SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA
FACULTY OF CHEMICAL AND FOOD TECHNOLOGY
IInstitute of Information Engineering, Automation, and Mathematics

Nonlinear Model Predictive Control of Rotary Inverted Pendulum

MASTER THESIS

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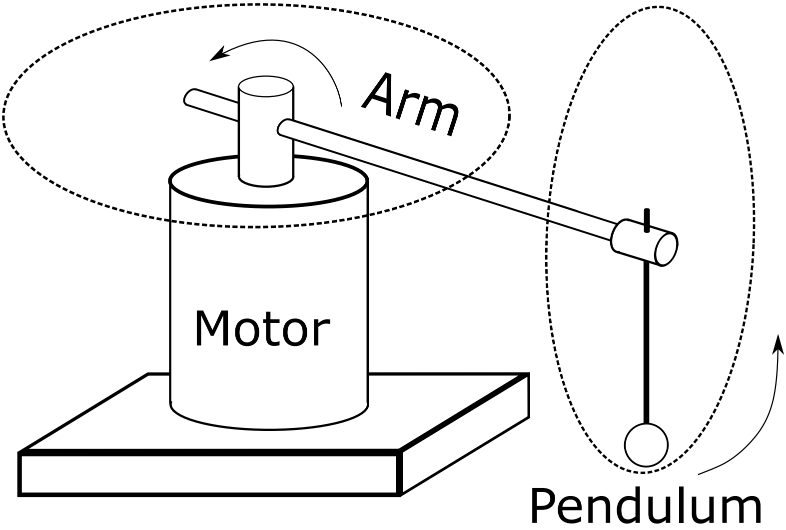
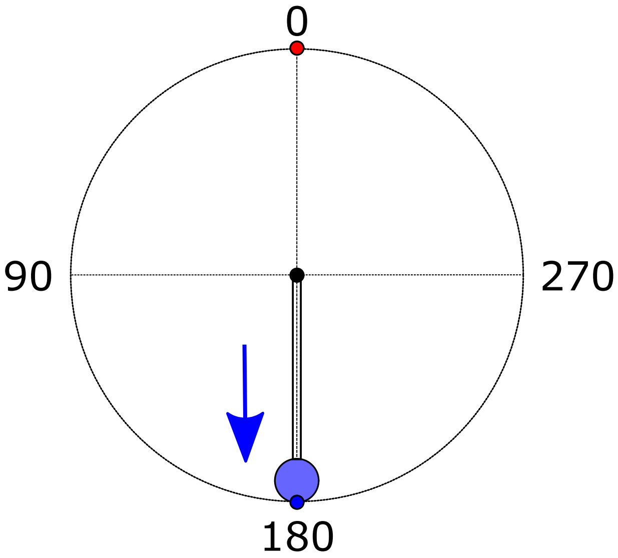
June, 2020

Furuta Pendulum Device



Swing-Up Control of the Pendulum



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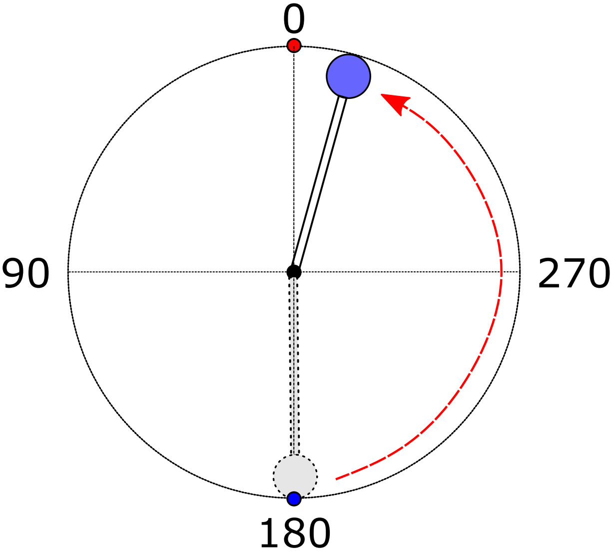
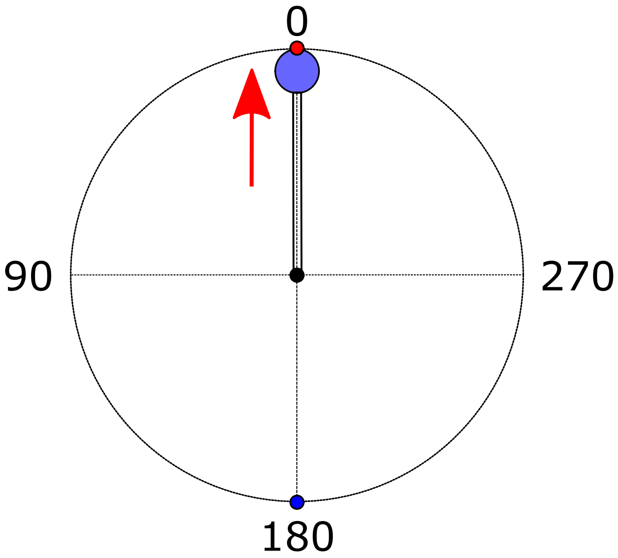


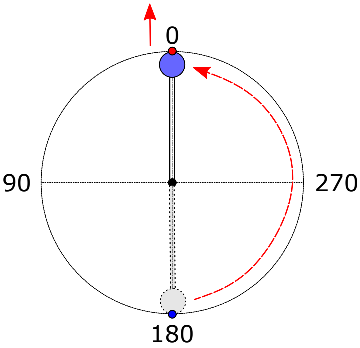
Swing-Up Control of the Pendulum



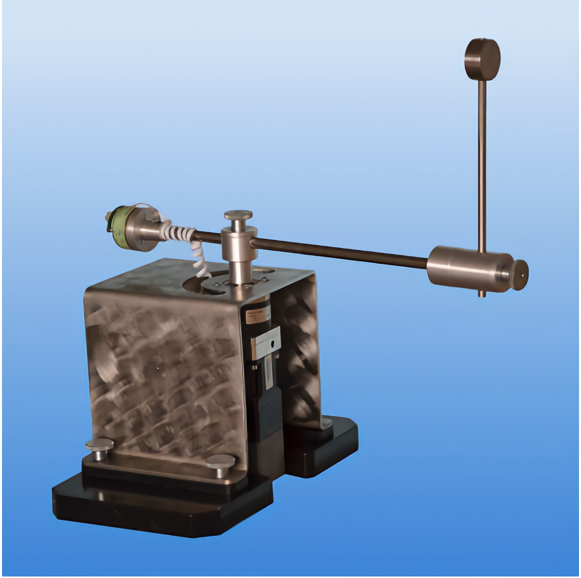
Swing-Up Control of the Pendulum

Main control strategies

1. Heuristic Swing-Up Control strategy
2. Optimal Swing-Up Control strategy



Furuta Pendulum Device



Derivation of the Mathematical Model

Lagrangian Formulation

Lagrangian

$$L = E_\mathsf{k} - E_\mathsf{p}$$

Euler-Lagrange Equations

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial L}{\partial \dot{\theta}_0}\right) - \frac{\partial L}{\partial \dot{\theta}_0} = \tau$$
$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \dot{\theta}_1} = 0$$

Derivation of the Mathematical Model

Lagrangian Formulation

Equations of motion

$$\ddot{\theta}_0 = \frac{\gamma(\epsilon\dot{\theta}_0^2 + \rho) - \delta(\tau + \beta\dot{\theta}_1^2 - \sigma\dot{\theta}_0\dot{\theta}_1)}{\gamma^2 - \alpha\delta}$$

$$\ddot{\theta}_1 = \frac{\gamma(\tau + \beta\dot{\theta}_1^2 - \sigma\dot{\theta}_0\dot{\theta}_1) - \alpha(\epsilon\dot{\theta}_0^2 + \rho)}{\gamma^2 - \alpha\delta}$$

Parametres

$$\alpha = I_0 + L_0^2 m_1 + l_1^2 m_1 \sin^2\theta_1$$
$$\beta = L_0 m_1 l_1 \sin\theta_1$$
$$\gamma = L_0 m_1 l_1 \cos\theta_1$$
$$\delta = I_1 + l_1^2 m_1$$
$$\epsilon = l_1^2 m_1 \sin\theta_1 \cos\theta_1$$
$$\rho = m_1 g l_1 \sin\theta_1$$
$$\sigma = 2 l_1^2 m_1 \sin\theta_1 \cos\theta_1$$

Derivation of the Mathematical Model

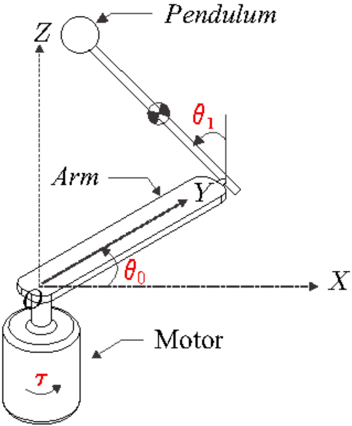
State-Space Representation

State variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \theta_0(t) \\ \dot{\theta}_0(t) \\ \theta_1(t) \\ \dot{\theta}_1(t) \end{bmatrix}$$

Control input

$$u(t) = \tau(t)$$



Derivation of the Mathematical Model

State-Space Representation

Non-linear model

$$\dot{x}(t) = f(x(t), u(t))$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} \dot{\theta}_0 \\ \frac{\gamma(\epsilon\dot{\theta}_0^2+\rho)-\delta(\tau+\beta\dot{\theta}_1^2-\sigma\dot{\theta}_0\dot{\theta}_1)}{\gamma^2-\alpha\delta} \\ \dot{\theta}_1 \\ \frac{\gamma(\tau+\beta\dot{\theta}_1^2-\sigma\dot{\theta}_0\dot{\theta}_1)-\alpha(\epsilon\dot{\theta}_0^2+\rho)}{\gamma^2-\alpha\delta} \end{bmatrix}$$

Derivation of the Mathematical Model

State-Space Representation

Linear model

$$\dot{x}(t) = Ax(t) + Bu(t)$$

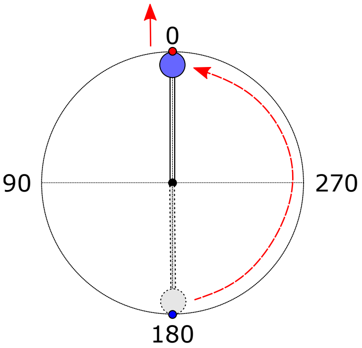
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-gL_0 l_1^2 m_1^2}{(m_1 L_0^2 + I_0)(m_1 l_1^2 + I_1) - L_0^2 l_1^2 m_1^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g l_1 m_1 (m_1 L_0^2 + I_0)}{(m_1 L_0^2 + I_0)(m_1 l_1^2 + I_1) - L_0^2 l_1^2 m_1^2} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{m_1 L_1^2 + I_1}{(m_1 L_0^2 + I_0)(m_1 l_1^2 + I_1) - L_0^2 l_1^2 m_1^2} \\ 0 \\ \frac{-L_0 l_1 m_1}{(m_1 L_0^2 + I_0)(m_1 l_1^2 + I_1) - L_0^2 l_1^2 m_1^2} \end{bmatrix}$$

Swing-Up Control of the Pendulum

Main control strategies

1. Heuristic Swing-Up Control strategy
2. Optimal Swing-Up Control strategy



Swing-Up Control of the Pendulum

Main control strategies

1. Heuristic Swing-Up Control strategy
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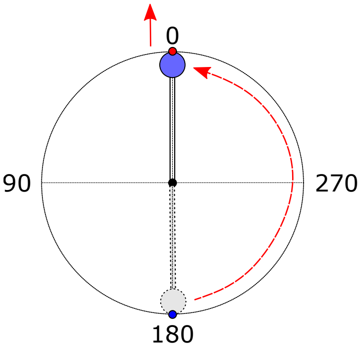


Heuristic Swing-Up Control

Two phases of control:

1. Initial excitation of the system
   (by Energy-Shaping controller)
2. Stabilisation of the pendulum
   (by Predictive controller)

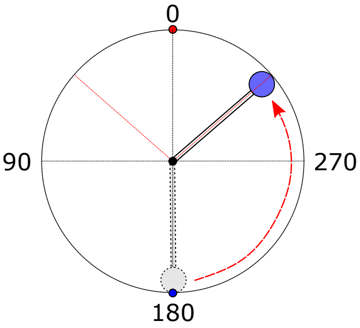
Heuristic Swing-Up Control

Development of control strategy:

1. Design of Energy-Shaping controller
2. Design of Model Predictive controller
3. Full-range control of the pendulum

First Phase of Control

Pendulum is controlled by Energy-Shaping controller.



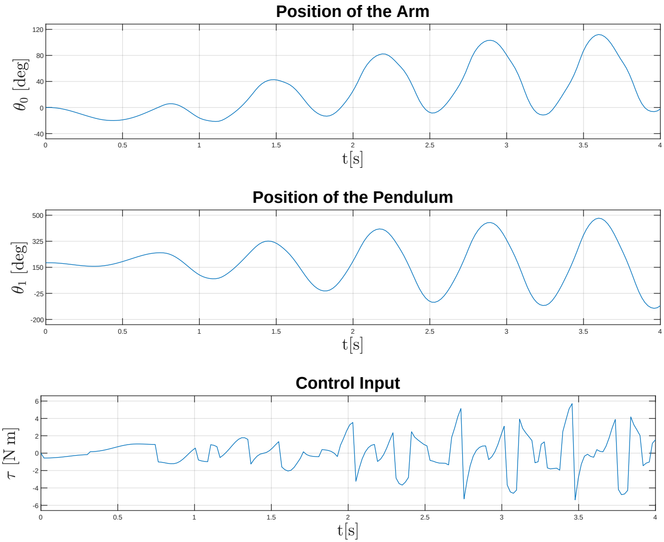
First Phase of Control

Energy-Shaping controller

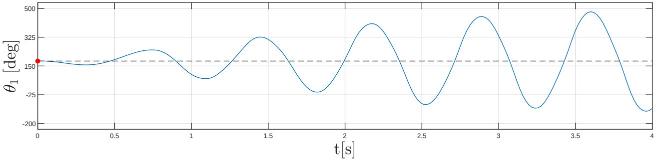
$$E = \frac{m_1 g l_1}{2} \left(\left(\frac{\dot{\theta}_1}{\omega_0} \right)^2 + \cos(\theta_1) - 1 \right)$$

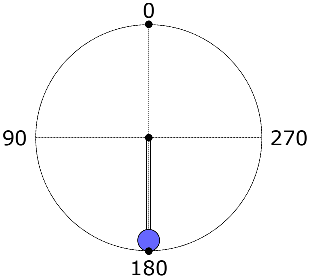
$$u = -4E \,\mathrm{sign}\left(\dot{\theta}_1 \cos(\theta_1) \right)$$

First Phase of Control

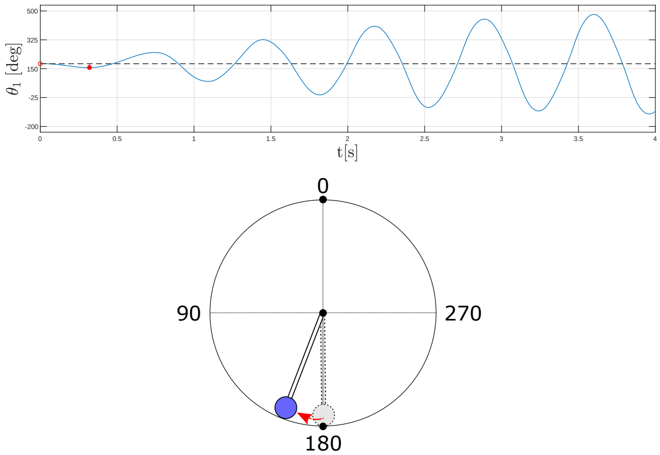


First Phase of Control

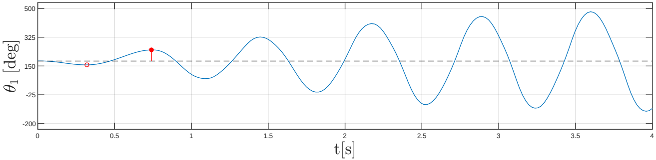


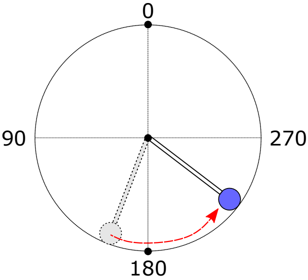


First Phase of Control

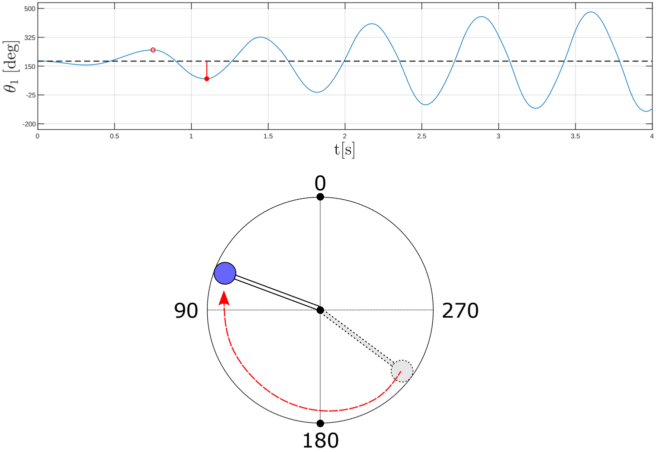


First Phase of Control



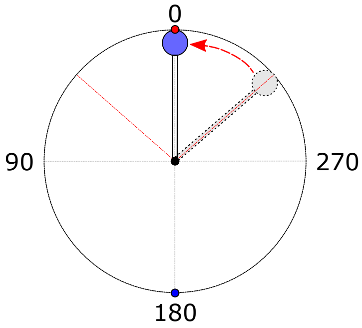


First Phase of Control



Second Phase of Control

Pendulum is controlled by Predictive controller.



Second Phase of Control

Model Predictive Control strategy

$$
\begin{aligned}
\min_{u_0,\ldots,u_{N-1}} \quad & \sum_{k=0}^{N-1} \left(x_k^{\mathsf{T}} Q_{\mathrm{x}} x_k + u_k^{\mathsf{T}} Q_{\mathrm{u}} u_k\right) \\
\text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k \qquad && k \in \mathbb{N}_0^{N-1} \\
& u_{\min} \leq u_k \leq u_{\max} \qquad && k \in \mathbb{N}_0^{N-1} \\
& x_0 = x(0)
\end{aligned}
$$

Parameter	Symbol	Value
Prediction horizon	N	20
Initial condition	x_0	$\left[0\ 0\ 28.5\ \ -40\right]^{\mathsf{T}}$
Constraint on control input-upper bound	$u_{\max}$	10
Constraint on control input-lower bound	$u_{\min}$	-10

Second Phase of Control

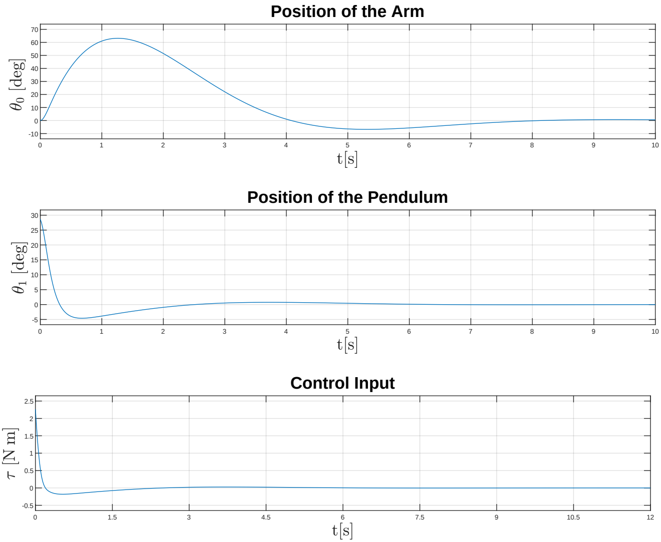
Model Predictive Control strategy

$$
\begin{aligned}
\min_{u_0,\dots,u_{N-1}} \quad & \sum_{k=0}^{N-1} \left(x_k^\mathsf{T} Q_\mathrm{x} x_k + u_k^\mathsf{T} Q_\mathrm{u} u_k\right) & \\
\text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k & k \in \mathbb{N}_0^{N-1} \\
& u_\mathrm{min} \le u_k \le u_\mathrm{max} & k \in \mathbb{N}_0^{N-1} \\
& x_0 = x(0) &
\end{aligned}
$$

Weight matrices

$$
Q_\mathrm{x} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0.08 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} \qquad Q_\mathrm{u} = 1 \times 10^{-3}
$$

Second Phase of Control

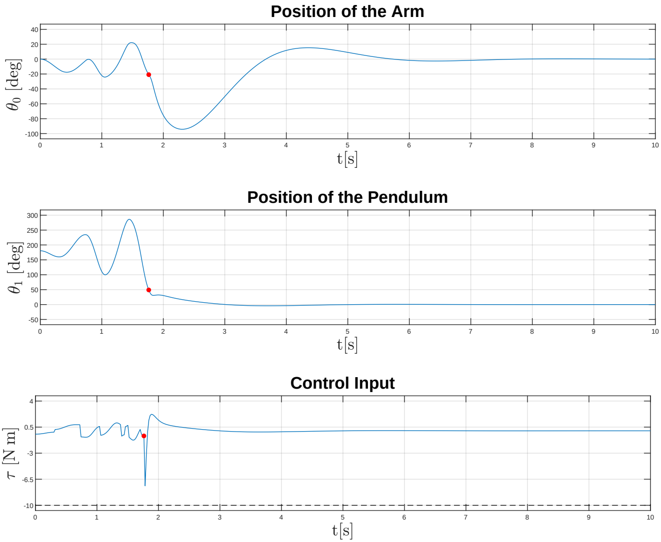


Full-Range Control

Algorithm 1 Heuristic Swing-Up Control strategy

1: **procedure** HEURISTIC SWING-UP CONTROL
2: $\theta_1 \leftarrow 180$ deg
3: Pendulum is controlled by Energy-Shaping controller
4: **if** $\theta_1 \in [-50,\ 50]$ deg **then**
5: Pendulum is controlled by Predictive controller
6: **end if**
7: **end procedure**

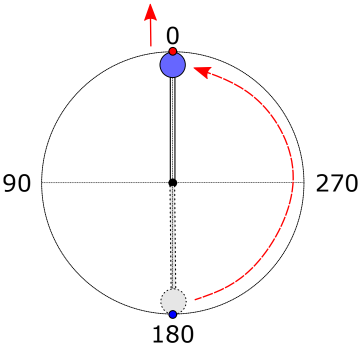
Full-Range Control



Swing-Up Control of the Pendulum

Main control strategies

1. Heuristic Swing-Up Control strategy
2. Optimal Swing-Up Control strategy



Optimal Control Strategy

Two phases of control:

1. Initial excitation of the system
   (by Non-linear Predictive controller)
2. Stabilisation of the pendulum
   (by Non-linear Predictive controller)

Non-Linerar Model Predictive Control

$$
\begin{aligned}
\min_{u_0,\ldots,u_{N-1}} \quad & \sum_{k=0}^{N-1} \left(x_k^\mathsf{T} Q_\mathrm{x} x_k + u_k^\mathsf{T} Q_\mathrm{u} u_k\right) & \\
\text{s.t.} \quad & x_{k+1} = f(x_k, u_k) & k \in \mathbb{N}_0^{N-1} \\
& u_{\min} \leq u_k \leq u_{\max} & k \in \mathbb{N}_0^{N-1} \\
& x_0 = x(0) &
\end{aligned}
$$

Parameter	Symbol	Value
Prediction horizon	N	20
Initial condition	x_0	$\begin{bmatrix}0 & 0 & 180 & 0\end{bmatrix}^\mathsf{T}$
Constraint on control input-upper bound	$u_{\max}$	10
Constraint on control input-lower bound	$u_{\min}$	-10

Non-Linerar Model Predictive Control

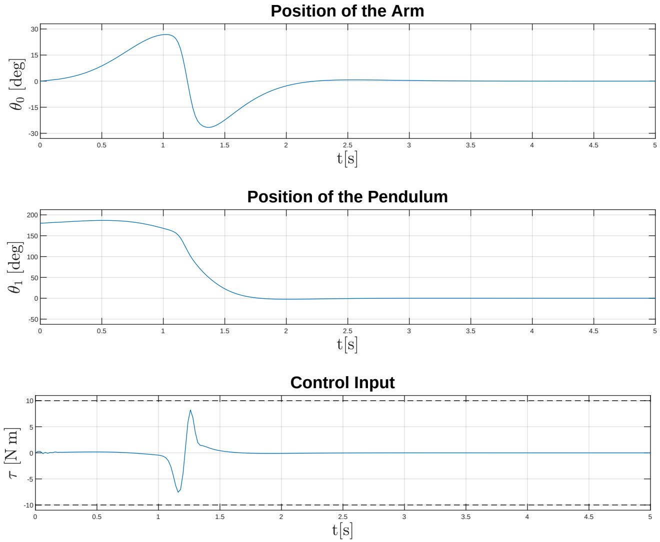
$$
\begin{aligned}
\min_{u_0,\dots,u_{N-1}} \quad & \sum_{k=0}^{N-1} \left(x_k^\mathsf{T} Q_\mathrm{x} x_k + u_k^\mathsf{T} Q_\mathrm{u} u_k \right) \\
\text{s.t.} \quad & x_{k+1} = f(x_k, u_k) && k \in \mathbb{N}_0^{N-1} \\
& u_\mathrm{min} \leq u_k \leq u_\mathrm{max} && k \in \mathbb{N}_0^{N-1} \\
& x_0 = x(0)
\end{aligned}
$$

Weight matrices

$$
Q_\mathrm{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \qquad Q_\mathrm{u} = 0.1
$$

Full-Range Control

Control of the arm

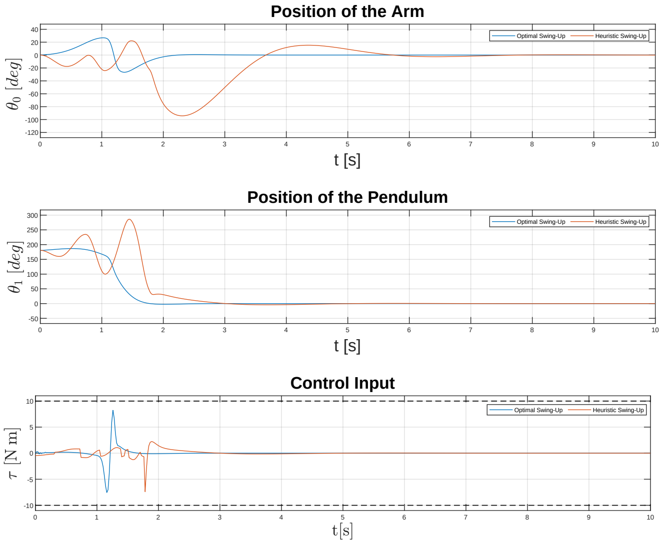


Discussion About Results

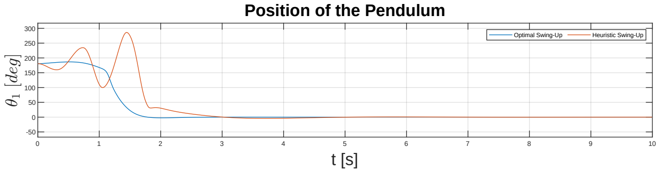
Quality of control criteria:

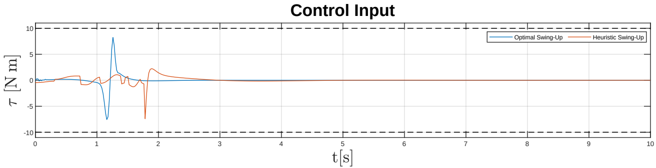
- Settling time
- Sum of abslolute error (SAE)
- Sum of squared control differences (SSCD)
- Solver solving time

Discussion About Results



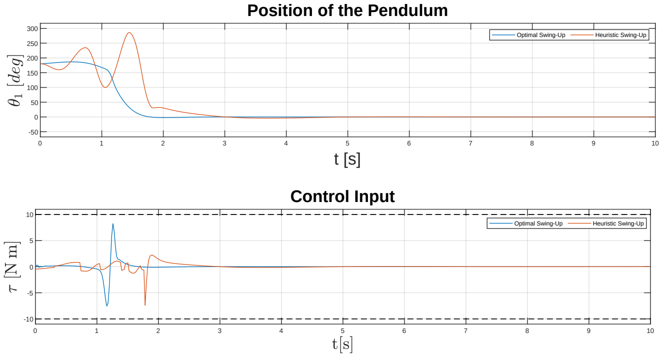
Discussion About Results





Strategy	Settling Time	SAE	SSCD	Solving Time		
				t_{min}	t_{avg}	t_{max}
Heuristic Swing-Up	7.51	305.15	86.77	0.0156	0.0234	0.0313
Optimal Swing-Up	3.20	205.56	88.94	0.0313	0.0469	0.0625

Discussion About Results



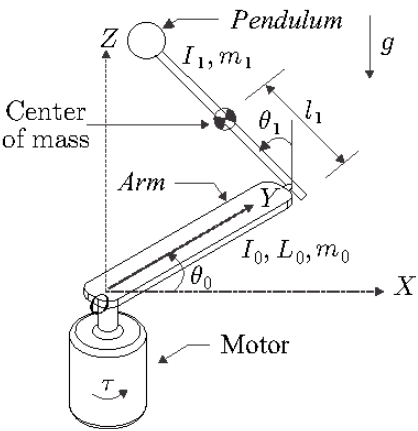
Strategy	Settling Time	SAE	SSCD	Solving Time		
				$t_{\min}$	t_{avg}	$t_{\max}$
Heuristic Swing-Up	100%	100%	100%	100%	100%	100%
Optimal Swing-Up	42.6%	67.4%	103%	201%	200%	200%

Summary

Main achievements of the thesis

- Design of Optimal Swing-Up control strategy
- Its comparisson against the heuristic method
- Evaluation of control quality and calculation time of individual approaches

Parametres of the Pendulum



g-gravitational acceleration [ms^{-2}]
m_0-mass of arm [kg]
m_1-mass of pendulum [kg]
L_0-length of arm [m]
L_1-length of pendulum [m]
l_0-arms center of mass [m]
l_1-pendulums center of mass [m]
I_0-arms moment of inertia [kg m^{-2}]
I_1-pendulums moment of inertia [kg m^{-2}]
θ_0-arm angle[rad]
θ_1-pendulum angle [rad]
τ-motor torque [N m]