An Alternative Proof of the Recursive Formula for Calculating the Chromatic Polynomial of a Graph by Vertex Deletion

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Abstract

This paper demonstrates a a double counting proof on the Recursive Formula for Calculating the Chromatic Polynomial of a Graph by Vertex Deletion.

1 Double Counting Proof

We will state Jin's main theorem in its original form, and then prove it by a double counting argument. For more details on the notation, see [1].

Theorem. *Let G be simple graph with p vertices. Let $u \in V(G)$ be such that $d(u) = p - k$, where $d(u)$ denotes the degree of u and $1 \leq k \leq p-1$. Let $V_u^* = \{v_1, v_2, v_3, \ldots, v_{k-1}\} \subset V(G)$ be the set of all vertices in $V(G)$ such that each vertex in V_u^* is not adjacent to u. Then we have the vertex-deleting formula for the chromatic polynomial of the graph G,*

$$P(G, \lambda) = \lambda P(G_u, \lambda - 1) + \lambda \sum_{H \subseteq V_u^*} P(G_{\{u\} \cup H}, \lambda - 1) \tag{1}$$

where the summation is extended over all independent sets $H \subseteq V_u^*$ with $1 \leq |H| \leq k-1$. Here G_J denotes the graph obtained from G by deleting all vertices in J.

Proof. The left hand side of equation (1) counts the number of proper colorings on the graph G with λ colors, so it suffices to show that the right hand side of equation (1) also counts the number of proper colorings on the graph G with λ colors. Let $\bar{H}$ be an arbitrary independent set that contains the vertex u. Since there are λ colors, pick one color to color all vertices in $\bar{H}$, delete all vertices in $\bar{H}$ including its associated edges, and finally use the remaining $\lambda - 1$ colors to properly color the graph $G_{\bar{H}}$. Then there are $\lambda P(G_{\bar{H}}, \lambda - 1)$ ways to do this. Since $\bar{H}$ is an arbitrary independent set that contains u, then we deduce that

$$P(G, \lambda) = \lambda \sum_{\bar{H}} P(G_{\bar{H}}, \lambda)$$

gives us the number of ways to properly color the graph G with λ colors. Observe that the above sum is summing over all independent sets that contain u. Additionally, we know that $\bar{H} = \{u\} \cup H$, where $H \subseteq V_u^*$, so the above sum can be simplified to equation (1).

□

References

[1] Xu Jin, *Recursive Formula for Calculating The Chromatic Polynomial of a Graph via Vertex Deletion*, Acta Mathematica Scientia, 577-582 (2004).