0.1 Diversity for the 0/1-Loss

Bregman divergences are certainly useful for regression tasks. Classification tasks can be approached by estimating class probabilities using divergences such as the KL-divergence and selecting the class with the highest estimated probability. In other settings, however, we are mainly concerned with whether the correct class is assigned or not.

Definition 0.1.1 *The 0/1-loss between two outcomes is*

$$\ell_{0/1}(Y, Y') =_{def} \mathbb{1}[Y \neq Y']$$

The ensemble combiner implied by the 0/1-loss is the plurality vote.

The proper term here would actually be *plurality* vote since, for a class to win the vote, it is required to have more than $\frac{1}{k}$ votes. Strictly speakin, a *majority* vote win requires the majority of all votes, i.e. more than $\frac{1}{2}$. For $k = 2$, majority and plurality voting is equivalent.

Definition 0.1.2 *(Majority/Plurality vote combiner) For a k-class classification problem, the majority vote combiner is defined as*

$$\bar{q}(X) = \arg\min_{z \in [k]} \mathbb{E}_\Theta \left[\ell_{0/1}(z, q_\Theta)\right]$$

This assumes that each member model predicts a single class, although this does not imply that member models must necessarily be trained based on the 0/1-loss, too. If a member predicts a class, we also say that the member *votes* for that class.

In the remainder of this section, we will analyse classification ensembles as measured by the 0/1-loss. A basic quantity for this will be the ratio of members that are incorrect for a given example-outcome pair.

Definition 0.1.3 *For a distribution of members constructed according to input data $D = (D_1, \ldots, D_M)$ and parameters $\Theta = (\Theta_1, \ldots, \Theta_M)$, the ratio of incorrect ("wrong") members is*

$$W(X, Y) =_{def} \mathbb{E}_{D,\Theta}\left[\ell_{0/1}(Y, q_{D,\Theta}(X))\right] \approx \frac{1}{M}\sum_{i=1}^{M} \ell_{0/1}(Y, q_i(X))$$

As with other variables, we sometimes omit explicitly stating the dependence on (X, Y). Further, we write $W_\Theta =_{def} \mathbb{E}_\Theta\left[\ell_{0/1}(Y, q_{D,\Theta})\right]$. For the complement, we write $\overline{W} =_{def} 1 - W$

A simple bound Using Markov's inequality, we can readily upper-bound the error of the ensemble in terms of expected errors of the members **[theisen_WhenAreEnsembles_2023]** [1] .

1: Markov's inequality states that for a nonnegative random variable X and $a > 0$
$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

$$0 \leq \mathbb{E}\left[\ell_{0/1}(Y, \bar{q})\right] = \mathbb{P}[W \geq \kappa] \leq \mathbb{P}\left[W \geq \frac{1}{2}\right] \leq 2\mathbb{E}[W]$$

While there exist examples for which this upper bound is tight **[theisen_WhenAreEnsembles_2023]**, it is reasonable to suspect that the ensemble being worse by a factor of two is only a pathological case and not relevant for practise.

0.1.1 Diversity in binary classification problems

We begin by considering binary classification problems which have $k = 2$ two possible outcomes. The special property of binary classification problems is that any vote which is not correct is automatically a vote for the single other class. In other words, diversity can be measured in terms of disagreement between members. This is not given for problems with $k > 2$ where an incorrect vote might correspond to any other class. We will see that, in this case, diversity will be measured in terms of differences in *correctness* of members.

The dichotomy of binary classification problems allows us to succinctly express the diversity-effect.

Lemma 0.1.1 *([**brown_GoodBadDiversity_2010**]) For a classification problem with $k = 2$ classes, let $y, \bar{q} \in \{-1, 1\}$. It then holds that*

$$\frac{1}{M}\sum_{i=1}^{M}\left[\ell_{0/1}(y, q_i) - \ell_{0/1}(y, \bar{q})\right] = (y \cdot \bar{q})\frac{1}{M}\sum_{i=1}^{M}\ell_{0/1}(\bar{q}, q_i) \in \{-1, 0, 1\}$$

Proof. Let $y, \bar{q} \in \{-1, 1\}$.

- Assume the ensemble is correct, i.e. $y = \bar{q}$. Then $\ell_{0/1}(y, \bar{q}) = 0$ and the left-hand-side equals $\frac{1}{M}\sum_{i=1}^{M}\ell_{0/1}(y, q_i) = \frac{1}{M}\sum_{i=1}^{M}\ell_{0/1}(\bar{q}, q_i)$. Further, $y \cdot \bar{q} = 1$.
- Assume the ensemble is incorrect, i.e. $y \neq \bar{q}$. Then $y \cdot \bar{q} = -1$ and, for the left-hand-side, we can write

$$\frac{1}{M}\sum_{i=1}^{M}\left[\ell_{0/1}(y, q_i)\right] - 1 = -\left(1 - \frac{1}{M}\sum_{i=1}^{M}\ell_{0/1}(y, q_i)\right) = -\left(\frac{1}{M}\sum_{i=1}^{M}\ell_{0/1}(\bar{q}, q_i)\right)$$

using that, since $y \neq \bar{q}$, $\left(1 - \frac{1}{M}\sum_{i=1}^{M}\ell_{0/1}(y, q_i)\right) = \frac{1}{M}\sum_{i=1}^{M}\ell_{0/1}(\bar{q}, q_i)$.

□

We can divide the range of X into two disjoint subsets. Let X_+ be the examples on which the ensemble is correct. Ambiguity on these points has a decreasing effect on the overall ensemble error. Let X_- be the examples on which the ensemble is incorrect. Ambiguity on these points has an increasing effect on the overall ensemble error. This yields a decomposition into *good* and *bad* diversity.

Corollary 0.1.2 *([**brown_GoodBadDiversity_2010**]) For a classification*

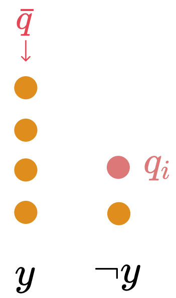

Figure 1: Example of the effect of a member's vote q_i on the diversity on a point for which the ensemble majority vote is correct. Example where q_i has positive contribution to the diversity effect term, i.e. $\ell_{0/1}(y, q_i) - \ell_{0/1}(y, \bar{q}) = 1$. The member q_i is incorrect but due to the discreteness of the majority vote combiner, the ensemble performance does not suffer – unless the majority vote is tipped. Any correct vote while the ensemble already is correct is effectively "wasted" and incorrect votes correspond to diversity.

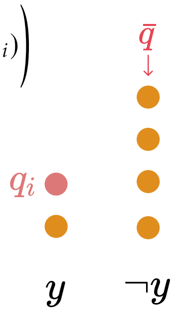

Figure 2: Example where q_i has negative contribution to the diversity effect term, i.e. $\ell_{0/1}(y, q_i) - \ell_{0/1}(y, \bar{q}) = -1$. Any further incorrect vote while the ensemble is already incorrect would be wasted. The negative effect here eventually results in the 0/1-loss of 1.

An intuition of this is also that of "wasted votes": Under the majority vote combiner, for the ensemble to be correct, we require only at least half of the members to be correct. Any higher ratio of correct ensemble members does not improve the ensemble performance on this point and these can be seen as "wasted". Likewise, the ensemble is incorrect if not more than half of the members are correct. Any positive votes do not influence the ensemble improvement and can be considered "wasted".

problem with $k = 2$ classes, let $y, \bar{q} \in \{-1, 1\}$. It then holds that

$$\begin{aligned}
\mathbb{E}_X\left[L(Y, \bar{q})\right] = \mathbb{E}_X &\left[\frac{1}{M}\sum_{i=1}^{M} \ell_{0/1}\left(Y, q_i\right)\right] \\
&- \underbrace{\mathbb{E}_{X_+}\left[\frac{1}{M}\sum_{i=1}^{M} \ell_{0/1}\left(\bar{q}, q_i\right)\right]}_{\text{"good" diversity}} \\
&+ \underbrace{\mathbb{E}_{X_-}\left[\frac{1}{M}\sum_{i=1}^{M} \ell_{0/1}\left(\bar{q}, q_i\right)\right]}_{\text{"bad" diversity}}
\end{aligned}$$

Note that this is a special case of the ambiguity-effect decomposition of theorem **??**. There is a tradeoff between average member error and diversity. Here, however, diversity is not always beneficial. On points where the ensemble is incorrect, disagreements have a negative effect on the overall ensemble error. In other words, for majority vote ensembles, diversity is only beneficial *on points at which the ensemble can actually afford to be diverse*.

Further, from corollary **??**, one can already see that the ensemble improvement (i.e. diversity-effect) in binary classification problems is only non-negative if good diversity outweights bad diversity.

Good and bad diversity can be expressed solely in terms of the ratio of incorrect members.

Lemma 0.1.3 ⋆ *Let y be the true outcome for a given example. Let $\neg y$ be an outcome that is not y. Write $\ell_{0/1}\left(q_i, \neg y\right) =_{def} \sum_{k \neq y} \ell_{0/1}\left(q_i, k\right)$ for the indication whether q_i is incorrect. Then the following identities hold.*

$$\begin{aligned}
\mathbb{E}_{X_+}\left[\frac{1}{M}\sum_{i=1}^{M} \ell_{0/1}\left(q_i, \bar{q}\right)\right] &= \mathbb{E}_{X_+}\left[W_1^M\right] \\
\mathbb{E}_{X_-}\left[\frac{1}{M}\sum_{i=1}^{M} \ell_{0/1}\left(q_i, \bar{q}\right)\right] &= \mathbb{E}_{X_-}\left[\frac{1}{M}\sum_{i=1}^{M} \ell_{0/1}\left(q_i, \neg y\right)\right] \\
&= \mathbb{E}_{X_-}\left[1 - W_1^M\right]
\end{aligned}$$

Analogous equalities hold in expectation over member parameter Θ.

0.1.2 Weak Learners

A common idea is that, in order for ensembling to be effective, the performance of individual members must not be too bad. In the remainder of this section, we will review some assumptions that imply ensemble effectivity and show that they are in fact tightly related.

Definition 0.1.4 *(Weak learner) need to check / reformulate this*

Theorem 0.1.4 *(*[**wood_UnifiedTheoryDiversity_2023**]*) In an ensemble of weak learners, diversity-effect is non-negative:*

$$\mathbb{E}_D\left[\mathbb{E}_Y\left[\frac{1}{M}\sum_{i=1}^{M}\ell_{0/1}(Y,q_i)-\ell_{0/1}(Y,\bar{q})\right]\right] \;\geq\; 0$$

Proof. ... essentially using jury theorem ☐

However, this is not a sufficient condition, as example **??** shows.

0.1.3 Competence in binary classification problems

[**theisen_WhenAreEnsembles_2023**] consider the question of ensemble improvement under the 0/1-loss. They define an assumption on the ratio of inccorect members.

Definition 0.1.5 (2*-competence,* [**theisen_WhenAreEnsembles_2023**]*) An ensemble is* 2*-competent iff*

$$\forall t\in\left[0,\frac{1}{2}\right]:\mathbb{P}_{(X,Y)}\left[W(X,Y)\in\left[t,\frac{1}{2}\right]\right]\;\geq\;\mathbb{P}_{(X,Y)}\left[W(X,Y)\in\left[\frac{1}{2},1-t\right]\right]$$

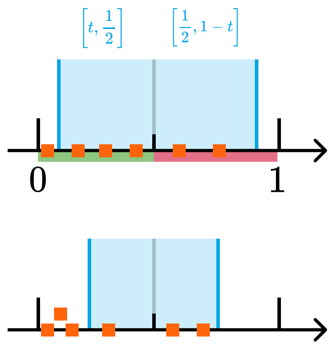

Figure 3: Illustration for the competence condition **??** for binary classification. Red squares correspond to pairs (X,Y) from the joint distribution of examples and outcomes. For each of these pairs, the average/expected member error $W_\Theta(X,Y)\approx\frac{1}{M}\sum_{i=1}^{M}\ell_{0/1}((,y),q_i)$ is the ratio of incorrect members. The center $\frac{1}{2}$ is the majority vote threshold. Informally, an ensemble is competent, if, for any two intervals defined by t left and right of the threshold, more examples are in the left part. For the upper example, this holds. For the lower example, even though many examples are classified correctly by many members, the ensemble is *not* competent.

The condition is illustrated in figure **??**. It is used to show two results:

- In 2-competent ensembles, diversity-effect is non-negative. [2]
- bounds on the generalisation error

One can see that competence is essentially determined by the distribution of examples (X,Y) over the range of $W(X,Y)$ which is divided by the majority vote threshold $\frac{1}{2}$. We have already seen that, similarly, diversity-effect in its apparent form of good and bad diversity is determined by just the same characteristics. How are these two related? In this section, we will argue that non-negative diversity-effect is in fact equivalent to a notion of competence generalised to $k>2$ classes. Unless otherwise noted, all expectations and probabilites are over the distribution of (X,Y).

While proving that 2-competence implies non-negative diversity-effect, [**theisen_WhenAreEnsembles_2023**] establish the following fact.

$$\text{ens. 2-competent} \;\leftrightarrow\; \mathbb{E}\left[W\,\mathbb{1}\left[W<\frac{1}{2}\right]\right]\geq\mathbb{E}\left[\overline{W}\,\mathbb{1}\left[\overline{W}\leq\frac{1}{2}\right]\right] \tag{0.1}$$

We can rearrange this into a more suggestive form. Recall that $W=\mathbb{E}_{D,\Theta}\left[\ell_{0/1}(Y,q_{D,\Theta}(X))\right]$. We now split off the expectation over D and instead consider $W_\Theta=\mathbb{E}_\Theta\left[\ell_{0/1}(Y,q_{D,\Theta}(X))\right]$. Rearranging the above and exploiting the linearity of expectation, we obtain

$$d=_{\text{def}}\mathbb{E}_{(X,Y),D}\left[W_\Theta\,\mathbb{1}\left[W_\Theta<\frac{1}{2}\right]\right]-\mathbb{E}_{(X,Y),D}\left[\overline{W_\Theta}\,\mathbb{1}\left[\overline{W_\Theta}\leq\frac{1}{2}\right]\right]\;\geq\;0$$

The indicator functions are mutually exclusive and can thus be understood as a case distinction. With slight abuse of notation, where the expectations

2: Although not acknowledging the role of diversity-effect as a component of the ensemble generalisation error and thus referring to it only as "ensemble improvement", one of the main results of [**theisen_WhenAreEnsembles_2023**] is that, in competent ensembles, ensemble improvement (i.e. diversity-effect) is non-negative.

are only over the subset of the distribution for which the case condition holds, we can write

$$d = \begin{cases} \mathbb{E}\left[W_\Theta\right] & \leftrightarrow W_\Theta < \frac{1}{2} \\ \mathbb{E}\left[\overline{W_\Theta}\right] = \mathbb{E}\left[1 - W_\Theta\right] & \leftrightarrow \overline{W_\Theta} \leq \frac{1}{2} \end{cases}$$

For $k = 2$, the majority vote threshold is $\frac{1}{2}$ and thus the conditions correspond exactly to the ensemble being either correct or incorrect.

$$k = 2 \ \longrightarrow \ \begin{cases} W_\Theta < \frac{1}{2} \ \leftrightarrow \ \bar{q}(X) = Y \\ \overline{W_\Theta} \leq \frac{1}{2} \ \leftrightarrow \ q(\bar{X}) \neq Y \end{cases} \tag{0.2}$$

Recalling the characterisation of good and bad diversity of lemma **??**, we can see that d is nothing else but the diversity-effect. This means that non-negative effect is exactly equivalent to 2-competence for a classification problem with $k = 2$ classes. Moreover, it is important to note that the gap of equation 0.1 and consequently the gap in the definition of 2-competence (**??**) is exactly the diversity-effect and hence exactly measures the ensemble improvement. In other words, one can now speak about the degree of competence of an ensemble.

0.1.4 Competence and Diversity in non-binary classification problems

However, for $k > 2$, the equivalence in equation **??** is not given. While it is sufficient (a class with more than $\frac{1}{2}M$ votes will win any plurality vote), it is not necessary: a plurality vote can be won with less than $\frac{1}{2}M$ votes. Thus, there are ensembles which have non-negative diversity-effect (ensemble improvement) that are not 2-competent.

The key difference is that for $k > 2$, the voting threshold for a pair (X, Y) is no longer the same value for all examples. Since a class wins if and only if it has more votes than any other class, the voting threshold depends on the distribution of class votes, which is potentially different for any pair (X, Y). Nevertheless, there is still *a* classification threshold, namely the ratio of votes for the next-best class. Because we will be considering the ratio of incorrect votes as a basic quantity, we will now define it from the reciprocal perspective:

$$\kappa(X, Y) = 1 - \max_{Z \neq Y} \mathbb{E}_\Theta \left[\mathbb{1}\left[q_\Theta = Z\right]\right]$$

and it holds that

$$\begin{aligned} W_\Theta &< \kappa & \leftrightarrow \ \bar{q}(X) = Y \\ \overline{W_\Theta} &\leq 1 - \kappa & \leftrightarrow \ \bar{q}(X) \neq Y \end{aligned}$$

Let

$$t =_{\text{def}} \max_{Z \neq Y} \mathbb{E}_\Theta \left[\mathbb{1}\left[q_\Theta = Z\right]\right]$$

. Then

$$\begin{aligned} \bar{q} = y \ &\leftrightarrow \ \overline{W} \geq t \\ &\leftrightarrow \ W = 1 - \overline{W} < 1 - t = \kappa \end{aligned}$$

and

$$\begin{aligned} \bar{q} \neq y \ &\leftrightarrow \ \overline{W} < t \\ &\leftrightarrow \ 1 - \overline{W} \geq 1 - t = \kappa \\ &\leftrightarrow \ 1 - (1 - \overline{W}) \leq 1 - \kappa \\ &\leftrightarrow \ \overline{W} \leq 1 - \kappa \end{aligned}$$

Definition 0.1.6 ★ *(k-competence) An ensemble is k-competent iff*

$$\forall t \in [0,1] : \mathbb{P}_{(X,Y)}\left[W \in [t, \kappa]\right] \geq \mathbb{P}_{(X,Y)}\left[W \in [1-\kappa, 1-t]\right]$$

for $\kappa =_{def} 1 - \max_{Z \neq Y} \mathbb{E}_\Theta \left[\, \mathbb{1}\left[q_\Theta = Z\right]\right]$.

For classification problems with $k = 2$, k-competence is exactly 2-competence as of definition **??** since the voting threshold is always $\frac{1}{2}$.

[**theisen_WhenAreEnsembles_2023**] showed that 2-competence implies non-negative diversity-effect. We now show that a very similar line of argument instead using k-competence actually holds in both directions.

Theorem 0.1.5 ★ *Consider an ensemble for a k-class classification problem. Then*

$$k\text{-competence} \quad \leftrightarrow \quad \text{diversity-effect} \geq 0$$

The main work lies in establishing the following lemma, which is a generalised form of equation 0.1.

Lemma 0.1.6 ★ *(Generalised from [**theisen_WhenAreEnsembles_2023**]) For an increasing function f with $f(0) = 0$, it holds that*

$$k\text{-competence} \leftrightarrow \mathbb{E}\left[f(W)\, \mathbb{1}\left[W < \kappa\right]\right] \geq \mathbb{E}\left[f(\overline{W})\, \mathbb{1}\left[\overline{W} \leq \kappa\right]\right]$$

where $\kappa =_{def} 1 - \max_{Z \neq Y} \mathbb{E}_\Theta \left[\, \mathbb{1}\left[q_\Theta = Z\right]\right]$.

Proof. We begin by observing that, for all $x \in [0,1]$

$$\begin{aligned} \mathbb{P}\left[W \in [x, \kappa]\right] \cdot\ \mathbb{1}\left[x \leq \kappa\right] &= \mathbb{P}\left[W\, \mathbb{1}\left[W < \kappa\right] \ \geq\ x\right] \\ \mathbb{P}\left[W \in [1-\kappa, 1-x]\right] \cdot\ \mathbb{1}\left[x \leq \kappa\right] &= \mathbb{P}\left[\overline{W}\, \mathbb{1}\left[\overline{W} \leq \kappa\right] \ \geq\ x\right] \end{aligned}$$

$$\begin{aligned} &\mathbb{P}\left[W \in [1-\kappa, 1-x]\right]\ \mathbb{1}\left[x \leq \kappa\right] \\ =&\mathbb{P}\left[W \in [1-\kappa, 1-x]\right]\ \mathbb{1}\left[1-x > 1-\kappa\right] \\ =&\mathbb{P}\left[W\, \mathbb{1}\left[W > 1-\kappa\right] \ <\ 1-x\right] \\ =&\mathbb{P}\left[W\, \mathbb{1}\left[\overline{W} \leq \kappa\right] \ <\ 1-x\right] \\ =&\mathbb{P}\left[\overline{W}\, \mathbb{1}\left[\overline{W} \leq \kappa\right] \ \geq\ x\right] \end{aligned}$$

where the first factors on the left-hand-side appear in the definition of k-competence. Since W is nonnegative, using that $\mathbb{E}[X] = \int \mathbb{P}[X \geq x]\, dx$, we can conclude that, for any $x \in [0,1]$

$$\begin{aligned} (k\text{-comp.}) \ \leftrightarrow\ & \mathbb{P}\left[W\, \mathbb{1}\left[W < \kappa\right] \ \geq x\right] \ \geq\ \mathbb{P}\left[\overline{W}\, \mathbb{1}\left[\overline{W} \leq \kappa\right] \ \geq x\right] \\ \leftrightarrow\ & \mathbb{E}\left[W\, \mathbb{1}\left[W < \kappa\right]\right] \ \geq\ \mathbb{E}\left[\overline{W}\, \mathbb{1}\left[\overline{W} \leq \kappa\right]\right] \end{aligned}$$

□

Using this, we can now directly prove theorem **??**.

Proof. (Theorem **??**, generalised from [**theisen_WhenAreEnsembles_2023**])

$$\begin{aligned} 0 &= \mathbb{E}\left[(W-1)\,\mathbb{1}\left[W \geq \kappa\right]\right] - \mathbb{E}\left[(W-1)\,\mathbb{1}\left[W \geq \kappa\right]\right] \\ &= \mathbb{E}\left[(W-1)\,\mathbb{1}\left[W \geq \kappa\right]\right] + \mathbb{E}\left[(1-W)\,\mathbb{1}\left[W \geq \kappa\right]\right] \\ &= \mathbb{E}\left[(W-1)\,\mathbb{1}\left[W \geq \kappa\right]\right] + \mathbb{E}\left[\overline{W}\,\mathbb{1}\left[\overline{W} < \kappa\right]\right] \end{aligned}$$

Applying lemma **??** for $f = \mathrm{id}$ to the second term yields

$$\begin{aligned} &\mathbb{E}\left[(W-1)\,\mathbb{1}\left[W \geq \kappa\right]\right] + \mathbb{E}\left[\overline{W}\,\mathbb{1}\left[\overline{W} < \kappa\right]\right] \\ \leq\ &\mathbb{E}\left[(W-1)\,\mathbb{1}\left[W \geq \kappa\right]\right] + \mathbb{E}\left[W\,\mathbb{1}\left[W < \kappa\right]\right] \end{aligned}$$

The above already is nothing but the diversity-effect:

$$\begin{aligned} 0 \leq &\mathbb{E}\left[(W-1)\,\mathbb{1}\left[W \geq \kappa\right]\right] + \mathbb{E}\left[W\,\mathbb{1}\left[W < \kappa\right]\right] \\ = &\mathbb{E}\left[W\right] - \mathbb{E}\left[\,\mathbb{1}\left[W \geq \kappa\right]\right] \\ = &\mathbb{E}\left[W\right] - \mathbb{P}\left[W \geq \kappa\right] \end{aligned}$$

The first term is the ratio of incorrect members in expectation over all examples and is equal to the error rate of an average member.

$$\mathbb{E}_{(X,Y)}\left[W\right] = \mathbb{E}_{(X,Y)}\left[\mathbb{E}_{D,\Theta}\left[\ell_{0/1}\left(Y, q_{D,\Theta}(X)\right)\right]\right] = \mathbb{E}_{D_\Theta}\left[\mathbb{E}_{(X,Y)}\left[\ell_{0/1}\left(Y, q_{D,\Theta}(X)\right)\right]\right]$$

The second term is the ensemble error. □

0.1.5 Bounds for competent ensembles

2-competence was used to show upper and lower bounds for the diversity-effect [**theisen_WhenAreEnsembles_2023**]. Now we show that, with minor adjustments, the same bounds can be derived from k-competence. Besides giving performance guarantees, these bounds are interesting due to that they are expressed in terms of disagreements between members, which until now we have only seen for Bregman divergences.

Theorem 0.1.7 *(Upper bound) In k-competent ensembles,*

$$\mathbb{E}\left[W\right] - \mathbb{P}\left[W \geq \kappa\right] \leq \mathbb{E}_{\rho,\rho'}\left[D(q_\rho, q_{\rho'})\right]$$

Proof. The proof can be found in [**theisen_WhenAreEnsembles_2023**]. It does not make use of competence and therefore still holds for k-competent ensembles. □

Theorem 0.1.8 *(Lower bound) In k-competent ensembles,*

$$\frac{2(k-1)}{k}\mathbb{E}\left[D(q_\rho, q_{\rho'})\right] - \frac{3k-4}{k}\mathbb{E}\left[W\right] \quad\leq\quad \mathbb{E}\left[W\right] - \mathbb{P}\left[W \geq \kappa\right]$$

Proof. Lemmas 3 and 4 hold without adjustments for k-competence and are shown in [**cited-by-theisen**] and [**theisen_WhenAreEnsembles_2023**], respectively.

$$
\begin{aligned}
&\mathbb{P}[W \geq \kappa] \\
&\leq 2\mathbb{E}\left[W^2\right] && \text{(Lemma ??)} \\
&= 2\mathbb{E}_{\rho,\rho'}\left[L(q_\rho, q_\rho)\right] && \text{(Lemma 3 in [\textbf{theisen\_WhenAreEnsembles\_2023}])} \\
&= \frac{4(k-1)}{k}\left(\mathbb{E}[W] - \frac{1}{2}\mathbb{E}_{\rho,\rho'}\left[D(q_\rho, q_{\rho'})\right]\right) && \text{(Lemma 4 in [\textbf{theisen\_WhenAreEnsembles\_2023}])}
\end{aligned}
$$

Rearranging the terms yields the statement. □

Lemma 0.1.9 ★ *(Generalised from [**theisen_WhenAreEnsembles_2023**])*
In k-competent ensembles it holds that

$$\mathbb{P}[W \geq \kappa] \leq 2\mathbb{E}\left[W^2\right]$$

Proof. Note that

$$\mathbb{P}[W \geq \kappa] \leq 2\mathbb{E}\left[W^2\right] \;\leftrightarrow\; \mathbb{P}[W \geq \kappa] - 2\mathbb{E}\left[W^2\right] \geq 0$$

$$2\mathbb{E}\left[W^2\right] - \mathbb{P}[W \geq \kappa] = \mathbb{E}\left[(2W^2 - 1)\,\mathbb{1}[W \geq \kappa]\right]$$

We will aim to show that this above expression is nonnegative. The final inequality is due to applying lemma **??** to the second term.

$$
\begin{aligned}
&W \geq \kappa \\
&\leftrightarrow 1 - W < 1 - \kappa \\
&\leftrightarrow \overline{W} \leq 1 - \kappa
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[2W^2\right] - \mathbb{P}[W \geq \kappa] \\
&= \mathbb{E}\left[2W^2\right] - \mathbb{E}\left[\,\mathbb{1}[W \geq \kappa]\right] \\
&= \mathbb{E}\left[(2W^2 - 1)\,\mathbb{1}[W \geq \kappa]\right] + \mathbb{E}\left[2W^2\,\mathbb{1}[W < \kappa]\right] \\
&\geq \mathbb{E}\left[(2W^2 - 1)\,\mathbb{1}[W \geq \kappa]\right] + \mathbb{E}\left[2\overline{W}^2\,\mathbb{1}\left[\overline{W} < \kappa\right]\right]
\end{aligned}
$$

Note that for $k \geq 2$, $\kappa > 1-\kappa$ and thus $\mathbb{E}\left[\,\mathbb{1}\left[\overline{W} \leq \kappa\right]\right] \geq \mathbb{E}\left[\,\mathbb{1}\left[\overline{W} \leq 1 - \kappa\right]\right]$, allowing us to continue

$$
\begin{aligned}
\ldots &\geq \mathbb{E}\left[(2W^2 - 1)\,\mathbb{1}[W \geq \kappa]\right] + \mathbb{E}\left[2\overline{W}^2\,\mathbb{1}\left[\overline{W} < 1 - \kappa\right]\right] \\
&= \mathbb{E}\left[1 - 4\overline{W} + 2\overline{W}^2\,\mathbb{1}\left[\overline{W} < 1 - \kappa\right]\right] \\
&\geq 0
\end{aligned}
$$

□