A SPACE TRAVEL SIMULATOR: PHYSICS AND ASTRONOMY BEHIND THE PYTHON PROJECT

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Abstract. In this document, we outline the physics and astronomy involved in the Python project 'a space travel simulator'. We consider a spaceship traveling along a straightline to a star of distance 10s or 100s of lightyears from the Earth. In the first half of the trip, the ship travels with a constant proper acceleration of $g = 9.8\ m \cdot s^{-2}$ and in the second half with $-g$. The simulation uses the Hipparcos and Tycho star catalog for right ascension, declination, parallax, magnitude and B-V color index of the stars to (1) compute the distance to the destination star, physical states of the ship (coordinate time, proper time, speed and Lorentz factor) during travel as well as (2) apparent position of the stars, including effective magnitude and apparent color.

Disclaimer: This is an amateur project. Most sources are available freely online. I will make an attempt to cite original source but will have some omissions. All the errors are my own.

1. History and Motivation

Since antiquity, the human race shared a passion for exploration. We have explored the surface of the Earth, the depth of the ocean. We orbited above the atmosphere and visited the moon. The furthest record a human has traveled from the Earth (as of 2024, see [3]) is the Apollo 13 crew on April 14th, 1970, at a distance of 400,171 km from the Earth. The furthest distance a man-made object has traveled is the Voyager-1 space probe [4], at a distance of 164.7 AU (one astronomical unit 1 AU $\approx 1.496 \times 10^8$ km, the average distance between the Earth and the Sun) from the Earth traveling at roughly 17 km/s as of August, 2024. The Voyager-1 probe has crossed the heliopause and entered into the interstellar space (defined to be where solar wind could not overcome stellar wind of other stars [5]).

Compared to the nearest stars however, the distance of hundreds of AU is insignificant. One of the closest star to the Sun, the Alpha Centauri, sits at $L = 4.3$ lightyears away from the Sun [6], or roughly 2.7×10^5 AU. At the current speed of Voyager-1 probe, it will take roughly 8×10^4 years to reach it, longer than then entierty of human history with written records (assuming it travels in a straight line with constant speed). The huge ratio L/AU implies that the trigonometric parallax – the annual motion of a star on the celestial sphere due to perspective at the opposite ends of the Earth's orbit – is rather small. It is in fact so tiny that it was for centuries not observable, and this lack of observation has long been an argument against heliocentrism. The first valid measurement of stellar parallax (61 Cygni) was made by Friedrich Bessel in 1838 [7].

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In 1905, Albert Einstein published his famous paper 'On the Electrodynamics of Moving Bodies' [1]. He argues that the Michelson-Morley experiment [8] in 1887 which intended to measure Earth's motion in aether, in fact verifies the astonishing fact that speed of light is a constant in any reference frame. Together with the work of Hendrik Lorentz [2], they established the theory of special relativity. The fastest speed in the universe is the speed of light $c = 299{,}792{,}458$ m/s and this is the same whichever reference frame you are in. The counter-intuitive fact that when you travel at $0.99c$ and point a flashlight forward, the light from it does not travel at $1.99c$ but still at c, is best understood in terms of the mathematics of Lorentz transformation and the accompanying notion of the Minkowski spacetime. That is, the space and time coordinates is no longer of completely different nature, but can be transformed into one another with a change of reference frame.

By the way, c is today a fixed integer value in the International System of Units [9] precisely because it is a constant – the less accurate standard of 1 meter was replaced with the more accurate standard of 1 second. A lightyear ly $= 9.4607304725808 \times 10^{12}$ km is roughly the distance traveled by light in a year (the value is precise, 'a year' is subject to different interpretations).

The Milky Way galaxy is roughly 8.74×10^4 ly across [21] and the fastest possible spaceship will take longer than 87 thousand years to traverse it. The best we could hope for theoretically, in terms of exploring exoplanets of a star up close, is to send a probe to send back photos of a star up to 50 ly away from the Sun, given the finite life-span of a modern human. However, there is good news if we wish to visit some star ourselves. Let us first recall the famous twin paradox [17] of special relativity.

Consider two twin brothers A and B. A stays on Earth and B travels into space. When B is traveling at speed v (we usually talk instead about the unitless quantity $\beta = v/c$), according to special relativity he experiences time slower than that of A, at a rate of the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}\,.$$

For v close to c, this is going to be a significant factor. However, since brother A is also traveling at speed v relative to B, the time experienced by A is also slower than that of B at the same rate γ. The question then is: when B returned to Earth, which twin brother is going to be older?

The key in the solution is the notion of proper time and world-line. There is no 'object' in the Minkowski spacetime, only events / instances. An object (any physical object with mass) moving around space (or at rest, depending on reference frame) traces out a world-line in spacetime naturally parametrized by the 'proper time' τ. The total time experienced by the object from point p to point q in spacetime is given by the integral of the proper time

$$\Delta\tau = \int_p^q d\tau = \int_p^q \frac{dt}{\gamma}$$

where t is the time coordinate of the rest frame (relative to Earth), which we also refer to as 'coordinate time'. $\Delta\tau_A$ is just the difference in coordinate time since $\gamma \equiv 1$ for brother A. On the other hand, $\Delta\tau_B < \Delta\tau_A$ since $\gamma > 1$ for at least some part of the world-line of B. The conclusion: brother B will be younger.

Stellar travel has long been in the imagination of popular culture: from the 'hyperspace' in Star Wars to the 'warp speed' in Star Trek and the 'FTL (faster-than-light) drive' in Battlestar Galactica. However, it is less known that one can in fact travel in the not-so-hypothetical sub-light speed, to almost the edge of the known universe and back in less than 100 years in proper time, whilst experiencing a proper acceleration of 9.8 m/s^2, with the same effect as gravity on Earth! This is what motivates me to work on this project that draws what the sky will look like during one of these trip – a good pedagogical toy to learn about the effects of Einstein's special relativity (SR). Wonder how different it's going to turn out from the classic 'flying through space' screensaver of Windows XP [1]? Read on!

2. Relativistic space travel

We consider a spaceship traveling along a straight line from the Earth to a nearby star. For most of the journey, the ship will accelerate with acceleration $g = 9.8$ $m{\cdot}s^{-2}$ (equal to Earth's gravity). The ship lifts off from the Earth and arrives at the destination star after slow-down. The ship will have constant **proper** acceleration g for the first half, followed by a brief maneuver (ignored in calculation) to point the engines backwards and accelerate at $-g$ for the second half.

Let T be the half-time of the travel and let L be the distance to destination. Proper acceleration is given by [22]

$$\alpha = \gamma^3 a$$

where a is acceleration in the rest frame. [2]. $\beta = v/c$ in the first half of travel $0 \le t \le T$ solves this first order differential equation:

$$\begin{cases} \dfrac{c\dot{\beta}}{(1-\beta^2)^{3/2}} = g\,, \\ \beta(0) = 0\,. \end{cases}$$

The solution is given by a direct integral:

$$\beta(t) = \frac{t/T_{gc}}{\sqrt{(t/T_{gc})^2 + 1}}\,.$$

where

$$T_{gc} = \frac{c}{g} = \frac{299792458 m \cdot s^{-1}}{9.8\, m \cdot s^{-2}} \approx 0.96937\,\mathrm{yr}$$

where 1 yr $= 3600 \times 24 \times 365.25 = 3.15576 \times 10^7$ s. The Lorentz factor for the first half is given by

$$\gamma(t) = \sqrt{(t/T_{gc})^2 + 1}\,.$$

Let $x(t)$ be the (coordinate) distance covered, we have

$$x(t) = \int_0^t c\beta(s)ds = L_{gc}\left(\sqrt{(t/T_{gc})^2 + 1} - 1\right)$$

where

$$L_{gc} = \frac{c^2}{g} \approx 0.96937\,\mathrm{ly}\,.$$

[1] or even earlier Windows, but XP is what I remembered it from as a kid

[2] I will later use α for another quantity

Note that $x(t) = \frac{1}{2}gt^2 + O(t^3)$ for small t, the non-relativistic form. Let

$$
\begin{aligned}
x_T &= x(T) = L_{gc}\left(\sqrt{(T/T_{gc})^2+1}-1\right) \\
\beta_T &= \frac{T/T_{gc}}{\sqrt{\left(\frac{T}{T_{gc}}\right)^2+1}}\,.
\end{aligned}
$$

For the second half $T \le t \le 2T$, $\beta(t)$ solves:

$$
\begin{cases}
\dfrac{c\dot\beta}{(1-\beta^2)^{3/2}} = -g\,, \\
\beta(0) = \beta_T\,.
\end{cases}
$$

We have

$$
\beta(t) = \frac{(2T-t)/T_{gc}}{\sqrt{((2T-t)/T_{gc})^2+1}}
$$

and

$$
\gamma(t) = \sqrt{((2T-t)/T_{gc})^2+1}\,.
$$

$x(t)$ for $T \le t \le 2T$ is given by $x_T + \int_T^t c\beta(s)ds$. We summarize:

$$
x(t) = \begin{cases}
L_{gc}\left(\sqrt{\left(\frac{t}{T_{gc}}\right)^2+1}-1\right) & 0 \le t \le T\,, \\
L_{gc}\left(2\sqrt{\left(\frac{T}{T_{gc}}\right)^2+1}-\sqrt{\left(\frac{2T-t}{T_{gc}}\right)^2+1}-1\right) & T < t \le 2T\,,
\end{cases}
$$

and

$$
\beta(t) = \begin{cases}
\dfrac{t/T_{gc}}{\sqrt{(t/T_{gc})^2+1}} & 0 \le t \le T\,, \\
\dfrac{(2T-t)/T_{gc}}{\sqrt{((2T-t)/T_{gc})^2+1}} & T < t \le 2T\,.
\end{cases}
$$

The Lorentz factor is given by

$$
\gamma(t) = \begin{cases}
\sqrt{(t/T_{gc})^2+1} & 0 \le t \le T\,, \\
\sqrt{((2T-t)/T_{gc})^2+1} & T < t \le 2T\,.
\end{cases} \tag{1}
$$

The maximum β and γ values are

$$
\beta_{\max} = \frac{T/T_{gc}}{\sqrt{(T/T_{gc})^2+1}}, \quad \gamma_{\max} = \sqrt{(T/T_{gc})^2+1}\,.
$$

We also have

$$
x(T) = L_{gc}\left(\sqrt{(T/T_{gc})^2+1}-1\right), \quad x(2T) = 2L_{gc}\left(\sqrt{(T/T_{gc})^2+1}-1\right)\,.
$$

Set $L = x(2T)$ we have

$$T = T_{gc}\sqrt{(L/(2L_{gc})+1)^2 - 1}\,.$$

The maximum Lorentz factor is given by

$$\gamma_{\max} = \frac{L}{2L_{gc}} + 1\,. \tag{2}$$

For long trips the term 1 is negligible and since L_{gc} is approximately 1 ly, a rule of thumb is $\gamma_{\max} \sim L/(2\,\mathrm{ly})$. From (1), the total proper time elapsed is

$$\Delta\tau = \int_0^{2T} \frac{1}{\gamma(t)}dt = 2T_{gc}\,\mathrm{arcsinh}\left(\frac{T}{T_{gc}}\right)\,.$$

We have

$$\Delta\tau = 2T_{gc}\log\left(\frac{L}{2L_{gc}} + 1 + \sqrt{\left(\frac{L}{2L_{gc}}+1\right)^2 - 1}\right) \tag{3}$$

2.1. **Examples.** Trip to 61 Cygni, 11.4 ly from Earth. By (3) and (2), total travel time for the crew is $\Delta\tau \approx 5.07$ years and the maximum Lorentz factor is $\gamma_{\max} = 6.88$.

Trip to the galactic center, 8kpc $= 2.6\times10^4$ ly from Earth. Total duration for the crew is only $\Delta\tau \approx 19.77$ years and the maximum Lorentz factor is $\gamma_{\max} = 1.34\times10^4$.

3. What does the stars look like during travel?

3.1. **What does the star look like on Earth?** The ancient Greek astronomer Hipparchus of Nicaea compiled the first known star catalogs in the western world [16], where he categorized visible stars into 6 scales of apparent magnitudes, from 1 the brightest to 6 the faintest. Modern astronomers have established a system that aligns roughly with that of Hipparchus. A difference of 5 magnitudes corresponds to 100 times difference in brightness [14]. (Brightness is the energy radiated per unit area per unit time. It has SI unit $J\cdot m^{-2}\cdot s^{-1}$.) The star Vega in the constellation Lyra [23] is traditionally used as a zero point for magnitude. The magnitude of a star m is related to its brightness I by

$$m = -2.5\log_{10}(I/I_{\mathrm{Vega}})$$

Note that brightness (therefore magnitude) varies by wavelength. The bolometric magnitude [15] corresponds to the entirety of the electromagnetic spectrum. The human eye, however, can perceive only wavelengths within 380-700 nm. In modern astronomy, we adopt systems of filters to measure magnitude of a star – a practice called **photometry**. One of the most common is the UBV system [10] (**U**ltraviolet, **B**lue and **V**isible, a.k.a. the **Johnson-Morgan system**). In most star catalogs, we see magnitude in one of these filters e.g. the V magnitude m_V.

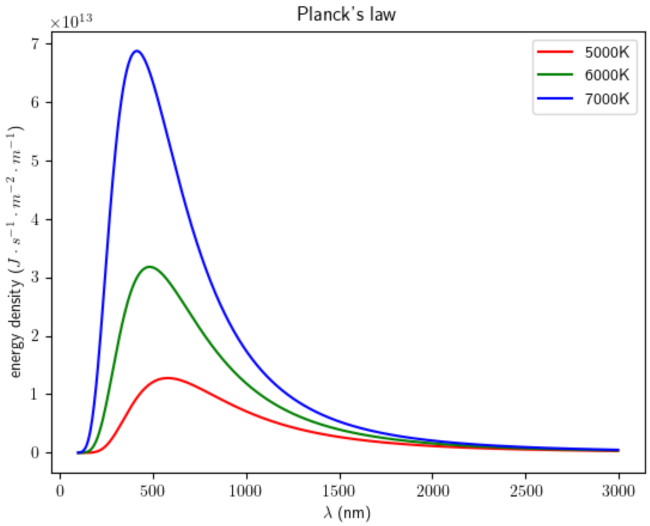
The spectrum energy density distribution of a star, to a good approximation, is that of a blackbody radiation of certain temperature T. It is governed by the Planck's law [12]:

$$I_T(\lambda)d\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}d\lambda$$

where $h \approx 6.626 \times 10^{-34} J \cdot s$ is the Planck's constant, $k_B \approx 1.38 \times 10^{-23} J \cdot K^{-1}$ is the Boltzman constant and T is temperature in Kelvin. Note that $I_T(\lambda)$ has SI unit $J \cdot m^{-2} \cdot s^{-1} \cdot m^{-1}$ (energy per unit area per unit time per unit wavelength interval). The 'bolometric' brightness of a blackbody is proportional to T^4, by changing variable and using $\int_0^\infty \frac{x^3}{e^x-1} dx = \pi^4/15$ we have:

$$\int_0^\infty I_T(\lambda) d\lambda = \sigma T^4$$

where $\sigma = \dfrac{2\pi^5}{15} \dfrac{k_B^4}{c^2 h^3} \approx 5.67 \times 10^{-8} W \cdot m^{-2} \cdot K^{-4}$ is the Stefan-Boltzman constant [13].



Consider a star of radius [3] R_{star}, temperature T at distance D from the Earth. Let $f_V(\lambda)$ be the normalized [4] profile of the V filter. We have

$$I = \frac{R_{\text{star}}^2}{D^2} \int_0^\infty f_V(\lambda) I_T(\lambda) d\lambda$$

and

$$\begin{aligned} m_V &= -2.5 \log_{10} \left(\frac{R_{\text{star}}^2}{D^2} \int_0^\infty f_V(\lambda) I_T(\lambda) d\lambda \right) + 2.5 \log_{10} I_{\text{Vega,V}} \\ &= -2.5 \log_{10} \left(\int_0^\infty f_V(\lambda) I_T(\lambda) d\lambda \right) + 5 \log_{10} D + (\text{const}) \end{aligned} \tag{4}$$

where '(const)' is a constant depending on the V intensity of Vega and R_{star}.

The shape of the blackbody radiation energy density spectrum allows us to gain information of temperature from intensity difference between e.g. Johnson B and V filters. This leads to the notion of 'color index' in astronomy. The following

[3]To be precise, this is radius of its photosphere, where light is radiated from

[4]Set $\int_0^\infty f_V(\lambda) d\lambda = 1$

approximation formula by Ballesteros [25] convers B-V color index to temperature:

$$T\,(\text{Kelvin}) = 4600\left(\frac{1}{0.92BV+1.7}+\frac{1}{0.92BV+0.62}\right).$$

3.2. **Digression: Minkowski spacetime, world-line, pseudometric and the Lorentz transform.** The usual Minkowski spacetime [18] is the 4-dimensional Euclidean space $\mathbb{R}^4$ 'with metric signature (+,-,-,-)', i.e. the tangent space to each point $T_{\vec{x}}\mathbb{R}^4$ is a vector space equipped with a pairing [5] of signature (+,-,-,-). Manifolds of this type is also called pseudo-Riemannian manifold $\mathbb{R}^{1,3}$. For two vectors $v^\mu = (v^0, \ldots, v^3)$ and $w^\mu = (w^0, \ldots, w^3) \in T_p\mathbb{R}^{1,3}$ the pairing yields [6]

$$\langle v, w\rangle = v^\mu w^\nu \eta_{\mu\nu} = v^0w^0 - v^1w^1 - v^2w^2 - v^3w^3\,.$$

where $\eta_{\mu\nu} = 1$ only for $(\mu,\nu) = (0,0)$ and -1 only for $(\mu,\nu) = (1,1)$, $(2,2)$, $(3,3)$. If $\langle v, v\rangle < 0$, v is called **time-like**. If $\langle v, v\rangle = 0$, v is called **null** (or **light-like**). If $\langle v, v\rangle > 0$, v is called **space-like**. A world-line is time-like (resp. null) if the tangent vector to each point is time-like (resp. null). Note that only the sign matters, so this definition is independent of parametrization. A physical particle with positive mass has time-like world-line whereas photons and particles with zero mass has null world-lines. For the following discussion, it suffices to consider the Minkowski spacetime $\mathbb{R}^{1,2}$. We set $c = 1$ (1 second on time axis is 299,792,458 meters on space axis).

In mathematics, a 'metric space' is a space equipped with a notion of distance between two points satisfying certain axioms (e.g. positivity and the triangle inequality). For a Minkowski spacetime, by integrating the 'metric tensor' there is a corresponding notion of 'distance' between two events, although it does not satisfy the same axioms for a metric space (e.g. distance can be negative or zero without the two points being the same). A Minkowski spacetime equipped with this notion of 'distance' is called a pseudo-metric space in mathematics. The Minkowski spacetime $\mathbb{R}^{1,n}$ come equipped with an atlas (a collection of coordinate systems) which physicists call **reference frames**, which are related by coordinate transformation called **Lorentz transformation** [20] (if the spatial origin agrees) or **Poincaré transformation** (if there is a spatial translation involved). The form of the metric tensor is invariant under such transformations, i.e. the transformations are 'isometries'. In fact in modern particle physics, elementary particles arrange neatly according to irreducible representations of the Lie group formed by such transformations – the Poincaré group [19].

The following is an example of a Lorentz transformation in $\mathbb{R}^{1,1}$:

$$\Lambda = (\Lambda^\mu_\nu) = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$$

where $\gamma = 1/\sqrt{1-\beta^2}$. Note that $\gamma^2 - \gamma^2\beta^2 = 1$, therefore there is $y \in \mathbb{R}$ such that

$$\Lambda = \begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix}$$

[5] which physicists call metric tensor

[6] The superscript is a physics convention for covariant vector, not to be confused with taking power

We see manifestly that $\Lambda \in O(1,1)$, i.e. Λ is an isometry $\Lambda^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This particular transformation is called a Lorentz boost [7]. See here for a nice animation of Lorentz boost in $\mathbb{R}^{1,1}$.

3.3. **Relativistic aberration of light.** Consider two reference frames S and S', both has spatial center at the origin with S at rest and S' moving with velocity [8] $\vec{v} = (\beta, 0)$. In S, consider a light source located at $(R\cos\theta, R\sin\theta)$. Its world-line is given by the parametrized curve $\mathbb{R} \to \mathbb{R}^{1,2}$: $\tau \mapsto (\tau, R\cos\theta, R\sin\theta)$. Note that the latter two, i.e. the spatial coordinates, are constants in parameter τ, because in S the light source is at rest. The world-line of the center of S' is given by the parametrized curve $\mathbb{R} \to \mathbb{R}^{1,2}$: $\tau \mapsto (\gamma\tau, \gamma\beta\tau, 0)$. We apply a Lorentz transform to get the world-line in frame S':

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that $\Lambda \cdot (\gamma\tau, \gamma\beta\tau, 0)^T = (\tau, 0, 0)$ as expected, i.e. the spatial center of S' is at rest in S' itself. By applying Λ we also see that the world-line of the light source is given in S' by:

$$\tau \mapsto \begin{pmatrix} \gamma\tau - R\gamma\beta\cos\theta \\ -\gamma\beta\tau + R\gamma\cos\theta \\ R\sin\theta \end{pmatrix}$$

Consider a null world-line in frame S given by[9] $s \mapsto (s, -s\cos\theta, -s\sin\theta)$. This intersects that of the light source as well as the origin of frame S at $s = 0$. It is the world-line of a photon emitted by the light source and observed at the origin by some observer traveling at velocity $\vec{v}$. In S' it is given by

$$\Lambda \cdot \begin{pmatrix} s \\ -s\cos\theta \\ -s\sin\theta \end{pmatrix} = \gamma(1+\cos\theta)s \begin{pmatrix} 1 \\ -\dfrac{\beta+\cos\theta}{1+\beta\cos\theta} \\ * \end{pmatrix} = \begin{pmatrix} s' \\ -s'\cos\theta' \\ -s'\sin\theta' \end{pmatrix}$$

The reason for the last equality is this: all coordinates are affine functions of the parameter s, therefore we may use the first (temporal) component as a new parameter s'. Note that the newly parametrized null world-line will still need to pass through $(0,0,0)$, thus each component is actually a linear function in s'. It is now clear that θ' is the apparent angle at which the light reaches the observer. We have

$$\cos\theta' = \frac{\beta + \cos\theta}{1 + \beta\cos\theta}$$

This is precisely the form of the formula in [1, p. 912]

$$\cos\phi' = \frac{\cos\phi - v/c}{1 - \cos\phi \cdot v/c}$$

[7] Some also call this a 'hyperbolic rotation'

[8] This is giving a description of the time-axis of S' as a coordinate system of $\mathbb{R}^{1,2}$

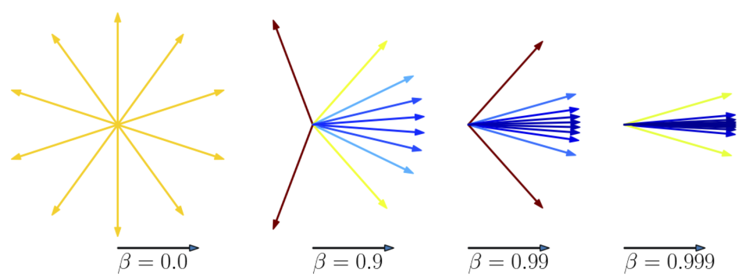
[9] Unlike time-like world-lines, there is no canonical way to parametrize null world-lines: there is no such thing as proper time experienced by a photon.

where ϕ resp. ϕ' in Einstein's original paper are angles between 'the wavefront normal and the connecting line source-observer', i.e. $\phi = -\theta$, $\phi' = -\theta'$ for us. This is known as **aberration** of light. In a not-so-precise metaphor in non-relativistic everyday life, imagine driving a car at very fast speed in the rain. The rain drops which falls directly downward will appear to the moving vehicle as tilted in the direction opposite of the travel.

Note for instance for $\theta = \pi/2$, we have $\cos\theta' = \beta$. All the light sources within the half-space in the rest frame S facing the direction of travel is seen in frame S' within a cone of angle arccosβ, which is rather small when $\beta = v/c$ is close to 1. It is not hard to convince oneself by extension that as we travel at near relativistic speed, the majority of the night sky will appear concentrated in a small cone along the direction of the travel. An equivalent statement is that a moving light source with relativistic speed has most of its intensity focused in a small cone along the moving direction. This is known as the **relativistic beaming effect** [24] or the **relativistic headlight effect**. As a result if a light source at rest emits at intensity I_0 to an observer as in the above set-up, the apparent intensity to the observer is given by

$$I = I_0\gamma^2(1 + \beta\cos\theta)^2$$

In fact, this formula can also be found on p.912 of [1]. The following is an illustration borrowed from Wikipedia.



3.4. **Relativistic Doppler effect.** On the same page of [1] is the formula for relativistic Doppler effect:

$$\nu' = \nu\gamma(1 + \beta\cos\theta)$$

where ν is the emitted frequency and ν' the observed frequency by the moving observer. (See last subsection for definition of θ.) We omit its derivation. The effect is similar to the non-relativistic version experienced when we hear the change in the pitch of the horn of an incoming or leaving train. For a light source moving toward us (resp. away from us), its light will appear blue shifted (resp. red shifted). We have in terms of wavelength:

$$\lambda' = \frac{\lambda}{\gamma(1 + \beta\cos\theta)}\,.$$

During our space travel for all but a small cone along opposite direction of travel, the star light will appear blue shifted.

3.5. **Relativistic effect on visible star light.** When it comes to appearance of a star seen by a moving *human* observer, another issue must be addressed: the spectral energy density of blackbody radiation has a specific shape that peaks at certain wavelength. At a temperature of about 5800K (temperature of the solar photosphere) this peak falls squarely in the visible range. However if we blue-shift the spectrum, the visible range will appear weaker relatively. This raises the question: what is the net effect on e.g. the V magnitude m_V?

By Einstein's Doppler effect formulas, the light emitted by the star from λ to $\lambda+\Delta\lambda$ with intensity $I_T(\lambda)\Delta\lambda$ is observed (before filtering) as ranging in wavelength from $\lambda' = \lambda/(\gamma(1+\beta\cos\theta))$ to $\lambda'+\Delta\lambda' = (\lambda+\Delta\lambda)/(\gamma(1+\beta\cos\theta))$ with intensity

$$\gamma^2(1+\beta\cos\theta)^2 I_T(\lambda)\Delta\lambda = \gamma^3(1+\beta\cos\theta)^3 I_T(\gamma(1+\beta\cos\theta)\lambda')\Delta\lambda'$$

It follows that the apparent V magnitude to the moving observer is given by

$$\begin{aligned}
m'_V - 5\log_{10} D' - (\text{const}) &= -2.5\log_{10}\left(\int_0^\infty \gamma^3(1+\beta\cos\theta)^3 f_V(\lambda) I_T(\gamma(1+\beta\cos\theta)\lambda)d\lambda\right)\\
&= -2.5\log_{10}\left(\frac{1}{\gamma^2(1+\beta\cos\theta)^2}\int_0^\infty f_V(\lambda) I_{\gamma(1+\beta\cos\theta)T}(\lambda)d\lambda\right)\\
&= -2.5\log_{10}\left(\int_0^\infty f_V(\lambda) I_{\gamma(1+\beta\cos\theta)T}(\lambda)d\lambda\right) + 5\log_{10}(\gamma(1+\beta\cos\theta))
\end{aligned}$$

where '(const)' is as in (4), D' is the distance from the star to the observer in the rest frame (of Earth), we used that for $a>0$

$$I_T(a\lambda) = a^{-5} I_{aT}(\lambda)$$

Define

$$\alpha(T) \propto \int_0^\infty f_V(\lambda) I_T(\lambda)d\lambda$$

normalized such that $\alpha(T_{\text{Sun}}) = 1$ where $T_{\text{Sun}} = 5800\text{K}$. We have from the above that

$$\begin{aligned}
m'_V = m_V &+ 2.5\log_{10}\alpha(T) - 2.5\log_{10}\alpha(\gamma(1+\beta\cos\theta)T) + 5\left(\log_{10} D' - \log_{10} D\right)\\
&+ 5\log_{10}(\gamma(1+\beta\cos\theta))
\end{aligned}$$

4. Hipparcos and Tycho catalog

The Hipparcos satellite, named after the Greek astronomer Hipparchus (but also an acronym for HIgh Precision PARallax COllecting Satellite), was launched by European Space Agency in 1989 and operated until 1993. It allowed compliation of the Hipparcos Catalog [27], containing highly accurate data of more than 118,200 stars, including magnitude, proper motion and parallaxes. The actual database used in the Python project is the one created in April 2000 by HEASARC (NASA's High Energy Astrophysics Science Archive Research Center, [28]) based on CDS Catalog I/239 file `hip_main.dat.gz`.

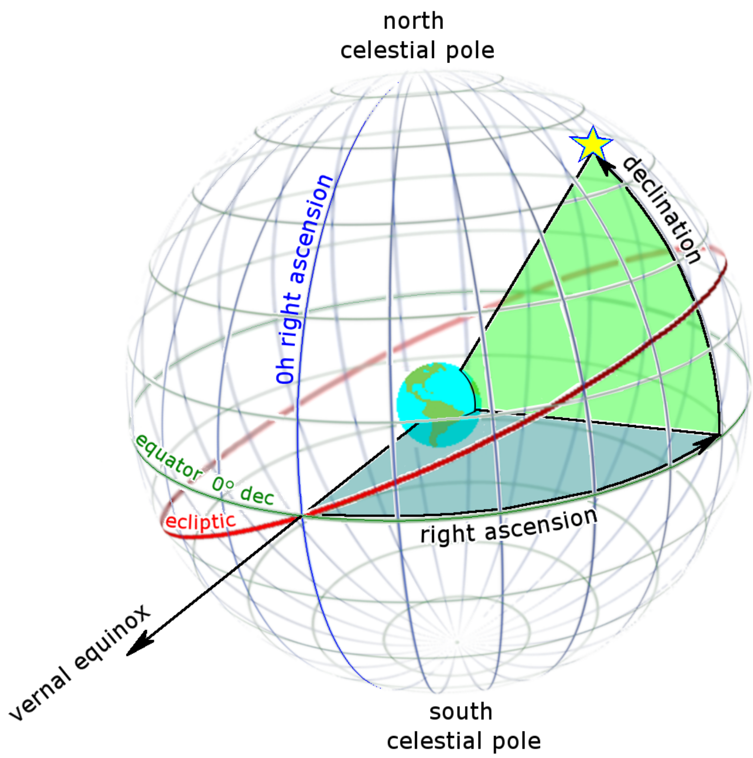
The Tycho catalog (TYC) compiled from raw data from the same mission contains more than 10^6 stars but with lower precision. The enhanced Tycho-2 catalog of 2.5×10^6 stars, however, lacks the parallax data. For the Python project, the user can choose between using Hipparcos + Tycho catalog or simply the smaller and more accurate Tycho catalog. Using the Tycho catalog results in slower performance due to its huge size.

Parallaxes of stars in these catalogs has unit mas = miliarcsecond. The distance to a star is computed as

$$D = 1000\cdot\left(\frac{\text{parallax}}{\text{mas}}\right)^{-1}\ \text{parsec}\,,$$

where 1 parsec is roughly 3.62156 ly. It is somewhat surprising that parallaxes of some of the brightest stars in the night sky are actually rather hard to measure accurately. For instance Betelgeuse (a.k.a. Alpha Orionis) in the constellation Orion with a V magnitude of 0.5 [31] and HIP number 27989. The listed parallax is 7.63 ± 1.64 mas, putting it at roughly 427 ly. More recent estimate of the parallax is $5.95 \begin{array}{c} +0.58 \\ -0.85 \end{array}$ mas [32].

The standard coordinate of the celestial sphere is right ascension [33] (RA) and declination (DE or Dec), defined as the azimuthal and altitude angle while facing the March equinox (one of the two intersection points of the big circles corresponding to the ecliptic plane – the plane of Earth's orbit, and the celestial equator – the plane of Earth's equator). The following illustration is from Wikipedia [33].



5. Conclusion

In conclusion, this project produces some nice simulation with full special relativity effects and serves as a good platform for education in physics, astronomy and geometry. It is fascinating to contemplate the possibility of exploring some exoplanets probably harboring alien life (e.g. HIP 38594 [10] [39]), a neutron star, a nebulae or an actual black hole (e.g. Cyg X-1), within a human life time. It is also fun to know that for any realistic space travel, the stars won't fly at you in the style of the classic Windows XP screen saver.

[10] Lots of more famous ones, but this is the only Hipparcos star in that table

There are of course a lot of factors that limit to what extent the simulation recovers an actual sky chart one sees out the forward window of the spaceship. Here is an incomplete list of deviations and limitations:

- The whole concept of a spaceship with seemingly limitless energy to use to achieve constant proper acceleration is highly theoretical. Maybe we could one day realize it using antimatter rocket [40] or something.
- The Hipparcos and Tycho star catalog does not include nebulae, nor does it include other galaxies, some of which are quite prominent e.g. the large and small Magellanic cloud and the Andromeda galaxy (Messier 31)
- The parallax (i.e. distance) data in the Hipparcos and Tycho star catalog, though already relatively accurate, could have some big error (see §4). This error should result in some inaccuracies in where stars appear (and how bright they are)
- Some stars are variable stars whose magnitude oscillates [37]. This data is available in the catalog but we choose to ignore this. It might be added in a future update.
- For travel of 100s of years, we may expect to see some stars going supernova. According to [38], the last directly observed supernovas in the Milky Way galaxy are Kepler's supernova in 1604 and Tycho's supernova in 1572. [11]
- Quite a lot of nearby stars are also missing from the Hipparcos and Tycho catalog, which could be supplemented in future updates. (E.g. we could include the Gliese catalog of nearby stars) These may become brighter during the trip. Other darker stars should be detectable during trip but not included, some could be missing from all catalogs possible (e.g. too obscured by interstellar dust that only reveals itself during travel past the dust)
- The interstellar dust reddens and obscures star light [35] in significant chunks of the Milky Way galaxy. This in fact is what gives the Milky Way its characteristic look in the night sky. If one actually travels thousands of light years, the shape of the interstellar dust cloud should be significantly different.
- The blue-shift factor $\gamma(1+\beta\cos\theta)$ for $\theta = 0$ is just the Lorentz factor. From our rule of thumb, we see that for a trip of total distance $L \sim 6000$ ly, $\gamma_{\max}$ is close to 3000 and this brings the CMB (Cosmic microwave background, [36]) radiation, a blackbody radiation of 2.7 Kelvin, to the visible range. This effect is not included. [12]
- Even though estimated proper motion of stars are available in Hipparcos and Tycho catalog, due to limited travel time in the rest frame, these are neglected. For travels of significantly longer distance, they should be included.
- We introduce 'saturation' to the brightness of a star to avoid the disk representing the star being too huge. This makes some sense physically. However, as stated in §5 the lack of range and resolution of pixel intensity of a computer screen means that the image itself has rather limited power to mimic the actual sky.

[11] and from what I gather, we cannot currently predict such event with any accuracy

[12] Frankly, it is not recommended to simulate trip to 1000s of lys since the huge Lorentz factor will compress nearly all the sky to a bright blue dot (and it does not look pretty).

- On average during travel, the star light are more powerful, more concentrated and more ultraviolet (or even X-ray and gamma-ray depending on $\gamma_{\max}$). An actual window of the traveling spaceship probably some sort of shielding to protect the crew.

Appendix I: color of the blackbody radiation

We saw that during travel, a star at an angle θ (in the rest frame) with the forward traveling direction will appear to emit blackbody radiation of temperature

$$T' = \gamma(1 + \beta \cos\theta)T$$

where T is the temperature at rest. The following is borrowed from this blog post and re-narrated in mathematical terms. For more detail see e.g. [11]

The human eyes perceive color by receiving neural signals from 3 types of cells on the back of the retina, corresponding very roughly to the three primary colors. An energy density $f(\lambda)d\lambda$ is perceived by convolution with 3 profile functions $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$ with compact support [13] and are determined from experiments and tabulated by the CIE (Commission internationale de l'éclairage).

The result is a triple of integrals (X, Y, Z), e.g.

$$X = \int f(\lambda)\bar{x}(\lambda)d\lambda\,.$$

Normalize them by $x = X/(X+Y+Z)$, $y = Y/(X+Y+Z)$ and $z = Z/(X+Y+Z)$. Note that now (x, y, z) lies on the plane $x + y + z = 1$.

A **color system** is a map from (x, y, z)-space to (R, G, B)-space to realized the color on a computer (or TV) screen. (e.g. The sRGB color system was created by HP and Microsoft in 1996 for monitors and printers [29].) It is determined by 4 points labeled 'red' (x_r, y_r, z_r), 'green' (x_g, y_g, z_g), 'blue' (x_b, y_b, z_b) and 'white' (x_w, y_w, z_w) in the triangle $\{(x, y, z) : x, y, z \geq 0, x + y + z = 1\}$ which we will call the big triangle Δ. The first 3 points makes another triangle which we will call the small triangle δ.

There is a unique linear transformation $T : \Delta \to \mathbb{R}^3$ such that $T(x_r, y_r, z_r) = (*, 0, 0)$, $T(x_g, y_g, z_g) = (0, *, 0)$, $T(x_b, y_b, z_b) = (0, 0, *)$, where $*$ stands for some positive number, and $T(x_w, y_w, z_w) = (1, 1, 1)$.

The problem here is that we expect (R, G, B) to be positive and between 0 and 1 (as per Python convention), but the region $\Delta - \delta$ is not mapped to within the first octant. The remedy is to apply two other transformations: (1) desaturation, (2) normalization.

Desaturation is a piece-wise linear map $D : T(\Delta) \to \mathbb{R}^3$, identity on the small triangle δ. The region $T(\Delta) - T(\delta)$ is cut along 3 rays (intersections with $x = y$, resp. $y = z$, $z = x$) and each projected along the 45 degree direction towards an appropriate choice of the xy, xz, yz-planes. In practice, $(x', y', z') \in T(\Delta) - T(\delta)$ is mapped to $(x', y', z') - \min(x', y', z')(1, 1, 1)$.

Normalization $N : D \circ T(\Delta) \to \mathbb{R}^3$ is given by

$$(x'', y'', z'') \mapsto (x'', y'', z'')/\max(x'', y'', z'')\,.$$

This is a non-linear map and the end result of composition of three steps is that the big triangle is mapped to the 3 faces of the cube $\{(r, g, b) : |r|, |g|, |b| \leq 1\}$ in the first octant, characterized by: one of the r, g or $b = 1$.

---

[13] meaning for λ too big or too small, their values will be 0

For the Python code in the project, we adopt an interpolation of the big table by M. Charity [30] using the CIE 1964 10 degree color matching functions.

Appendix II: Shape of star image

Ever wondered why bright stars on astronomical photos has spikes? The reason lies in the optical phenomenon called 'diffraction' and it has a lot to do with the wave nature of light itself. The **point spread function** is the profile of the diffraction patter of a point source when received by an optical system. The spikes are a result of the point spred function caused by e.g. support struts of the telescope. [34]

What is the point-spread function of the human eye? Recall that the Fraunhofer diffraction occurs when a (monochromatic) plane wave light hits some obstacle (like an aperture) and the resulting wave propagates to 'far field' before being observed. [14] The diffraction pattern can be derived by a spatial Fourier transformation of the characteristic function of the aperture. Doing so for a disk shaped aperture results in the (squared modulus of) $J_1(x)/x$ as the intensity profile where J_1 is the first Bessel function and $x = ka \sin\theta$, $k = 2\pi/\lambda$ is the wave number and a here is the radius of the aperture. This is known as the **Airy diffraction pattern** [15] The Airy disk is defined to be the disk whose boundary corresponds to the first zero of the function $x \mapsto J_1(x)/x$. The usual approximate formula is

$$\theta \sim 1.22\lambda/d$$

where d is diameter of the aperture. The human pupil size dilates to about 8-9mm in the dark. Using the wavelength range for visible light 380-700nm, we have that the Airy disk has angular radius of 5×10^{-5} to 10^{-4} radians. (See also this Wikipedia article [41], the angular resolution of the human eye is 3×10^{-4} radian or about 1 arcminute which is directed related to the sizes of the Airy disk)

The Airy disk profile $(J_1(x)/x)^2$ may be approximated by a Gaussian with standard deviation $\sigma \sim 0.42\lambda/d$. However, for sky chart on the computer, the Airy disk is going to be far too small to plot (with the above estimate for a picture of dimension 2000 pixels showing a range of angle from -45 to 45 degree, i.e. about 22.22 pixels per degree. The Airy disk is at 10^{-5} pixel!).

In the actual Python code, we use size parameter (which is defined proportional to area, i.e. radius squared) in `Matplotlib`

$$\propto 10^{-m_V/2.5}$$

which is directly proportional to intensity. That is, we simulate brightness of a star by the size of the disk representing it. Note that this is NOT related to the Airy disk pattern, it is the unfortunate artificial design that the computer screen pixel has rather limited range (and resolution) of intensity. [16]

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[14] Actually the human eyeball is far too small for the far field condition, but people use the Airy disk radius formula nonetheless. I think the estimate is still somewhat valid

[15] not to be confused with the Airy function!

[16] In a CCD (charge-coupled device), the light is transformed into charge in cells and if a cell receives too much light, it may overcome the potential well to propagate to nearby cells. For this reason, bright stars may appear larger on a CCD image. However I could not find a source of the relationship between magnitude and apparent size of the star in CCD photos

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