Optimal resolvable $2 \times c$ grid-block coverings

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Abstract

The notion of grid-block designs was introduced by Fu *et al.*[*J. Statist. Plann. Inference*, 119(2):225–236, 2004] for the applications to DNA library screenings based on non-adaptive group testing. This paper is concentrated on resolvable grid-block coverings with a minimal number of parallel classes, which are called optimal. In particular, we characterize optimal $2 \times c$ resolvable coverings, and prove that an optimal resolvable 2×3 covering of order v exists if and only if $v \equiv 0 \pmod 6$.

Keywords. Grid-block; resolvable; optimal covering; Cartesian product; excess graph; DNA library screening

AMS subject classifications. Primary 05B30, Secondary 05C70.

1 Introduction

Let V be a finite set of cardinality v. Let $\mathcal{B}$ be a collection of $r \times c$ arrays with rc distinct entries in V. The elements of V and $\mathcal{B}$ are called *points* and *grid-blocks* respectively. Two points are *collinear* in a grid-block $\mathsf{B} \in \mathcal{B}$ if they lie in the same row or in the same column of B. A pair $(V, \mathcal{B})$ is said to be a $(v, r \times c, 1)$ *grid-block design (resp., packing, covering)* if any pair of distinct points of V is collinear in exactly (resp., at most, at least) one grid-block of $\mathcal{B}$. A grid-block design (resp., packing, covering) $(V, \mathcal{B})$ is *resolvable*, if $\mathcal{B}$ can be partitioned into *parallel classes*, where a parallel class is a collection of grid-blocks in which each point of V appears exactly once. In particular, when $v = rc$ and $r = c$ hold, $(V, \mathcal{B})$ is called a *lattice square design.*

The study of lattice square designs can be traced back to Yates [12] since 1940s, which was motivated for agricultural experiments. Hereafter, from the aspect of combinatorics, Hwang [6] proved that an $r \times c$ grid-block design of order rc exists if and only if $r = c$ (i.e., a lattice square design) is odd, and $r - 1$ mutually orthogonal Latin squares (MOLS) of order r exist. Since then, Fu *et al.* [3] formally introduced the notion of *grid-block designs* and discussed their further applications to DNA library screenings based on non-adaptive group testing. We refer the readers to the monograph [2] for further information on group testing and its applications to DNA sequencing. As to other generalizations of lattice square designs, various array type designs have also been investigated by statisticians for decades, such as Youden squares, Latin squares, row-column designs, and nested row-column designs (see [10]).

However, for practical applications, "designs" do not always exist, especially when *resolvability* is desired. When the considerations similar to (nested) row-column designs are taken into account, grid-block designs are not only applicable for group testing, but also can be used for designs of experiments. In this case, designs, coverings, or packings with small grid-block sizes such as 2×3 and 3×3 are also useful.

Moreover, in order that an experiment containing all the treatments (viz. a parallel class) could be performed simultaneously with each other, *resolvability* should be taken into account. Such situations often happen in design of experiments or group testing to require the efficiency of testing process.

In the case when $r = c$ is odd, Mutoh, Jimbo, and Fu [9] proposed a construction of resolvable grid-block designs via cyclic grid-block designs with mutually disjoint base grid-blocks. Recently, Lu, Satoh, and Jimbo [8] generalized this construction and showed the following theorem for resolvable 3×3 grid-block designs.

Theorem 1.1. *There exists a resolvable 3×3 grid-block design of order $9v$ if every prime factor p of v satisfies $p \equiv 1 \pmod{36}$.*

Whereas, in the case of $r = 2$, we necessarily have $v \equiv 1 \pmod{c}$ for a $(v, 2 \times c, 1)$ grid-block design. But the resolvability requires v to be a multiple of $2c$. Thus resolvable $2 \times c$ grid-block designs do not exit. In this case, grid-block coverings or packings are interested. Li and Yin [7] proposed several constructions for resolvable $2 \times c$ grid-block packings and proved that a resolvable 2×3 grid-block packing of order v with a maximal number of parallel classes exists if and only if $v \equiv 0 \pmod{6}$.

However, in the application to group testing for identifying all defective items, it is usually desired to test every pair of items at least once. Thus coverings are more desirable than packings. Accordingly, the construction of resolvable grid-block designs has been generalized to coverings by Lu, Satoh, and Jimbo [8]. In general, for a resolvable $(v, r \times c, 1)$ grid-block covering, the number of parallel classes ρ should satisfy $\rho \geq \lceil \frac{v-1}{r+c-2} \rceil$.

Moreover, if the equality holds, the covering is called *optimal.* Note that the resultant coverings in [8] are not optimal if $r \neq c$.

In this paper, we intend to concentrate on resolvable $2 \times c$ grid-block coverings with $c \geq 3$, where the necessary condition for the existence follows straightforwardly from the resolvability.

Lemma 1.2. *If a resolvable* $2 \times c$ *grid-block covering of order* v *exists, then* $v \equiv 0 \pmod{2c}$.

Let $v = 2ch$, where h is the number of grid-blocks in a parallel class. Then an *optimal* $(v, 2 \times c, 1)$ grid-block covering should have exactly $\rho = \lceil \frac{v-1}{c} \rceil = 2h$ parallel classes.

In Section 2, we will characterize the optimal resolvable $2 \times c$ grid-block coverings for any $c > 2$. Then in Section 3, we will prove that an optimal resolvable 2×3 grid-block covering of order v exists if and only if $v \equiv 0 \pmod{6}$.

2 Graphical characterization of optimal resolvable $2 \times c$ grid-block coverings

Grid-block designs (resp., packings, covering) can be naturally presented as graph designs. Let H denote a (finite, simple, and undirected) graph. A collection $\mathcal{A}$ of subgraphs of H is said to be a *decomposition* of H, if each edge of H appears in exactly one subgraph in $\mathcal{A}$. Moreover, if every graph in $\mathcal{A}$ is isomorphic to a graph G, then $\mathcal{A}$ is said to be a G-*decomposition* of H. In particular, if $H = K_v$, namely, the complete graph on v vertices, the a G-decomposition is also known as a G-*design* of order v. Accordingly, under the assumption that every graph in $\mathcal{A}$ is isomorphic to G, if each edge of K_v appears in at least (resp., at most) one of the graphs in $\mathcal{A}$, then $(V, \mathcal{A})$ is said to be a G-covering (resp., G-packing), where V denotes the vertex set of K_v. When $(V, \mathcal{A})$ is a G-covering, the *excess graph* of $(V, \mathcal{A})$ is the multigraph (V, E), where each edge $\{x, y\}$ occurs with multiplicity $|\{A \in \mathcal{A} \mid \{x, y\} \text{ is an edge in } A\}| - 1$.

Let $L_{r,c}$ denote the Cartesian product graph of the complete graphs K_r and K_c. An $r \times c$ grid-block design (resp., packing, covering) is nothing but an $L_{r,c}$-design (resp., packing, covering). We simply refer to a resolvable $L_{2,c}$-covering as an $L_{2,c}$-RC or a $(v, L_{2,c}, 1)$-RC.

Lemma 2.1. *Let* $(V, \mathcal{A})$ *be a* $(2ch, L_{2,c}, 1)$*-RC. Then* $(V, \mathcal{A})$ *is optimal if and only if its excess graph forms a* 1*-factor of* K_{2ch} *over* V.

Proof. Since every grid-block contains c^2 edges, the number of edges in the excess graph of an optimal $L_{2,c}$-RC is $2h \cdot h \cdot c^2 - \binom{2ch}{2} = hc$. For a fixed vertex x, there are c edges adjacent to x in a grid-block. The total degree of x in an optimal $L_{2,c}$-RC is $2h \cdot c$. Whereas, the degree of x in K_{2ch} is $2ch-1$. Since $2ch-(2ch-1)=1$, the excess graph in an optimal $L_{2,c}$-RC forms a 1-factor of K_{2ch}. Conversely, if the excess graph of $(V,\mathcal{A})$ forms a 1-factor of K_{2ch}, then the total number of grid-blocks in $\mathcal{A}$ is clearly $2h^2$, i.e., $\rho = 2h$. $\square$

It is notable that, unlike the leave of an optimal packing, which is known to be a $(c-1)$-factor (see [7] Theorem 2.1), the excess graph of an optimal covering is much more smaller and independent with c. Especially when c is large, it becomes more strict for a covering to reach the optimality. In particular, an optimal covering with the smallest possible order does not always exist.

Lemma 2.2. *There exists an optimal* $(2c, L_{2,c}, 1)$*-RC if and only if* $c \le 4$.

Proof. Let $V = \mathbb{Z}_c \times \mathbb{Z}_2 = \{x_i \mid x \in \mathbb{Z}_c, i \in \mathbb{Z}_2\}$. An optimal $(2c, L_{2,c}, 1)$-RC should consist of two grid-blocks, say B_1 and B_2. Without loss of generality, let

$$\mathsf{B}_1 = \begin{bmatrix} 0_0 & 1_0 & \cdots & (c-1)_0 \\ 0_1 & 1_1 & \cdots & (c-1)_1 \end{bmatrix}.$$

Let $M_1 = (\mathbb{Z}_c \setminus \{0\}) \times \{1\}$. In order to cover the pairs of the form $\{0_0, \mu_1\}$ with $\mu_1 \in M_1$, at least $c-2$ points in M should be collinear with 0_0 in B_2. In this case, at least a $(c-2)$-clique is contained in the excess graph, which contradicts Lemma 2.1 when $c \ge 5$. Therefore, an optimal $(2c, L_{2,c}, 1)$-RC does not exists for $c \ge 5$.

For $c=4$, let $\mathsf{B}_1 = \begin{bmatrix} 0_0 & 1_0 & 2_0 & 3_0 \\ 0_1 & 1_1 & 2_1 & 3_1 \end{bmatrix}$ and $\mathsf{B}_2 = \begin{bmatrix} 0_0 & 1_0 & 3_1 & 2_1 \\ 1_1 & 0_1 & 2_0 & 3_0 \end{bmatrix}$. For $c=3$, let $\mathsf{B}_1 = \begin{bmatrix} 0_0 & 1_0 & 2_0 \\ 0_1 & 1_1 & 2_1 \end{bmatrix}$ and $\mathsf{B}_2 = \begin{bmatrix} 0_0 & 1_1 & 2_1 \\ 0_1 & 2_0 & 1_0 \end{bmatrix}$. Then $(V, \{\mathsf{B}_1, \mathsf{B}_2\})$ is an optimal $(2c, L_{2,c}, 1)$-RC for $c \in \{3,4\}$. $\square$

In design theory, RGDDs (resolvable group divisible designs) and frames are commonly used for constructions of new designs. For the standard notions of design theory, the reader may refer to [1, 4]. The "blocks" of RGDDs and frames can be generalized to "graphs".

Let $K_{g_1,g_2,\ldots,g_u}$ denote a complete u-partite graph with vertex set V whose partite sets are $G_1, G_2, \ldots, G_u$ with $|G_i| = g_i$ for every $1 \le i \le u$. Suppose $\mathcal{A}$ is an $L_{2,c}$-decomposition of $K_{g_1,g_2,\ldots,g_u}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_u\}$. Then $(V, \mathcal{G}, \mathcal{A})$ is defined to be an $L_{2,c}$ *group divisible design* (*GDD*) of type $g_1 g_2 \cdots g_u$, where G_i is referred to as a *group* for every $1 \le i \le u$. Moreover, if $\mathcal{A}$ can be partitioned into parallel classes, then $(V, \mathcal{G}, \mathcal{A})$ is called a *resolvable group divisible design* (*RGDD*). If $g = g_1 = g_2 = \cdots = g_u$, the GDD is said to be uniform. In this case, $(V, \mathcal{G}, \mathcal{A})$ is called an $L_{2,c}$-GDD of type g^u for convenience. Clearly, a (resolvable) $L_{2,c}$-GDD of type 1^v is nothing but a (resolvable) $(v, L_{2,c}, 1)$ design (see also [7]).

Proposition 2.3 ([7] Lemma 2.2)**.** There are exactly $\frac{g(u-1)}{c}$ parallel classes in any $L_{2,c}$-RGDD of type g^u.

Let $(V, \mathcal{G}, \mathcal{A})$ be an $L_{2,c}$-GDD with $\mathcal{G} = \{G_1, G_2, \dots, G_u\}$. A partial parallel class of $(V, \mathcal{G}, \mathcal{A})$ is a collection of subgraphs of $\mathcal{A}$ whose vertex sets are mutually disjoint. If $\mathcal{A}$ can be partitioned into partial parallel classes, each of which forms a partition of $V \setminus G_i$ for some $G_i \in \mathcal{G}$, then $(V, \mathcal{G}, \mathcal{A})$ is called an $L_{2,c}$-frame of type $(g_1, g_2, \dots, g_u)$, where $|G_i| = g_i$ for every $1 \le i \le u$. If $g = g_1 = g_2 = \cdots = g_u$, then $(V, \mathcal{G}, \mathcal{A})$ is called an $L_{2,c}$-frame of type g^u for convenience.

Proposition 2.4 ([7] Lemma 2.7)**.** Let $(V, \mathcal{G}, \mathcal{A})$ be an $L_{2,c}$-frame. For any $G_i \in \mathcal{G}$, the number of partial parallel classes over V missing G_i is $|G_i|/c$.

Some fundamental constructions for an optimal $L_{2,c}$-RC by using $L_{2,c}$-RGDDs and $L_{2,c}$-frames are stated as follows.

Construction 2.5. Let u be an even positive integer. If there exists an $L_{2,c}$-RGDD of type c^u, then an optimal $(cu, L_{2,c}, 1)$-RC exists.

Proof. Since u is even, let $(V, \mathcal{G}, \mathcal{A})$ be an $L_{2,c}$-RGDD of type c^u with $\mathcal{G} = \{G_1, G_2, \dots, G_{u/2}\} \cup \{H_1, H_2, \dots, H_{u/2}\}$. Let $G_i = \{g_1^{(i)}, g_2^{(i)}, \dots, g_c^{(i)}\}$ and $H_i = \{h_1^{(i)}, h_2^{(i)}, \dots, h_c^{(i)}\}$ for every $1 \le i \le u/2$. Then, we have $u/2$ new grid-blocks given by

$$\mathsf{F}_i = \begin{bmatrix} g_1^{(i)} & g_2^{(i)} & \cdots & g_c^{(i)} \\ h_1^{(i)} & h_2^{(i)} & \cdots & h_c^{(i)} \end{bmatrix} \text{ for every } 1 \le i \le u/2.$$

F_i covers all edges in two complete graphs whose vertex sets are G_i and H_i, and

$$\mathcal{F}_i = \left\{ \{g_j^{(i)}, h_j^{(i)}\} \mid 1 \le j \le c \right\}.$$

Then $(V, \mathcal{A} \cup \{\mathsf{F}_1, \mathsf{F}_2, \dots, \mathsf{F}_{u/2}\})$ is an $L_{2,c}$-RC, in which $\{\mathsf{F}_1, \mathsf{F}_2, \dots, \mathsf{F}_{u/2}\}$ forms one more parallel class. Moreover, $\bigcup_{i=1}^{u/2} \mathcal{F}_i$ is a partition of V into pairs, which is nothing but the excess graph of $(V, \mathcal{A} \cup \{\mathsf{F}_1, \mathsf{F}_2, \dots, \mathsf{F}_{u/2}\})$. By Lemma 2.1, $(V, \mathcal{A} \cup \{\mathsf{F}_1, \mathsf{F}_2, \dots, \mathsf{F}_{u/2}\})$ is a desired optimal $L_{2,c}$-RC. $\square$

Construction 2.6. Let g be a multiple of $2c$. Suppose a $(g, L_{2,c}, 1)$-RC exists. If there exists an $L_{2,c}$-RGDD of type g^u, then an optimal $(gu, L_{2,c}, 1)$-RC exists.

Proof. Let $(V, \mathcal{G}, \mathcal{A})$ be an $L_{2,c}$-RGDD of type g^u with $\mathcal{G} = \{G_1, G_2, \dots, G_u\}$. Suppose $(G_i, \mathcal{B}_i)$ is an optimal $(g, L_{2,c}, 1)$-RC for every $1 \le i \le u$. Then every $(G_i, \mathcal{B}_i)$ has the same number of parallel classes, because $|G_i|$ is identical with each other for $1 \le i \le u$. Hence $(V, \bigcup_{i=1}^{u} \mathcal{B}_i)$ is resolvable, where $V = \bigcup_{i=1}^{u} G_i$. Clearly, $(V, \bigcup_{i=1}^{u} \mathcal{B}_i \cup \mathcal{A})$ is an $L_{2,c}$-RC as well. Moreover, by Lemma 2.1, the excess graph of $(G_i, \mathcal{B}_i)$ forms a 1-factor of K_g on G_i for each $1 \le i \le u$. The excess graph of $(V, \bigcup_{i=1}^{u} \mathcal{B}_i \cup \mathcal{A})$ is obviously the union of 1-factors, each of which consists of a partition of G_i. Thus, $(V, \bigcup_{i=1}^{u} \mathcal{B}_i \cup \mathcal{A})$ is also optimal by Lemma 2.1. $\square$

Construction 2.7. Let g be a multiple of $2c$. Suppose a $(g, L_{2,c}, 1)$-RC exists. If there exists an $L_{2,c}$-frame of type g^u, then an optimal $(gu, L_{2,c}, 1)$-RC exists.

Proof. Let $h = \frac{g}{2c}$. Let $(V, \mathcal{G}, \mathcal{A})$ be an $L_{2,c}$-frame of type g^u with $\mathcal{G} = \{G_1, G_2, \dots, G_u\}$. By Proposition 2.4, the number of partial parallel classes over V missing G_i is $2h$ for every $G_i \in \mathcal{G}$. On the other hand, it is shown in the proof of Lemma 2.1 that the number of parallel classes in any $(g, L_{2,c}, 1)$-RC is also $2h$. For $1 \le i \le u$, let $(G_i, \mathcal{B}_i)$ be a $(g, L_{2,c}, 1)$-RC. Clearly, $(V, \bigcup_{i=1}^{u} \mathcal{B}_i \cup \mathcal{A})$ is an $L_{2,c}$ covering. By combining a parallel class in $(G_i, \mathcal{B}_i)$ with a partial parallel classes of $(V, \mathcal{G}, \mathcal{A})$ missing G_i, a new parallel class over V can be obtained. Proceeding similarly for each $1 \le i \le u$, we obtain $2h$ parallel classes over V. Therefore, $(V, \bigcup_{i=1}^{u} \mathcal{B}_i \cup \mathcal{A})$ is resolvable. Finally, it is similar to the proof of Construction 2.6 that the excess graph of $(V, \bigcup_{i=1}^{u} \mathcal{B}_i \cup \mathcal{A})$ forms a 1-factor of K_{gu} on V. By Lemma 2.1, $(V, \bigcup_{i=1}^{u} \mathcal{B}_i \cup \mathcal{A})$ is an optimal $(gu, L_{2,c}, 1)$-RC. $\square$

Li and Yin [7] described several recursive constructions of $L_{2,c}$-RGDDs and $L_{2,c}$-frames, which generalize the classical RGDDs, frames, and RBIBDs. We will use the following two theorems from [7].

Theorem 2.8 ([7] Construction 2.9)**.** *Suppose there exists an $L_{2,c}$-frame of type $(mg)^h$. If an $L_{2,c}$-RGDD of type g^{m+1} exists, then an $L_{2,c}$-RGDD of type g^{1+mh} exits.*

Theorem 2.9 ([7] Construction 2.13)**.** *Suppose there exists a k-frame of type g^u. If an $L_{2,c}$-RGDD of type m^k exists, then an $L_{2,c}$-frame of type $(mg)^u$ exits.*

3 Optimal $L_{2,3}$-RC

Lemma 3.1. *There exists an optimal $(12, L_{2,3}, 1)$-RC.*

Proof. The grid-blocks of an optimal $(12, L_{2,3}, 1)$-RC over $\mathbb{Z}_{12}$ are shown as follows, consisting of 4 parallel classes.

$$\begin{bmatrix} 0 & 9 & 2 \\ 8 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 10 & 6 & 11 \\ 1 & 7 & 5 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 6 & 1 \\ 11 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 9 & 10 & 4 \\ 5 & 8 & 7 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 10 & 7 \\ 5 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 11 & 4 \\ 9 & 8 & 6 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 3 & 10 \\ 4 & 6 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 1 & 2 \\ 9 & 11 & 7 \end{bmatrix}.$$

□

Lemma 3.2. *There exists an $L_{2,3}$-RGDD of type 3^u for $u \in \{8, 12, 14\}$.*

Proof. Let $X_{3u} = I_3 \times (\mathbb{Z}_{u-1} \cup \{\infty\})$, where $I_3 = \{1, 2, 3\}$. For each $u \in \{8, 12, 14\}$, we define $u/2$ base grid-blocks, say $\mathsf{B}_1, \mathsf{B}_2, \ldots, \mathsf{B}_{u/2}$, shown in Table 1. Let $\mathcal{G}_{3u} = \big\{I_3 \times \{x\} \mid x \in \mathbb{Z}_{u-1} \cup \{\infty\}\big\}$ and $\mathcal{B}_{3u} = \{\mathsf{B}_i + (*, j) \mid 1 \le i \le u/2, j \in \mathbb{Z}_{u-1}\}$, where the second component is deduced cyclically modulo $u-1$ leaving ∞ fixed. Then $\bigcup_{i=1}^{u/2} \{\mathsf{B}_i + (*, j)\}$ is a parallel class for every $j \in \mathbb{Z}_{u-1}$. Therefore $(X_{3u}, \mathcal{G}_{3u}, \mathcal{B}_{3u})$ is a desired $L_{2,3}$-RGDD of type 3^u. □

Lemma 3.3. *There exists an $L_{2,3}$-RGDD of type 6^u for $u \in \{3, 5\}$.*

Proof. An $L_{2,3}$-RGDD of type 6^3 is given in [7] Lemma 3.6. For $u = 5$, take $X_{30} = I_3 \times (\mathbb{Z}_8 \cup \{\infty_1, \infty_2\})$ and $\mathcal{G}_{30} = \big\{I_3 \times \{x, x+4\} \mid 0 \le x \le 3\big\} \cup \big\{I_3 \times \{\infty_1, \infty_2\}\big\}$, where $I_3 = \{1, 2, 3\}$. Let

$$\mathsf{B}_1 = \begin{bmatrix} (3,0) & (1,5) & (1,\infty_2) \\ (1,2) & (1,3) & (2,0) \end{bmatrix}, \mathsf{B}_2 = \begin{bmatrix} (2,1) & (1,0) & (2,2) \\ (2,\infty_1) & (3,1) & (1,7) \end{bmatrix},$$
$$\mathsf{B}_3 = \begin{bmatrix} (1,1) & (3,6) & (3,\infty_2) \\ (1,6) & (3,\infty_1) & (2,5) \end{bmatrix}, \mathsf{B}_4 = \begin{bmatrix} (3,7) & (2,4) & (2,6) \\ (3,5) & (3,2) & (2,3) \end{bmatrix},$$
$$\mathsf{B}_5 = \begin{bmatrix} (2,\infty_2) & (2,7) & (3,4) \\ (1,4) & (1,\infty_1) & (3,3) \end{bmatrix},$$

and $\mathcal{B}_{30} = \{\mathsf{B}_i + (*, j) \mid 1 \le i \le 5, j \in \mathbb{Z}_8\}$, where the second component is deduced cyclically modulo 8 leaving ∞_1 and ∞_2 fixed. Clearly, $\bigcup_{i=1}^{5} \{\mathsf{B}_i + (*, j)\}$ is a parallel class for every $j \in \mathbb{Z}_8$. Then $(X_{30}, \mathcal{G}_{30}, \mathcal{B}_{30})$ is a desired $L_{2,3}$-RGDD of type 6^5. □

Lemma 3.4 ([7] Lemma 3.7)**.** *For any integer $u \ge 4$ and $u \notin \{8, 12, 14, 18\}$, an $L_{2,3}$-frame of type 12^u exists.*

Lemma 3.5. *An $L_{2,3}$-frame of type 24^h exists for $h \in \{4, 6, 7, 9\}$.*

Table 1: $L_{2,3}$-RGDD of type 3^u

u	$\mathsf{B}_1, \mathsf{B}_2, \ldots, \mathsf{B}_{u/2}$
$u = 8$	
	$\mathsf{B}_1 = \begin{bmatrix} (3,0) & (1,2) & (3,4) \\ (2,5) & (2,\infty) & (3,3) \end{bmatrix}, \mathsf{B}_2 = \begin{bmatrix} (3,6) & (2,0) & (3,\infty) \\ (3,1) & (2,4) & (1,0) \end{bmatrix},$
	$\mathsf{B}_3 = \begin{bmatrix} (1,1) & (3,5) & (1,\infty) \\ (1,4) & (2,2) & (2,3) \end{bmatrix}, \mathsf{B}_4 = \begin{bmatrix} (1,6) & (2,1) & (3,2) \\ (1,5) & (2,6) & (1,3) \end{bmatrix}.$
$u = 12$	
	$\mathsf{B}_1 = \begin{bmatrix} (3,0) & (1,2) & (1,3) \\ (1,8) & (2,7) & (1,6) \end{bmatrix}, \mathsf{B}_2 = \begin{bmatrix} (2,8) & (3,1) & (3,9) \\ (2,4) & (3,10) & (3,3) \end{bmatrix},$
	$\mathsf{B}_3 = \begin{bmatrix} (1,1) & (3,6) & (2,9) \\ (1,5) & (3,7) & (1,0) \end{bmatrix}, \mathsf{B}_4 = \begin{bmatrix} (2,10) & (3,4) & (1,\infty) \\ (2,0) & (2,6) & (1,4) \end{bmatrix},$
	$\mathsf{B}_5 = \begin{bmatrix} (1,9) & (3,8) & (2,1) \\ (3,2) & (1,7) & (2,\infty) \end{bmatrix}, \mathsf{B}_6 = \begin{bmatrix} (2,5) & (1,10) & (2,3) \\ (2,2) & (3,\infty) & (3,5) \end{bmatrix}.$
$u = 14$	
	$\mathsf{B}_1 = \begin{bmatrix} (3,0) & (3,12) & (2,1) \\ (3,7) & (2,3) & (3,4) \end{bmatrix}, \mathsf{B}_2 = \begin{bmatrix} (3,3) & (3,11) & (2,6) \\ (1,2) & (2,4) & (2,7) \end{bmatrix},$
	$\mathsf{B}_3 = \begin{bmatrix} (2,2) & (2,9) & (1,3) \\ (2,11) & (1,0) & (1,7) \end{bmatrix}, \mathsf{B}_4 = \begin{bmatrix} (1,\infty) & (3,5) & (2,10) \\ (1,11) & (2,\infty) & (2,12) \end{bmatrix},$
	$\mathsf{B}_5 = \begin{bmatrix} (3,6) & (3,2) & (2,0) \\ (1,10) & (1,8) & (2,5) \end{bmatrix}, \mathsf{B}_6 = \begin{bmatrix} (1,12) & (1,9) & (3,1) \\ (3,10) & (3,8) & (1,4) \end{bmatrix},$
	$\mathsf{B}_7 = \begin{bmatrix} (1,6) & (1,1) & (3,9) \\ (1,5) & (2,8) & (3,\infty) \end{bmatrix}.$

Proof. The cases when $h \in \{6, 7, 9\}$ is shown in [7] Lemma 3.8. For $h = 4$, take a 3-frame of type 4^4 (see [5] IV.5.30 for the existence). Then apply Theorem 2.9 with $c = 3$, $k = 3$, $g = 4$, $u = 4$, and $m = 6$ to the $L_{2,3}$-RGDD of type 6^3 in Lemma 3.3 to complete the proof. □

Lemma 3.6. *There exists an optimal* $(24, L_{2,3}, 1)$*-RC.*

Proof. Applying Construction 2.5 to the $L_{2,3}$-RGDD of type 3^8 in Lemma 3.2, we can meet the claim. □

Theorem 3.7. *There exists an optimal* $(12u, L_{2,3}, 1)$*-RC for any positive integer* u*.*

Proof. This holds for $u = 1, 2$ by Lemmas 3.1 and 3.6. By Construction 2.7, Lemmas 3.1, and 3.6, it suffices to find $L_{2,3}$-frames of type 12^u or 24^u. Lemma 3.4 gives the $L_{2,3}$-frame of type 12^u for any integer $u \geq 4$ and $u \notin \{8, 12, 14, 18\}$. Lemma 3.5 gives the $L_{2,3}$-frame of type 24^h for $2h \in \{8, 12, 14, 18\}$. For $u = 3$, by applying Construction 2.5 to the $L_{2,3}$-RGDD of type 3^{12} in Lemma 3.2, we obtain an optimal $(36, L_{2,3}, 1)$-RC. □

Theorem 3.8. *There exists an optimal* $(12u + 6, L_{2,3}, 1)$*-RC for any positive integer* u*.*

Proof. By Construction 2.6 and Lemma 2.2, it suffices to find $L_{2,3}$-RGDDs of type 6^u. An $L_{2,3}$-RGDD of type 6^u exists for $u \in \{3, 5\}$ by Lemma 3.3. By applying Theorem 2.8 with $c = 3$, $g = 6$, and $m = 2$ to the $L_{2,3}$-frames of type 12^u in Lemma 3.4, one can obtain an $L_{2,3}$-RGDD of type 6^{2u+1} for any positive integer u with $u \notin \{2, 3, 8, 12, 14, 18\}$. Similarly, by applying Theorem 2.8 with $c = 3$, $g = 6$, and $m = 4$ to the $L_{2,3}$-frames of type 24^h in Lemma 3.5, an $L_{2,3}$-RGDD of type 6^{4h+1} can be obtained for any $h \in \{1, 4, 6, 7, 9\}$. For $u = 3$, by applying Construction 2.5 to the $L_{2,3}$-RGDD of type 3^{14} in Lemma 3.2, we obtain the desired RC. □

We are now in a position to give a complete solution for the existence of an optimal $L_{2,3}$-RC, namely an optimal resolvable 2×3 grid-block covering, by combining Theorems 3.7 and 3.8.

Theorem 3.9. *Let* v *be a positive integer. An optimal resolvable* 2×3 *grid-block covering of order* v *exists if and only if* $v \equiv 0 \pmod 6$*.*

Finally, we remark that optimal resolvable 2×4 grid-block coverings could be constructed similarly, but the computation for small designs needs more efforts and strategies. For the cases when $c \geq 5$, since there is no optimal $(2c, L_{2,c}, 1)$-RC (see Lemma 2.2), the recursive constructions need to be improved based on the recent methods. We leave these directions for future research work.

Acknowledgements. The designs in Lemmas 3.1, 3.2, and 3.3 are computed with the aid of a SAT-based constraint solver – Sugar (cf. [11]; see http://bach.istc.kobe-u.ac.jp/sugar/). The author would like to thank Professor Naoyuki Tamura who provided the information of Sugar, Professor Masakazu Jimbo for his advice, and the anonymous referees for their helpful comments. This work was supported in part by JSPS under Grant-in-Aid for JSPS Fellows No. 26011700.

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