg_{\mathrm{in}}(\sigma(u)) = \deg_{\mathrm{in}}(u)$ and $\deg_{\mathrm{out}}(\sigma(u)) = \deg_{\mathrm{out}}(u)$.

Hence if D is vertex–transitive then D is regular. To see this, consider $v \in V$. By the Handshaking Lemma for Digraphs and vertex–transitivity, we have that

$$n \cdot \deg_{\mathrm{in}}(v) = \sum_{u \in V} \deg_{\mathrm{in}}(u) = \sum_{u \in V} \deg_{\mathrm{out}}(u) = n \cdot \deg_{\mathrm{out}}(v)$$

and hence $\deg_{\mathrm{in}}(v) = \deg_{\mathrm{out}}(v)$.

Let Γ be a group and let S be a subset of Γ. The *Cayley digraph* $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ is a digraph with vertex set Γ and arc set defined as:

$$\bigcup_{g \in \Gamma} \{(g, gs) \colon s \in S\}$$

where $gs \in \Gamma$ by closure in Γ. The set S is often termed as the *connecting set* of the Cayley digraph. For any $s \in S$ and $g \in \Gamma$, we say that the arc (g, gs) in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ is *labelled* s.

Recall that in a graph G, we can consider an edge $\{u, v\}$ as the pair of arcs (u, v) and (v, u). Furthermore, given a subset X of a group Γ, the set X^{-1} is defined as $\{x^{-1} \colon x \in X\}$. Hence, in the case that the connecting set is inverse–closed *ie.* $S = S^{-1}$, an arc (u, v) is in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ if, and only if, the arc (v, u)

is in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$. The *Cayley graph* $\mathsf{Cay}(\Gamma; S)$ is then defined as the Cayley digraph $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ with every pair of arcs (u, v) and (v, u) replaced by a single edge $\{u, v\}$. Hence the vertex set is Γ and the edge set is $\{\{u, v\} : u, v \in \Gamma, uv^{-1} \in S\}$.

The follow proposition summarises a number of textbook results for Cayley graphs and digraphs, respectively, which will be useful throughout this work.

Proposition 1.3.1. *Let S be a subset of a group Γ. Then,*

i. $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ *is strongly connected if, and only if, the connecting set S generates Γ;*

ii. $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ *is vertex–transitive;*

iii. $1 \in S$ *if, and only if, every vertex of* $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ *has a loop;*

iv. If $S = S^{-1}$ then for every arc (u, v) in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$, the arc (v, u) is also in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$.

Similarly, i. – iii. hold for the Cayley graph $\mathsf{Cay}(\Gamma; S \cup S^{-1})$.

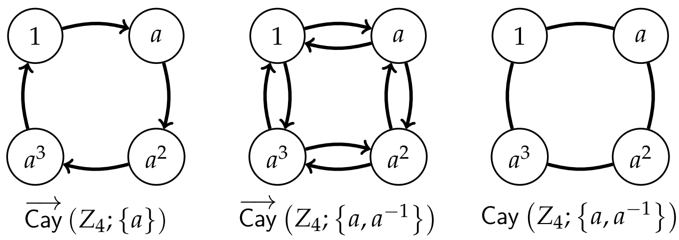


Figure 1.2: Different Cayley (di)graphs associated with the cyclic group Z_4 of order 4.

In the context of (oriented) Hamiltonian paths and cycles, disconnected (di)graphs are of zero interest. More so, loops have no effect on the Hamiltonicity of a (di)graph. Therefore, in light of Proposition 1.3.1, unless otherwise stated we shall assume that the connecting set S of a Cayley (di)graph is a generating set, and that $1 \notin S$.

Example 1.3.2. It is worth noting that not every vertex–transitive (di)graph is a Cayley (di)graph. Consider the (well known) *Petersen graph*, which is vertex–

transitive[3]. Suppose that there exists some group Γ with generating set S such that $\mathrm{Cay}\left(\Gamma; S \cup S^{-1}\right)$ is isomorphic to the Petersen graph, that is, suppose the Petersen graph is a Cayley graph.

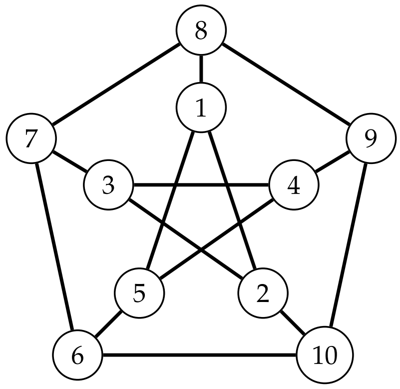


Figure 1.3: The Petersen graph.

The Petersen graph has 10 vertices and is 3–regular, hence $|\Gamma| = 10$ and $\left|S \cup S^{-1}\right| = 3$. Since the Petersen graph is connected then S must be a generating set, and since it is loop–less then $1 \notin S$. Hence either $S = \{a, b\}$ where b is an involution and $a \neq a^{-1}$ (and hence $S \cup S^{-1} = \{a, a^{-1}, b\}$), where as a consequence of Lagrange's Theorem we have that a has order 5, or $S = S \cup S^{-1} = \{x, y, z\}$ for three distinct involutions x, y and z.

Now, as a consequence of the classification of small groups, the only two groups of order 10 are the cyclic group Z_{10} and the dihedral group D_5. In the case that Γ is Z_{10}, there is only one involution and therefore $S = \{a, b\}$. Then we have that

$$(1)(1a)(1ab)(1aba^{-1})(1aba^{-1}b)$$

is a 4–cycle in $\mathrm{Cay}(\Gamma; S)$ since $1aba^{-1}b = 1$ by the Abelian condition.

Consider the case when Γ is D_5. We first show that no product of 5 involutions in D_5 is equal to 1. Suppose that s and r are two generators for D_5 satisfying the relations $s^5 = r^2 = 1$ and $rs = s^{-1}r$. From these relations, D_5 has 5

[3] This can be verified manually or using a computational tool such as *Mathematica* by running the command `GroupOrbits[GraphAutomorphismGroup[PetersenGraph[5, 2]]]`, which should return a single–orbit (and hence showing that the Petersen graph is vertex–transitive).

involutions, namely the reflections r, rs, rs^2, rs^3 and rs^4. The product of 5 involutions in D_5 then takes the form $rs^{j_1}rs^{j_2}rs^{j_3}rs^{j_4}rs^{j_5}$, which is equal to rs^j for $j = j_1 - j_2 + j_3 - j_4 + j_5$, by the relations $rs = s^{-1}r$ and $r^2 = 1$. Hence, for any integer j, $rs^j \neq 1$ and therefore the product of 5 involutions is not 1 in D_5.

Since the Petersen graph has a 5–cycle and no product of 5 involutions in D_5 is equal to 1, then it cannot be the case that $S = \{x, y, z\}$ as otherwise $\mathrm{Cay}(\Gamma; S)$ does not have any 5–cycles. Hence $S = \{a, b\}$ and since $a^5 = b^2 = 1$ then by the presentation of the group we have that $(ab)^2 = 1$ and therefore

$$(1)(1a)(1ab)(1aba)(1abab)$$

is a 4–cycle in $\mathrm{Cay}(\Gamma; S)$.

However, as one can observe from Figure 1.3, the Petersen graph does not have a 4–cycle. Hence the Petersen graph is a vertex–transitive graph which is not a Cayley graph.

In the context of Cayley (di)graphs, walks can be described as a series of multiplications of the elements in the connecting set. We shall use one of the various notations common in the literature, used in recent works by Witte. Given a group Γ and a connecting set $S \subseteq \Gamma$, by $(s_i)_{i=1}^{r}$ we denote the walk

$$(1)\,(s_1)\,(s_1s_2)\ldots(s_1s_2\ldots s_r)\,,$$

where $s_i \in S$ with repetition allowed. For any $g \in \Gamma$, $[g]\,(s_i)_{i=1}^{r}$ denotes the walk

$$(g)\,(gs_1)\,(gs_1s_2)\ldots(gs_1s_2\ldots s_r)$$

and given another walk $(t_i)_{i=1}^{k}$, $(s_i)_{i=1}^{r} + (t_i)_{i=1}^{k}$ denotes the walk

$$(1)\,(s_1)\ldots(s_1s_2\ldots s_r)\,(s_1s_2\ldots s_rt_1)\ldots(s_1s_2\ldots s_rt_1\ldots t_k)$$

where $s_i, t_i \in S$ and repetition is allowed as before. While we shall for the most part describe walks (and in particular paths) explicitly for the sake of clarity, for some of the techniques that we shall see, describing walks in such a manner simplifies the work greatly.

1.4 | Lovász's conjecture

We are now in a position to introduce one of the most active and intriguing problems related to Hamiltonicity. Since, in particular, a vertex–transitive graph is regular, then Theorem 1.2.5 gives that with high probability every vertex–transitive graph has a Hamiltonian path. Indeed, Theorem 1.2.6 extends this to vertex–transitive digraphs. We have also seen how the Petersen graph is vertex–transitive, and with some observation one can conclude from Figure 1.3 that the Petersen graph has a Hamiltonian path, but not a Hamiltonian cycle.

There are, in fact, only five known examples of vertex–transitive graphs without a Hamiltonian cycle. All of which, however, have a Hamiltonian path. This is summarised in the following conjecture, commonly attributed to Lovász. A recent survey by Pak and Radoičić (2009) however suggests that the origins of this conjecture trace back to the work of Rankin.

Conjecture 1.4.1 (Lovász's conjecture)**.** *Every finite connected vertex–transitive graph has a Hamiltonian cycle, besides the five known counterexamples which only have a Hamiltonian path: the complete graph on two vertices, the Petersen graph, the Coxeter graph, and the two graphs derived from the Petersen and Coxeter graphs by replacing each vertex with a 3–cycle.*

One of the five counterexamples, the Petersen graph, as we have seen is not a Cayley graph. Indeed, all of the five examples stated are not Cayley graphs. Given this observation and the fact that Cayley graphs are vertex–transitive, the following conjecture arises.

Conjecture 1.4.2. *Every finite connected Cayley graph has a Hamiltonian cycle.*

While the class of vertex–transitive graphs considered is now more restricted and also more appealing (given that Hamiltonian cycles can be seen as sequences of generators which do not yield repeated elements), the conjecture still remains wide open. Indeed, most of the work done towards Lovász's conjecture has been carried out in this setting of Cayley graphs. As we shall see, there are many strong results which suggest that Conjecture 1.4.2 holds. However, it is worth noting that Babai (1996) has conjectured a sharp contradiction, namely that for some $\epsilon > 0$, there exist infinitely many connected Cayley graphs on n vertices whose longest cycles do not exceed $(1-\epsilon)n$ in length.

We shall primarily concern ourselves with this more appealing and approachable conjecture, studying the development made towards an affirmative answer throughout the years. However, given that Cayley graphs are vertex–transitive, we shall at times also briefly consider Conjecture 1.4.1.

1.5 | Overview

In Chapter 2 we shall introduce a number of techniques and ideas that underpin both earlier results in this area of research, as well as modern day results. We shall also consider some immediate consequences of these ideas. Chapters 3 through 5 shall evaluate further on these techniques, giving a number of prominent results. Most notably we shall consider at length groups which have a generating set of size 2 and Abelian groups. One of the strongest results that we shall see in Chapter 2, due to Marušič (1983), being that for any finite Abelian group of order at least 3 and any generating set S of the group, the corresponding Cayley graph with connecting set $S \cup S^{-1}$ has a Hamiltonian cycle.

A natural question to ask is whether the conjecture can be restated in terms of Cayley *digraphs*. Consider $\mathbb{Z}_{+\ \mathrm{mod}\ 12}$ (the additive group of integers modulo 12), for which $a = 3$ and $b = 4$ are generators. Suppose that there is a potential oriented Hamiltonian cycle H in $\overrightarrow{\mathsf{Cay}}\left(\mathbb{Z}_{+\ \mathrm{mod}\ 12}; \{3, 4\}\right)$. Observe that for all i in $\left\langle ab^{-1} \right\rangle = \langle 11 \rangle$, either $(i, i+3)$ is in H and $(i, i+4)$ is not, or vice–versa. Since 11 generates the group, all the arcs in H must be of this form; but neither 3 nor 4 generate the group, so H must be the union of at least 2 disjoint oriented cycles.

Consequently then, H cannot be an oriented Hamiltonian cycle, and hence counterexamples to the digraph case do exist. Since the existence of a Hamiltonian cycle in $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S\right)$ implies the existence of a Hamiltonian cycle in the corresponding $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$, it is still worth studying the Hamiltonicity of Cayley digraphs in view of Conjecture 1.4.2. More so, the existence of a Hamiltonian path in a Cayley digraph implies the existence of a Hamiltonian path in some corresponding vertex–transitive graph (that being the corresponding Cayley graph), which in view of Conjecture 1.4.1 is also worth considering.

In Chapter 4 we provide a comprehensive treatment of the Hamiltonicity of Cayley digraphs for Abelian groups. In particular, we show that we can always find a generating set S for any Abelian group Γ guaranteeing that the Cayley digraph $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ has a Hamiltonian cycle.

In Chapter 5 we shall in particular discuss one of the more recent cutting edge results in this area, due to Pak and Radoičić (2009), namely that for any group Γ, we can construct a generating set S of size $|S| \leq \log_2 |\Gamma|$ such that the Cayley graph $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle.

We shall then tie everything together in Chapter 6, giving a comprehensive overview of the results presented, along with a number of open problems and directions for future research. More so, we shall also briefly (and non–exhaustively) survey a number of other prominent results in this area of research, which had to be left out for the sake of brevity.

In this manner, we hope to give the reader a concise overview of this exciting and active area of research, by following the development in techniques and results throughout the years, trying as much as possible to highlight how certain ideas permeate across a vast number of the results presented. Needless to say, we cannot state that this survey is exhaustive – however we hope that it serves the reader as a springboard for further research.

The Magical Mystery Tour
Is waiting to take you away
Waiting to take you away

John Lennon & Paul McCartney,
Magical Mystery Tour

2

Techniques and Hamiltonicity of Cayley graphs for all Abelian groups

In this chapter we shall discuss three techniques which underpin a number of the results that we shall see. We begin with a discussion on the quotient digraph of a Cayley digraph, and we study how for a group Γ generated by S and a subgroup Ψ of Γ, under certain conditions which we shall discuss later, the Hamiltonicity of $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S\right)/\Psi$ can be *'lifted'* to $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S\right)$. The technique, in particular, lends itself naturally to inductive constructions of Hamiltonian paths and cycles.

This is then followed by a discussion on the ideas of Rankin, most notably the *arc–forcing subgroup* $\left\langle S^{-1}S\right\rangle$ for a generating set S. As we shall see, the arc–forcing subgroup is one choice of subgroup that one can make to apply a variant of the aforementioned lifting technique, namely through Witte's *skewed generators argument*.

Lastly, we introduce the *generator extension* technique for the generating sets of Abelian groups, pioneered by Marušič (1983), which as an argument crops up in many constructive proofs for the existence of Hamiltonian paths and cycles in Cayley graphs and digraphs on Abelian groups.

2.1 | Vertex partitions and quotient digraphs

Let D be a digraph and let $\mathcal{V}$ be a partition of the vertex set V. The *quotient digraph* $\mathsf{D}/\mathcal{V}$ is defined as the digraph with vertex set $\mathcal{V}$, such that for any $V_1, V_2 \in \mathcal{V}$, (V_1, V_2) is an arc in $\mathsf{D}/\mathcal{V}$ if, and only if, for every $v_1 \in V_1$ there exists $v_2 \in V_2$ such that (v_1, v_2) is an arc in D. Note that this definition allows for loops, but not multi–edges.

Let Ψ be a subgroup of some group Γ. It is well known that the collection of right cosets $\mathcal{H} = \{\Psi g \colon g \in \Gamma\}$ is a partition of Γ. Hence, for any generating set S of Γ, we can consider the quotient digraph $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S\right)/\mathcal{H}$. With some abuse of notation[1], whenever there's no room for ambiguity, we shall write $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S\right)/\Psi$ instead of $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S\right)/\mathcal{H}$.

Let Λ be a normal subgroup of a group Γ and let S generate Γ. Note that we call the set $S^{(\Lambda)} := \{s\Lambda \colon s \in S\}$ the *projection* of S in Γ/Λ, where $S^{(\Lambda)}$ generates Γ/Λ. Note that if $S \cap \Lambda \neq \emptyset$, then $\Lambda \in S^{(\Lambda)}$ and hence for quotient digraphs we will allow the identity element to be part of the generating set (in accordance to our allowance of loops in the definition of a quotient digraph). In particular, if $S \cap \Lambda \neq \emptyset$ and $S^{(\Lambda)}$ is the connecting set for a Cayley digraph on Γ/Λ, the identity is in the connecting set. We have the following result which we state without proof.

Lemma 2.1.1. *Let Γ be a group and let Λ be a normal subgroup of Γ. Let S be a generating set for Γ. Then $\overrightarrow{\mathsf{Cay}}\left(\Gamma/\Lambda; S^{(\Lambda)}\right) \simeq \overrightarrow{\mathsf{Cay}}\left(\Gamma; S\right)/\Lambda$.*

2.1.1 | The 'lifting' lemma

It is relatively straightforward to see that if a finite vertex–transitive digraph D has a Hamiltonian path with source vertex u, then in particular for every vertex v, D has a Hamiltonian path with source vertex v. This is by virtue of the fact that there exists an automorphism mapping u to v while preserving adjacencies. In the case of a graph, v is a pendant vertex in at least one such path.

[1] The reason why shall become more apparent when we consider quotient groups Γ/Λ, for normal subgroups Λ of Γ.

Lemma 2.1.2. *Let* D *be a finite vertex–transitive digraph with an oriented Hamiltonian path, and let v be a vertex of* D*. Then v is a source vertex in some oriented Hamiltonian path in* D*.*

The following result on the Hamiltonicity of quotient digraphs, when paired with Lemma 2.1.1, shall give a useful corollary which we call the 'lifting' lemma, since it demonstrates how for a group Γ and a normal subgroup Ψ generated by S and $S_\Psi \subseteq S$, respectively, if $\overrightarrow{\mathsf{Cay}}\,(\Psi; S_\Psi)$ and $\overrightarrow{\mathsf{Cay}}\left(\Gamma/\Psi; S^{(\Psi)}\right)$ have an oriented Hamiltonian path then the oriented Hamiltonian path in $\overrightarrow{\mathsf{Cay}}\left(\Gamma/\Psi; S^{(\Psi)}\right)$ can be 'lifted' to an oriented Hamiltonian path in $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$.

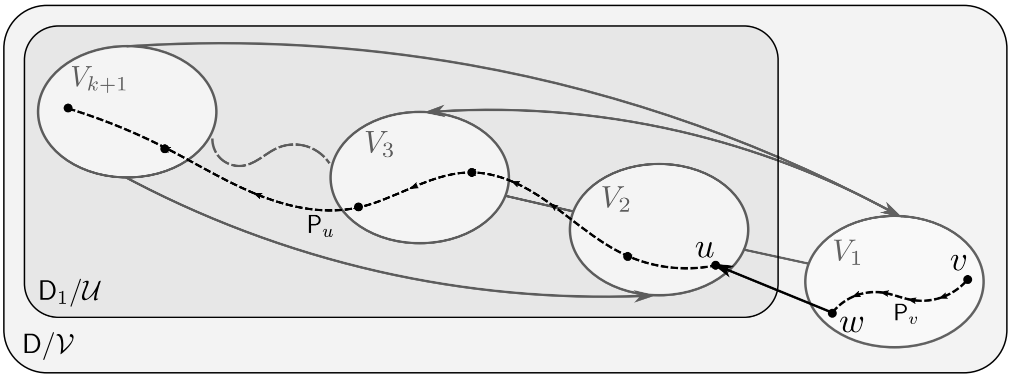


Figure 2.1: Inductive construction of the oriented Hamiltonian path in Proposition 2.1.3.

Proposition 2.1.3. *Let* D *be a digraph and let $\mathcal{V}$ be a partition of V, such that the subdigraph in* D *induced by each block is connected, vertex–transitive, and has an oriented Hamiltonian path. If* $\mathsf{D}/\mathcal{V}$ *has an oriented Hamiltonian path, then* D *has an oriented Hamiltonian path.*

Proof. We will show that if, in particular, $\mathsf{D}/\mathcal{V}$ has an oriented Hamiltonian path with source vertex V_1, then for any $v \in V_1$, D has an oriented Hamiltonian path with source vertex v. We proceed by induction on $N = |\mathcal{V}|$. For the base case when $N = 1$, the result follows by Lemma 2.1.2, since $\mathcal{V} = \{V\}$ and $\mathsf{D}[V] = \mathsf{D}$ is given to have an oriented Hamiltonian path.

Suppose the statement holds for some $N = k, k \in \mathbb{N}$. Let $\mathcal{V}$ be a partition of V with $N = k + 1$ blocks, where the subdigraph in D induced by each block is vertex–transitive and has an oriented Hamiltonian path. Furthermore, let $\mathsf{D}/\mathcal{V}$ have an oriented Hamiltonian path, which must be of length k. Let V_1 be the source vertex of this Hamiltonian path, with an arc to some other vertex V_2.

Let $v \in V_1$. By Lemma 2.1.2, the subdigraph in D induced by V_1 has an oriented Hamiltonian path P_v with v as a source vertex. Let $w \in V_1$ be the sink vertex of this oriented path P_v. By the definition of $\mathsf{D}/\mathcal{V}$, w must have an arc to some u in V_2.

Now, $\mathsf{D}_1 = \mathsf{D} - V_1$ is a digraph for which $\mathcal{U} = \mathcal{V} - \{V_1\}$ is a partition of its vertex set into k blocks. Consequently, $\mathsf{D}_1/\mathcal{U} = (\mathsf{D}/\mathcal{V}) - V_1$ and hence $\mathsf{D}_1/\mathcal{U}$ has an oriented path of length $k-1$ with source vertex V_2. By the inductive hypothesis, there exists an oriented Hamiltonian path P_u in D_1 with source vertex u.

It follows then that P_u and P_v joined by the arc (w, u) is a oriented Hamiltonian path in D with source vertex v. The result follows. □

Corollary 2.1.4 ('Lifting' lemma)**.** *Let Γ be a group and let Ψ be a normal subgroup of Γ. Let $S_\Psi \subseteq S$ such that S_Ψ and S generate Ψ and Γ, respectively. If $\overrightarrow{\mathsf{Cay}}(\Psi; S_\Psi)$ and $\overrightarrow{\mathsf{Cay}}\left(\Gamma/\Psi; S^{(\Psi)}\right)$ have an oriented Hamiltonian path, then so does $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$.*

Proof. Firstly note that the subdigraph in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ induced by each coset of Ψ is isomorphic to $\overrightarrow{\mathsf{Cay}}(\Psi; S_\Psi)$. Hence, the result follows by Lemma 2.1.1 and Proposition 2.1.3, noting by Proposition 1.3.1 that a Cayley digraph is vertex–transitive, and connected whenever the connecting set is a generating set. □

2.2 | Rankin's *'A campanological problem in group theory'*

Campanology is the art of bell ringing, with *change ringing* specifically being the practice of ringing n bells in *rounds*, where in each round every bell is rung exactly once (a permutation, termed as a *change*). There are further constraints to change ringing, namely:

i. The first and the last rounds must consist of the same change, corresponding to the permutation $(1\,2\ldots n)$.

ii. Except for the first and last, no two rounds must consist of the same change.

iii. From one change to the next, only bells which are neighbours in one change can be permuted to give the next change.

An interesting mathematical problem arises from this: What is the longest sequence of changes (or cycle) that can be rung? Observe the relation with Hamilton's problem, where here we are considering cycles amongst the elements of the symmetric group S_n, starting and ending at the cycle $(1\,2\ldots n)$. Rankin (1948, 1966) formally treated this problem from a group–theoretical perspective in his two seminal papers, *'A campanological problem in group theory'* and *'A campanological problem in group theory II'*.

In this chapter, we introduce Rankin's ideas in a graph theoretical context. As noted by Witte and Gallian (1984), and as will be exhibited by the proofs given in later chapters, the work of Rankin underpins many results on the Hamiltonicity of Cayley (di)graphs.

Let Γ be a group generated by $S = \{a, b\}$. We say that a vertex v in $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$ *travels* by a with respect to a spanning subdigraph H if the arc (v, va) is in H but (v, vb) is not. Similarly, we say that a subset of Γ travels by a with respect to a spanning subdigraph H if every element in the subset travels by a in H.

The essence of Rankin's ideas is the following: by a careful choice of a subgroup Ψ of Γ, such as the *arc–forcing subgroup* $\langle ab^{-1} \rangle$ or the cyclic group $\langle b \rangle$, under certain constraints, we consider by which elements cosets of Ψ travel in some subdigraph H of $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$, and construct Hamiltonian cycles in the Cayley digraph from these arcs.

2.2.1 | Arc–forcing subgroups

Let Γ be a group and let $S = \{a, b\}$ generate the group, where possibly $a = b^{-1}$ *ie.* $S = S^{-1}$. The *arc–forcing subgroup* of Γ associated with $S = \{a, b\}$ is the cyclic group $\langle ab^{-1} \rangle$ generated by ab^{-1}.

More generally, we can extend this definition of the arc–forcing subgroup to any generating set S, not necessarily of size 2. In this case, we define the arc–forcing subgroup as $\langle S^{-1}S \rangle = \langle a^{-1}b | a, b \in S \rangle$. Note that for any $s \in S$, we have that $\langle S^{-1}S \rangle = \langle s^{-1}S \rangle$.

Lemma 2.2.1. *Let* Γ *be a group generated by* S*. Then for any* $s \in S$, $\langle S^{-1}S \rangle = \langle s^{-1}S \rangle$.

Proof. By definition of $S^{-1}S$, $s^{-1}S \subseteq S^{-1}S$ and hence $\langle s^{-1}S \rangle \subseteq \langle S^{-1}S \rangle$. We show that $\langle S^{-1}S \rangle \subseteq \langle s^{-1}S \rangle$; by closure, it suffices to show that $S^{-1}S \subseteq \langle s^{-1}S \rangle$.

Consider $r^{-1}t \in S^{-1}S$, where $r^{-1} \in S^{-1}$ and $t \in S$. Then $s^{-1}r, s^{-1}t \in s^{-1}S$. Hence $r^{-1}s \in \langle s^{-1}S \rangle$ and by closure we have that $r^{-1}t = (r^{-1}s)(s^{-1}t)$ is in $\langle s^{-1}S \rangle$. Therefore $S^{-1}S \subseteq \langle s^{-1}S \rangle$ and the result follows. □

Example 2.2.2. To begin understanding the consequences of the arc–forcing subgroup, consider the dihedral group $D_4 = \left\langle a, b \middle| a^4 = b^2 = (ab^{-1})^2 = 1 \right\rangle$ as an example. In this case, with respect to the generating set $\{a, b\}$, the arc–forcing subgroup is $\langle ab^{-1} \rangle = \langle ab \rangle$, since $b = b^{-1}$. The left cosets of the arc–forcing subgroup in this case are:

$$\langle ab \rangle = \{1, ab\} \quad a \langle ab \rangle = \{a, a^2b\}$$
$$b \langle ab \rangle = \{b, a^3\} \quad a^2 \langle ab \rangle = \{a^2, a^3b\}$$

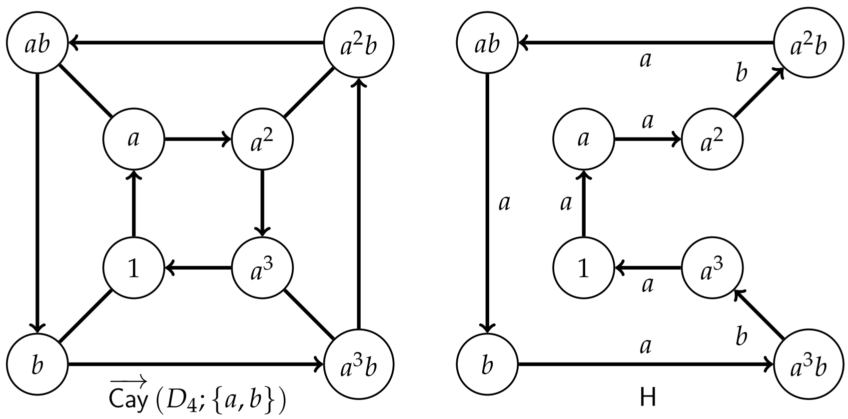


Figure 2.2: $\overrightarrow{\mathsf{Cay}}(D_4; \{a, b\})$ and a spanning cycle H of $\overrightarrow{\mathsf{Cay}}(D_4; \{a, b\})$, with arcs labelled with the generator by which the source vertex travels.

Consider the oriented Hamiltonian cycle H in $\overrightarrow{\mathsf{Cay}}\,(D_4;\{a,b\})$ as shown in Figure 2.2. Observe that for any left coset of the arc–forcing subgroup $\langle ab\rangle$, the elements in the coset all travel in H by the same generator: the elements of $\langle ab\rangle$, $a\,\langle ab\rangle$ and $b\,\langle ab\rangle$ travel by a in H, and the elements of $a^2\,\langle ab\rangle$ travel by b in H.

Suppose this was not the case; *eg.* consider $b\,\langle ab\rangle$ such that b travels by a and a^3 travels by b. Then a^3b would have in–degree at least 2 (since a^3b can be written in at least two ways, namely $b * a = a^3b = a^3 * b$), contradicting that every vertex of an oriented Hamiltonian cycle has in/out–degree 1.

As we shall see in the following chapter, it can be shown that if $\{a,b\}$ generates Γ and $\overrightarrow{\mathsf{Cay}}\,(\Gamma;\{a,b\})$ has an oriented Hamiltonian cycle H, all the vertices belonging to the same left coset of the arc–forcing subgroup $\langle ab^{-1}\rangle$ travel by the same generator, either a or b, relative to H. More so, as a consequence of the definition of the arc–forcing subgroup, if for every left coset all the vertices travel by the same generator a or b on a connected spanning subdigraph H of Γ, then H is an oriented Hamiltonian cycle in $\overrightarrow{\mathsf{Cay}}\,(\Gamma;\{a,b\})$.

2.2.1.1 | Arc–forcing subgroups and the 'lifting' lemma

The strength of the arc–forcing subgroup and the ideas of Rankin become more apparent when one considers them in conjunction with the ideas of Witte (1986), namely through a variant of the 'lifting' technique known as the *'skewed generators argument'*. For the sake of brevity, we shall not discuss Witte's technique to its full extent here. However as an intermediate step we consider the following simpler variant, given in Lemma 2.6 of Morris (2011).

Theorem 2.2.3 (Morris (2011))**.** *Let Γ be a group generated by S and let $\Psi = \langle S^{-1}S\rangle$ be the arc–forcing subgroup of Γ with respect to S. If for any generating set T of Ψ, $\overrightarrow{\mathsf{Cay}}\,(\Psi;T)$ has an oriented Hamiltonian path and there is an oriented Hamiltonian cycle in $\overrightarrow{\mathsf{Cay}}\,(\Gamma;S)\,/\Psi$, then $\overrightarrow{\mathsf{Cay}}\,(\Gamma;S)$ has an oriented Hamiltonian path.*

Proof. Let $(\Psi s_i)_{i=1}^{m}$ be an oriented Hamiltonian cycle in $\overrightarrow{\mathsf{Cay}}\,(\Gamma;S)\,/\Psi$, for $s_i \in S$, amongst the $m-1$ cosets of Ψ. Then $s_1s_2\ldots s_m = 1$ and hence $s_m^{-1} = s_1s_2\ldots s_{m-1}$. By Lemma 2.2.1, it follows that $\Psi = \langle s_m^{-1}S\rangle$. In particular then,

$\overrightarrow{\mathsf{Cay}}\left(\Psi; s_m^{-1}S\right)$ has an oriented Hamiltonian path $\left(s_m^{-1}t_i\right)_{i=1}^{r}$. Consequently, under concatenation (recalling the notation introduced at the end of Section 1.3),

$$\left(\sum_{i=1}^{r-1}(s_1, s_2, \ldots, s_{m-1}, t_i)\right) + (s_1, s_2, \ldots, s_{m-1})$$

is an oriented Hamiltonian path in $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S\right)$, as required. □

Observe how the idea here differs from the previous lifting technique which we have seen; namely, for any $h \in \Psi$ we are traversing the path $h \cdot (s_i)_{i=1}^{m-1}$ *ie.* we are sequentially visiting each right coset of Ψ once relative to the same element h, then travelling to some other element ht_j in Ψ and repeating. This differs from the lifting technique we saw in Corollary 2.1.4, where we were first travelling across *all* vertices belonging to the same coset sequentially and then visiting another coset and repeating.

More so, in this case we drop the requirement that the subgroup is normal. Indeed, the arc–forcing subgroup need not necessarily be normal; consider $\langle ab \rangle$ for D_4 in our previous example. We have that:

$$a\langle ab \rangle = \left\{a, a^2 b\right\} \neq \{a, b\} = \langle ab \rangle a$$

and hence $\langle ab \rangle$ is not normal in D_4.

2.3 | The ‘generator extension’ lemma

In this chapter and Chapter 4, we shall give an in–depth treatment of the Hamiltonicity of Cayley (di)graphs on Abelian groups. Marušič (1983) pioneered the work towards positive results in this area, by considering for a non–empty proper subset T of a generating set S for an Abelian group, under what conditions the Hamiltonicity of $\mathsf{Cay}\left(\langle T \rangle; T \cup T^{-1}\right)$ extends to the Hamiltonicity of $\mathsf{Cay}\left(\langle S \rangle; S \cup S^{-1}\right)$. We shall consider the ideas of Marušič (1983) in a more general context, namely that of Cayley digraphs.

Example 2.3.1. Consider $\Gamma = \left\langle a, b | a^4 = b^2 = 1, ab = ba \right\rangle$. Let $S = \{a, b\}$ and $T = \{b\}$. Let $\Psi = \langle T \rangle$. Since Ψ is cyclic, $\overrightarrow{\mathsf{Cay}}\left(\Psi; T\right)$ has an oriented Hamiltonian cycle. Let H_i be the subdigraph of $\overrightarrow{\mathsf{Cay}}(\Gamma;S)$ induced by the coset Ψa^i, where each H_i is isomorphic to $\overrightarrow{\mathsf{Cay}}(\Psi;T)$ and hence also have an oriented Hamiltonian cycle, as shown in Figure 2.3.

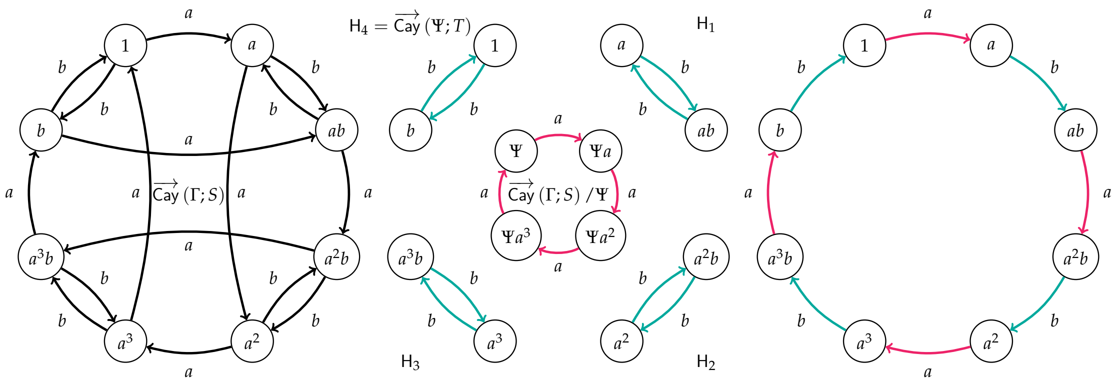


Figure 2.3: Construction of an oriented Hamiltonian cycle in $\overrightarrow{\mathsf{Cay}}(\Gamma;S)$ from the Hamiltonicity of $\overrightarrow{\mathsf{Cay}}(\Psi;T)$ and $\overrightarrow{\mathsf{Cay}}(\Gamma;S)/\Psi$, where: $\Gamma = \langle a,b|a^4 = b^2 = 1, ab = ba\rangle$, $S = \{a,b\}$, $T = \{b\}$ and $\Psi = \langle T\rangle$.

Observe in Figure 2.3 how the quotient digraph $\overrightarrow{\mathsf{Cay}}(\Gamma;S)/\Psi$ also has an oriented Hamiltonian cycle. Suppose we start at b in H_4; by Hamiltonicity we know that we can visit all the vertices in H_4 exactly once – namely, we have an arc from b to 1. By a we can traverse to another coset, hence we can visit H_1 from H_4 by the arc from 1 to a. Repeating, we can visit all the vertices in H_1 exactly once. Indeed by the Hamiltonicity of $\overrightarrow{\mathsf{Cay}}(\Gamma;S)/\Psi$, we can visit all the cosets via a exactly once, and traverse all their vertices exactly once, respectively.

In this manner, we have an oriented Hamiltonian path; however observe in Figure 2.3 that when traversing from one coset to another by a, we have an arc from some element of the form a^jb^k to an element of the form $a^{j+1}b^k$. We then visit all the other vertices in the coset, namely $a^{j+1}b^{k+1}$, from which we can traverse to another coset by a. This yields the element $a^{j+2}b^{k+1}$; the other element in this coset is $a^{j+2}b^{k+2} = a^{j+2}b^k$. Observe that since the order of b is 2, after visiting two cosets via a, we have traversed from a^jb^k to $a^{j+2}b^k$, where $a^{j+2}b^k$ has an out–going arc (via a) to another coset.

Consequently, if the order of b divides the number of cosets of Ψ (since if a in general has order m, then after travelling between m cosets we go from $a^j b^k$ to $a^{j+m} b^k$), then we start and end at the same vertex and hence we have an oriented Hamiltonian cycle. This is illustrated in Figure 2.3.

Knowing that both $\overrightarrow{\mathsf{Cay}}\,(\Psi; T)$ and $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)\,/\Psi$ are Hamiltonian, the ‘lifting’ lemma allows us to immediately conclude that there is an oriented Hamiltonian path in $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$. However, what we have shown is even stronger – the existence of an oriented Hamiltonian cycle. Marušič (1983) observed that given any Abelian group Γ with a generating set S, $s \in S$ and $T = S - \{s\}$, if the order of $\Psi = \langle T \rangle$ divides the order of s and $\overrightarrow{\mathsf{Cay}}\,(\Psi; T)$ has an oriented Hamiltonian cycle, then so does $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$.

We summarise these ideas of Marušič (1983) in the lemmas below, which we refer to as the *‘generator extension’ lemmas* (for obvious reasons). The proof technique developed here is a significant departure from that used for the ‘lifting’ lemma, even if both seemingly bear a relation – this is however necessary in order to show the existence of (oriented) Hamiltonian cycles.

In their original paper, Marušič (1983) consider these ideas for Cayley graphs of Abelian groups, however Gallian (2012) gives (through a series of proofs and examples) a generalisation to Cayley digraphs. We shall first give a detailed proof of the digraph case, filling in any ‘gaps’ which might not be obvious to non–experts. In a similar manner, a proof for the case of a graph will immediately follow.

Lemma 2.3.2 (‘Generator extension’ lemma for digraphs, Gallian (2012))**.**
Let Γ be an Abelian group and let S be a generating set of Γ. Let $s \in S$ be of order $m \in \mathbb{N}$, and define $T := S - \{s\}$. Let Ψ be the subgroup of Γ generated by T. If $\overrightarrow{\mathsf{Cay}}\,(\Psi; T)$ has an oriented Hamiltonian cycle, then:

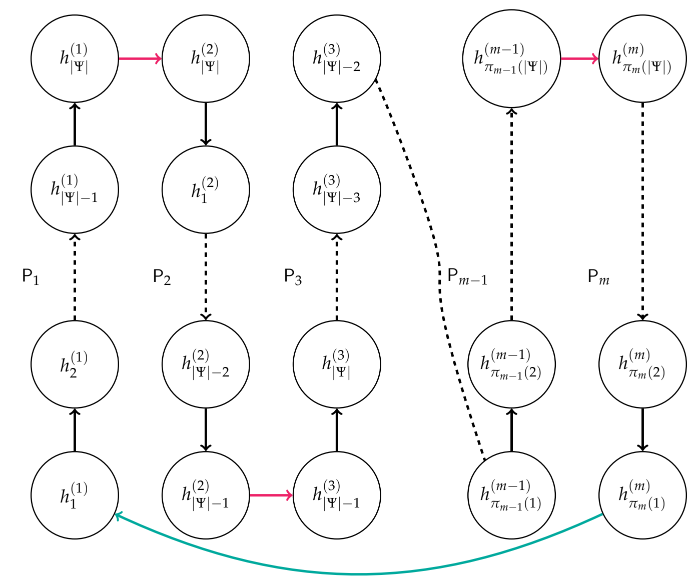
i. $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$ has an oriented Hamiltonian path;

ii. Furthermore, if $|\Psi|$ divides m then $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$ has an oriented Hamiltonian cycle.

Proof. Let the oriented Hamiltonian cycle in $\overrightarrow{\mathsf{Cay}}\,(\Psi; T)$ be $h_1 h_2 \ldots h_{|\Psi|} h_1$. Since Γ is Abelian, the cosets of Ψ are $\Psi s, \ldots, \Psi s^{m-1}$, and Ψs^m which form the quotient group Γ/Ψ and hence partition Γ. For $1 \le i \le m$, let H_i be the subdigraph of $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ induced by Ψs^i. We shall adopt some shorthand notation, namely we shall write hs^i as $h^{(i)}$. Also define $\phi_i \colon \Psi \to \Psi s^i$, $h \mapsto h^{(i)}$, where ϕ_i is an isomorphism such that H_i is isomorphic to $\overrightarrow{\mathsf{Cay}}(\Psi; T)$, with adjacency is preserved by commutativity. Consequently, $h_1^{(i)} h_2^{(i)} \dots h_{|\Psi|}^{(i)} h_1^{(i)}$ is an oriented Hamiltonian cycle in H_i.

For $1 \le k \le |\Psi|$, define the permutations $\pi_1(k) := k$ and for $2 \le i \le m$,

$$\pi_i(k) := (\pi_{i-1}(|\Psi|) + k - 1) \mod |\Psi|.$$



a

Figure 2.4: Construction of Hamiltonian paths in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$; if $|\Psi|$ is a factor of m then the arc in blue can be added to extend these paths to a Hamiltonian cycle.

We construct a Hamiltonian path in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ as follows:

i. Consider the oriented Hamiltonian paths $\mathsf{P}_i := h^{(i)}_{\pi_i(1)} h^{(i)}_{\pi_i(2)} h^{(i)}_{\pi_i(3)} \dots h^{(i)}_{\pi_i(|\Psi|)}$ in H_i. Note that by the cyclic nature of π_i, then P_i is indeed an oriented Hamiltonian path in H_i, by virtue of the oriented Hamiltonian cycles in H_i.

ii. For $1 \le i < m - 1$, the oriented Hamiltonian paths P_i and P_{i+1} can be extended to an oriented Hamiltonian path on the subdigraph of $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$

induced by $\Psi s^i \cup \Psi s^{i+1}$, by adding the arc $h^{(i)}_{\pi_i(|\Psi|)} \to h^{(i+1)}_{\pi_{i+1}(1)}$ between the sink of P_i and the source of P_{i+1}.

These arcs are indeed in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$; firstly note that

$$\pi_{i+1}(1) = (\pi_i(|\Psi|) + 1 - 1) \mod |\Psi| = \pi_i(|\Psi|),$$

and hence $h^{(i)}_{\pi_i(|\Gamma|)}$ and $h^{(i+1)}_{\pi_{i+1}(1)}$ correspond to the same element h in Ψ. In other words, we must check if the arc $hs^i \to hs^{i+1}$ is in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$. This is indeed the case, since s is in the generating set S of the Cayley digraph. By adding these arcs, we get an oriented Hamiltonian path as shown in Figure 2.4. This proves the first part of the result.

Now suppose that there exists $q \in \mathbb{N}$ such that $m = q|\Psi|$. We show that $\pi_m(|\Psi|) = \pi_1(1)$. Indeed, observe that by a simple recurrence argument we get that $\pi_i(|\Psi|) = (1 - i) \mod |\Psi|$ and hence:

$$\pi_m(|\Psi|) = (1 - m) \mod |\Psi| = (1 - q|\Psi|) \mod |\Psi| = 1 \mod |\Psi| = \pi_1(1),$$

as required. Hence $h^{(m)}_{\pi_m(|\Psi|)}$ and $h^{(1)}_{\pi_1(1)}$ correspond to the same element h in Ψ, and since s is in the generating set S of the Cayley digraph then the arc $hs^m \to hs$ is in $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$. Hence the oriented Hamiltonian path in the first part of the proof can be extended to an oriented Hamiltonian cycle by the arc $hs^m \to hs$, proving the second part of the result. $\square$

The proof in the case of a graph follows similarly; we sketch the proof below, based on the notation and technique developed in the digraph case.

Lemma 2.3.3 (‘Generator extension’ lemma for graphs, Marušič (1983)). *Let Γ be an Abelian group and let S be a generating set of Γ. Let $s \in S$ be of order $m \in \mathbb{N}$, and define $T := S - \{s\}$. Let Ψ be the subgroup of Γ generated by T. If* $\mathsf{Cay}\left(\Psi; T \cup T^{-1}\right)$ *has a Hamiltonian cycle, then so does* $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$.

Proof. Firstly note that in this case we shall ignore the orientation of the chosen Hamiltonian cycles in H_i. Secondly, we have that the edges

$$\mathcal{E} = \left\{ \left\{ h^{(2k-1)}_{|\Psi|}, h^{(2k)}_{|\Psi|} \right\} : 1 \le k \le \left\lfloor \frac{m}{2} \right\rfloor \right\} \bigcup \left\{ \left\{ h^{(2k)}_{2}, h^{(2k+1)}_{2} \right\} : 1 \le k \le \left\lfloor \frac{m-1}{2} \right\rfloor \right\}$$

are in $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ since $s \in S$ and we have $h_j^{(i)} s = h_j^{(i+1)}$ for $1 \leq i < m$ and $1 \leq j \leq |\Psi|$. Define $\mathsf{P}' = h_1^{(1)} h_1^{(2)} \ldots h_1^{(m)}$ and

$$\mathsf{P}_i = \begin{cases} h_1^{(1)} h_2^{(1)} \ldots h_{|\Psi|}^{(1)}, & i = 1 \\ h_2^{(i)} h_3^{(i)} \ldots h_{|\Psi|}^{(i)}, & 2 \leq i < m \\ h_1^{(m)} h_2^{(m)} \ldots h_{|\Psi|}^{(m)}, & i = m, m \text{ even} \\ h_2^{(m)} \ldots h_{|\Psi|}^{(m)} h_1^{(m)}, & i = m, m \text{ odd.} \end{cases}$$

Then, as illustrated in Figure 2.5, $\mathsf{P}' \cup \left(\bigcup_{i=1}^{m} \mathsf{P}_i \right) + \mathcal{E}$ is a Hamiltonian cycle in $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$, completing the proof. $\square$

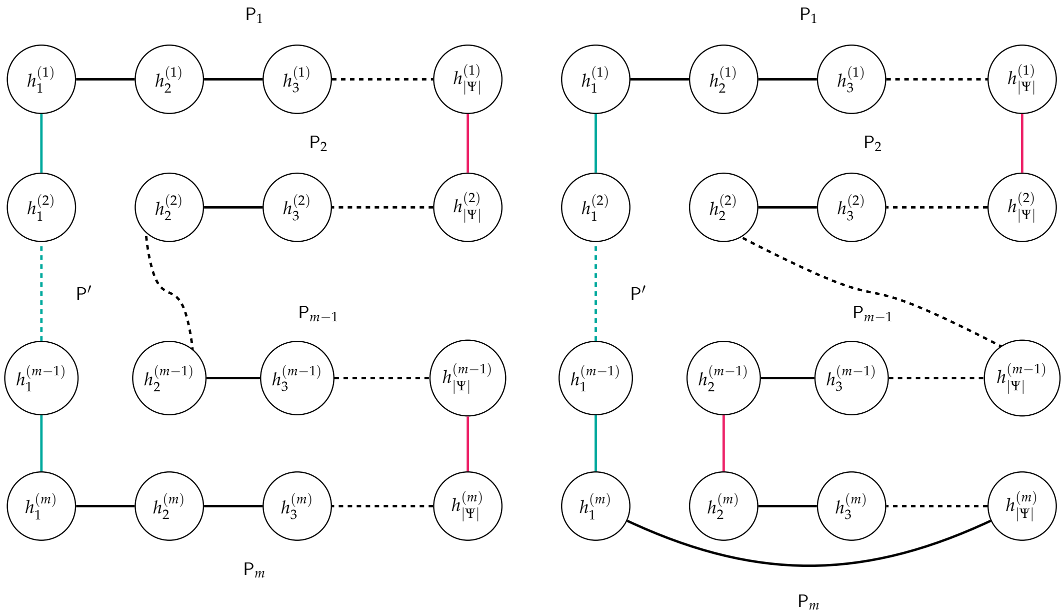


Figure 2.5: Construction of Hamiltonian cycles in $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ for when m is even and m is odd. The edges in $\mathcal{E}$ are highlighted in red, while the path P' is highlighted in blue.

As an immediate consequence of the 'generator extension' lemma for graphs (Lemma 2.3.3), we have the following result due to Marušič (1983) – arguably one of the strongest in this area of research, proven in an elegant manner.

Theorem 2.3.4 (Marušič (1983))**.** *For any finite Abelian group* Γ *of order at least* 3 *and any generating set* S *of* Γ*,* $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ *has a Hamiltonian cycle.*

Proof. Suppose that S contains an element a of order at least 3. By the cyclic nature of $\langle a \rangle$, there exists a Hamiltonian cycle in $\mathsf{Cay}\left(\langle a \rangle; \left\{a, a^{-1}\right\}\right)$. Consider the case when S does not have an element of order at least 3; then since Γ has order at least 3, S must contain at least two (distinct) elements a and b of order 2. Observe that $\mathsf{Cay}\left(\langle a, b \rangle; \{a, b\}\right)$ has a Hamiltonian cycle, namely (by commutativity) the cycle

$$(1)(a)(ab)(aba)(abab).$$

In either case, by the recursive application of the ‘generator extension’ lemma for graphs (Lemma 2.3.3), the result follows by extension of the Hamiltonian cycles in these subgraphs to Hamiltonian cycles in $\mathsf{Cay}\left(\Gamma; S\right)$. $\square$

Two of us wearing raincoats
Standing solo in the sun
You and me chasing paper
Getting nowhere on our way back home

Paul McCartney,
Two of Us

3

Hamiltonicity of $\overrightarrow{\mathsf{Cay}}\left(\Gamma;S\right)$ and $\mathsf{Cay}\left(\Gamma;S\cup S^{-1}\right)$ with $|S|=2$

3.1 | Arc–forcing subgroups revisited

We begin by formalising some of the consequences discussed in the previous chapter about Rankin's arc–forcing subgroups, in the context of groups with a generating set of size 2. We shall assume that whenever a generating set has size 2, then the two generators are distinct *ie.* if $S=\{a,b\}$, then $a\neq b$. However, unless otherwise stated, it is possible that $a=b^{-1}$ (in which case the group is cyclic). Throughout this chapter, we shall in particular prove a number of results given in Rankin's two seminal papers, using techniques from graph and group theory. In other cases, we shall state and discuss consequences of Rankin's work. We begin by formalising the ideas behind the example given in Figure 2.2.

Proposition 3.1.1 (Rankin (1948))**.** *Let Γ be a finite group generated by $\{a,b\}$ and $\Psi=\langle ab^{-1}\rangle$ the arc–forcing subgroup of Γ with respect to $\{a,b\}$. If $\overrightarrow{\mathsf{Cay}}\left(\Gamma;\{a,b\}\right)$ has an oriented Hamiltonian cycle H, then every left coset of Ψ travels by either a or b in H.*

Proof. Let $g\in\Gamma$ and consider the left coset $g\Psi$. Let $m\in\mathbb{N}$. Suppose, without loss of generality, that $g\left(ab^{-1}\right)^{m}$ travels by b in H. Then H has an arc from

$g\left(ab^{-1}\right)^m$ to $g\left(ab^{-1}\right)^m b$. Now,

$$g\left(ab^{-1}\right)^m b = g\left(ab^{-1}\right)^{m-1} ab^{-1}b = g\left(ab^{-1}\right)^{m-1} a$$

where $g\left(ab^{-1}\right)^m$ and $g\left(ab^{-1}\right)^{m-1}$ are distinct as otherwise $a = b$.

Since H is an oriented Hamiltonian cycle, hence a spanning subdigraph such that each vertex has in–degree 1, it follows that $g\left(ab^{-1}\right)^{m-1}$ must travel by b as well, as otherwise $g\left(ab^{-1}\right)^m b = g\left(ab^{-1}\right)^{m-1} a$ would have in–degree at least 2. Repeating the argument, it follows that every vertex in $g\Psi$ must travel by b in H. A similar argument follows for a. $\square$

The following Proposition 3.1.2, for which we give a new proof, gives a 'partial converse' of sorts to the previous Proposition 3.1.1.

Proposition 3.1.2 (Rankin (1948))**.** *Let Γ be a finite group generated by $\{a, b\}$, $\Psi = \langle ab^{-1}\rangle$ the arc–forcing subgroup of Γ with respect to $\{a, b\}$, and H a spanning subdigraph of $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$. If every left coset of Ψ travels by either a or b in H, then each component of H is an oriented cycle. In particular if H is connected, then $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$ has an oriented Hamiltonian cycle.*

Proof. We show that the in–degree and out–degree in H of every vertex is 1. Note that in this case, $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$ is 2–regular and hence in H the in–degree and out–degree of any vertex is bounded above by 2.

Since any vertex of H must belong to a left coset of Ψ and every such coset travels by either a or b in H, then every vertex must travel by either a or b in H, which is to say that for any g in Γ, $\deg_{\text{out}}(g) = 1$ in H. Consequently by the Handshaking Lemma for Digraphs, the sum of in/out–degrees in H gives

$$\sum_{g\in\Gamma} \deg_{\text{in}}(g) = |\Gamma|.$$

Letting n_0 and n_2 be the number of vertices of in–degree 0 and 2 in H, respectively; we have that:

$$\sum_{g\in\Gamma} \deg_{\text{in}}(g) = 0 \cdot n_0 + 1 \cdot (|\Gamma| - n_0 - n_2) + 2 \cdot n_2 = |\Gamma| - n_0 + n_2$$

where if we substitute in the previous equation and simplify, we get that $n_0 = n_2$. Hence the number of vertices with in–degree 0 and in–degree 2 in H must be equal.

Suppose there exists a vertex with in–degree 2 in H. Since every vertex belongs to a left coset of Ψ, then there exist $g \in \Gamma$ and $m \in \mathbb{N}$ such that $g\left(ab^{-1}\right)^m$ has in–degree 2 in H. In particular then, there exist $g_1, g_2 \in \Gamma$ and $m_1, m_2 \in \mathbb{N}$ such that:

$$g_1\left(ab^{-1}\right)^{m_1} a = g\left(ab^{-1}\right)^{m} = g_2\left(ab^{-1}\right)^{m_2} b$$

and consequently $g_1\left(ab^{-1}\right)^{m_1}$ and $g_2\left(ab^{-1}\right)^{m_2}$ travel by a and b in H, respectively, to $g\left(ab^{-1}\right)^m$. Note that since elements in the same left coset of Ψ can only travel by a single generator in H, it follows that $g_1 \neq g_2$.

Rearranging, we have that $g_1\left(ab^{-1}\right)^{m_1-m_2+1} = g_2$ and hence g_2 is an element of the left coset $g_1\Psi$. Since cosets are either equal or disjoint, it follows that $g_1\Psi = g_2\Psi$ and therefore this coset travels by both a and b, a contradiction. Hence no such vertex with in–degree 2 may exist in H. It follows that $n_0 = n_2 = 0$.

Consequently, every vertex in H must have in–degree 1 and, as previously shown, out–degree 1. Since oriented cycles are characterised by having in–degree and out–degree 1 for every vertex, it follows that the components of H are oriented cycles. In the case that H has exactly one component, then $\overrightarrow{\mathrm{Cay}}\left(\Gamma;\{a,b\}\right)$ has an oriented Hamiltonian cycle since H spans $\overrightarrow{\mathrm{Cay}}\left(\Gamma;\{a,b\}\right)$. $\square$

3.2 | Generating sets $\{a, b\}$ with $(ab)^2 = 1$

In this section we shall be concerned with groups, such as the dihedral groups D_n, which have a generating set $\{a, b\}$ such that ab is an involution. Note that in this case, by the definition of an involution, $a \neq b^{-1}$. In particular, we shall constructively prove (by means of an algorithm) a result of Rankin that for such groups having such a generating set $S = \{a, b\}$, $\mathrm{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle. The proof is based on the ideas of Pak and Radoičić (2009); we fill in a number of gaps and prove an intermediary result, Lemma 3.2.3, to clarify the proof argument. Employing similar techniques, the authors also prove the following results.

Theorem 3.2.1 (Pak and Radoičić (2009)). *Let Γ be a finite group generated by $S = \{a, b, a^{-1}ba\}$ such that $b^2 = 1$. Then $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle.*

Theorem 3.2.2 (Pak and Radoičić (2009); Rapaport-Strasser (1959)). *Let Γ be a finite group generated by $\{a, b, c\}$ such that $a^2 = b^2 = c^2 = 1$ and $ab = ba$. Then $\mathsf{Cay}(\Gamma; \{a, b, c\})$ has a Hamiltonian cycle.*

Let $X \subseteq \Gamma$. Define $\partial_g(X) = \{g' \in \Gamma - X \colon g' = xg, x \in X\} = Xg - X$, where $Xg = \{xg \colon x \in X\}$ (using coset notation; however X need not be a subgroup of Γ, but rather it can be any subset of Γ).

Lemma 3.2.3. *Let Γ be a finite group generated by $\{a, b\}$ and let X be a union of left cosets of $\langle b \rangle$. If $\partial_a(X) = \partial_{a^{-1}}(X) = \emptyset$, then $X = \Gamma$.*

Proof. Let $x \in X$. Since X is the union of cosets of $\langle b \rangle$, then there exists $y \in \Gamma$ such that $x = yb^m$ for some $m \in \mathbb{N}$. In particular then, $y\langle b \rangle \subseteq X$. By closure in $\langle b \rangle$, xb and xb^{-1} are in $y\langle b \rangle \subseteq X$. Moreover, since $\partial_a(X) = \partial_{a^{-1}}(X) = \emptyset$, it follows that xa and xa^{-1} are also in X.

Hence, all the vertices incident to and from x in $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$ are in X. By the arbitrariness of $x \in X$, it follows that the subdigraph induced by X in $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$ is in fact a component of $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$. However, by Proposition 1.3.1, a Cayley digraph with the group generating set as the connecting set is always connected, therefore the component induced by X is in fact $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$. Since the vertex set of $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$ is Γ, it follows that $X = \Gamma$. $\square$

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of sets and let $X = \bigcup_{i \in \mathbb{N}} X_i$. We write $X_i \uparrow X$ whenever $(X_i)_{i \in \mathbb{N}}$ is an *increasing* sequence, in the sense that $X_i \subseteq X_{i+1}$, which 'converges' to X.

Theorem 3.2.4 (Pak and Radoičić (2009); Rankin (1966)). *Let Γ be a finite group generated by $S = \{a, b\}$ such that ab is an involution. Then $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle.*

Proof. By the relation between the arcs of $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S \cup S^{-1}\right)$ and the edges of $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$, if $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S \cup S^{-1}\right)$ has an oriented Hamiltonian cycle, then $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle. Hence we shall consider $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S \cup S^{-1}\right)$.

We will proceed by '*iteratively*' constructing a sequence of sets (X_i) such that $(X_i) \uparrow \Gamma$, $X_1 = \langle b \rangle$, and for $i \geq 2$: X_i is the (disjoint) union of X_{i-1} and some coset of $\langle b \rangle$. At each i^{th} iteration, we show that the subgraph of $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S \cup S^{-1}\right)$ induced by X_i contains an oriented Hamiltonian cycle C_i, the labels of which are either b or a^{-1} (known as the '*label conditions*'). Note that since Γ is finite, then so is the number of cosets of $\langle b \rangle$, and hence '*termination*' is guaranteed in a finite number of iterations.

Let $X_1 = \langle b \rangle$. Since $\langle b \rangle$ is isomorphic to a cyclic group, then the subdigraph of $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S \cup S^{-1}\right)$ induced by X_1 has an oriented Hamiltonian cycle, call it C_1. Furthermore all the labels on C_1 are b, satisfying the label conditions.

Suppose that the construction holds up to the i^{th} iteration. Consider $\partial_a(X_i) \neq \emptyset$ and let $y = xa \in \partial_a(X_i)$. We show that the construction holds for the $(i+1)^{\text{th}}$ iteration. Note that the arc incident to $x \in X_i$ in C_i cannot have the labels a and b^{-1} by the label conditions, and neither the label a^{-1} (otherwise the arc is (y, x) where $y \notin X_i$ since $\partial_a(X_i) \subseteq \Gamma - X_i$). Hence this arc incident to x must be labelled b, and is therefore the arc $\left(xb^{-1}, x\right) \in \mathsf{C}_i$.

Now consider the oriented cycle C in the subdigraph of $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S \cup S^{-1}\right)$ induced by $y\langle b \rangle$. Observe that:

$$x \xrightarrow{a} xa = y \xrightarrow{b} xab = yb \xrightarrow{a} yba = xaba = xb^{-1} \xrightarrow{b} x$$

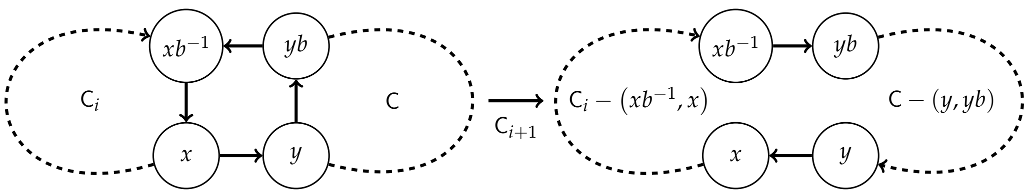


Figure 3.1: Iterative enlargement of the oriented cycle C_i to C_{i+1}.

is an oriented 4–cycle connecting C_i and C (see Figure 3.1), by virtue of ab being an involution. In particular, we can then consider the arcs $\left(xb^{-1}, yb\right)$ and (y, x), since the connecting set $S \cup S^{-1}$ is inverse–closed. Inheriting the orientation of C_i and defining:

$$\mathsf{C}_{i+1} = \mathsf{C}_i \cup \mathsf{C} + (y, x) + \left(xb^{-1}, yb\right) - \left(xb^{-1}, x\right) - (y, yb),$$

we have an oriented Hamiltonian cycle on $X_{i+1} = X_i \cup y\langle b\rangle$. Since the arcs in C_i satisfy the label conditions, the arcs $\left(xb^{-1}, yb\right)$ and (y, x) are labelled a^{-1} and the arcs in C are labelled b, then C_{i+1} satisfies the label conditions as required.

In the case that $\partial_a(X_i) = \emptyset$ and $\partial_a(X_i) \neq \partial_{a^{-1}}(X_i)$, we consider $y = xa^{-1} \in \partial_{a^{-1}}(X_i)$. The proof for this case follows similarly, where instead we consider the arc incident *from* (instead of *to*) x in C_i.

Lastly, if $\partial_a(X_i) = \partial_{a^{-1}}(X_i) = \emptyset$ then $X_i = \Gamma$ by Lemma 3.2.3, and C_i is an oriented Hamiltonian cycle in $\overrightarrow{\mathrm{Cay}}\left(\Gamma; S \cup S^{-1}\right)$. This concludes the proof. $\square$

You say, "Yes", I say, "No"
You say, "Stop" and I say, "Go, go, go"
Oh no

You say, "Goodbye" and I say, "Hello, hello, hello"

Paul McCartney,
Hello, Goodbye

4

Hamiltonicity of Cayley digraphs on Abelian groups

The work of Marušič (1983) on the Hamiltonicity of all Cayley graphs for Abelian groups, as mentioned earlier, strongly supports the conjecture of Lovász on the Hamiltonicity of all Cayley graphs. Unsurprisingly, this spurred numerous attempts to extend the work, wherever possible, of Marušič (1983) to the Hamiltonicity of Cayley *digraphs* on Abelian groups. Consequently, in this chapter we shall give an in–depth treatment of the Hamiltonicity of Cayley digraphs on Abelian groups.

4.1 | A potpourri of non–Hamiltonian Cayley digraphs

We begin by outlining a consequence of Rankin's techniques with regards to finite Abelian groups having a generating set of size 2, as considered by Rankin (1948) himself. In particular, given a finite Abelian group Γ and a generating set $\{a, b\}$ where $a \neq b$, Rankin gave a complete characterisation of which such Cayley digraphs $\overrightarrow{\mathsf{Cay}}(\Gamma; \{a, b\})$ have an oriented Hamiltonian cycle. Consequently, this gives rise to an infinitely large class of non–Hamiltonian Cayley digraphs.

Theorem 4.1.1 (Rankin (1948)). *Let Γ be a finite Abelian group generated by $\{a, b\}$, where $a \neq b$. Let $m = \dfrac{|\Gamma|}{|\langle ab^{-1} \rangle|}$ and let $s \in \mathbb{Z}$ such that $b^m = \left(ab^{-1}\right)^s$. Then $\overrightarrow{\mathsf{Cay}}\left(\Gamma; \{a, b\}\right)$ has an oriented Hamiltonian cycle if, and only if, there exists $k \in \mathbb{Z}$ such that $\gcd\left(k, \circ\left(ab^{-1}\right)\right) = 1$ and $s \leq k \leq s + m$.*

Example 4.1.2. Indeed, consider $\mathbb{Z}_{+ \bmod 12}$ once again, with generators $a = 3$ and $b = 4$. Then $b^{-1} = 8$ and hence $\langle ab^{-1} \rangle = \langle 11 \rangle = \mathbb{Z}_{+ \bmod 12}$. We will once again show, this time using Rankin's result on the arc–forcing subgroup, that $\overrightarrow{\mathsf{Cay}}\left(\mathbb{Z}_{+ \bmod 12}; \{3, 4\}\right)$ does not have an oriented Hamiltonian cycle. Let m, s and k have the same meaning as in Theorem 4.1.1. Then $m = 1$ and for $s = 8$ we have $b = \left(ab^{-1}\right)^s$. Hence the possible values of k are either 8 or 9. Consequently, $\gcd(8, 12) = 4$ and $\gcd(9, 12) = 3$ and therefore by Theorem 4.1.1, $\overrightarrow{\mathsf{Cay}}\left(\mathbb{Z}_{+ \bmod 12}; \{3, 4\}\right)$ does not have an oriented Hamiltonian cycle.

More so, we can also construct an infinite class of digraphs which do not have an oriented Hamiltonian path, let alone an oriented Hamiltonian cycle.

Theorem 4.1.3 (Morris (2013)). *For every $n \in \mathbb{N}$, there exists a group Γ with generating set $\{a, b\}$ such that the order a and b is at least n, respectively, and $\overrightarrow{\mathsf{Cay}}\left(\Gamma; \{a, b\}\right)$ does not have an oriented Hamiltonian path.*

Since Proposition 1.3.1 asserts that every Cayley digraph is connected when the connecting set is a generating set, and also vertex–transitive, then Theorems 4.1.1 and 4.1.3 assert that there are infinitely large classes of finite Cayley digraphs which are non–Hamiltonian. In particular then, there exist infinitely large classes of finite connected vertex–transitive digraphs which are non–Hamiltonian. Because of the existence of these instances, variants of Lovász's conjecture cannot be stated in terms of Cayley digraphs; however no such examples are known to exist for Cayley graphs and hence Conjecture 1.4.2 is still open and constitutes of an active area of research.

4.2 | Hamiltonicity of Cayley digraphs on Abelian groups

In our treatment of the 'generator extension' lemma, we have already seen one of the most celebrated results in the study of the Hamiltonicity of Cayley graphs, due to Marušič (1983), namely that a Cayley graph on an Abelian group (of order at least 3) has a Hamiltonian cycle whenever *any* generating set of the group is used as the connecting set. Observe also how this result strongly supports (even if it requires the additional constraint of commutativity) Conjecture 1.4.2 in the positive, in particular since Abelian groups constitute a large class of non–trivial groups.

While the existence of an oriented Hamiltonian cycle in the Cayley digraph on an Abelian group is not always necessarily guaranteed, the existence of a Hamiltonian path is guaranteed for any choice of generating set as the connecting set. Indeed, this is summarised in the following (stronger) result by Holsztyński and Strube (1978)[1]. Firstly note that a *Dedekind group* is a group such that every subgroup is normal. Every Abelian group is Dedekind, however the converse is not necessarily true. Indeed, non–Abelian Dedekind groups are called *Hamiltonian groups*, with the smallest such example being the quaternion group

$$Q_8 = \left\langle a, b | a^4 = 1, a^2 = b^2, ba = a^{-1}b \right\rangle.$$

Theorem 4.2.1 (Holsztyński and Strube (1978))**.** *For any finite Dedekind group Γ and any generating set S of Γ, $\overrightarrow{\mathrm{Cay}}\,(\Gamma; S)$ has an oriented Hamiltonian path.*

Proof. We proceed by induction on the size $|S|$ of S. If $|S| = 1$, then Γ is generated by a single element s and hence $\overrightarrow{\mathrm{Cay}}\,(\Gamma; \{s\})$ has an oriented Hamiltonian path.

Suppose that the result holds for some $|S| = k \in \mathbb{N}$. Let $|S| = k + 1$ and fix $s \in S$. Define $S' = S - \{s\}$ and $\Lambda = \langle S' \rangle$. Since Λ is a subgroup of Γ, then any subgroup Ψ of Λ is also a subgroup of Γ. Hence since Γ is Dedekind, Ψ is a normal subgroup of Γ and hence also of Λ. Therefore Λ is also Dedekind.

[1] Note that in the original paper, this result is attributed to Holsztyński and Nathanson. It is worth noting that the setting of the paper is primarily in group and number theory.

Since $|S'| = k$, by the inductive hypothesis, $\overrightarrow{\text{Cay}}(\Lambda; S')$ has an oriented Hamiltonian path. Moreover, since Λ is normal in Γ, we can consider the quotient group Γ/Λ which is generated by $\{s\Lambda\}$. Hence, as before, $\overrightarrow{\text{Cay}}(\Gamma/\Lambda; \{s\Lambda\})$ has an oriented Hamiltonian cycle, and in particular an oriented Hamiltonian path. The result follows by the 'lifting lemma' of Corollary 2.1.4. □

More could be said however for cyclic groups of prime–power order; indeed, Holsztyński and Strube (1978) have shown that, for any generating set as the connecting set, the Cayley digraph on a cyclic group has an oriented Hamiltonian cycle if, and only if, the group has prime–power order. We will restrict ourselves to proving one direction of this statement.

Proposition 4.2.2 (Holsztyński and Strube (1978))**.** *Let p be prime and $n \in \mathbb{N}$. Then for any generating set S of Z_{p^n}, $\overrightarrow{\text{Cay}}\left(Z_{p^n}; S\right)$ has an oriented Hamiltonian cycle.*

Proof. Let $r = |S|$. If $r = 1$, then the result follows immediately. Hence suppose $r \geq 2$. Each element s of S must have order p^m for some $m \leq n$. Hence we can enumerate the elements of S as $s_1, s_2, \ldots, s_r$ such that each s_i has order p^{m_i} and for $1 \leq i < r$ we have that p^{m_i} divides $p^{m_{i+1}}$.

Let $\Psi_i = \langle s_1, s_2, \ldots, s_i \rangle$ for $1 \leq i < r$, which in particular is cyclic (since the subgroups of a cyclic group are cyclic). Then Ψ_i is generated by a single element; in particular, by the Abelian condition, any element in Ψ_i can be written as $s_1^{k_1} s_2^{k_2} \ldots s_i^{k_i}$. Since for $1 \leq j \leq i$ we have that p^{m_j} divides p^{m_i}, then

$$\left(s_1^{k_1} s_2^{k_2} \ldots s_i^{k_i}\right)^{p^{m_i}} = 1$$

and therefore the order of every element in Ψ_i does not exceed p^{m_i}. Hence, since Ψ_i is cyclic and $s_i \in \Psi_i$, we have that $|\Psi_i| = p^{m_i}$.

Therefore we have that $|\Psi_i|$ divides the order of s_{i+1}. The result follows by iterative application of the 'generator extension' lemma (Lemma 2.3.2). □

4.2.1 | Connecting sets guaranteeing Hamiltonicity

In the previous chapter, we have seen how the Cayley digraph $\overrightarrow{\text{Cay}}(\mathbb{Z}_{+ \bmod 12}; \{3, 4\})$ does not have an oriented Hamiltonian cycle. However, since 1 has order 12 in the group, the Cayley digraph $\overrightarrow{\mathsf{Cay}}\,(\mathbb{Z}_{+\ \mathrm{mod}\ 12}; \{1\})$ does have an oriented Hamiltonian cycle. Observe then that the Hamiltonicity of a Cayley digraph also depends on the choice of connecting set. In this section, we shall define a framework within which we can construct an appropriate generating set S for any Abelian group Γ, such that $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$ has an oriented Hamiltonian cycle. What is of interest is how the result of Holsztyński and Strube (1978) that we have seen in the previous section, namely that the Cayley digraph on any Abelian group has a Hamiltonian path whenever the connecting set is a generating set, does not extend to the existence of Hamiltonian cycles.

Remark 4.2.3. We briefly outline the extension of generating sets for two groups Γ and Ψ, to a generating set of their direct product $\Gamma \times \Psi$. Let S_Γ and S_Ψ be two generating sets for the groups $\Gamma = \left\langle S_\Gamma | r_1^\Gamma = 1, \ldots, r_s^\Gamma = 1 \right\rangle$ and $\Psi = \left\langle S_\Psi | r_1^\Psi = 1, \ldots, r_t^\Psi = 1 \right\rangle$, respectively. Define the set product $\times_{\mathsf{id}}$ as:

$$S_\Gamma \times_{\mathsf{id}} S_\Psi := (S_\Gamma \times \{1_\Psi\}) \cup (\{1_\Gamma\} \times S_\Psi)\,.$$

Note that by the properties of the Cartesian product and union of sets, $\times_{\mathsf{id}}$ is associative. One can then observe that the direct product group $\Gamma \times \Psi$ is generated by $S_\Gamma \times_{\mathsf{id}} S_\Psi$.

Proposition 4.2.4. *Let Γ be an Abelian group generated by S and let Z_n be a cyclic group of order $n \in \mathbb{N}$ generated by $\{a\}$. If $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$ has a Hamiltonian cycle, then:*

i. $\overrightarrow{\mathsf{Cay}}\,(\Gamma \times Z_n; S \times_{\mathsf{id}} \{a\})$ *has a Hamiltonian path.*

ii. If $|\Gamma|$ divides n, then $\overrightarrow{\mathsf{Cay}}\,(\Gamma \times Z_n; S \times_{\mathsf{id}} \{a\})$ *has a Hamiltonian cycle.*

Proof. Firstly note that a^n is the identity in Z_n, hence $\Gamma \times \{a^n\}$ is isomorphic to Γ and is generated by $S \times \{a^n\}$. Consequently, $\overrightarrow{\mathsf{Cay}}\,(\Gamma \times \{a^n\}\,; S \times \{a^n\})$ is isomorphic to $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$. Hence $\overrightarrow{\mathsf{Cay}}\,(\Gamma \times \{a^n\}\,; S \times \{a^n\})$ has a Hamiltonian cycle. Now $S \times_{\mathsf{id}} \{a\} = S \times \{a^n\} \cup \{(1_\Gamma, a)\}$ *ie.* the extension of $S \times \{a^n\}$ by the additional generator $(1_\Gamma, a)$ yields a generating set for $\Gamma \times Z_n$. Hence by the 'generator extension' lemma for digraphs (Lemma 2.3.2), the result follows. □

Before constructively proving that there exists a generating set S for any Abelian group Γ such that $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$ has an oriented Hamiltonian cycle, we state without proof the following textbook result from group theory[2].

Theorem 4.2.5 (Fundamental Theorem of Finite Abelian Groups)**.** *Any finite Abelian group is isomorphic to a direct product of cyclic groups*

$$Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_k}$$

where m_i divides m_{i+1} for $1 \leq i < k$.

Theorem 4.2.6. *For any Abelian group Γ, there exists a generating set S such that $\overrightarrow{\mathsf{Cay}}\,(\Gamma; S)$ has an oriented Hamiltonian cycle.*

We base our proof on that given by Stelow (2017), as well as the examples given in Gallian (2012).

Proof. By the Fundamental Theorem of Finite Abelian Groups (Theorem 4.2.5), it suffices to prove the result for the direct product of $k \in \mathbb{N}$ cyclic groups

$$Z_{m_1} \times Z_{m_2} \cdots \times Z_{m_k}$$

, where m_i divides m_{i+1} for $1 \leq i < k$. We shall proceed by induction on k.

Firstly, note that for any of the cyclic groups Z_{m_i}, there exists an element a_i of order m_i which generates the group. Define the set

$$S := \{a_1\} \times_{\mathsf{id}} \{a_2\} \times_{\mathsf{id}} \cdots \times_{\mathsf{id}} \{a_k\}$$

which by Remark 4.2.3 generates $Z_{m_1} \times \cdots \times Z_{m_k}$. We show that S is a generating set such that $\overrightarrow{\mathsf{Cay}}\,(Z_{m_1} \times \cdots \times Z_{m_k}; S)$ has an oriented Hamiltonian cycle.

The base case for when $k = 1$ holds, since $\overrightarrow{\mathsf{Cay}}\,(Z_{m_1}; \{a_1\})$ is an oriented Hamiltonian cycle. Suppose the result holds up to some $k \in \mathbb{N}$; we show that it holds for $k + 1$.

Let $\Gamma = Z_{m_1} \times \cdots \times Z_{m_k}$. Define the set $S' := \{a_1\} \times_{\mathsf{id}} \{a_2\} \times_{\mathsf{id}} \cdots \times_{\mathsf{id}} \{a_k\}$, which generates Γ. Also define the set $S := S' \times_{\mathsf{id}} \{a_{k+1}\}$, which generates

[2] See Armstrong (1997) for a proof and discussion on this result.

$\Gamma \times Z_{m_{k+1}}$. By the inductive hypothesis, $\overrightarrow{\mathrm{Cay}}\,(\Gamma; S')$ has an oriented Hamiltonian cycle, and since m_i divides m_{i+1} for $1 \leq i \leq k$, it follows that $|\Gamma| = \prod_{i=1}^{k} m_i$ is a factor of m_{k+1}. Hence by Proposition 4.2.4, it follows that $\overrightarrow{\mathrm{Cay}}\,(\Gamma \times Z_{m_{k+1}}; S)$ has an oriented Hamiltonian cycle. The result follows. $\square$

More generally, Curran and Witte (1985) have made the following conjecture for *minimal* generating sets of Abelian groups. A minimal generating set S is a generating set of Γ such that for any proper subset T of S, T does not generate Γ.

Conjecture 4.2.7 (Curran and Witte (1985))**.** *For any finite Abelian group Γ and any minimal generating set S of Γ with $|S| \geq 3$, $\overrightarrow{\mathrm{Cay}}\,(\Gamma; S)$ has an oriented Hamiltonian cycle.*

The long and winding road
That leads to your door
Will never disappear
I've seen that road before
It always leads me here
Lead me to you door

John Lennon & Paul McCartney,
The Long and Winding Road

5

Existence of small connecting sets guaranteeing Hamiltonian paths

This chapter shall be dedicated to one of the most recent 'cutting edge' developments, due to Pak and Radoičić (2009), in the study of Hamiltonicity of Cayley graphs. The proof relies on ideas both old and new, and will unify a number of the ideas presented in the previous chapters. Without further ado, we state the result in full.

Theorem 5.1 (Pak and Radoičić (2009)) *Every finite group Γ of size $|\Gamma| \geq 3$ has a generating set S of size $|S| \leq \log_2 |\Gamma|$, such that $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle.*

We shall give our proof of the following weaker version of the above.

Theorem 5.1 (Weak Version) (Pak and Radoičić (2009)) *Every finite group Γ has a generating set S of size $|S| \leq \log_2 |\Gamma|$, such that $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian path.*

Example 5.0.1. We have indeed already seen examples of such generating sets. Recall how in Theorem 4.2.6, we explicitly constructed for any finite Abelian group Γ a generating set S such that $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ has an oriented Hamiltonian cycle, and consequently $\mathsf{Cay}(\Gamma; S \cup S^{-1})$ has a Hamiltonian cycle, when $|\Gamma| \geq 3$. We will show that $|S| \leq \log_2 |\Gamma|$.

By the Fundamental Theorem of Finite Abelian Groups, Γ is isomorphic to a direct product of cyclic groups $Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_k}$, where m_i divides m_{i+1} for $1 \leq i < k$. Recall that S was constructed by associating a generator with each of these cyclic groups, hence $|S| = k$.

The *prime omega function* $\Omega(n)$ counts, for any natural number n, the number of prime factors of n up to multiplicity. More formally, if the (unique) prime factorisation of n is $\prod_{i=1}^{r} p_i^{\alpha_i}$ for distinct primes p_i, then $\Omega(n) = \sum_{i=1}^{r} \alpha_i$. Observe that $\Omega(n) \leq \log_2 n$.

Since $|\Gamma| = \prod_{i=1}^{k} m_i$, then $k \leq \Omega(|\Gamma|)$ and therefore $|S| \leq \log_2 |\Gamma|$, as required. Hence Theorem 4.2.6 gives an explicit construction of a generating set in accordance with Theorem 5.1.

5.1 | Subnormal series of groups

Let Γ be a group and let $(\Psi_i)_{i=0}^{r}$ be a sequence of subgroups of Γ. We say that $(\Psi_i)_{i=0}^{r}$ is a *subnormal series of* Γ if $\Psi_0 = \{1\}$, $\Psi_r = \Gamma$, and for $0 \leq i < r$: Ψ_i is a normal subgroup of Ψ_{i+1}. More compactly, we write

$$\{1\} = \Psi_0 \lhd \Psi_1 \lhd \Psi_2 \lhd \cdots \lhd \Psi_r = \Gamma.$$

Note that only Ψ_r need be a normal subgroup of Γ; in the case that every Ψ_i is a normal subgroup of Γ, we term $(\Psi_i)_{i=0}^{r}$ as a *normal series of* Γ.

We say that a subgroup Ψ of Γ is *proper* if Ψ is a proper subset of Γ. Moreover, a proper subgroup Ψ of Γ is said to be *maximal* if there does not exist another proper subgroup $\Delta \neq \Psi$ of Γ containing Ψ, in other words there for does exist a subgroup Δ satisfying $\Psi \subset \Delta \subset \Gamma$. Consequently, we have that for a normal subgroup Λ of Γ, if Λ is maximal, then the quotient group Γ/Λ is *simple*[1] (meaning that its normal subgroups are only the trivial ones: $\{\Lambda\}$ and Γ/Λ itself).

For a subnormal series $(\Lambda_i)_{i=0}^{r}$ of a group Γ, if for $0 \leq i < r$ we have that Λ_i is a maximal subgroup of Λ_{i+1}, then the series is, in particular, termed as a *composition series*. By our previous remark, the quotient groups Λ_{i+1}/Λ_i are simple, and we term them as the *composition factors* of the series.

[1] Indeed, the converse is also true: If Γ/Λ is simple, then Λ is maximal.

It is straightforward to see that every group Γ has a subnormal series; for example $\{1\} \lhd \Gamma$ (which for a simple group is the only subnormal series, and is in particular a composition series). Indeed, we have the following result which we state without proof.

Proposition 5.1.1. *Every finite group Γ has a composition series.*

More so, we have the following classical result on composition series and factors of groups.

Theorem 5.1.2 (Jordan–Hölder Theorem)**.** *If $(\Lambda_i)_{i=0}^{r}$ and $(\Psi_j)_{j=0}^{t}$ are two composition series of a finite group Γ, then $r = t$ and for every composition factor Λ_{i+1}/Λ_i there exists a composition factor Ψ_{j+1}/Ψ_j such that the two are isomorphic (and hence composition factors are unique up to permutation).*

Remark 5.1.3. For a composition series $(\Lambda_i)_{i=0}^{r}$ of Γ, let $\mathcal{K}(\Lambda_i)_{i=0}^{r}$ and $\mathcal{L}(\Lambda_i)_{i=0}^{r}$ denote the sets of Abelian and non–Abelian composition factors; similarly, let $k(\Lambda_i)_{i=0}^{r}$ and $l(\Lambda_i)_{i=0}^{r}$ be the number of Abelian and non–Abelian composition factors. For brevity, we shall simply write $\mathcal{K}, \mathcal{L}, k$ and l.[2] Observe that as a consequence of Lagrange's Theorem, we have that $\left(\prod_{K\in\mathcal{K}} |K|\right)\left(\prod_{L\in\mathcal{L}} |L|\right) = |\Gamma|$. Moreover, as a consequence of the *Classification of Small Finite Groups*, since the smallest non–Abelian group has order 6, we have that every $L \in \mathcal{L}$ has size at least 4. Hence,

$$2^{k+2l} = 2^k 4^l \leq \left(\prod_{K\in\mathcal{K}} |K|\right)\left(\prod_{L\in\mathcal{L}} |L|\right) = |\Gamma|$$

and therefore, for any composition series of Γ, we have that $k + 2l \leq \log_2 |\Gamma|$.

5.2 | Hamiltonicity of Cayley graphs on non–Abelian finite simple groups

We shall briefly note a consequence of the *Classification of Finite Simple Groups*, a decades long effort to classify all finite simple groups, which will be central

[2]Indeed, observe that k and l are invariant for any composition series of Γ, as a consequence of the Jordan–Hölder Theorem.

in the proof of Theorem 5.1. By invoking the ideas of Rankin, we shall see that every non–Abelian finite simple group Γ has a generating set $S = \{a, b\}$ such that $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle.

Theorem 5.2.1 (King (2017)). *Every non–Abelian finite simple group Γ is generated by an involution and an element of prime order.*

Corollary 5.2.2. *Every non–Abelian finite simple group Γ has a generating set $S = \{a, b\}$ such that $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle.*

Proof. By Theorem 5.2.1, let $\{a, c\}$ be a generating set of Γ such that a is of prime order and c is an involution. Let $b = a^{-1}c$. Then $ab = c$ and hence ab is an involution. By the definition of b and the fact that $\{a, c\}$ generates Γ, then $S = \{a, b\}$ also generates Γ. As a consequence of Rankin's work, by Theorem 3.2.4, it follows that $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle. □

5.3 | Proof of Theorem 5.1 (Weak Version)

Before giving a rigorous proof, we shall first begin with a simple example to illustrate the main ideas of the techniques employed. In essence, for a group Γ we will consider a composition series $(\Lambda_i)_{i=0}^{r}$ of length r, and its factors.

We also consider, for $1 \leq i \leq r$, the careful selection of a generating set S_i for each maximal normal subgroup Λ_i in the composition series, such that $|S_i| \leq \log_2 |\Gamma|$. These generating sets will be determined from the generating sets of the respective composition factors.

We will then consider the Hamiltonicity of the associated Cayley graphs $\mathsf{Cay}\left(\Lambda_i; S_i \cup S_i^{-1}\right)$ along with the respective quotient digraphs associated with the composition factors. Using our 'lifting' lemma we will iteratively 'lift' a Hamiltonian path from $\mathsf{Cay}\left(\Lambda_i; S_i \cup S_i^{-1}\right)$ to $\mathsf{Cay}\left(\Lambda_{i+1}; S_{i+1} \cup S_{i+1}^{-1}\right)$, eventually obtaining a Hamiltonian path in $\mathsf{Cay}\left(\Gamma; S_r \cup S_r^{-1}\right)$.

Example 5.3.1. Consider the alternating group A_4 of even permutations on $\{1, 2, 3, 4\}$, which is not Abelian since $(1\ 2\ 3)$ and $(1\ 2)(3\ 4)$ do not commute.

One can check that $\Lambda_2 = \{(), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is a maximal normal subgroup of A_4. More over, $\Lambda_1 = \{(), (1\,2)(3\,4)\}$ is a maximal normal subgroup of Λ_2. Therefore,

$$\Lambda_0 = \{()\} \lhd \Lambda_1 \lhd \Lambda_2 \lhd A_4 = \Lambda_3$$

is a composition series of A_4.

Part I: Generating sets for the composition factors

The composition factors Λ_1/Λ_0 and Λ_2/Λ_1 both have order 2, while Λ_3/Λ_2 has order 3. Since groups of prime order are cyclic, it follows that each composition factor is Abelian and can be generated by a single element. Indeed, one can check that Λ_1/Λ_0 is generated by $T_0 = \{(1\,2)(3\,4)\Lambda_0\}$, Λ_2/Λ_1 is generated by $T_1 = \{(1\,3)(2\,4)\Lambda_1\}$, and Λ_3/Λ_2 is generated by $T_2 = \{(1\,2\,3)\Lambda_2\}$.

By the cyclic nature of each composition factor and the respective choice of generating set, each $\mathrm{Cay}\left(\Lambda_{i+1}/\Lambda_i; T_i \cup T_i^{-1}\right)$ has a Hamiltonian path.

Part II: Existence of a generating set S_3 of size $\leq \log_2 |A_4|$

Consider the elements of $T_i = \{s_i\Lambda_i\}$ where $s_i \in \Lambda_{i+1}$. With each T_i associate the set $R_i = \{s_i\}$; so consider $R_0 = \{(1\,2)(3\,4)\}$, $R_1 = \{(1\,3)(2\,4)\}$ and $R_2 = \{(1\,2\,3)\}$. We can then define $S_1 = R_1$ and $S_{i+1} = S_i \dot\cup R_{i+1}$ for $1 \leq i < 3$. Hence $S_1 = \{(1\,2)(3\,4)\}$, $S_2 = \{(1\,2)(3\,4), (1\,3)(2\,4)\}$, and $S_3 = \{(1\,2)(3\,4), (1\,3)(2\,4), (1\,2\,3)\}$.

With some checking, one can easily see that each S_i is a generating set for Λ_i, $1 \leq i \leq 3$. In particular, S_3 is a generating set for A_4, where

$$|S_3| = 3 < \log_2 |A_4| = \log_2 12 \simeq 3.585$$

and hence by considering suitably small generating sets for each of the composition factors, we have constructed a generating set for A_4 with size $\leq \log_2 |A_4|$, as required.

Remark 5.3.2. Observe that each T_i is the projection of S_{i+1} in Λ_{i+1}/Λ_i, with Λ_i removed, which is to say $T_i = S_{i+1}^{(\Lambda_i)} - \{\Lambda_i\}$. Hence the Cayley graph associated with T_i is the Cayley graph associated with $S_{i+1}^{(\Lambda_i)}$ without the loops due to Λ_i, and therefore the Hamiltonicity of one is related to that of the other.

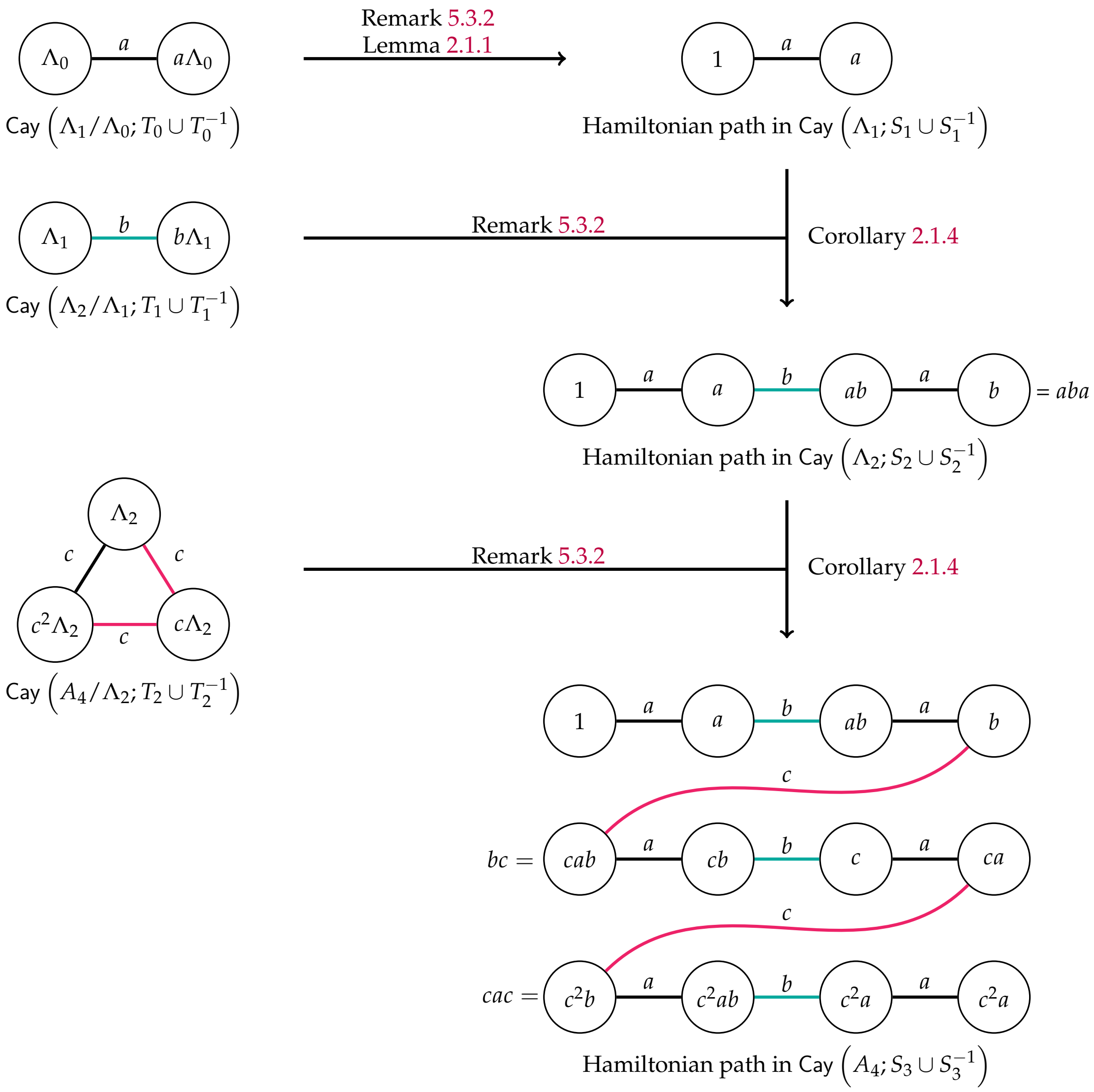


Figure 5.1: The iterative construction of a generating set $S_3 = \{a, b, c\}$ for A_4 and a Hamiltonian path in Cay $\left(A_4; S_3 \cup S_3^{-1}\right)$, from a composition series of A_4.

Part III: Existence of a Hamiltonian path in Cay $\left(A_4; S_3 \cup S_3^{-1}\right)$

What remains is the construction of a Hamiltonian path in Cay $\left(A_4; S_3 \cup S_3^{-1}\right)$. This is illustrated in Figure 5.1, where for brevity $a = (1\ 2)(3\ 4)$, $b = (1\ 3)(2\ 4)$ and $c = (1\ 2\ 3)$.

In particular, it follows from the repeated application of Remark 5.3.2 and of the 'lifting' lemma (Corollary 2.1.4), that if $\mathsf{Cay}\left(\Lambda_{i+1}/\Lambda_i; T_i \cup T_i^{-1}\right)$ and $\mathsf{Cay}\left(\Lambda_i; S_i \cup S_i^{-1}\right)$ have a Hamiltonian path, then so does $\mathsf{Cay}\left(\Lambda_{i+1}; S_{i+1} \cup S_{i+1}^{-1}\right)$.

Note that for simplicity, we carefully chose our example such that all the composition factors are Abelian, however this need not be the case. Indeed, in the proof for Theorem 5.1 we must consider the case when a composition factor is non–Abelian. We are now in a position to prove our main result for this chapter. In particular, the proof we present aims at filling in a number of 'gaps' in the original proof which are not necessarily obvious to non–experts.

Theorem 5.1 (Weak Version) (Pak and Radoičić (2009)) *Every finite group Γ has a generating set S of size $|S| \leq \log_2 |\Gamma|$, such that* $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ *has a Hamiltonian path.*

Proof. Fix a composition series $(\Lambda_i)_{i=0}^r$ of Γ. Then, by Remark 5.1.3, we have that $k + 2l \leq \log_2 |\Gamma|$. Let $0 \leq i < r$ and consider the composition factor Λ_{i+1}/Λ_i.

Part I: Generating sets for the composition factors

If Λ_{i+1}/Λ_i is Abelian, then in particular it is cyclic since it is simple (otherwise, if the group is generated by at least two generators, each generator corresponds to a proper normal subgroup by the Abelian condition – a contradiction).

Hence for every Abelian composition factor Λ_{i+1}/Λ_i, we can associate a generating set T_i, $|T_i| = 1$, such that $\mathsf{Cay}\left(\Lambda_{i+1}/\Lambda_i; T_i \cup T_i^{-1}\right)$ has a Hamiltonian path. Since only Λ_0 has size 1, it follows that the generator is of order at least 2 and hence the identity Λ_i is not in T_i.

Otherwise, if Λ_{i+1}/Λ_i is non–Abelian, then by Corollary 5.2.2 we can associate with the composition factor a generating set T_i, $|T_i| = 2$, such that $\mathsf{Cay}\left(\Lambda_{i+1}/\Lambda_i; T_i \cup T_i^{-1}\right)$ has a Hamiltonian path. Note that $\Lambda_i \notin T_i$, as otherwise Λ_{i+1}/Λ_i can be generated by a single element, contradicting that it is non–Abelian.

Part II: Existence of a generating set S_r of size $\leq \log_2 |\Gamma|$

Consider the elements of $T_i = \left\{s_1^{(i+1)}\Lambda_i, \ldots, s_{m_i}^{(i+1)}\Lambda_i\right\}$, where each $s_j^{(i+1)}$ is in Γ_{i+1}. With each set T_i associate the set $R_i = \left\{s_1^{(i+1)}, \ldots, s_{m_i}^{(i+1)}\right\}$. We can then define $S_1 = R_1$, and $S_{i+1} = S_i \dot{\cup} R_{i+1}$ for $1 \le i < r$. In this manner, each S_i is a generating set for Λ_i, S_1 and $S_1^{(\{1\})}$ have the same size, and $|S_{i+1}| \le |S_i| + |T_i|$.

Observe that $\sum_{i=0}^{r-1} |T_i| = k + 2l \le \log_2 |\Gamma|$. Consequently then, combining all these inequalities, we get that:

$$|S_r| \le \sum_{i=0}^{r-1} |T_i| = k + 2l \le \log_2 |\Gamma|$$

where S_r is a generating set of $\Lambda_r = \Gamma$.

Part III: Existence of a Hamiltonian path in $\mathsf{Cay}\left(\Gamma; S_r \cup S_r^{-1}\right)$

Now note that $\Lambda_1/\Lambda_0 \simeq \Lambda_1$, and hence $\mathsf{Cay}\left(\Lambda_1; S_1 \cup S_1^{-1}\right)$ has a Hamiltonian cycle, as a consequence of Lemma 2.1.1 and Remark 5.3.2 paired with the fact that $\mathsf{Cay}\left(\Lambda_1/\Lambda_0; T_0 \cup T_0^{-1}\right)$ has a Hamiltonian cycle. Since $\mathsf{Cay}\left(\Lambda_2/\Lambda_1; T_1 \cup T_1^{-1}\right)$ also has a Hamiltonian cycle, then in particular we can 'lift' a Hamiltonian path in $\mathsf{Cay}\left(\Lambda_1; S_1 \cup S_1^{-1}\right)$ to a Hamiltonian path in $\mathsf{Cay}\left(\Lambda_2; S_2 \cup S_2^{-1}\right)$, as a consequence of the 'lifting' lemma (Corollary 2.1.4) and Remark 5.3.2. Repeating this lifting argument iteratively, $\mathsf{Cay}\left(\Gamma; S_r \cup S_r^{-1}\right)$ has a Hamiltonian path. This completes the proof.

□

Remark 5.3.3. When $|\Gamma| \ge 3$, note that we can easily find a generating set S of size $|S| \le 1 + \log_2 |\Gamma|$ such that $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle. Consider the generating set S_n constructed in the proof of the weak version of Theorem 5.1 and some Hamiltonian path P in $\mathsf{Cay}\left(\Gamma; S_r \cup S_r^{-1}\right)$. Suppose the end vertices of this Hamiltonian path are g and h. Then defining $S = S_r \cup \{g^{-1}h\}$, its follows that $\{g, h\}$ is an edge in $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ and therefore $\mathsf{P} + \{g, h\}$ is a Hamiltonian cycle in $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$. This is the main idea behind the extension of the proof of the weak version of Theorem 5.1 to a proof of the full statement.

Now the moon begins to shine
Good night sleep tight
Dream sweet dreams for me
Dream sweet dreams for you

John Lennon,
Good Night

6

Conclusion

Lovász's conjecture was originally stated with regards to the existence of Hamiltonian paths in finite connected vertex–transitive graphs, and in particular on Hamiltonian cycles in any such graph except for 5 known exceptions, as outlined in Conjecture 1.4.1. As we have seen in Proposition 1.3.1, every connected Cayley graph on a finite group with an inverse–closed generating set as the connecting set is, in particular, a finite connected vertex–transitive graph. Really and truly then, Lovász's conjecture for Cayley graphs as given in Conjecture 1.4.2 is a restriction of Conjecture 1.4.1 to a specific class of vertex–transitive graphs, namely one which allows us to use tools and machinery from group theory in conjunction with those from graph theory. Yet, even with so much more machinery at our disposal, as we have seen, the problem is still very hard to solve.

Throughout this dissertation, we have considered the Hamiltonicity of Cayley graphs and digraphs, in light of Lovász's conjecture for Cayley graphs. In Chapter 2 we developed and outlined a number of techniques that form the backbone to a number of the results considered in subsequent chapters. More so, the results presented make a compelling case for the conjecture to hold in the positive – which is the primary reason we believe, even after so many decades, why this area of research is still highly active to this day.

The strongest of these results is Theorem 2.3.4, which asserts the existence of a Hamiltonian cycle in any Cayley graph on an Abelian group. This leaves the Cayley graphs on non–Abelian groups for consideration. In Chapter 3 we considered groups with a generating set S of size 2, and outlined a number of criteria such that $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle.

In particular, in light of Theorem 5.2.1, we saw in Corollary 5.2.2 that every non–Abelian simple group has a generating set of size 2 such that the corresponding Cayley graph has a Hamiltonian cycle. Albeit this result only asserts Hamiltonicity with respect to a *particular* non–trivial generating set (and hence a particular Cayley graph), it supports Lovász's conjecture for Cayley graphs with respect to the Hamiltonicity of *any* finite connected Cayley graph on a finite non–Abelian simple group.

However, Pak and Radoičić (2009) showed that we can do even better than this, especially with regards to finite non–Abelian groups (not necessarily simple). As a consequence of the results we've just mentioned, namely Theorem 2.3.4 and Corollary 5.2.2, they showed in Theorem 5.1 that *every* finite group Γ has a generating set S of small order (namely $|S| \leq \log_2 |\Gamma|$) such that $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle. Once again, albeit only asserting Hamiltonicity with respect to a particular non–trivial generating set, Theorem 5.1 strongly supports Lovász's conjecture for Cayley graphs with respect to the Hamiltonicity of *any* finite connected Cayley graph on *any* finite group.

Note that it suffices to prove Lovász's conjecture for Cayley graphs with respect to only minimal generating sets. This adds further to the significance of Theorem 5.1, since given a finite group Γ, a minimal generating set S of Γ has size at most $\log_2 |\Gamma|$. To see this, let $S = \left\{s_1, s_2, \ldots, s_{|S|}\right\}$ be a minimal generating set. Then by minimality, $1 \notin S$ and given $1 \leq r < |S|$ for $\Gamma_r = \langle s_1, s_2, \ldots, s_r\rangle$ we have that $s_{r+1} \notin \Gamma_r$ (otherwise s_{r+1} can be removed from S and S would still generate Γ – contradicting minimality). Then Γ_r and $s_{r+1}\Gamma_r$ are two distinct cosets of Γ_r in Γ_{r+1}, satisfying $|\Gamma_{r+1}| \geq 2|\Gamma_r|$. It follows that $|\Gamma| \geq 2^{|S|}$ and hence $|S| \leq \log_2 |\Gamma|$.

Since the Hamiltonicity of a Cayley digraph $\overrightarrow{\mathsf{Cay}}\left(\Gamma; S\right)$ implies the Hamiltonicity of the Cayley graph $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$, we also considered the existence of oriented Hamiltonian paths and cycles in Cayley digraphs. In particular, in Chapter 4 we considered an in–depth treatment of the Hamiltonicity of Cayley digraphs on Abelian groups, starting with Theorem 4.1.1 which gave rise to an infinite class of Cayley digraphs on Abelian groups without an oriented Hamiltonian cycle. However, we subsequently showed in Theorem 4.2.1 that for any Dedekind group Γ, and in particular any Abelian group, and any generating set S of Γ, the Cayley digraph $\overrightarrow{\mathrm{Cay}}\,(\Gamma; S)$ has an oriented Hamiltonian path.

6.1 | Survey of further results and problems

Having provided an in–depth treatment of a number of techniques and their consequent results, we shall now shift gears and spare any unnecessary details; to conclude this dissertation, we shall give a non–exhaustive overview of a number of further results and open problems in this area of research, with the aim being to provide a more holistic and complete picture of where the problem presently stands. We suggest that interested readers look at the surveys given in Curran and Gallian (1996); Curran and Witte (1985); Pak and Radoičić (2009).

6.1.1 | Hamiltonicity of Cayley (di)graphs on groups of special order

The results considered throughout this dissertation were generally based on some algebraic or structural property of the group in question, rather than on the order of the group. We have seen recent results by Pak and Radoičić (2009), most notably Theorem 5.1, concerning the existence of generating sets guaranteeing Hamiltonicity, where the generating sets in question are of small size relative to the order of the group. Yet, we have not seen results which explicitly state that any Cayley (di)graph on any group of some special order must be Hamiltonian.

The majority of the work carried out in the past couple of decades however concerns such results – finding orders of groups such that the Cayley (di)graph of any group of such an order is Hamiltonian. Given the endless scope of this area of research, it has been impossible to touch upon every possible type of result. However, we shall briefly review here some of these results concerning special orders of groups.

One of the most beautiful and strong result of this type is due to Witte (1986), concerning p–groups. A p–group is a group whose order is some positive integer power of a prime number p.

Theorem 6.1.1 (Witte (1986)). *Every connected Cayley digraph on a p–group has an oriented Hamiltonian cycle.*

The proof relies on the *skewed–generators argument*, which we have seen a variant of when developing the lifting and arc–forcing subgroup techniques in Chapter 2. It is for this reason that we favoured thoroughly developing the chosen techniques, as they underpin a number of the results presented here.

In the more general context of vertex–transitive graphs, the following result by Marušič was one of the earliest results on the Hamiltonicty of vertex–transitive graphs whose vertex set has a special order, namely $2p^2$ for p prime.

Theorem 6.1.2 (Marušič (1987)). *Let p be a prime number. Every connected vertex–transitive graph on $2p^2$ vertices has a Hamiltonian cycle.*

Consequently, every connected Cayley graph on a group Γ with order $2p^2$ has a Hamiltonian cycle. The following theorem surveys the current landscape for the Hamiltonicity of Cayley graphs on groups whose order have few prime factors. The references given correspond to the most recently proved cases at the time of writing; references within the given reference list cover the other cases.

Theorem 6.1.3. *Let Γ be a group with generating set S. Let p, q, r, s be distinct primes and k a positive integer. If $|\Gamma|$ has one of the following forms:*

	$\|\Gamma\|$	Conditions
Kutnar et al. (2012); Morris and Wilk (2020)	kp	$k \leq 47$
Kutnar et al. (2012)	kp^2	$k \leq 4$
Kutnar et al. (2012)	$2p^3$	
Witte (1986)	p^k	
Kutnar et al. (2012); Maghsoudi (2021); Morris (2015)	kqr	$k \leq 9, k \neq 8$
Kutnar et al. (2012)	pqr	
Morris (2021)	$pqrs$	$pqrs$ is odd

then Cay $\left(\Gamma; S \cup S^{-1}\right)$ *has a Hamiltonian cycle.*

Another class of results concerns the order of the *commutator subgroup* $[\Gamma, \Gamma]$ of a Γ, rather than the order of the group Γ itself. Given any two elements $g, h \in \Gamma$, their *commutator* is the element $[g, h] = g^{-1}h^{-1}gh$. The commutator subgroup $[\Gamma, \Gamma]$ is then the subgroup generated by all commutators in Γ.

In particular, we have the following result analogous to that for p–groups given by Witte (1986).

Theorem 6.1.4 (Keating and Witte (1985))**.** *Every connected Cayley graph on a group whose commutator is a cyclic p–group has a Hamiltonian cycle.*

The following theorem surveys the current landscape for the Hamiltonicity of Cayley graphs of groups whose commutator subgroup has an order with few prime factors.

Theorem 6.1.5. *Let Γ be a group with generating set S. Let p and q be distinct primes, k and r positive integers. If $|[\Gamma, \Gamma]|$ has one of the following forms:*

	$\lvert[\Gamma, \Gamma]\rvert$	Conditions
Morris (2018)	$2p$	$p \geq 2$
Morris (2015)	pq	$\lvert\Gamma\rvert$ odd
Morris (2015)	$p^k q^r$	$\lvert\Gamma\rvert$ odd, $[\Gamma, \Gamma]$ cyclic

then Cay $\left(\Gamma; S \cup S^{-1}\right)$ *has a Hamiltonian cycle.*

6.1.2 | Hamiltonicity of Cayley (di)graphs on special classes of groups

Throughout this dissertation we have given due consideration to a number of classes of groups, namely Abelian and 2–generated groups, as well as Dedekind and non–Abelian simple groups. We shall briefly consider some other results on the Hamiltonicity of Cayley (di)graphs on other classes of groups. We begin with the following result for dihedral groups D_n, where n is even.

Theorem 6.1.6 (Alspach et al. (2010)). *Every connected Cayley graph on the dihedral group D_{2k}, $k \in \mathbb{N}$, has a Hamiltonian cycle.*

In the case of dihedral groups D_n where n is odd, we have the following for when n in particular is prime.

Theorem 6.1.7 (Holsztyński and Strube (1978)). *Every connected Cayley graph on the dihedral group D_p, p prime, has a Hamiltonian cycle.*

Furthermore, for every odd prime p, we have that every connected Cayley graph on the group $D_p \times D_p$ has a Hamiltonian cycle.

Theorem 6.1.8 (Curran and Gallian (1996)). *Every connected Cayley graph on the direct product of dihedral groups $D_p \times D_p$, p an odd prime, has a Hamiltonian cycle.*

Other classes of groups which are of interest are the alternating groups A_n and the symmetric groups S_n. A *transposition* is a permutation of the form $(\alpha\ \beta)$. We have the following result for generating sets S of S_n consisting of only transpositions.

Theorem 6.1.9 (Kompel'makher and Liskovets (1975)). *Let S be a generating set of S_n consisting of only transpositions. Then $\mathsf{Cay}\left(S_n; S \cup S^{-1}\right)$ has a Hamiltonian cycle.*

Let $B_n = \{(1\ j\ n) : 2 \le j \le n-1\}$. It can be shown that B_n is a minimal generating set for A_n. We have the following result due to Gould and Roth (1987).

Theorem 6.1.10 (Gould and Roth (1987)). *The Cayley digraph $\overrightarrow{\mathsf{Cay}}\left(A_n; B_n\right)$ has an oriented Hamiltonian cycle for $n = 3$ and $n \ge 5$; $\overrightarrow{\mathsf{Cay}}\left(A_4; B_4\right)$ has an oriented Hamiltonian path, but not a cycle.*

6.1.3 | Hamiltonicity of the Cartesian product of oriented cycles

The Cartesian product $\mathsf{D}_1 \square \mathsf{D}_2$ of two digraphs D_1 and D_2 is another digraph with vertex set $V\left(\mathsf{D}_1\right) \times V\left(\mathsf{D}_2\right)$ such that there is an arc from (u, u') to (v, v') if, and only if, either $u = v$ and (u', v') is in the arc set of D_2, or $u' = v'$ and (u, v) is in the arc set of D_1.

Remark 6.1.11. Consider the direct product of $k \in \mathbb{N}$ cyclic groups

$$Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_k}$$

where $n_i > 1$. For each Z_{n_i}, let a_i be the element that generates the group. Define the set

$$S := \{a_1\} \times_{\mathsf{id}} \cdots \times_{\mathsf{id}} \{a_k\}$$

which by Remark 4.2.3 generates $Z_{n_1} \times \cdots \times Z_{n_k}$. It is easy to see that $\overrightarrow{\mathsf{Cay}}\,(Z_{m_1} \times \cdots \times Z_{m_k}; S)$ is isomorphic to the Cartesian product of the oriented cycles on $n_1, n_2, \ldots n_k$ vertices, respectively.

The choice of cyclic groups in the proof of Theorem 4.2.6 arises from the Abelian requirement (namely the Fundamental Theorem of Finite Abelian Groups); indeed this requirement is not redundant, since the Cartesian product of any two oriented cycles $\overrightarrow{\mathsf{C}_n} \,\square\, \overrightarrow{\mathsf{C}_m}$ need not necessarily have a Hamiltonian cycle, and hence neither does $\overrightarrow{\mathsf{Cay}}\,(Z_n \times Z_m; S)$ by our previous remark.

The following result gives necessary and sufficient conditions for the Hamiltonicity of $\overrightarrow{\mathsf{C}_n} \,\square\, \overrightarrow{\mathsf{C}_m}$.

Theorem 6.1.12 (Trotter Jr. and Erdös (1978))**.** *The Cartesian product $\overrightarrow{\mathsf{C}_n} \,\square\, \overrightarrow{\mathsf{C}_m}$ of oriented cycles has an oriented Hamiltonian cycle if, and only if, $d = \gcd(n, m) \geq 2$ and there exists positive integers d_n and d_m such that $d_n + d_m = d$ and $\gcd(n, d_n) = 1 = \gcd(m, d_m)$.*

Even more surprisingly, Curran and Witte (1985) showed that the Cartesian product of 3 or more oriented cycles is always Hamiltonian.

Theorem 6.1.13 (Curran and Witte (1985))**.** *The Cartesian product of $k \geq 3$ oriented cycles has an oriented Hamiltonian cycle.*

6.1.4 | Open problems

While we have covered with significant detail the Hamiltonicity of Cayley (di)graphs on Abelian groups, there are still open problems in the case of digraphs, such as the following.

Problem 6.1.14 (Curran and Witte (1985)). For any finite Abelian group Γ and any minimal generating set S of Γ with $|S| \geq 3$, does $\overrightarrow{\mathsf{Cay}}(\Gamma; S)$ have an oriented Hamiltonian cycle?

A problem related to the existence of Hamiltonian cycles is that of *Hamiltonian decompositions*. A Hamiltonian decomposition of a graph is a partitioning of its edge set into edge disjoint Hamiltonian cycles.

Problem 6.1.15 (Alspach et al. (1990)). Let Γ be an Abelian group and S a generating set of Γ such that $|S| = k$. Does $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ have a Hamiltonian decomposition of size k?

There are also a number of recent problems arising from the highly active study of special group orders.

Problem 6.1.16. Let Γ be a group with generating set S. Let p, q, r be distinct odd primes. If $|\Gamma| = 2pqr$, does $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ have a Hamiltonian cycle?

Morris (2021) showed that for any 4 distinct primes p, q, r, s such that $|\Gamma| = pqrs$ and $pqrs$ is odd (and so none of p, q, r or s is equal to 2), then $\mathsf{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle. Problem 6.1.16 considers the case when one of the distinct primes is 2, and hence $pqrs$ is not odd.

Problem 6.1.17. For which other values of k can the results in Theorem 6.1.5 be extended to?

Other problems can be posed for specific classes of groups. In the case of the dihedral group D_n, we have seen that every connected Cayley graph on D_n has a Hamiltonian cycle whenever n is even or n is prime. Hamiltonicity for dihedral groups has been considered in other cases, however the case for when n is odd and not prime remains open.

Problem 6.1.18. Does every connected Cayley graph on the dihedral group D_n, n odd and not prime, have a Hamiltonian cycle?

In Theorem 6.1.9, we have seen that for the symmetric group S_n, the Cayley graph $\mathsf{Cay}\left(S_n; S \cup S^{-1}\right)$ has a Hamiltonian cycle whenever S consists of transpositions only. Note that every transposition is an involution; hence we have the following open problem, of which Theorem 6.1.9 is a special case.

Problem 6.1.19 (Curran and Gallian (1996))**.** Let S be a generating set of S_n, consisting of only involutions. Does $\mathsf{Cay}\left(S_n; S \cup S^{-1}\right)$ have a Hamiltonian cycle?

It is not known how the Hamiltonicity of a finite collection of Cayley graphs can be extended to the respective Cayley graph on the direct product of the underlying groups. This poses the following interesting problem.

Problem 6.1.20. Let $\mathcal{G}$ be a family of finite groups. Does the existence of a Hamiltonian cycle in every connected Cayley graph on groups in $\mathcal{G}$ imply the existence of a Hamiltonian cycle in every connected Cayley graph on the direct product of a finite collection of groups in $\mathcal{G}$?

References

B. Alspach, J.-C. Bermond, and D. Sotteau. *Decomposition into Cycles I: Hamilton Decompositions*, pages 9–18. Springer Netherlands, Dordrecht, 1990. ISBN 978-94-009-0517-7. doi: 10.1007/978-94-009-0517-7_2.

B. Alspach, C. C. Chen, and M. Dean. Hamilton paths in Cayley graphs on generalized dihedral groups. *ARS MATHEMATICA CONTEMPORANEA*, 3:29–47, 2010.

M.A. Armstrong. *Groups and Symmetry*. Undergraduate Texts in Mathematics. Springer New York, 1997.

L. Babai. *Automorphism Groups, Isomorphism, Reconstruction*, page 1447–1540. MIT Press, 1996.

W.W. Rouse Ball. *Mathematical Recreations & Essays: 4th Edition*. Macmillan, 1905.

P. J. Cameron. *Permutation Groups*. London Mathematical Society Student Texts. Cambridge University Press, 1999. doi: 10.1017/CBO9780511623677.

C. Cooper, A. Frieze, and M. Molloy. Hamilton cycles in random regular digraphs. *Combinatorics, Probability and Computing*, 3(1):39–49, 1994. doi: 10.1017/S096354830000095X.

S. J. Curran and J. A. Gallian. Hamiltonian cycles and paths in Cayley graphs and digraphs – a survey. *Discrete Mathematics*, 156(1):1–18, 1996. ISSN 0012-365X. doi: https://doi.org/10.1016/0012-365X(95)00072-5.

Stephen J. Curran and David Witte. Hamilton paths in Cartesian products of directed cycles. *North-holland Mathematics Studies*, 115:35–74, 1985.

R. Diestel. *Graph Theory*. Springer-Verlag Berlin Heidelberg, 5th edition, 2017.

P. Erdős and A. Rényi. On the evolution of random graphs. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 5:17–61, 1960.

J.A. Gallian. *Contemporary Abstract Algebra*. Brooks/Cole, 2012. ISBN 9781133606758.

R. J. Gould and R. Roth. Cayley digraphs and $(1, j, n)$-sequencings of the alternating groups a_n. *Discrete Math.*, 66(1–2):91–102, aug 1987. ISSN 0012-365X.

W. Holsztyński and R.F.E. Strube. Paths and circuits in finite groups. *Discrete Mathematics*, 22(3): 263–272, 1978.

K. Keating and D. Witte. On Hamilton cycles in Cayley graphs in groups with cyclic commutator subgroup. In B.R. Alspach and C.D. Godsil, editors, *Annals of Discrete Mathematics (27): Cycles in Graphs*, volume 115 of *North-Holland Mathematics Studies*, pages 89–102. North-Holland, 1985. doi: https://doi.org/10.1016/S0304-0208(08)72999-2.

C. S. H. King. Generation of finite simple groups by an involution and an element of prime order. *Journal of Algebra*, 478:153–173, 2017. ISSN 0021-8693.

V. L. Kompel'makher and V. A. Liskovets. Sequential generation of arrangements by means of a basis of transpositions. *Cybernetics*, 11(3):362–366, 1975. doi: https://doi.org/10.1007/BF01069459.

K. Kutnar, D. Marušič, D. W. Morris, J. Morris, and P. Šparl. Hamiltonian cycles in Cayley graphs whose order has few prime factors. *ARS MATHEMATICA CONTEMPORANEA*, 5: 27–71, 2012.

J. Lauri and R. Scapellato. *Topics in Graph Automorphisms and Reconstruction*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2 edition, 2016. doi: 10.1017/CBO9781316669846.

F. Maghsoudi. Cayley graphs of order $6pq$ and $7pq$ are Hamiltonian. *The Art of Discrete and Applied Mathematics*, 2021.

D. Marušič. Hamiltonian cycles in vertex symmetric graphs of order $2p^2$. *Discrete Math.*, 66(1–2): 169–174, aug 1987. doi: 10.1016/0012-365X(87)90129-4.

D. Marušič. Hamiltonian circuits in Cayley graphs. *Discrete Mathematics*, 46(1):49–54, 1983.

D. W. Morris. 2-generated Cayley digraphs on nilpotent groups have Hamiltonian paths. *arXiv:1103.5293*, 2011.

D. W. Morris. On Cayley digraphs that do not have Hamiltonian paths. *International Journal of Combinatorics*, 2013, 2013.

D. W. Morris. Odd–order Cayley graphs with commutator subgroup of order pq are Hamiltonian. *ARS MATHEMATICA CONTEMPORANEA*, 8:1–28, 2015.

D. W. Morris. Cayley graphs on groups with commutator subgroup of order $2p$ are Hamiltonian. *The Art of Discrete and Applied Mathematics*, 1, 2018.

D. W. Morris. On Hamiltonian cycles in Cayley graphs of order $pqrs$. *arXiv e-prints*, art. arXiv:2107.14787, July 2021.

D. W. Morris and K. Wilk. Cayley graphs of order kp are Hamiltonian for $k < 48$. *The Art of Discrete and Applied Mathematics*, 3, 2020.

I. Pak and R. Radoičić. Hamiltonian paths in Cayley graphs. *Discrete Mathematics*, 309(17):5501–5508, 2009.

R. A. Rankin. A campanological problem in group theory. *Mathematical Proceedings of the Cambridge Philosophical Society*, 44(1):17–25, 1948. doi: 10.1017/S030500410002394X.

R. A. Rankin. A campanological problem in group theory. ii. *Mathematical Proceedings of the Cambridge Philosophical Society*, 62(1):11–18, 1966. doi: 10.1017/S0305004100039451.

E. Rapaport-Strasser. Cayley color groups and Hamilton lines. *Scripta Math.*, 24:51–58, 1959.

R. W. Robinson and N. C. Wormald. Almost all cubic graphs are Hamiltonian. *Random Structures & Algorithms*, 3(2):117–125, 1992. doi: https://doi.org/10.1002/rsa.3240030202.

R. W. Robinson and N. C. Wormald. Almost all regular graphs are Hamiltonian. *Random Structures & Algorithms*, 5(2):363–374, 1994. doi: https://doi.org/10.1002/rsa.3240050209.

M. Stelow. Hamiltonicity in Cayley graphs and digraphs of finite Abelian groups. 2017.

W. T. Trotter Jr. and P. Erdös. When the Cartesian product of directed cycles is Hamiltonian. *Journal of Graph Theory*, 2(2):137–142, 1978.

R.J. Wilson. A brief history of Hamiltonian graphs. In L. D. Andersen, I. T. Jakobsen, C. Thomassen, B. Toft, and P. D. Vestergaard, editors, *Graph Theory in Memory of G.A. Dirac*, volume 41 of *Annals of Discrete Mathematics*, pages 487–496. Elsevier, 1988. doi: https://doi.org/10.1016/S0167-5060(08)70484-9.

D. Witte. Cayley digraphs of prime-power order are Hamiltonian. *Journal of Combinatorial Theory, Series B*, 40(1):107–112, 1986. doi: https://doi.org/10.1016/0095-8956(86)90068-7.

D. Witte and J. A. Gallian. A survey: Hamiltonian cycles in Cayley graphs. *Discrete Mathematics*, 51(3):293–304, 1984. doi: https://doi.org/10.1016/0012-365X(84)90010-4.

D. R. Woodall. Sufficient Conditions for Circuits in Graphs. *Proceedings of the London Mathematical Society*, S3-24(4):739–755, 05 1972. ISSN 0024-6115. doi: 10.1112/plms/s3-24.4.739.

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