AMS526: Numerical Analysis I (Numerical Linear Algebra for Computational and Data Sciences)

Lecture 14: Eigenvalue Problems; Eigenvalue Revealing Factorizations

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Outline

Eigenvalue and Eigenvectors

- *Eigenvalue problem* of $n \times n$ matrix A is

$$Ax = \lambda x$$

  with eigenvalues λ and eigenvectors x (nonzero)
- The set of all the eigenvalues of A is the *spectrum* of A
- Eigenvalue are generally used where a matrix is to be compounded iteratively
- Eigenvalues are useful for algorithmic and physical reasons
  - Algorithmically, eigenvalue analysis can reduce a coupled system to a collection of scalar problems
  - Physically, eigenvalue analysis can be used to study resonance of musical instruments and stability of physical systems

Eigenvalue Decomposition

- *Eigenvalue decomposition* of A is

$$A = X\Lambda X^{-1} \text{ or } AX = X\Lambda$$

  with eigenvectors x_i as columns of X and eigenvalues λ_i along diagonal of Λ. Alternatively,

$$Ax_i = \lambda_i x_i$$

- Eigenvalue decomposition is change of basis to "eigenvector coordinates"

$$Ax = b \quad \rightarrow \quad (X^{-1}b) = \Lambda(X^{-1}x)$$

- Note that eigenvalue decomposition may not exist
- Question: How does eigenvalue decomposition differ from SVD?

Geometric Multiplicity

- Eigenvectors corresponding to a single eigenvalue λ form an *eigenspace* $E_\lambda \subseteq \mathbb{C}^{n\times n}$
- Eigenspace is *invariant* in that $AE_\lambda \subseteq E_\lambda$
- Dimension of E_λ is the maximum number of linearly independent eigenvectors that can be found
- *Geometric multiplicity* of λ is dimension of E_λ, i.e., $\dim(\text{null}(A - \lambda I))$

Algebraic Multiplicity

- The *characteristic polynomial* of A is degree m polynomial

$$p_A(z) = \det(zI - A) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$$

  which is *monic* in that coefficient of z^n is 1
- λ is eigenvalue of A iff $p_A(\lambda) = 0$
  - If λ is eigenvalue, then by definition, $\lambda x - Ax = (\lambda I - A)x = 0$, so $(\lambda I - A)$ is singular and its determinant is 0
  - If $(\lambda I - A)$ is singular, then for $x \in \text{null}(\lambda I - A)$ we have $\lambda x - Ax = 0$
- *Algebraic multiplicity* of λ is its multiplicity as a root of p_A
- Any matrix $A \in \mathbb{C}^{n\times n}$ has n eigenvalues, counted with algebraic multiplicity
- Question: What are the eigenvalues of a triangular matrix?
- Question: How are geometric multiplicity and algebraic multiplicity related?

Similarity Transformations

- The map $A \to Y^{-1}AY$ is a *similarity transformation* of A for any nonsingular $Y \in \mathbb{C}^{n\times n}$
- A and B are *similar* if there is a similarity transformation $B = Y^{-1}AY$

Theorem

If Y is nonsingular, then A and $Y^{-1}AY$ have the same characteristic polynomials, eigenvalues, and algebraic and geometric multiplicities.

1. For characteristic polynomial:

$$\det(zI - Y^{-1}AY) = \det(Y^{-1}(zI - A)Y) = \det(zI - A)$$

   so algebraic multiplicities remain the same

2. If $x \in E_\lambda$ for A, then $Y^{-1}x$ is in eigenspace of $Y^{-1}AY$ corresponding to λ, and vice versa, so geometric multiplicities remain the same

Algebraic Multiplicity $\geq$ Geometric Multiplicity

- Let k be be geometric multiplicity of λ for A. Let $\hat{V} \in \mathbb{C}^{n\times k}$ constitute of orthonormal basis of the E_λ
- Extend $\hat{V}$ to unitary $V \equiv [\hat{V}, \tilde{V}] \in \mathbb{C}^{n\times n}$ and form

$$B = V^*AV = \begin{bmatrix} \hat{V}^*A\hat{V} & \hat{V}^*A\tilde{V} \\ \tilde{V}^*A\hat{V} & \tilde{V}^*A\tilde{V} \end{bmatrix} = \begin{bmatrix} \lambda I & C \\ 0 & D \end{bmatrix}$$

- $\det(zI - B) = \det(zI - \lambda I)\det(zI - D) = (z - \lambda)^k \det(zI - D)$, so the algebraic multiplicity of λ as an eigenvalue of B is $\geq k$
- A and B are similar, so the algebraic multiplicity of λ as an eigenvalue of A is at least $\geq k$
- Examples:

$$A = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix}$$

Their characteristic polynomial is $(z-2)^3$, so algebraic multiplicity of $\lambda = 2$ is 3. But geometric multiplicity of A is 3 and that of B is 1.

Defective and Diagonalizable Matrices

- An eigenvalue of a matrix is *defective* if its algebraic multiplicity $>$ its geometric multiplicity
- A matrix is *defective* if it has a defective eigenvalue. Otherwise, it is called *nondefective*.

Theorem

An $n \times n$ matrix A is nondefective iff it has an eigenvalue decomposition $A = X\Lambda X^{-1}$.

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- ($\Leftarrow$) Λ is nondefective, and A is similar to Λ, so A is nondefective.
- ($\Rightarrow$) A nondefective matrix has n linearly independent eigenvectors. Take them as columns of X to obtain $A = X \Lambda X^{-1}$.
- Nondefective matrices are therefore also said to be *diagonalizable*.

Determinant and Trace

- Determinant of A is $\det(A) = \prod_{j=1}^{n} \lambda_j$, because

$$\det(A) = (-1)^n \det(-A) = (-1)^n p_A(0) = \prod_{j=1}^{n} \lambda_j$$

- Trace of A is $\mathrm{tr}(A) = \sum_{j=1}^{n} \lambda_j$, since

$$p_A(z) = \det(zI - A) = z^n - \sum_{j=1}^{n} a_{jj} z^{n-1} + O(z^{n-2})$$

$$p_A(z) = \prod_{j=1}^{n} (z - \lambda_j) = z^n - \sum_{j=1}^{n} \lambda_j z^{n-1} + O(z^{n-2})$$

- Question: Are these results valid for defective or nondefective matrices?

Outline

Unitary Diagonalization

- A matrix A is *unitarily diagonalizable* if $A = Q\Lambda Q^*$ for a unitary matrix Q
- A Hermitian matrix is unitarily diagonalizable, with real eigenvalues
- A matrix A is *normal* if $A^*A = AA^*$
  - Examples of normal matrices include Hermitian matrices, skew Hermitian matrices
  - Hermitian $\Leftrightarrow$ matrix is normal and all eigenvalues are real
  - skew Hermitian $\Leftrightarrow$ matrix is normal and all eigenvalues are imaginary
  - If A is both triangular and normal, then A is diagonal
- Unitarily diagonalizable $\Leftrightarrow$ normal
  - "$\Rightarrow$" is easy. Prove "$\Leftarrow$" by induction using Schur factorization next

Schur Factorization

- *Schur factorization* is $A = QTQ^*$, where Q is unitary and T is upper triangular

Theorem

Every square matrix A has a Schur factorization.

Proof by induction on dimension of A. Case $n = 1$ is trivial.
For $n \geq 2$, let x be any unit eigenvector of A, with corresponding eigenvalue λ. Let U be unitary matrix with x as first column. Then

$$U^*AU = \begin{bmatrix} \lambda & w^* \\ 0 & C \end{bmatrix}.$$

By induction hypothesis, there is a Schur factorization $\tilde{T} = V^*CV$. Let

$$Q = U \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}, \quad T = \begin{bmatrix} \lambda & w^*V \\ 0 & \tilde{T} \end{bmatrix},$$

and then $A = QTQ^*$.

Eigenvalue Revealing Factorizations

- Eigenvalue-revealing factorization of square matrix A
  - Diagonalization $A = X\Lambda X^{-1}$ (nondefective A)
  - Unitary Diagonalization $A = Q\Lambda Q^*$ (normal A)
  - Unitary triangularization (Schur factorization) $A = QTQ^*$ (any A)
  - Jordan normal form $A = XJX^{-1}$, where J block diagonal with

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

- In general, Schur factorization is used, because
  - Unitary matrices are involved, so algorithm tends to be more stable
  - If A is normal, then Schur form is diagonal