応用数学特論 II (集中講義)

Day 4 Error-Correcting Codes

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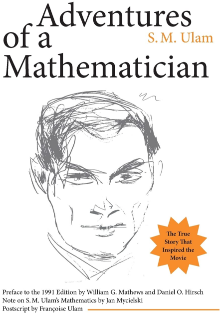
Outline

Ulam's game

Ulam's game (Rényi, 1961; Ulam, 1976)

Given N, guess a number n $(0 \le n < N)$ by asking a series of yes-or-no questions. The respondent is permitted to lie at most once.

Stanisław Ulam (1909–1984)



Adventures of a Mathematician

(1st ed. published in 1976)

Adventures of a Mathematician

(film released in 2020)

Example of Ulam's game with $N = 16$

- Please think of a number n with $0 \leq n \leq 15$.
- Please answer the following questions.
  - You are permitted to lie to at most one question.
  - Truth-telling for all the questions is allowed.

1. Is $n \geq 8$?
2. Is $n \in \{4, 5, 6, 7, 12, 13, 14, 15\}$?
3. Is $n \in \{2, 3, 6, 7, 10, 11, 14, 15\}$?
4. Is n odd?
5. Is $n \in \{1, 2, 4, 7, 9, 10, 12, 15\}$?
6. Is $n \in \{1, 2, 5, 6, 8, 11, 12, 15\}$?
7. Is $n \in \{1, 3, 4, 6, 8, 10, 13, 15\}$?

1	Yes	1
2	Yes	1
3	No	0
4	Yes	1
5	No	0
6	Yes	0
7	No	0

- According to your answers, I guess

$$n = 5$$

I guess that you lied to question

No. 1.

Catch the liar

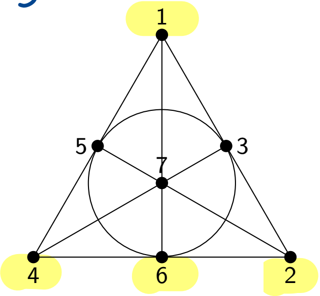
$y = (1\ 1\ 0\ 1\ 0\ 0\ 0)$

$x = (1\ 1\ 0\ 1\ 0\ 1\ 0)$

- The answers can be represented by a binary vector $\mathbf{x}$ in $\{0,1\}^7$, where 1=yes, 0=no.
- The number of 1s is called the weight (重み) of $\mathbf{x}$, denoted by $\mathrm{wt}(\mathbf{x})$.

$\mathrm{wt}(x) = 4$. $\mathrm{wt}(y) = 3$

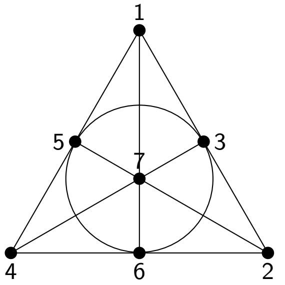
1. If $\mathrm{wt}(\mathbf{x}) = 0$, no lie.
2. If $\mathrm{wt}(\mathbf{x}) = 1$, the lie = the position of 1.
3. If $\mathrm{wt}(\mathbf{x}) = 2$, two positions of 1s are lying in a unique line ℓ in Fano plane. The lie = the third point on ℓ.
4. If $\mathrm{wt}(\mathbf{x}) = 3$, if three positions of 1s are lying in a unique line, then no lie. Otherwise, the four positions of 0s contains three point which form a line and one other point P. The lie = P.
5. If $\mathrm{wt}(\mathbf{x}) \geq 4$, apply the above rules for positions of 0s.



Catch the liar

- The answers can be represented by a binary vector $\mathbf{x}$ in $\{0,1\}^7$, where 1=yes, 0=no.
- The number of 1s is called the weight (重み) of $\mathbf{x}$, denoted by $\mathrm{wt}(\mathbf{x})$.

- $\mathbf{x} = (0,0,0,0,0,0,1)$. The lie $= 7$.
- $\mathbf{x} = (0,1,0,0,0,0,1)$. There is a line $\{2,5,7\}$, so the lie $= 5$.
- $\mathbf{x} = (0,1,0,1,0,0,1)$. $\{2,4,7\}$ does not form a line. $\{1,3,5,6\}$ contains a line $\{3,5,6\}$. So the lie $= 1$.
- $\mathbf{x} = (0,1,0,1,0,1,1)$. $\{1,3,5\}$ does not form a line. $\{2,4,6,7\}$ contains a line $\{2,4,6\}$. So the lie $= 7$.
- $\mathbf{x} = (0,1,0,1,1,1,1)$. There is a line $\{1,2,3\}$, so the lie $= 2$.
- $\mathbf{x} = (1,1,0,1,1,1,1)$. The lie $= 3$.



Correct the error

- Flip the bit ($0 \to 1$, $1 \to 0$) on the liar position. Denote the corrected vector by $\tilde{\mathbf{x}}$.
- Take the first four bits of $\tilde{\mathbf{x}}$ (a binary number) and convert it to decimal.

- $\mathbf{x} = (0, 0, 0, 0, 0, 0, 1)$. The lie $= 7$. $(0000)_2 \to (0)_{10}$.
- $\mathbf{x} = (0, 1, 0, 0, 0, 0, 1)$. The lie $= 5$. $(0100)_2 \to (4)_{10}$.
- $\mathbf{x} = (0, 1, 0, 1, 0, 0, 1)$. The lie $= 1$. $(1101)_2 \to (13)_{10}$. $= 8 + 4 + 1 = 13$
- $\mathbf{x} = (0, 1, 0, 1, 0, 1, 1)$. The lie $= 7$. $(0101)_2 \to (5)_{10}$.
- $\mathbf{x} = (0, 1, 0, 1, 1, 1, 1)$. The lie $= 2$. $(0001)_2 \to (1)_{10}$.
- $\mathbf{x} = (1, 1, 0, 1, 1, 1, 1)$. The lie $= 3$. $(1111)_2 \to (15)_{10}$.

$x = (1101010)$. lie $= 1$. $\tilde{x} = (0101010)$. $n = 4 + 1 = 5$.

Aim of today's lecture

The most important aim of today's lecture

To understand the "coding theory" for winning Ulam's game.

- Basic tools: linear algebra (matrix multiplication, linear mapping), finite field (mainly $\mathbb{F}_2$)
- Basic ideas: finite projective geometry, combinatorial designs

Outline

1. Ulam's game: A toy example for error-correction

2. Linear codes
   - Basics on codes and linear codes
   - Syndrome ~~coding~~ decoding for linear codes

3. Hamming codes and projective geometry

4. Perfect codes and MDS codes

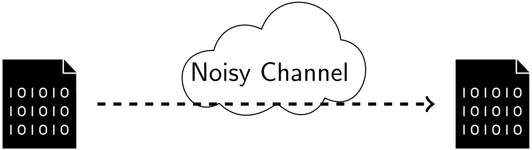
Outline

Codes

- A code (符号) $\mathcal{C}$ of length n over an alphabet Ω is a subset of vectors in

$$\Omega^n = \{(x_1, x_2, \dots, x_n) : x_i \in \Omega, 1 \leq i \leq n\}$$

- The vectors in a code $\mathcal{C}$ are called codewords (符号語).
- Usually, we consider Ω to be a finite field $\mathbb{F}_q$ or finite ring (e.g. $\mathbb{Z}_4$ for DNA encoding).
- For most applications in electronic communications, $\Omega = \mathbb{F}_2 = \{0, 1\}$.



Hamming distance

- As the "space" of codes, consider the notion of distance in Ω^n.

Hamming distance

The Hamming distance (ハミング距離) between any two codewords $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ is the number of positions where they differ, i.e.

$$d_H(\mathbf{x}, \mathbf{y}) = \#\{i : x_i \neq y_i, 1 \leq i \leq n\}.$$

Example (Hamming distance)

For $\mathbf{x} = (00111)$, $\mathbf{y} = (10101)$, we have $d_H(\mathbf{x}, \mathbf{y}) = 2$.

10101

Hamming distance is a distance

Proposition

$d_H(x, y)$

The Hamming distance is a metric (aka, distance function) on the space Ω^n, namely, it satisfies the following conditions:

1. (nonnegativity; 非負性) $d_H(\mathbf{x}, \mathbf{y}) \geq 0$ for any $\mathbf{x}$ and $\mathbf{y}$.
2. (identity of indiscernibles; 同一律) $d_H(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
3. (symmetry; 対称性) $d_H(\mathbf{x}, \mathbf{y}) = d_H(\mathbf{y}, \mathbf{x})$ for any $\mathbf{x}$ and $\mathbf{y}$.
4. (triangle inequality; 三角不等式) $d_H(\mathbf{x}, \mathbf{y}) \leq d_H(\mathbf{x}, \mathbf{z}) + d_H(\mathbf{z}, \mathbf{y})$ for any $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

$$x \xrightarrow[d_H(x,z)]{} z \xrightarrow[d_H(z,y)]{} y$$

bits are changed   bits are changed

Minimum distance

Minimum distance

The minimum Hamming distance (最小ハミング距離) $d_{\mathcal{C}}$ of a code $\mathcal{C}$ is the smallest distance between any two distinct vectors,

$$d_{\mathcal{C}} = \min\{d_H(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in \mathcal{C}\}.$$

- If a code is of length n, has M codewords, and minimum distance d, then the code is said to be an (n, M, d) code.

Linear codes over $\mathbb{F}_q$

- Now suppose $\Omega = \mathbb{F}_q$, the finite field of order q, where q is a prime power.

Linear code

$\mathcal{C} \subseteq \mathbb{F}_q^n$ is a linear code (線形符号) if the following conditions hold:

- If $\mathbf{v}$, $\mathbf{u} \in \mathcal{C}$ then $\mathbf{v} + \mathbf{u} \in \mathcal{C}$.
- If $\mathbf{v} \in \mathcal{C}$ and $\alpha \in \mathbb{F}_q$ then $\alpha\mathbf{v} \in \mathcal{C}$.

Example (Repetition code)

Let p be a prime. The following code

$$\mathcal{C} = \{(0, 0, \ldots, 0), (1, 1, \ldots, 1), \ldots, (p-1, p-1, \ldots, p-1)\}$$

is a linear code, called the repetition code (反復符号).

Minimum weight of linear codes

Hamming weight

The Hamming wight (ハミング重み) of a codeword $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{F}_q^n$ is the number of its non-zero elements, i.e.

$$\mathrm{wt}(\mathbf{x}) = \#\{i : x_i \neq 0, 1 \leq i \leq n\}.$$

The minimum weight (最小重み) of a code is the smallest non-zero weight in the code.

zero codeword $= (0, 0, \cdots, 0)$

Proposition

For a linear code $\mathcal{C}$, the minimum weight of the code is the minimum distance.

Minimum weight of linear codes

Hamming weight

The Hamming wight (ハミング重み) of a codeword $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{F}_q^n$ is the number of its non-zero elements, i.e.

$$\mathrm{wt}(\mathbf{x}) = \#\{i : x_i \neq 0, 1 \leq i \leq n\}.$$

The minimum weight (最小重み) of a code is the smallest non-zero weight in the code.

Proposition

For a linear code $\mathcal{C}$, the minimum weight of the code is the minimum distance.

Number of codewords in a linear code

Proposition

Let V be a vector space of dimension k over $\mathbb{F}_q$, then $|V| = q^k$.

Theorem

A linear code of length n over $\mathbb{F}_q$ is a subspace of $\mathbb{F}_q^n$. Hence, if $\mathcal{C}$ is a linear code over $\mathbb{F}_q$, then $|\mathcal{C}| = q^k$ for some k.

$|\Omega| = q$

- If a code of length n, with M codewords, and minimum distance d is an $(n, M, d)_q$ code.
- A linear code of length n, dim. k, and min. distance d over $\mathbb{F}_q$ is an $[n, k, d]_q$ code.
- When $q = 2$, we usually omit the subscription q and simple say an $[n, k, d]$ code.
- If the minimum distance d is unknown, we say it is an $[n, k]_q$ code.

Basis of a linear code

- $\mathcal{C}$ is an $[n, k]_q$ code $\iff$ $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_q^n$
- $\mathcal{C}$ has a basis (基底) of k vectors. By elementary row / column operations (行列の行・列における基本変形), the matrix of basis can be transformed to $\begin{bmatrix} I_k & A \end{bmatrix}$.

Example

Consider $\{(0,1,1,0), (1,1,1,1), (0,0,0,1)\}$ as a basis over $\mathbb{F}_2$. Then (k=3)

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{(2)-(1)-(3)}]{\text{row}} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{(2,\ 1,\ 4,\ 3) \\ (1\ 2\ 3\ 4)}]{\text{column}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Generator matrix of linear code

Generator matrix

For an $[n,k]_q$ code $\mathcal{C}$, the matrix of a basis is a generator matrix (生成行列) of $\mathcal{C}$.

- A generator matrix of the form $\begin{bmatrix} I_k & A \end{bmatrix}$ is commonly used.

Proposition

A linear code $\mathcal{C}$ with generator matrix G can be obtained by

$$\mathcal{C} = \{\mathbf{x}G : \mathbf{x} \in \mathbb{F}_q^k\}.$$

Remark:

- In this lecture, we consider row vectors as codewords.
- If we consider column vectors as codewords, a generator matrix is of the form $\begin{bmatrix} I_k \\ A \end{bmatrix}$.

Exercise 1: generator matrix of linear code

Exercise 1

1. Find all the codewords of the linear code $\mathcal{C} \subseteq \mathbb{F}_2^4$ defined by the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2. Find another generator matrix of $\mathcal{C}$ that is different from G.

Minimum distance decoding

ball: 球

sphere: 球面

Hamming ball

An n-dimensional Hamming ball (ハミング球) of radius r with center $\mathbf{v} \in \mathcal{C}$, where $\mathcal{C} \subseteq \mathbb{F}_q^n$ is a code, is the set of all the vectors in $\mathbb{F}_q^n$ having Hamming distance $\leq r$, i.e.

$$B_r(\mathbf{v}) = \{\mathbf{x} \in \mathbb{F}_q^n : d_H(\mathbf{x}, \mathbf{v}) \leq r\}$$

Minimum distance decoding

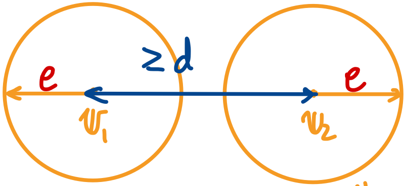
When a vector $\mathbf{w}$ is received, it is decoded to the vector $\mathbf{v} \in \mathcal{C}$ that is closest to it. This method is called minimum distance decoding (最小距離復号) or nearest neighbor decoding.

Error-correcting ability of linear codes

Theorem

Let $\mathcal{C}$ be an $[n,k,d]_q$ code, then $\mathcal{C}$ can detect $d-1$ errors. If $d=2e+1$ then $\mathcal{C}$ can correct e errors, and $\mathcal{C}$ is said to be an e-error correcting code (誤り訂正能力 e の符号; e 誤り訂正符号).

$v_1, v_2 \in \mathcal{C}$.



$d=2e+1$

Hamming ball of radius e.

Example (Repetition code)

The code $\mathcal{C}$ of length n over $\mathbb{F}_q$ generated by $(1,1,\ldots,1)^\top$ has minimum distance of n. Hence, it can detect $n-1$ errors and correct $\lfloor (n-1)/2 \rfloor$ errors.

Outline

Dual code of linear code

- Let $\langle \mathbf{w}, \mathbf{v} \rangle$ denote the inner product (内積) of $\mathbf{w}, \mathbf{v} \in \mathbb{F}_q^n$.

Dual code

For a linear code $\mathcal{C}$ over $\mathbb{F}_q$, let

$$\mathcal{C}^\perp = \{\mathbf{w} : \langle \mathbf{w}, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{v} \in \mathcal{C}\}.$$

Then $\mathcal{C}^\perp$ is called the dual code (双対符号) of $\mathcal{C}$.

Proposition

If $\mathcal{C}$ is an $[n, k]_q$ code, then $\mathcal{C}^\perp$ is an $[n, n-k]_q$ code.

Example (Dual code of binary repetition code)

Let $\mathcal{C}$ be the code of length n over $\mathbb{F}_2$ generated by $(1, 1, \ldots, 1)^\top$. Then $\mathcal{C}^\perp$ has dimension $n - 1$ and consists of all vectors of length n with even weight.

Generator matrix of dual code

Theorem

If a linear code $\mathcal{C}$ is generated by $\begin{bmatrix} I_k & A \end{bmatrix}$, then $\mathcal{C}^\perp$ is generated by $\begin{bmatrix} -A^\top & I_{n-k} \end{bmatrix}$.

Exercise 1

3. For the linear code $\mathcal{C} \subseteq \mathbb{F}_2^4$ generated by $G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, produce all the codewords of its dual code $\mathcal{C}^\perp$.
4. Produce a generator matrix of $\mathcal{C}^\perp$. (Hint: you can use the above theorem.)

Self-orthogonal and self-dual codes

Self-orthogonal and self-dual codes

A code $\mathcal{C}$ is self-orthogonal (自己直交) if $\mathcal{C} \subseteq \mathcal{C}^{\perp}$ and it is self-dual (自己双対) if $\mathcal{C} = \mathcal{C}^{\perp}$.

Exercise 1

5. Check whether $\mathcal{C}^{\perp}$ is a self-orthogonal code and briefly state the reason why it is or it is not.

Parity check matrix and syndrome

- Let $\mathcal{C}$ be an $[n, k]_q$ code and suppose $\mathcal{C}^{\perp}$ has a generator matrix $H = \begin{bmatrix} I_{n-k} & B \end{bmatrix}$.
- Then $\mathbf{x} \in \mathcal{C} \iff H\mathbf{x}^{\top} = \mathbf{0}$. $\iff$ x is orthogonal with codewords in $\mathcal{C}^{\perp}$.
- This matrix H is called the parity check matrix (パリティ検査行列) of $\mathcal{C}$.

奇偶性

Syndrome

Let $\mathcal{C}$ be a code in $\mathbb{F}_q$ with parity check matrix H. Then the syndrome (シンドローム[1].) of a vector $\mathbf{v} \in \mathbb{F}_q^n$ is $S(\mathbf{v}) = H\mathbf{v}^{\top}$. ← column vector

- For $\mathbf{x} \in \mathcal{C}$, we have $S(\mathbf{x} + \mathbf{e}) = H\mathbf{x}^{\top} + H\mathbf{e}^{\top} = H\mathbf{e}^{\top}$.

受信系列 v, 誤り e

[1] In Chinese, "校正子" or "校驗子"

Binary Hamming code: The winning strategy for Ulam's game (1/4)

- Consider the linear code $\mathcal{C}$ generate by

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

- Its dual code $\mathcal{C}^{\perp}$ has generator matrix

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

which is the parity check matrix of $\mathcal{C}$.

Binary Hamming code: The winning strategy for Ulam's game (2/4)

- Suppose that $n = 14$ is chosen, whose binary expression is (1110).
- The codeword is $|\mathcal{C}| = 2^4 = 16$

$$\mathbf{x} = \begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 1 & 1\\0 & 1 & 0 & 0 & 1 & 0 & 1\\0 & 0 & 1 & 0 & 1 & 1 & 0\\0 & 0 & 0 & 1 & 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0 & 0\end{bmatrix}.$$

G

- Now sent the message (1110000). If it is received as $\mathbf{x} = (1110000)$ then it is considered to be correct (with no lie). Because, the syndrome is $\mathbf{0}$, i.e.,

$$H\mathbf{x}^\top = \begin{bmatrix}0 & 1 & 1 & 1 & 1 & 0 & 0\\1 & 0 & 1 & 1 & 0 & 1 & 0\\1 & 1 & 0 & 1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0 & 0\end{bmatrix}^\top = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

H

Binary Hamming code: The winning strategy for Ulam's game (3/4)

- Precisely, $\mathcal{C}$ has 16 codewords, in which (1110000) is a codeword.

$$
\begin{array}{cc}
(0,0,0,0,0,0,0) & (1,0,0,0,0,1,1) \\
(0,0,0,1,1,1,1) & (1,0,0,1,1,0,0) \\
(0,0,1,0,1,1,0) & (1,0,1,0,1,0,1) \\
(0,0,1,1,0,0,1) & (1,0,1,1,0,1,0) \\
(0,1,0,0,1,0,1) & (1,1,0,0,1,1,0) \\
(0,1,0,1,0,1,0) & (1,1,0,1,0,0,1) \\
(0,1,1,0,0,1,1) & (1,1,1,0,0,0,0) \\
(0,1,1,1,1,0,0) & (1,1,1,1,1,1,1)
\end{array}
$$

- In general, it is not smart to search through an entire code, because usually the codes contain an extremely large amount of codewords.

Binary Hamming code: The winning strategy for Ulam's game (4/4)

- Assume that it is received as $\mathbf{x}' = (1110010)$. We have

$$H\mathbf{x'}^\top = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

  The syndrome is nothing but the 6th column of H. So we conclude that the 6th bit has an error (tells a lie).

- Assume that it is received as $\mathbf{x}' = (1010000)$. We have

$$H\mathbf{x'}^\top = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

  So we conclude that the 2nd bit has an error (tells a lie).

Errors that $[7,4]_2$ Hamming code cannot correct

- Assume that it is received as $\mathbf{x}' = (1100010)$. We have

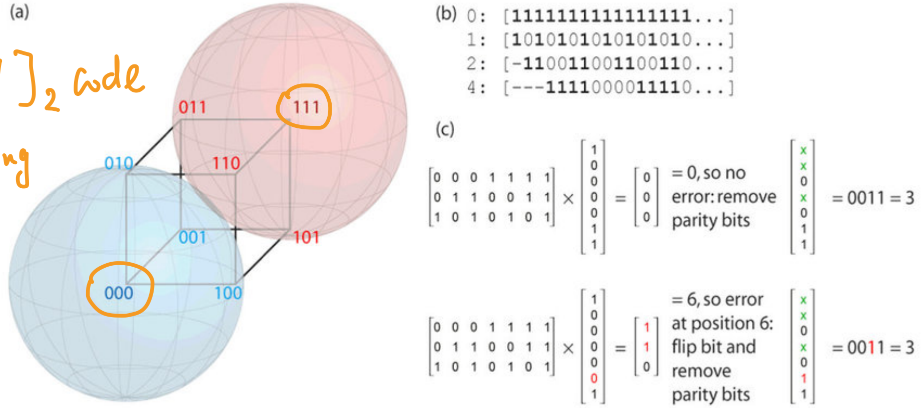
$$H\mathbf{x}'^{\top} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^{\top} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

  So we found that the 5th bit has an error (tells a lie). Then we "correct" it to (1100110). Wrong answer!
- If you told two lies or more, I could not win (by using the $[7,4]_2$ code).

An illustration for a Hamming code

$[3,1]_2$ code

Hamming



$$\begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}\times\begin{bmatrix}1\\0\\0\\0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}\times\begin{bmatrix}1\\0\\0\\0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Figure from

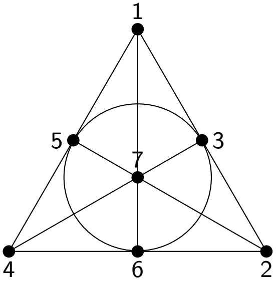
Hamady, Micah, et al. Error-correcting barcoded primers allow hundreds of samples to be pyrosequenced in multiplex. Nature Methods 5(3): 235–237., 2008.

Outline

Fano plane (revisit)

- Fano plane is $\mathrm{PG}(2, \mathbb{F}_2)$ with 7 points $\{1, 2, \ldots, 7\}$ and 7 lines.
- The characteristic vectors (特性ベクトル) of lines are the red ones among the codewords of the $[7, 4]_2$ Hamming code.

$(0,0,0,0,0,0,0)$	$(1,0,0,0,0,1,1)$
$(0,0,0,1,1,1,1)$	$(1,0,0,1,1,0,0)$
$(0,0,1,0,1,1,0)$	$(1,0,1,0,1,0,1)$
$(0,0,1,1,0,0,1)$	$(1,0,1,1,0,1,0)$
$(0,1,0,0,1,0,1)$	$(1,1,0,0,1,1,0)$
$(0,1,0,1,0,1,0)$	$(1,1,0,1,0,0,1)$
$(0,1,1,0,0,1,1)$	$(1,1,1,0,0,0,0)$
$(0,1,1,1,1,0,0)$	$(1,1,1,1,1,1,1)$



Some more geometric terms

极大

- A k-arc (弧) is a set of k points in a plane such that no three are collinear. An arc of maximal size is said to be an oval (卵形).
- The maximum 最大 size of an arc in a plane of order n is $n+2$ (called a hyperoval) if n is even and $n+1$ (called a oval) if n is odd.
- Any four points in Fano plane that does not contain a line form a hyperoval.
- The complement of any line in Fano plane is a hyperoval.

For the $[7,4]_2$ Hamming code $\mathcal{C}$,

$$\mathcal{C} = \{\text{lines in } \mathrm{PG}(2,\mathbb{F}_2)\} \cup \{\text{hyperovals in } \mathrm{PG}(2,\mathbb{F}_2)\} \\ \cup \{\text{all-one codeword}, \text{all-zero codeword}\}$$

Why does the geometric winning strategy work?

1. If $\mathrm{wt}(\mathbf{x}) = 1$, the lie $=$ the position of 1. [nearest neighbor to all-zero codeword]
2. If $\mathrm{wt}(\mathbf{x}) = 2$, two positions of 1s are lying in a unique line ℓ in Fano plane. The lie $=$ the third point on ℓ. [nearest neighbor to a line]
3. If $\mathrm{wt}(\mathbf{x}) = 3$, if three positions of 1s are lying in a unique line, then no lie. Otherwise, the four positions of 0s contains three point which form a line and one other point P. The lie $= P$. [a line or nearest neighbor to a hyperoval]
4. If $\mathrm{wt}(\mathbf{x}) = 4$, [a hyperoval or nearest neighbor to a line]
5. If $\mathrm{wt}(\mathbf{x}) = 5$, [nearest neighbor to a hyperoval]
6. If $\mathrm{wt}(\mathbf{x}) = 6$, [nearest neighbor to all-one codeword]

We were looking for the ~~nearest neighbor to~~ some codeword!

as nearest neighbor of $\mathbf{x}$

Some more algebra ... Coset decomposition

Proposition

For an $[n, k]_q$ code $\mathcal{C}$ and a vector $\mathbf{w} \in \mathbb{F}_q^n$,

$$\mathcal{C} + \mathbf{w} = \{\mathbf{x} + \mathbf{w} : \mathbf{x} \in \mathcal{C}\}.$$

is called a coset (コセット, 剰余類 [2]) of $\mathcal{C}$.

- For $\mathbf{w} \in \mathbb{F}_q^n$, $|\mathcal{C}| = |\mathcal{C} + \mathbf{w}|$.
- For $\mathbf{w}, \mathbf{u} \in \mathbb{F}_q^n$, either $\mathcal{C} + \mathbf{w} = \mathcal{C} + \mathbf{u}$ or $(\mathcal{C} + \mathbf{w}) \cap (\mathcal{C} + \mathbf{u}) = \emptyset$.
- There exist distinct vectors $\mathbf{w}_0, \ldots, \mathbf{w}_{q^{n-k}-1}$ such that

$$\mathbb{F}_q^n = \bigcup_{i=0}^{q^{n-k}-1} (\mathcal{C} + \mathbf{w}_i)$$

[2] In Chinese, "陪集".

Final explanation for winning strategy of Ulam's game

Proposition

Let $\mathcal{C}$ be an $[n,k]_q$ code with parity check matrix H. Let $\mathbf{v},\mathbf{w} \in \mathbb{F}_q^n$. Then $S(\mathbf{v}) = S(\mathbf{w})$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are in the same coset of $\mathcal{C}$.

- Each coset of the $[7,4]_2$ Hamming code $\mathcal{C}$ has 16 codewords.
- Let $\mathbf{e}_i$ $(1 \le i \le 7)$ denote the vector in $\mathbb{F}_2^7$ whose ith entry is 1 and the others are 0.
- The cosets $\mathcal{C} + \mathbf{e}_i$ are distinct for distinct i. Hence,

$e_1 = (1, 0, 0, \ldots, 0)$
$e_2 = (0, 1, 0, \ldots, 0)$

$$\mathbb{F}_2^7 = \bigcup_{i=1}^{7} (\mathcal{C} + \mathbf{e}_i)$$

- For $\mathbf{x}' = \mathbf{x} + \mathbf{e}_i$ (one lie is told for the ith question),

$$H{\mathbf{x}'}^\top = H(\mathbf{x} + \mathbf{e}_i)^\top = H\mathbf{e}_i^\top = i\text{th column vector of } H.$$

Summary on Hamming codes

Binary Hamming code

$[7,4]_2$ code ← ($m=3$ のとき)

A binary Hamming code is a $[n=2^m-1, k=2^m-m-1]_2$ code with parity check matrix H whose ith $(1 \le i \le 2^m-1)$ column is the binary representation of i.

Proposition

単一誤り訂正符号

A binary Hamming code corrects all single errors, i.e., its minimum distance is $d=3$.

Theorem (generalized Hamming code)

Let $n=(q^m-1)/(q-1)$. *Let* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *be distinct points in* $\mathrm{PG}(m-1, \mathbb{F}_q)$, *i.e. the vectors in* $\mathbb{F}_q^m$. *Let* H *be the* $m \times n$ *matrix whose column vectors are* $\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n$. *Let* $\mathcal{H}(m,q)$ *be the code having* H *as its parity check matrix. Then* $\mathcal{H}(m,q)$ *is called a* *generalized Hamming code* *and it is a* $[\frac{q^m-1}{q-1}, \frac{q^m-1}{q-1}-m, 3]_q$ *code.*

q-ary (q元H.C.)

Outline

Sphere packing bound

Proposition

- For any vector $\mathbf{v} \in \mathbb{F}_q^n$ there are $\binom{n}{s}(q-1)^s$ vectors in $\mathbb{F}_q^n$ that have Hamming distance s from $\mathbf{v}$.
- For any vector $\mathbf{v} \in \mathbb{F}_q^n$ there are $\sum_{s=0}^{t}\binom{n}{s}(q-1)^s$ vectors in the sphere of radius t centered at $\mathbf{v}$.

(Hamming ball)

Theorem (sphere packing bound) 球充填の限界式

Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a code with minimum weight $2t+1$. Then

$$|\mathcal{C}| \leq \frac{q^n}{\sum_{s=0}^{t}\binom{n}{s}(q-1)^s}.$$

$\leftarrow |\mathbb{F}_q^n|$

$\leftarrow$ # vectors in a Hamming ball

Perfect codes

Perfect codes

A code with equality in the sphere packing bound is said to be a perfect code.

- The generalized Hamming codes are perfect codes.

Figure from https://en.wikipedia.org/wiki/Sphere_packing

Golay codes

- Golay's $[11, 6, 5]_3$ codes and Golay's $[23, 12, 7]_2$ codes are perfect codes (Golay, 1949).
- The binary Golay code is closely related to the Witt 4- and 5-designs and the Mathieu groups.

Notes on Digital Coding*

The consideration of message coding as a means for approaching the theoretical capacity of a communication channel, while reducing the probability of errors, has suggested the interesting number theoretical problem of devising lossless binary (or other) coding schemes serving to insure the reception of a correct, but reduced, message when an upper limit to the number of transmission errors is postulated.

An example of lossless binary coding is treated by Shannon[1] who considers the case of blocks of seven symbols, one or none of which can be in error. The solution of this case can be extended to blocks of 2^n-1-binary symbols, and, more generally, when coding schemes based on the prime number p are employed, to blocks of $p^n-1/p-1$ symbols which are transmitted, and received with complete equivocation of one or no symbol, each block comprising n redundant symbols designed to remove the equivocation. When encoding the message, the n redundant symbols x_m are determined in terms of the message symbols Y_k from the congruent relations

$$E_m \equiv X_m + \sum_{k=1}^{k=(p^n-1)/(p-1)-n} a_{mk} Y_k \equiv 0 \pmod{p}.$$

In the decoding process, the E's are recalculated with the received symbols, and their ensemble forms a number on the base p which determines univocally the mistransmitted symbol and its correction.

In passing from n to $n+1$, the matrix with n rows and $p^n-1/p-1$ columns formed with the coefficients of the X's and Y's in the expression above is repeated p times horizontally, while an $(n+1)$ st row added, consisting of $p^n-1/p-1$ zeroes, followed by as many one's etc. up to $p-1$; an added column of n zeroes with a one for the lowest term completes the new matrix for $n+1$.

If we except the trivial case of blocks of $2S+1$ binary symbols, of which any group comprising up to S symbols can be received in error which equal probability, it does not appear that a search for lossless coding schemes, in which the number of errors is limited but larger than one, can be systematized so as to yield a family of solutions. A necessary but not sufficient condition for the existence of such a lossless coding scheme in the binary system is the existence of three or more first numbers of a line of Pascal's triangle which add up to an exact power of 2. A limited search has revealed two such cases; namely, that of the first three numbers of the 90th line, which add up to 2^{12} and that of the first four numbers of the 23rd line, which add up to 2^{11}. The first case does not correspond to a lossless coding scheme, for, were such a scheme to exist, we could designate by r the number of E_m ensembles corresponding to one error and having an odd number of 1's and by $90-r$ the remaining (even) ensembles. The odd ensembles corresponding to two transmission errors could be formed by re-entering term by term all the conbinations of one even and one odd ensemble corresponding each to one error, and would number $r(90-r)$. We should have $r+r(90-r)=2^{11}$, which is impossible for integral values of r.

On the other side, the second case can be coded so as to yield 12 sure symbols, and the a_{mk} matrix of this case is given in Table I. A second matrix is also given, which is that of the only other lossless coding scheme encountered (in addition to the general class mentioned above) in which blocks of eleven ternary symbols are transmitted with no more than 2 errors, and out of which six sure symbols can be obtained.

It must be mentioned that the use of the ternary coding scheme just mentioned will always result in a power loss, whereas the coding scheme for 23 binary symbols and a maximum of three transmission errors yields a power saving of $1\frac{1}{2}$ db for vanishing probabilities of errors. The saving realized with the coding scheme for blocks of 2^n-1 binary symbols approaches 3 db for increasing n's and decreasing probabilities of error, but a loss is always encountered when $n=3$.

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TABLE I

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}
X_1	1	0	0	1	1	1	0	0	0	1	1	1
X_2	1	0	1	0	1	1	0	1	1	0	0	1
X_3	1	0	1	1	0	1	1	0	1	0	1	0
X_4	1	0	1	1	1	0	1	1	0	1	0	0
X_5	1	1	0	0	1	1	1	0	1	1	0	0
X_6	1	1	0	1	0	1	1	1	0	0	0	1
X_7	1	1	0	1	1	0	0	1	1	0	1	0
X_8	1	1	1	0	0	1	0	1	0	1	1	0
X_9	1	1	1	0	1	0	1	0	0	0	1	1
X_{10}	1	1	1	1	0	0	0	0	1	1	0	1
X_{11}	0	1	1	1	1	1	1	1	1	1	1	1

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
X_1	1	1	1	2	2	0
X_2	1	1	2	1	0	2
X_3	1	2	1	0	1	2
X_4	1	2	0	1	2	1
X_5	1	0	2	2	1	1

* Received by the Institute, February 23, 1949.
[1] C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. Jour.*, vol. 27, p. 418; July, 1948.

Reprinted from *Proc. IRE*, vol. 37, p. 657, June 1949.

M. J. E. Golay, Notes on digital coding. Proc. IEEE 37, p. 657, 1949.

by E. R. Berlekamp, the "best single published page" in coding theory.

Singleton bound and MDS codes

Theorem (Singleton bound)

Let $\mathcal{C}$ be a code of length n over an alphabet Ω of size q with minimum Hamming distance d and q^k elements. Then

$$d \leq n - k + 1.$$

MDS codes

A code with equality in the singleton bound is said to be a Maximum Distance Separable (MDS) code.

MDS conjecture

If $\mathcal{C}$ is an $[n, k, n-k+1]_q$ MDS code then $n \leq q + 1$.

MDS codes and combinatorial structures

Theorem

A set of s *MOLS of order* q *is equivalent to an* $[s+2, 2, s+1]_q$ *MDS code.*

Theorem

An $[n, k, n-k+1]_q$ *MDS code is equivalent to an* n*-arc in* $\mathrm{PG}(n-k-1, \mathbb{F}_q)$.

Homework assignments (レポート課題) for 4th day

Exercise 1

▸▸ (1) (2), ▸▸ (3) (4), ▸▸ (5)

Exercise 2

$2^4-1=15$ $(m=4)$

1. Produce the parity check matrix for the Hamming code of length 15.
2. Use the parity check matrix produced above to decode the vectors (111001101101101) and (001100110011010).

- You are encouraged to use computer programs.
- Deadline: 6th Sept., 23:59:59