Nonlinear Model Predictive Control of Rotary Pendulum

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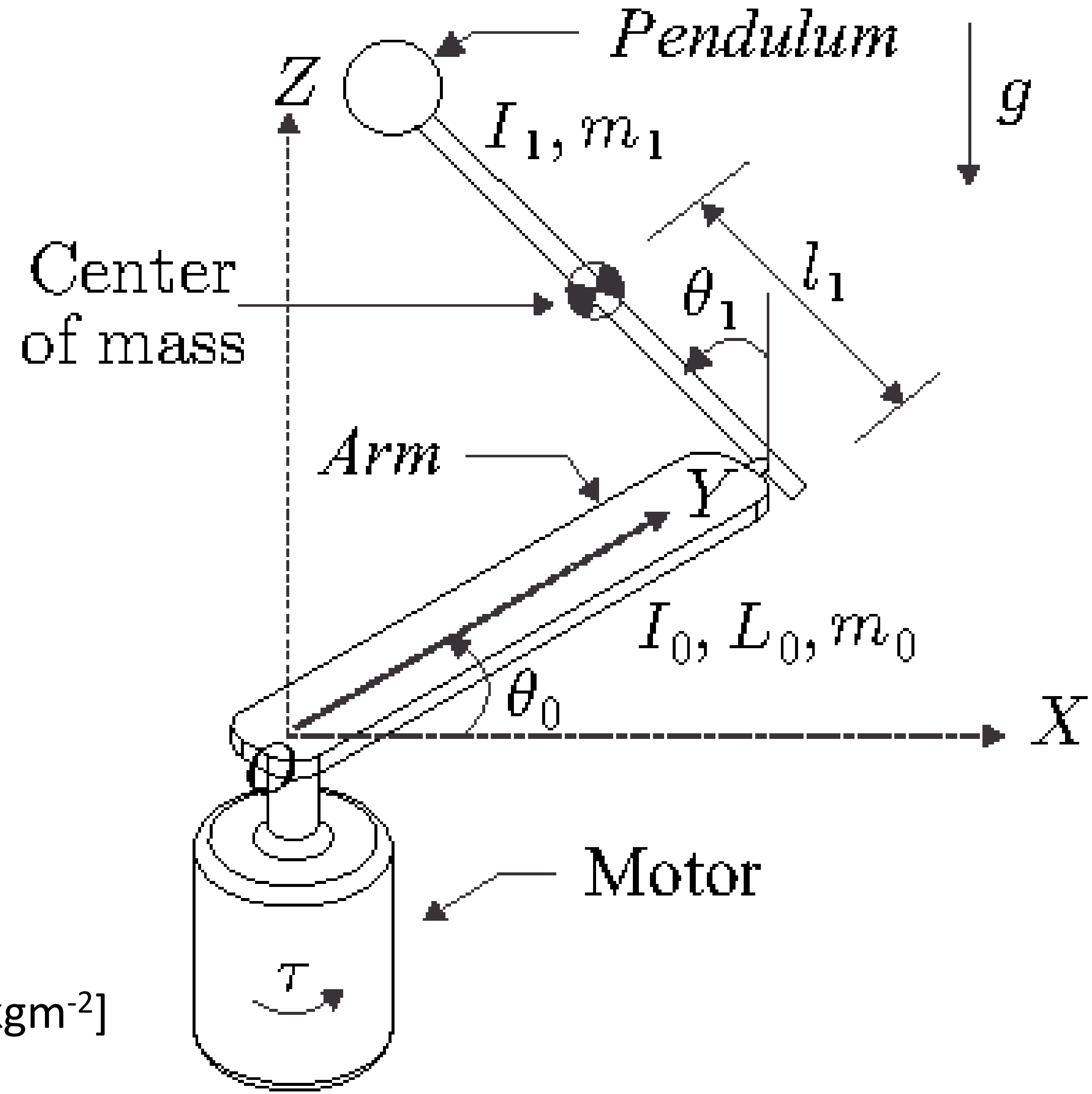
Bratislava 2020

Furuta Pendulum

- θ_0 - arm angle [rad]
- θ_1 - pendulum angle [rad]
- τ - motor torque [V]
- g - gravitational acceleration [ms^{-2}]
- m_0 - mass of arm [kg]
- m_1 - mass of pendulum [kg]
- L_0 - length of arm [m]
- L_1 - length of pendulum [m]
- l_1 - location of the
  pendulums center of mass [m]
- I_0 - moment of inertia of arm [kgm^{-2}]
- I_1 - moment of inertia of pendulum [kgm^{-2}]



Dynamic Model

State variables:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \\ \theta_1 \\ \dot{\theta}_1 \end{bmatrix}$$

Control variables:

$$u = \tau$$

Output variable:

$$y = x_3$$

Dynamic Model

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} \dot{\theta}_0 \\ \dfrac{\gamma\left(\epsilon\dot{\theta_0}^2 + \rho\right) - \delta(\tau + \beta\dot{\theta_1}^2 - \sigma\dot{\theta}_0\dot{\theta}_1)}{\gamma^2 - \alpha\delta} \\ \dot{\theta}_1 \\ \dfrac{\gamma\left(\tau + \beta\dot{\theta_1}^2 - \sigma\dot{\theta}_0\dot{\theta}_1\right) - \alpha(\epsilon\dot{\theta_0}^2 + \rho)}{\gamma^2 - \alpha\delta} \end{bmatrix}$$

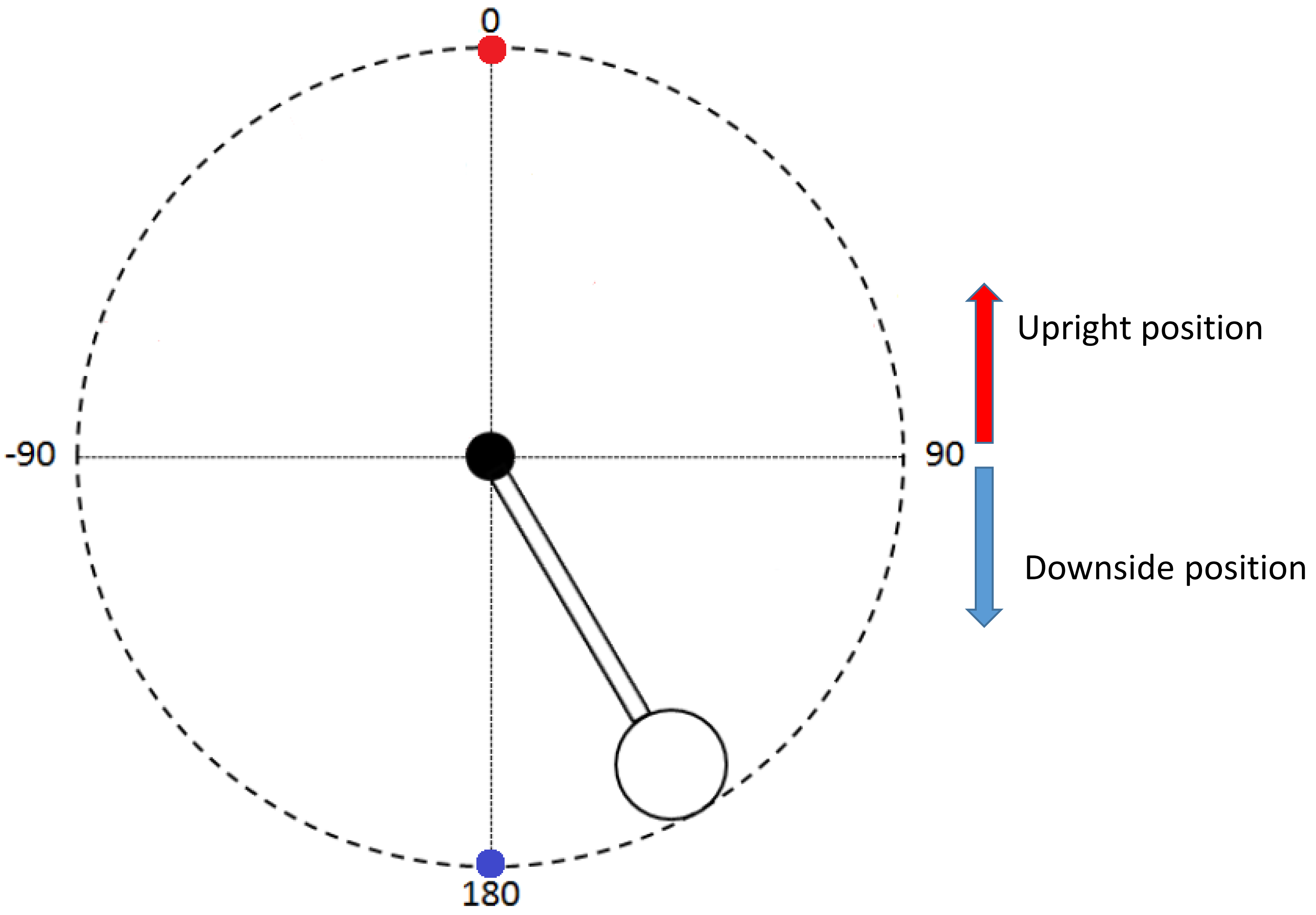
$$\alpha = I_0 + L_0^2 m_1 + l_1^2 m_1 \sin^2\theta_1$$
$$\beta = L_0 m_1 l_1 \sin\theta_1$$
$$\gamma = L_0 m_1 l_1 \cos\theta_1$$
$$\delta = I_1 l_1^2 m_1$$
$$\epsilon = l_1^2 m_1 \sin\theta_1 \cos\theta_1$$
$$\rho = m_1 g l_1 \sin\theta_1$$

Dynamic Model

$$\dot{x} = f(x,u) = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \\ f_3(x,u) \\ f_4(x,u) \end{bmatrix} \rightarrow \ \dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \frac{\partial f_1(x,u)}{\partial x_2} & \frac{\partial f_1(x,u)}{\partial x_3} & \frac{\partial f_1(x,u)}{\partial x_4} \\ \frac{\partial f_2(x,u)}{\partial x_1} & \frac{\partial f_2(x,u)}{\partial x_2} & \frac{\partial f_2(x,u)}{\partial x_3} & \frac{\partial f_2(x,u)}{\partial x_4} \\ \frac{\partial f_3(x,u)}{\partial x_1} & \frac{\partial f_3(x,u)}{\partial x_2} & \frac{\partial f_3(x,u)}{\partial x_3} & \frac{\partial f_3(x,u)}{\partial x_4} \\ \frac{\partial f_4(x,u)}{\partial x_1} & \frac{\partial f_4(x,u)}{\partial x_2} & \frac{\partial f_4(x,u)}{\partial x_3} & \frac{\partial f_4(x,u)}{\partial x_4} \end{bmatrix} \quad B = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial u} \\ \frac{\partial f_2(x,u)}{\partial u} \\ \frac{\partial f_3(x,u)}{\partial u} \\ \frac{\partial f_4(x,u)}{\partial u} \end{bmatrix}$$

Operating Points



Dynamic Model

$$\dot{x} = f(x,u) = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \\ f_3(x,u) \\ f_4(x,u) \end{bmatrix} \rightarrow \dot{x} = Ax + Bu$$

Upright operating point

$x_0 = [0 \quad 0 \quad 0 \quad 0]^T$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -15.88 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 79.19 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 17.64 \\ 0 \\ -38.94 \end{bmatrix}$$

Downside operating point

$x_0 = [0 \quad 0 \quad \pi \quad 0]^T$

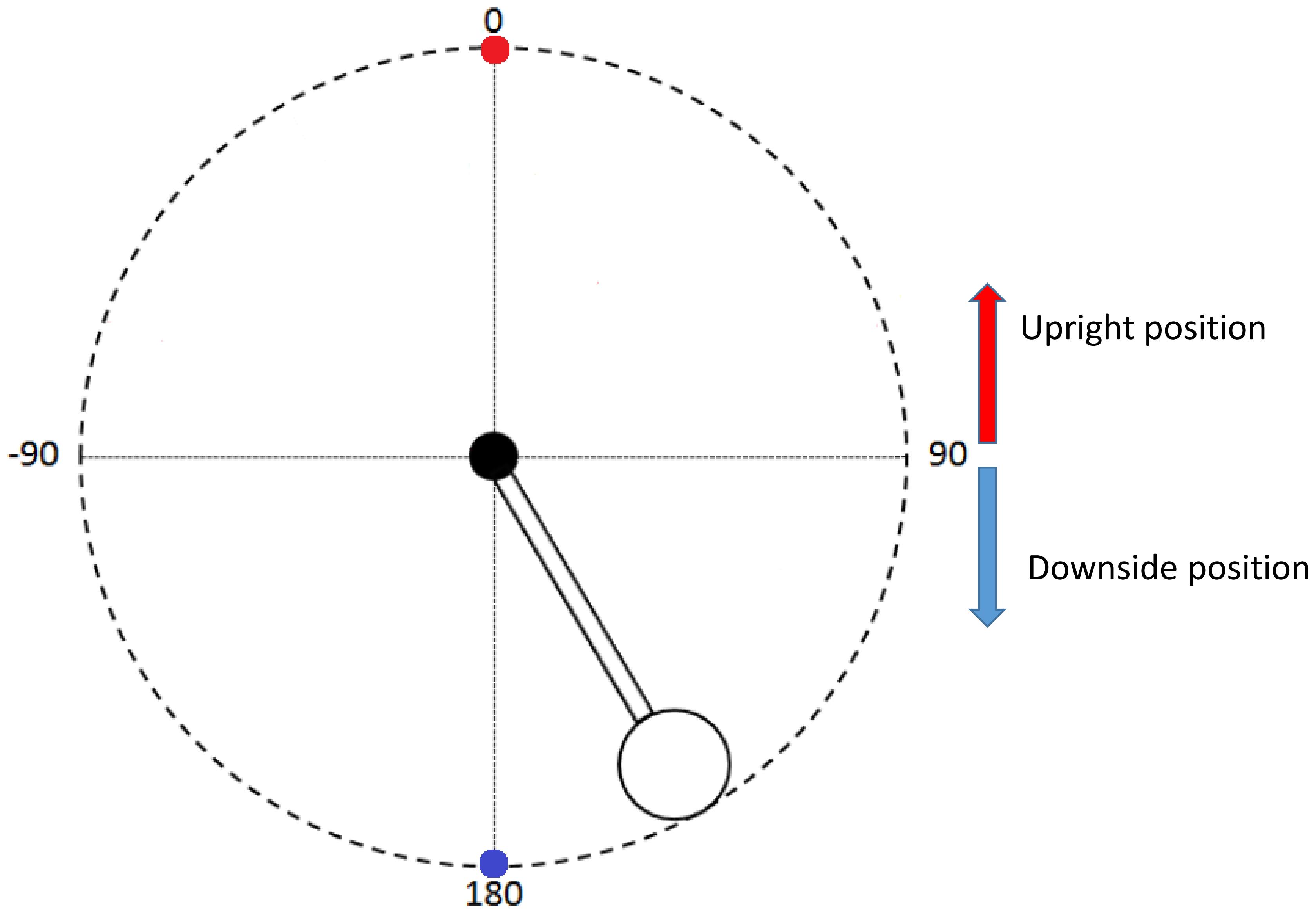
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -15.88 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -75.88 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 17.64 \\ 0 \\ 38.94 \end{bmatrix}$$

Controllers

Upright position:
- LQR
- MPC

Downside position:
- Swing-up

Full range:
- NMPC



LQR Design

LQR formulation:

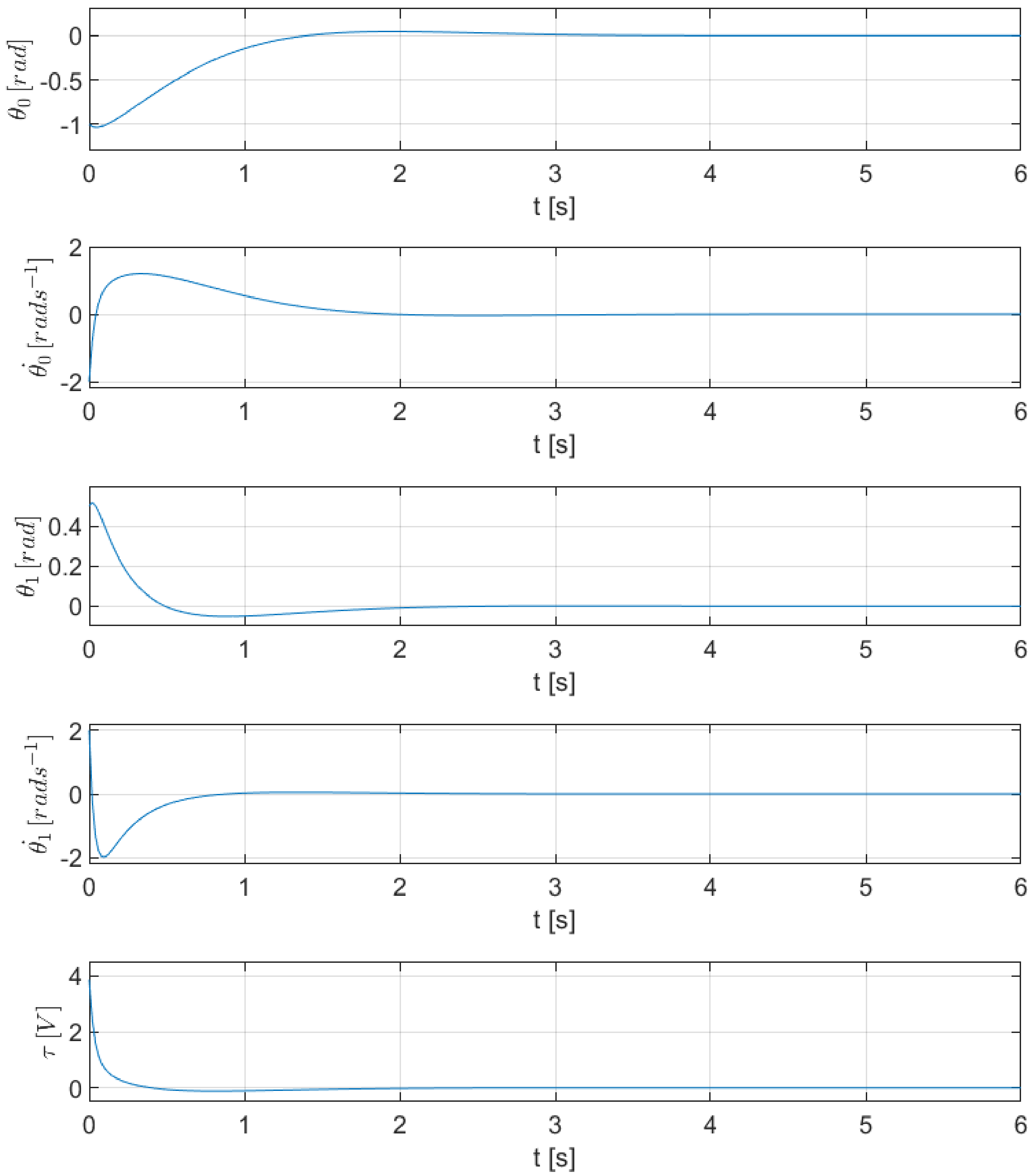
$$J = \min_{u} \frac{1}{2} \int_{t_0}^{t_f} x^T Q\, x + u^T R u\, dt$$

$$u^* = -R^{-1} B^T P\, x$$

LQR setup:

$$Q = \begin{bmatrix} 0.6 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \qquad R = 1$$

Simulations:



MPC: Design

MPC formulation:

$$J = \min_u \sum_{k}^{k+N} x^T Q_x x + u^T Q_u u \, dt$$

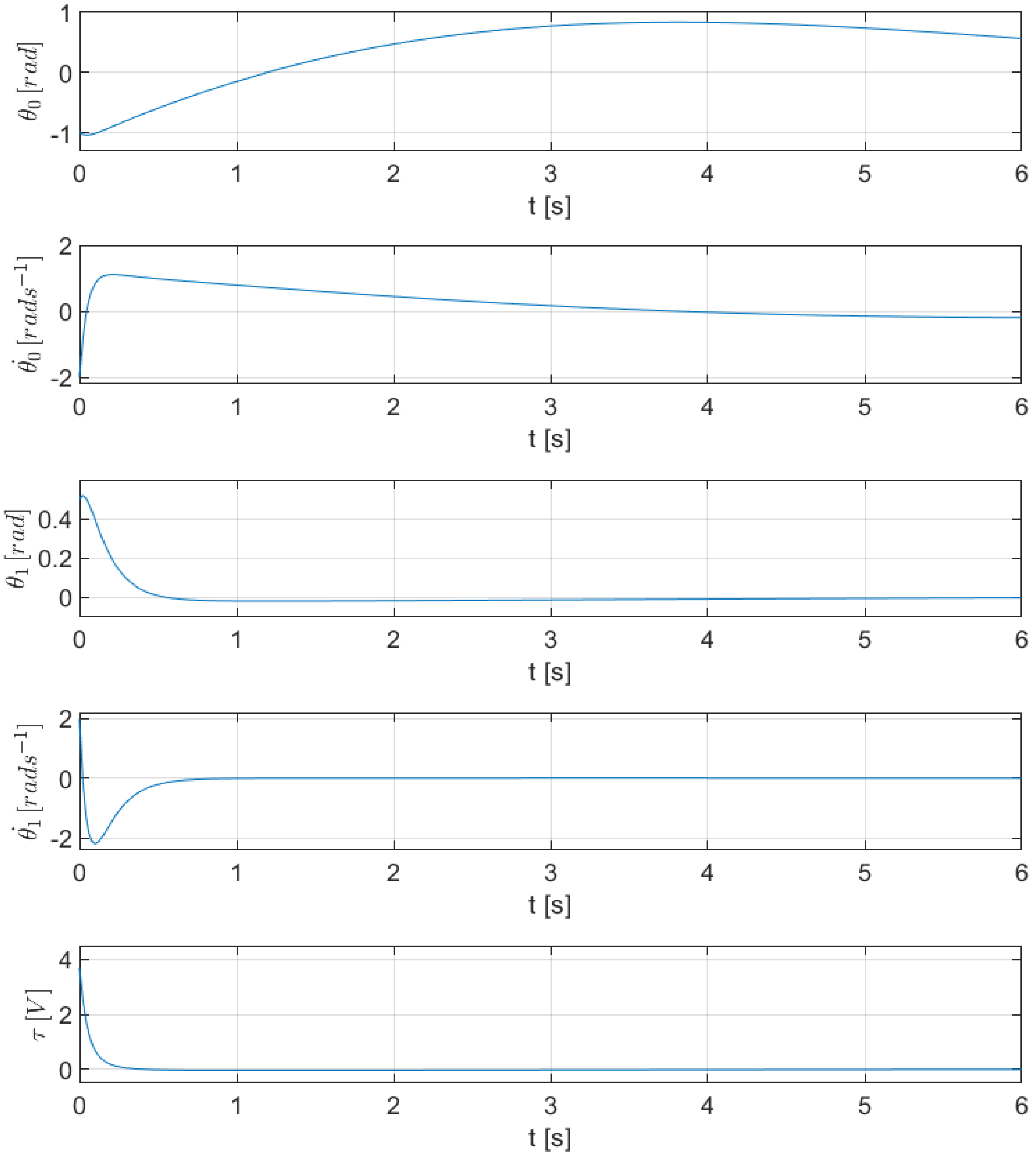
$$\begin{aligned} s.t.\ x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \\ u_{min} &\le u_k \le u_{\max} \end{aligned}$$

MPC setup:

$$Q_X = \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 0.08 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} \quad Q_u = 1$$

$$N = 20$$
$$u_{min} = -5$$
$$u_{max} = 5$$

Simulations:

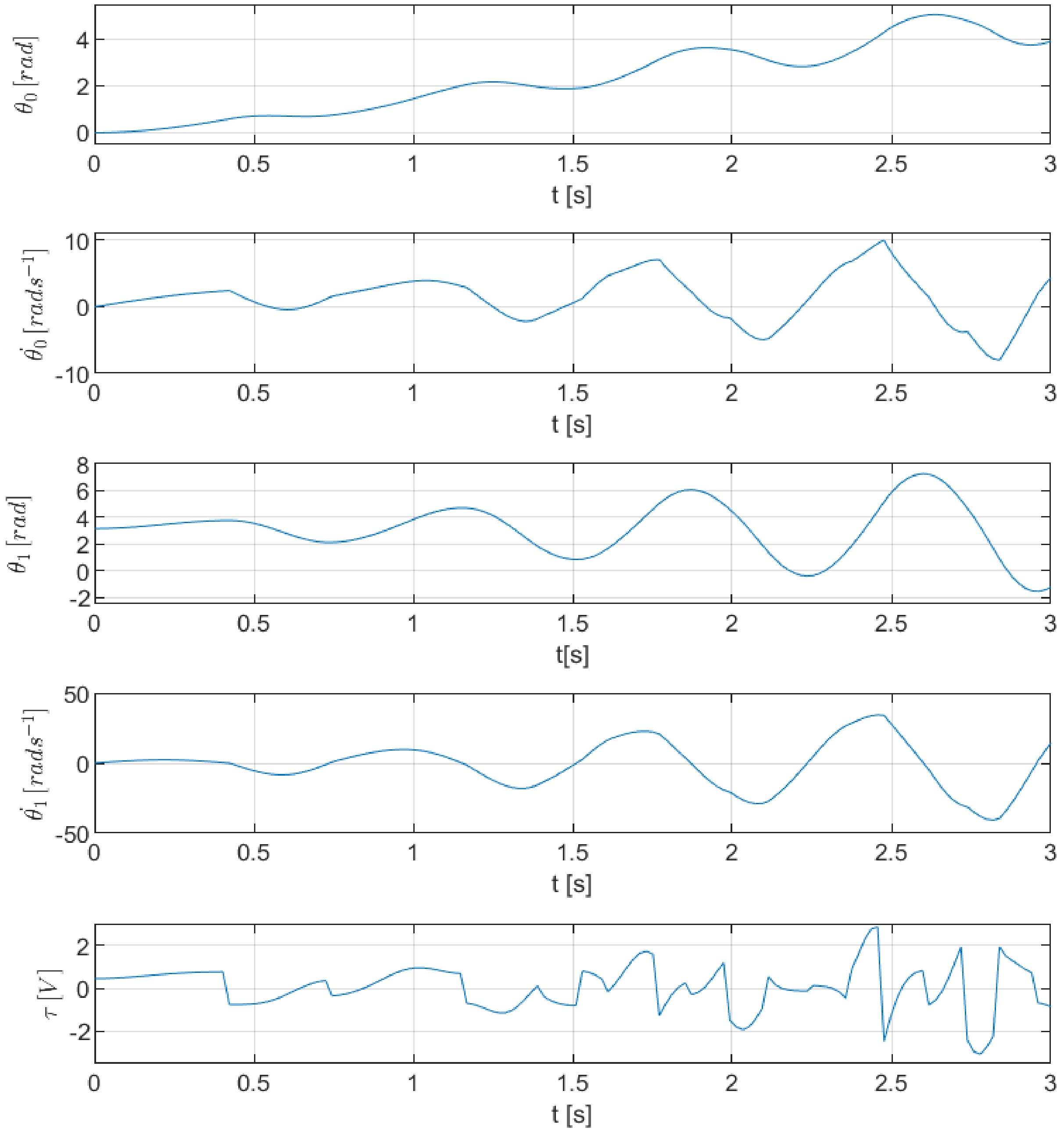


Swing-up Controller: Design

$$u = \ k_v E sign(\dot{\theta}_1 \cos \theta_1)$$

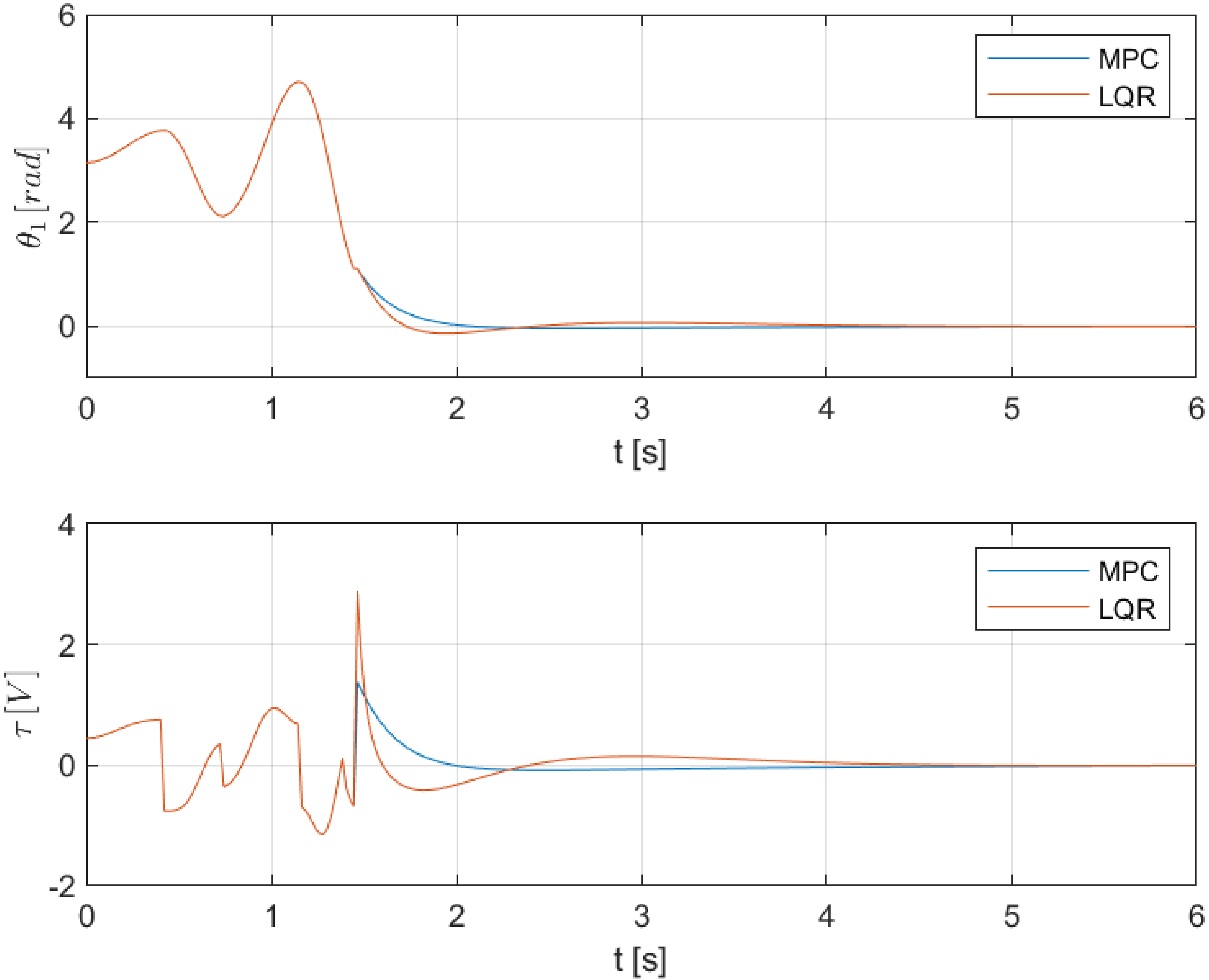
$$E = \ \frac{m_1 g l_1}{2}\left(\left(\frac{\dot{\theta}_1}{\sqrt{\frac{m_1 g l_1}{I_1}}}\right)^2 + \cos \theta_1 - 1\right)$$

Simulations:



Combined Control Strategies

- Swing-up + LQR
- Swing-up + MPC



NMPC: Design

Software used:

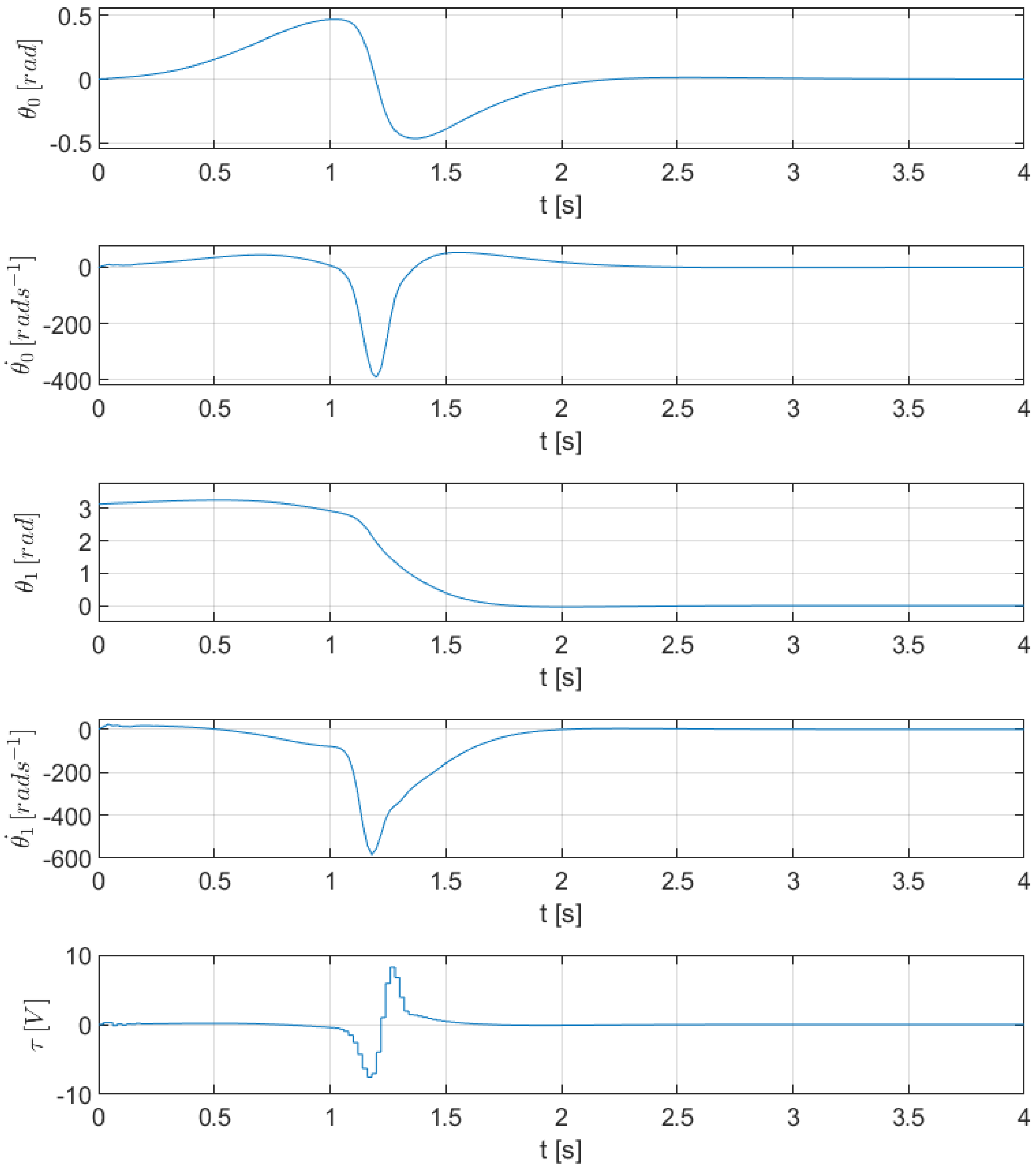
- ***MATMPC*** toolbox
- ***CasAdi*** toolbox
- ***MinGW-w64 C/C++*** Compiler

NMPC setup

$$Q_X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \quad Q_u = 0.1$$

$$N = 20$$
$$u_{min} = -10$$
$$u_{max} = 10$$

Simulations:



Conclusions

Controllers design:

- LQR
- MPC controller
- Swing-up controller
- NMPC controller

Full-range control:

- Swing-up + LQR
- Swing-up + MPC
- NMPC