8/27 応用数学特論II

Affine plane

An affine plane (アフィン平面) is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ such that

1. For any two points $p_1, p_2 \in \mathcal{P}$, there exists a unique line $\ell \in \mathcal{L}$ passing through both p_1 and p_2, i.e. $(p_1, \ell), (p_2, \ell) \in \mathcal{I}$.
2. If $p \in \mathcal{P}$ is not lying on $\ell \in \mathcal{L}$, i.e. $(p, \ell) \notin \mathcal{I}$, then there exists a unique line ℓ_P passing through P, i.e. $(P, \ell_P) \in \mathcal{I}$, and ℓ_P parallel to ℓ.
3. There exists at least three non-collinear points.

Proposition C

There are four points such that any three of them are not lying on a same line.

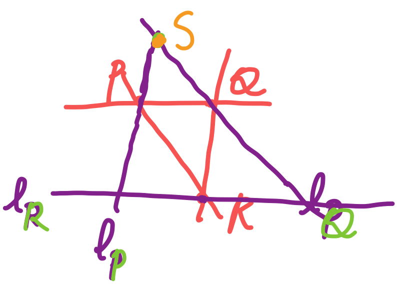
Proof: By Axiom 3, let P, Q, R be three non-collinear points. Let

- ℓ_P: the unique line passing P that is $//\overline{QR}$
- ℓ_Q: the unique line passing Q that is $//\overline{PR}$
- ℓ_R: the unique line passing R that is $//\overline{PQ}$

By transitivity of parallelism, ℓ_P, ℓ_Q, ℓ_R are not parallel to each other. (*)
Suppose ℓ_P and ℓ_Q intersect at S.

(*) Assume $\ell_P // \ell_R$

$\because \left.\begin{array}{l} \ell_P // \overline{QR}, \quad \ell_R // \overline{PQ} \end{array}\right\} \Rightarrow \overline{QR} // \overline{PQ} \Rightarrow$ contradiction.



ℓ_P and ℓ_Q intersect at S

Claim: P, Q, R, S の中の任意の3点は共線しない.

Proposition F

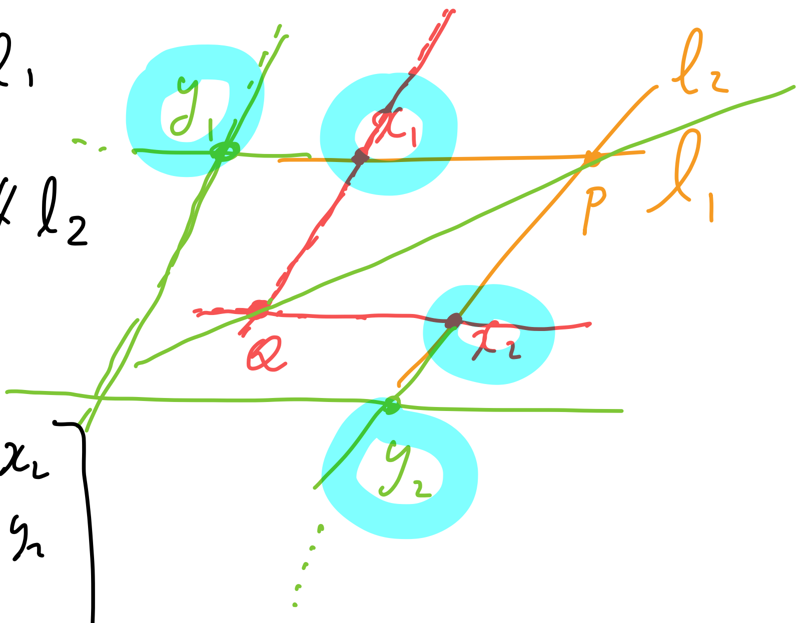
In an affine plane, all the lines contain the same number of points.

(Sketch) $\ell_1, \ell_2 \in \mathcal{L}$

Built a bijection $\left(\text{pts on } \ell_1 \leftrightarrow \text{pts on } \ell_2 \right)$
(1-1 mapping)

Case ① $\ell_1 \not\parallel \ell_2$

Case ② $\ell_2 \parallel \ell_1$



Case ① $\ell_1 \not\parallel \ell_2$

$$\left[\begin{array}{c} x_1 \leftrightarrow x_2 \\ y_1 \leftrightarrow y_2 \\ \vdots \end{array} \right]$$

Case ② $\ell_1 \parallel \ell_2$

$$\begin{bmatrix} x_1 & \leftrightarrow & x_2 \\ y_1 & \leftrightarrow & y_2 \\ & \vdots & \end{bmatrix}$$