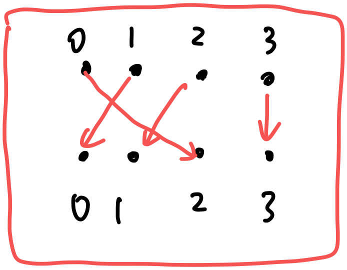
8/25 応用数学特論II

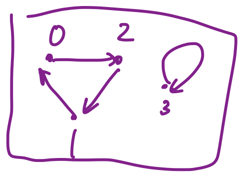
- Permutation:

- $\sigma = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 3 \end{pmatrix}$

$1^\sigma = 0, \quad 2^\sigma = 1, \quad 0^\sigma = 2, \quad 3^\sigma = 3$

σ:

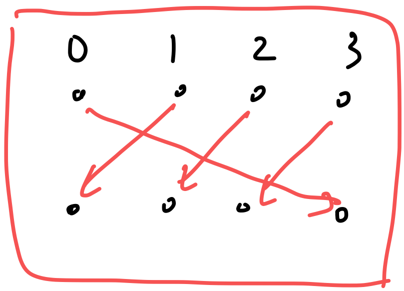


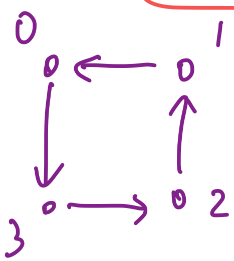


$\sigma = (0\ 1\ 2)(3)$
$= (0\ 1\ 2)$

- $\tau = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix}$

τ:





$\tau = (0\ 1\ 2\ 3)$

- $\sigma \circ \tau = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \end{pmatrix}$

Corollary

If $|S| = k$ for each $S \in \mathcal{S}$, then $\mathcal{S}$ (said to be k-uniform) has an SDR.

Proof: For any $W \subseteq \mathcal{S}$ with

$$|W| = w \geq 2$$

Then

$$\left| \bigcup_{A \in W} W \right| \geq k + (w-1) > |W|$$

$\Longleftrightarrow$ Hall's condition is satisfied.

$\overset{\text{Hall's thm}}{\Longleftrightarrow}$ $\mathcal{S}$ has SDR.

Theorem

$N(n) \le n - 1.$

Fact 1: By replacing symbols of L,

a new (equivalent) LS is obtained.

Fact 2: If LS_1, LS_2 are two orth. LS.

then, $(LS_1)^\sigma$, $(LS_2)^\tau$ are orth.

for any $\sigma, \tau \in S_n$.

Fact 3: Every LS can be written in

"standard form", in which

the 1st row & 1st col. of LS

are $[1, 2, 3, \dots, n]$.

$$L_1 = \begin{bmatrix} 1 & 2 & \cdots & n \\ \boxed{x} & & & \\ & & & \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & 2 & \cdots & n \\ \boxed{y} & & & \\ & & & \end{bmatrix}$$

$$L_1 \boxplus L_2 = \begin{bmatrix} 11 & 22 & 33 & \cdots & nn \\ \underline{xy} & & & & \\ & \cancel{xx} & & & \end{bmatrix}$$

$$x \in \{2, \cdots, n\}$$

$$y \in \{2, \cdots, n\} \setminus \{x\} \leftarrow (n-2) \text{ possible symbols}$$

$\Longleftrightarrow$ The number of possible LS which is orthogonal to L_1 is $\underline{n-2}$.

$$\Longleftrightarrow \quad N(n) \le n-1. \qquad \blacksquare$$

Theorem

- A_1, A_2: *MOLS of order* n
- B_1, B_2: *MOLS of order* m

Then $A_1 \otimes B_1$ *and* $A_2 \otimes B_2$ *are MOLS of order* mn*, where* $\otimes$ *denotes Kronecker product.*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{m1} & \cdots & b_{mm} \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a_{11} \otimes B & \cdots & a_{1n} \otimes B \\ \vdots & & \\ a_{n1} \otimes B & \cdots & a_{nn} \otimes B \end{bmatrix}, \text{ where}$$

$$a_{11} \otimes B = \begin{bmatrix} (a_{11}, b_{11}), & \cdots & , (a_{11}, b_{1m}) \\ \vdots & & \vdots \\ (a_{11}, b_{1m}), & \cdots, & (a_{11}, b_{mm}) \end{bmatrix}$$

$A \otimes B$: order mn

In $(A_1 \otimes B_1) \boxplus (A_2 \otimes B_2)$, each combination in $([n] \times [m]) \times ([n] \times [m])$ appears once.

Theorem

For any prime power q, $N(q) = q - 1$.

$\Rightarrow$ $N(3)=2$, $N(4)=3$, $N(5)=4$, ...

For any odd prime p, $N(p)=p-1$.

Theorem

For any $n \equiv 0, 1, 3 \pmod 4$, $N(n) \geq 2$.

① $n \equiv 0 \pmod 4$, $n = 2^m \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_t^{\alpha_t}$,

where $p_1, \ldots, p_t$: odd prime, & $m \geq 2$

$$\begin{aligned} N(n) &= N(2^m \cdot p_1^{\alpha_1} \cdots p_t^{\alpha_t}) \\ &\geq \min\{\underbrace{N(2^m)}_{\geq 4}, \underbrace{N(p_1^{\alpha_1})}_{\geq 3}, \ldots, \underbrace{N(p_t^{\alpha_t})}_{\geq 3}\} \\ &\geq N(3) = 2. \end{aligned}$$

② n : odd, $N(n) \geq \min\{N(p_1^{\alpha_1}), \ldots, N(p_t^{\alpha_t})\}$

$$\geq N(3) = 2.$$

Theorem (Wanless, Webb, 2006)

The Latin square generated by the cyclic group $\mathbb{Z}_n$ has no transversal if n is even.

L: Cayley table of $\mathbb{Z}_n$: (n: even)

$$L(x,y) = x+y \pmod n$$

$$\Delta(x,y,L(x,y)) = x+y-L(x,y) \mod n$$
$$= x+y-(x+y)$$
$$= 0 \pmod n$$

$\therefore$ T: transversal.

$$\sum_{e\in T} \Delta(e) = 0. \neq \frac{n}{2}.$$

$\Rightarrow$ A contradiction to Lemma.

$\therefore$ L has no transversal.

Statistical model for experiments using OA(4, 3, 2, 2)

(4 exp. & 3 factors)

$$\begin{cases} y_{112} = \mu + A_1 + B_1 + C_2 + \varepsilon_{112} \\ y_{121} = \mu + A_1 + B_2 + C_1 + \varepsilon_{121} \\ y_{211} = \mu + A_2 + B_1 + C_1 + \varepsilon_{211} \\ y_{222} = \mu + A_2 + B_2 + C_2 + \varepsilon_{222}. \end{cases}$$

where $\begin{cases} A_1 + A_2 = 0 \\ B_1 + B_2 = 0 \\ C_1 + C_2 = 0 \end{cases}$

A_i: factor 1 level i の主効果 (main effect)

y_{***}: yield (収量)

B_i: m.e. of factor 2 level i

ε_{***}: error (誤差)

C_i: m.e. of factor 3 level i.

μ: general mean (総平均) $\left[\begin{array}{l} \text{not a} \\ \text{rand.} \\ \text{var.} \end{array}\right]$

$$\bar{y} = \frac{1}{4}(y_{112} + y_{121} + y_{211} + y_{222})$$

$$= \mu + \frac{1}{4}(\varepsilon_{112} + \varepsilon_{121} + \varepsilon_{211} + \varepsilon_{222})$$

$\bar{y}$は

μの推定量 (estimator)

$$\bar{y}_{A_1} = \frac{1}{2}(y_{112} + y_{121})$$

$$= \mu + A_1 + \frac{1}{2}(\varepsilon_{112} + \varepsilon_{121})$$

$$\bar{y}_{A_2} = \frac{1}{2}(y_{211} + y_{222})$$

$$= \mu + A_2 + \frac{1}{2}(\varepsilon_{211} + \varepsilon_{222})$$

$$\hat{A}_1 = \bar{y}_{A_1} - \frac{1}{2}(\varepsilon_{112} + \varepsilon_{121}) - (\bar{y} - \bar{\varepsilon})$$

$$\boxed{\mathrm{Var}(\hat{A}_1)} = \mathrm{Var}\left(\frac{1}{4}(\varepsilon_{112}+\varepsilon_{121}+\varepsilon_{211}+\varepsilon_{222}) - \frac{1}{2}(\varepsilon_{112}+\varepsilon_{121})\right)$$

(i.i.d.)
ε_{***}
$\sim N(0,\sigma^2)$

$$= \mathrm{Var}\left(\frac{1}{4}(-\varepsilon_{112}-\varepsilon_{121}+\varepsilon_{211}+\varepsilon_{222})\right)$$

$$= \left(\frac{1}{4}\right)^2 \cdot (\sigma^2+\sigma^2+\sigma^2+\sigma^2)$$

$$\boxed{= \frac{1}{4}\sigma^2.}$$

$$\mathrm{Var}(\hat{A}_i) = \mathrm{Var}(\hat{B}_i) = \mathrm{Var}(\hat{C}_i)$$

$$= \frac{1}{4}\sigma^2.$$

不偏(unbiased)推定.