Atomic Types T $::=$ $A \rightarrow B$ Function type
$\quad$ | $\top$ Top type
$\quad$ | $\forall \alpha * B. A$ Universal quantification

Types A, B, C, D $::=$ α Type variable
$\quad$ | $A \cap B$ Intersection type
$\quad$ | T Atomic type

Expressions e $::=$ x Variable
$\quad$ | $\top$ Top
$\quad$ | $\lambda(x{:}A). e$ Lambda
$\quad$ | $e_1\ e_2$ Application
$\quad$ | $\Lambda \alpha * A. e$ Big lambda
$\quad$ | $e\ A$ Type application
$\quad$ | $e_1,, e_2$ Merge

Contexts Γ $::=$ ϵ
$\quad$ | $\Gamma, \alpha * A$
$\quad$ | $\Gamma, x{:}A$

Figure 1. Syntax.

$\boxed{\Gamma \vdash A \bot B}$

$$\frac{\alpha * B \in \Gamma}{\Gamma \vdash \alpha \bot B}\ \text{DisjointRefl} \qquad \frac{\alpha * A \in \Gamma}{\Gamma \vdash A \bot \alpha}\ \text{DisjointSym}$$

$$\frac{\Gamma \vdash A \bot C \quad \Gamma \vdash B \bot C}{\Gamma \vdash A \& B \bot C}\ \text{DisjointSub1}$$

$$\frac{\Gamma \vdash A \bot B \quad \Gamma \vdash A \bot C}{\Gamma \vdash A \bot B \& C}\ \text{DisjointSub2}$$

$$\frac{T_1 \not<: T_2 \quad T_2 \not<: T_1}{\Gamma \vdash T_1 \bot T_2}\ \text{DisjointAtomic}$$

Figure 3. Disjointness.

$\boxed{A <: B\ \hookrightarrow F}$

$$\frac{}{\alpha <: \alpha\ \hookrightarrow \lambda(x{:}|\alpha|). x}\ \text{SubVar}$$

$$\frac{}{A <: \top\ \hookrightarrow \lambda(x{:}|A|). ()}\ \text{SubTop}$$

$$\frac{\tau_3 <: \tau_1\ \hookrightarrow C_1 \quad \tau_2 <: \tau_4\ \hookrightarrow C_2}{\tau_1 \rightarrow \tau_2 <: \tau_3 \rightarrow \tau_4\ \hookrightarrow \lambda(f{:}|\tau_1 \rightarrow \tau_2|). \lambda(x{:}|\tau_3|). C_2\ (f\ (C_1\ x))}\ \text{SubFun}$$

$$\frac{\tau_1 <: [\alpha_1/\alpha_2]\tau_2\ \hookrightarrow C}{\forall \alpha_1 * \tau_3. \tau_1 <: \forall \alpha_2 * \tau_3. \tau_2\ \hookrightarrow \lambda(f{:}|\forall \alpha_1 * \tau_3. \tau_1|). \Lambda(.\ \alpha)C\ (f\ \alpha)}\ \text{SubForall}$$

$$\frac{\tau_1 <: \tau_2\ \hookrightarrow C_1 \quad \tau_1 <: \tau_3\ \hookrightarrow C_2}{\tau_1 <: \tau_2 \cap \tau_3\ \hookrightarrow \lambda(x{:}|\tau_1|). (C_1\ x, C_2\ x)}\ \text{SubAnd}$$

$$\frac{\tau_1 <: \tau_3\ \hookrightarrow C}{\tau_1 \cap \tau_2 <: \tau_3\ \hookrightarrow \lambda(x{:}|\tau_1 \cap \tau_2|). C\ (\text{proj}_1 x)}\ \text{SubAnd}_1$$

$$\frac{\tau_2 <: \tau_3\ \hookrightarrow C}{\tau_1 \cap \tau_2 <: \tau_3\ \hookrightarrow \lambda(x{:}|\tau_1 \cap \tau_2|). C\ (\text{proj}_2 x)}\ \text{SubAnd}_2$$

Figure 2. Subtyping.

$\boxed{\Gamma \vdash e : A\ \hookrightarrow E}$

$$\frac{x{:}A \in \Gamma}{\Gamma \vdash x : A\ \hookrightarrow x}\ \text{TyVar} \qquad \frac{}{\Gamma \vdash \top : \top\ \hookrightarrow ()}\ \text{TyTop}$$

$$\frac{\Gamma \vdash A\ \text{type} \quad \Gamma, x{:}A \vdash e : B\ \hookrightarrow E}{\Gamma \vdash \lambda(x{:}A). e : A \rightarrow B\ \hookrightarrow \lambda(x{:}|A|). E}\ \text{TyLam}$$

$$\frac{\Gamma \vdash e_1 : A_1 \rightarrow A_2\ \hookrightarrow E_1 \quad \Gamma \vdash e_2 : A_3\ \hookrightarrow E_2 \quad A_3 <: A_1\ \hookrightarrow C}{\Gamma \vdash e_1\ e_2 : A_2\ \hookrightarrow E_1\ (C\ E_2)}\ \text{TyApp}$$

$$\frac{\Gamma, \alpha * B \vdash e : A\ \hookrightarrow E}{\Gamma \vdash \Lambda \alpha * B. e : \forall \alpha * B. A\ \hookrightarrow \Lambda \alpha. E}\ \text{TyBLam}$$

$$\frac{\Gamma \vdash e : \forall \alpha * C. B\ \hookrightarrow E \quad \Gamma \vdash A \bot C \quad \Gamma \vdash A\ \text{type}}{\Gamma \vdash e\ A : [A/\alpha]B\ \hookrightarrow E\ |A|}\ \text{TyTApp}$$

$$\frac{\Gamma \vdash e_1 : A\ \hookrightarrow E_1 \quad \Gamma \vdash e_2 : B\ \hookrightarrow E_2 \quad \Gamma \vdash A \bot B}{\Gamma \vdash e_1,, e_2 : A \cap B\ \hookrightarrow (E_1, E_2)}\ \text{TyMerge}$$

Figure 4. Typing.

$$\epsilon \vdash 1,, 2 : (\text{Int} * \text{Int}) \Rightarrow \text{Int} \cap \text{Int}$$

Definition 1. (Disjointness) Two sets S and T are *disjoint* if there does not exist an element x, such that $x \in \mathsf{S}$ and $x \in \mathsf{T}$.

Definition 2. (Disjointness) Two types A and B are *disjoint* if there does not exist an expression e, which is not a merge, such that $\epsilon \vdash e : A'$, $\epsilon \vdash e : B'$, $A' <: A$, and $B' <: B$.

Definition 3. (Disjointness) $A \perp B = \nexists C.A <: C \wedge B <: C$

Two types A and B are *disjoint* if their least common supertype is $\top$.