

# Disjoint Intersection Types

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## Abstract

Dunfield has shown that a simply typed core calculus with intersection types and a merge operator is able to capture various programming language features. While his calculus is type-safe, it is known that it is not *coherent*: different derivations for the same expression can lead to different results. The lack of coherence is an important disadvantage for adoption of his core calculus in implementations of programming languages, as the semantics of the programming language becomes implementation-dependent.

This paper presents  $\lambda_i$ : a core calculus with a variant of *intersection types* and a *merge operator*. The semantics  $\lambda_i$  is both type-safe and coherent. Coherence is achieved by ensuring that intersection types are *disjoint*. Formally, two types are disjoint if they do not share a common supertype. **BRUNO: Abstract needs to be fixed, disjointness is no longer the absence of a common supertype!** We present a type system that prevents intersection types that are not disjoint, as well as an algorithmic specification to determine whether two types are disjoint. Moreover, we show the applicability of this calculus to express a simple, yet powerful form of dynamically composable traits, paving the way for new designs of object-oriented programming languages. **BRUNO: Drop sentence about traits and replace by other contributions?**

**Categories and Subject Descriptors** CR-number [subcategory]: third-level

**General Terms** Design, Languages, Theory

**Keywords** Intersection Types, Polymorphism, Type System

## 1. Introduction

Previous work by Dunfield [13] has shown the usefulness of type systems with intersection types and a merge operator. The presence of a merge operator in a core calculus provides significant expressiveness, allowing encodings for many other language constructs as syntactic sugar. For example single-field records are easily encoded as types with a label, and multi-field records are encoded as the concatenation of single-field records. Concatenation of records is expressed using intersection types at the type-level and the corresponding merge operator at the term level. Dunfield formalized a simply typed lambda calculus with intersection types and a merge operator. He showed how to give a semantics to the calculus by

a type-directed translation to a simply typed lambda calculus extended with pairs. The type-directed translation is elegant and type-safe.

While Dunfield's calculus is type-safe, it lacks the property of *coherence*: different derivations for the same expression can lead to different results. The lack of coherence is an important disadvantage for adoption of his core calculus in implementations of programming languages, as the semantics of the programming language becomes implementation dependent. Although Dunfield mentioned the possibility of extending the type system to allow only disjoint intersection types, he did not formalize or further pursue this approach.

This paper presents  $\lambda_i$ : a core calculus with a variant of *intersection types* and a *merge operator*. The semantics of  $\lambda_i$  is both type-safe and coherent. Coherence is achieved by ensuring that intersection types are *disjoint*. **BRUNO: introduction needs to be rewritten at least from this point!**

Given two types  $A$  and  $B$ , two types are disjoint  $(A * B)^1$  if there is no type  $C$  such that both  $A$  and  $B$  are subtypes of  $C$ . Formally this definition is captured as follows:

$$A * B \equiv \neg C. A <: C \wedge B <: C$$

With this definition of disjointness we present a formal specification of a type system that prevents intersection types that are not disjoint. However, the formal definition of disjointness does not lend itself directly to an algorithmic implementation. Therefore, we also present an algorithmic specification to determine whether two types are disjoint. Moreover, this specification is shown to be sound and complete with respect to the formal definition of disjointness. We have mechanized the many of the interesting metatheoretical results using the Coq proof assistant, and the main proof of type-safety and the coherence theorem can be found in the full version of the paper<sup>2</sup>.

As an application of  $\lambda_i$ , we show that  $\lambda_i$  can serve as a foundation for a trait-based [21] object-oriented language. Due to the type system of  $\lambda_i$ , the language supports common features of traits such as commutative composition, explicit conflict detection and resolution. In addition, our language also supports instantiating and composing traits dynamically, which cannot be achieved in many existing statically typed object-oriented languages.

In summary, the contributions of this paper are:

- **Disjoint Intersection Types:** A new form of intersection type where only disjoint types are allowed. A sound and complete algorithmic specification of disjointness (with respect to the corresponding formal definition) is presented.
- **Dynamically Composable Traits:** As an application of this work, we show how to build a trait-based object-oriented lan-

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<sup>1</sup>The notation  $A * B$  is inspired by the *separating conjunction* construct in Reynolds' separation logic [20].

<sup>2</sup><https://github.com/zhiyuanshi/intersection/blob/master/esop-16/paper-full.pdf>

guage by leveraging the type system of  $\lambda_i$ . The resulting source language enjoys many features described in the original traits proposal for free.

- **Formalization of  $\lambda_i$  and Proof of Coherence:** An elaboration semantics of  $\lambda_i$  into  $\lambda$ -calculus is given. Type-soundness and coherence are proved and formalized. Several key theorems are mechanically formalized using the Coq theorem prover.
- **Implementation:** An implementation of an extension of  $\lambda_i$ , as well as encodings of the examples presented in the paper, are publicly available online<sup>3</sup>.

**BRUNO:** Records and record operations missing in the appendix.

## 2. Overview

This section introduces  $\lambda_i$  and its support for intersection types and the merge operator. It then discusses the issue of coherence and shows how the notion of disjoint intersection types achieve a coherent semantics.

Note that this section uses some syntactic sugar, as well as standard programming language features, to illustrate the various concepts in  $\lambda_i$ . Although the minimal core language that we formalize in Section 4 does not present all such features, our implementation supports them.

### 2.1 Intersection Types and the Merge Operator

Intersection types date back as early as Coppo *et al.*'s work [8]. Since then various researchers have studied intersection types, and some languages have adopted them in one form or another.

**Intersection Types.** The intersection of type A and B (denoted as  $A \& B$  in  $\lambda_i$ ) contains exactly those values which can be used as either values of type A or of type B. For instance, consider the following program in  $\lambda_i$ :

```
let x : Int & Bool = ... in -- definition omitted
let succ (y : Int) : Int = y+1 in
let not (y : Bool) : Bool = if y then False else True in
(succ x, not x)
```

If a value  $x$  has type  $\text{Int} \& \text{Bool}$  then  $x$  can be used as an integer and as a boolean. Therefore,  $x$  can be used as an argument to any function that takes an integer as an argument, and any function that take a boolean as an argument. In the program above the functions `succ` and `not` are the simple functions on integers and characters, respectively. Passing  $x$  as an argument to either one (or both) of the functions is valid.

**Merge Operator.** In the previous program we deliberately did not show how to introduce values of an intersection type. There are many variants of intersection types in the literature. Our work follows a particular formulation, where intersection types are introduced by a *merge operator*. As Dunfield [13] has argued a merge operator adds considerable expressiveness to a calculus. The merge operator allows two values to be merged in a single intersection type. In  $\lambda_i$  (following Dunfield's notation), the merge of two values  $v_1$  and  $v_2$  is denoted as  $v_1, v_2$ . For example, an implementation of  $x$  is constructed in  $\lambda_i$  as follows:

```
let x : Int & Boolean = 1, True in ...
```

**Merge Operator and Pairs.** The merge operator is similar to the introduction construct on pairs. An analogous implementation of  $x$  with pairs would be:

```
let xPair : (Int, Bool) = (1, True) in ...
```

The significant difference between intersection types with a merge operator and pairs is in the elimination construct. With pairs there are explicit eliminators (`fst` and `snd`). These eliminators must be used to extract the components of the right type. For example, in order to use `succ` and `not` with pairs, we would need to write a program such as:

```
(succ (fst xPair), not (snd xPair))
```

In contrast the elimination of intersection types is done implicitly, by following a type-directed process. For example, when a value of type  $\text{Int}$  is needed, but an intersection of type  $\text{Int} \& \text{Bool}$  is found, the compiler generates code that uses `fst` to extract the corresponding value at runtime.

### 2.2 (In)Coherence

Coherence is a desirable property for the semantics of a programming language. A semantics is said to be coherent if any *valid program* has exactly one meaning [18] (that is, the semantics is not ambiguous). In contrast a semantics is said to be *incoherent* if there are multiple possible meanings for the same valid program.

**Incoherence in Dunfield's calculus** A problem with Dunfield's calculus [13] is that it is incoherent. Unfortunately the implicit nature of elimination for intersection types built with a merge operator can lead to incoherence. The merge operator combines two terms, of type A and B respectively, to form a term of type  $A \& B$ . For example, `1, True` is of type  $\text{Int} \& \text{Bool}$ . In this case, no matter if `1, True` is used as  $\text{Int}$  or  $\text{Bool}$ , the result of evaluation is always clear. However, with overlapping types, it is not straightforward anymore to see the intended result. For example, what should be the result of the following program, which asks for an integer (using a type annotation) out of a merge of two integers:

```
(1, 2) : Int
```

Should the result be 1 or 2?

Dunfield's calculus [13] accepts the program above, and it allows that program to result in 1 or 2. In other words the results of the program are incoherent. **BRUNO:** rephrase better?

**Getting Around Incoherence** In a real implementation of Dunfield calculus a choice has to be made on which value to compute. For example, one potential option is to always take the left-most value matching the type in the merge. Similarly, one could always take the right-most value matching the type in the merge. Either way, the meaning of a program will depend on a biased implementation choice, which is clearly unsatisfying from the theoretical point of view. Dunfield suggests some other possibilities, such as the possibility of restricting typing of merges so that a merge has type A only if exactly one branch has type A. He also suggested another possibility, which is to allow only for *disjoint* types in an intersection. This is the starting point for us and the approach that we will investigate in this paper.

### 2.3 Disjoint Intersection Types and their Challenges

$\lambda_i$  requires that the two types in an intersection to be *disjoint*. Informally saying that two types are disjoint means that the set of values of both types are disjoint. Disjoint intersection types are potentially useful for coherence, since they can rule out ambiguity when looking up a value of a certain type in an intersection. However there are several issues that need to be addressed first in order to design a calculus with disjoint intersection types and that ensures coherence. The key issues and the solutions provided by our work are discussed next. We emphasize that although Dunfield has mentioned disjointness as an option to restore coherence, he has not studied the approach further or addressed the issues discussed next.

<sup>3</sup> <https://github.com/hkuplg/fcore/tree/develop/examples/traits>

**Simple disjoint intersection types** Looking back at the expression 1, 2 in Section 2.2, we can see that the reason for incoherence is that there are multiple, overlapping, integers in the merge. Generally speaking, if both terms can be assigned some type  $C$ , both of them can be chosen as the meaning of the merge, which leads to multiple meanings of a term. A natural option is to try to forbid such overlapping values of the same type in a merge. Thus, for atomic types such as `Int` and `Bool`, it is easy to see that disjointness holds when the two types are different. Intersections such as `Int & Bool` and `String & Bool` are clearly disjoint.

**Disjointness of non-atomic types** While it is easy to see how disjointness works for atomic types, it is more challenging to see what disjointness mean for non-atomic types. For example, consider the following intersection types of functions:

1.  $(\text{Int} \rightarrow \text{Int}) \& (\text{String} \rightarrow \text{String})$
2.  $(\text{String} \rightarrow \text{Int}) \& (\text{String} \rightarrow \text{String})$
3.  $(\text{Int} \rightarrow \text{String}) \& (\text{String} \rightarrow \text{String})$

Which of those intersection types are disjoint? It seems reasonable to expect that the first intersection type is disjoint: both the domain and codomain of the two functions in the intersection are different. However, it is less clear whether the two other intersection types are disjoint or not. This paper proposes the following rule for determining the disjointness of two function types:

$$\frac{\vdash A_2 *_i B_2}{\vdash A_1 \rightarrow A_2 *_i B_1 \rightarrow B_2} \text{DIS-FUN}$$

Here  $*_i$  denotes disjointness. According to the rule the first two intersection types are disjoint, whereas the last one is not. This rule is justified in Section ?? **BRUNO: Should we stop here; or briefly argue about some justification**

**Formalizing disjointness** Another challenge is how to formalize the notion of disjointness? So far all discussion about disjointness has been based on an informal understanding of what disjointness means. Clearly a formal definition of disjointness is desirable to design a calculus with disjoint intersection types. As we shall see the particular notion of disjointness can be quite sensitive to the language features that are allowed in a language.

A first attempt at a definition for disjointness is to require that, given two types  $A$  and  $B$ , both types are not subtypes of each other. Thus, denoting disjointness as  $A * B$ , we would have:

$$A * B \equiv A \not\prec B \wedge B \not\prec A$$

At first sight this seems a reasonable definition and it does prevent merges such as 1, 2. However some moments of thought are enough to realize that such definition does not ensure disjointness. For example, consider the following merge:

$(1, , 'c') \text{ } , , (2, , \text{True})$

This merge has two components which are also intersection types. The first component  $(1, , 'c')$  has type `Int&Char`, whereas the second component  $(2, , \text{True})$  has type `Int&Bool`. Clearly,

$$\text{Int}\&\text{Char} \not\prec \text{Int}\&\text{Bool} \wedge \text{Int}\&\text{Bool} \not\prec \text{Int}\&\text{Char}$$

Nevertheless the following program still leads to incoherence:

`succ ((1, , 'c'), (2, , True))`

as both 2 or 3 are possible outcomes of the program. Although this attempt to define disjointness failed, it did bring us some additional insight: although the types of the two components of the merge are not subtypes of each other, they share some types in common.

In order for two types to be truly disjoint, they must not have any subcomponents sharing the same type. In a simply typed calculus

with intersection types (and without a  $\top$  type) this can be ensured by requiring the two types not to share a common supertype:

**Definition 1** (Simple disjointness). Given two types  $A$  and  $B$ , two types are disjoint (written  $A * B$ ) if there is no type  $C$  such that both  $A$  and  $B$  are subtypes of  $C$ :

$$A * B \equiv \nexists C. A <: C \wedge B <: C$$

This definition of disjointness prevents the problematic  $(1, , 'c') \text{ } , , (2, , \text{True})$  merge. Since `Int` is a common supertype of both `Int&Char` and `Int&Bool`, those two types are not disjoint, according to this simple notion of disjointness.

The simple definition of disjointness is the basis for the first calculus presented in this paper: a simply typed lambda calculus with intersection (but without a  $\top$  type). This variant of  $\lambda_i$  is useful to study many important issues arising from disjoint intersections, without the additional complications of  $\top$ .

**Disjointness in the presence of  $\top$  types** In the presence of a  $\top$  type the simple definition of disjointness is useless. In that case there *always* exists a common supertype (for example, the  $\top$  type). So adding a  $\top$  type requires some adaptations on the notion of disjointness. This work studies two additional variants of  $\lambda_i$  with  $\top$  types. In both variants the definition of disjointness is revised as follows:

**Definition 2** (Disjointness with  $\top$ ). Given two types  $A$  and  $B$ , two types are disjoint (written  $A * B$ ) if the following two conditions are met:

1.  $\neg A_\top$  and  $\neg B_\top$
2.  $\forall C.$  if  $A <: C$  and  $B <: C$  then  $C_\top$

In this definition the idea is to embrace the presence of certain common supertypes. Instead of requiring that two types do not have any common supertype, we require that the only supertypes are *top-like*. The unary relation  $A_\top$  denotes such top-like types. As shown in this paper allowing such supertypes does not cause incoherence.

$\lambda_i$ 's type system only accepts programs that use disjoint intersection types. As shown in Section 5 disjoint intersection types will play a crucial rule in guaranteeing that the semantics is coherent.

**Is disjointness sufficient to ensure coherence?** A final question is whether disjoint intersection types are sufficient to ensure coherence. Consider the following example:

`(succ, , not) (3, , True)`

Here there are two merges. The first merge contains two functions (`succ` and `not`). The second merge contains two values that can serve as input to the functions. The two merges are disjoint. However what should be the result of this program? Should it be 4 or `False`?

At first this program appears to lead to incoherence, even though it only uses disjoint merges. However, a closer look reveals that there are two possible types for this program: `Int` or `Bool`. Once the type of the program is fixed, there is only one possible result: if the type is `Int` the result of the program is 4; if the type is `Bool` then the result of the program is `False`. Like other programming language features (for example type classes `[]`), types play a fundamental role in determining the result of a program, and the semantics of the language is not independent from types.

Disjointness is indeed sufficient to ensure coherence. The key point is that given a program *and its type* there is always a unique result for that program. However, for certain programs type annotations are necessary to rule out *type ambiguity*. Similar type ambiguity issues arise in other type-directed mechanisms such as type-classes `[]`.

In our implementation, which uses bi-directional type-checking techniques [], the ambiguity problem is solved by annotating the term being applied. For example:

```
((succ,,not) : Int → Int) (3,,True)
```

is a valid program and results in 4.

## 2.4 Restoring Coherence: Disjoint Intersection Types

However he did not present a formal definition of disjointness and did not pursue further that approach.

*Searching for a Definition of Disjointness.*

## 3. Dynamically Composable Traits

**GEORGE: after the new keyword, types should be specified.**

As an application of disjoint intersection types, we show how to model a simple, yet expressive form of dynamically composable traits [21] in  $\lambda_i$ . Traits provide a mechanism of code reuse in object-oriented programming, that can be used as an alternative to multiple inheritance. The interesting aspect about traits is the way conflicts that typically arise in multiple-inheritance are dealt with. Instead of trying to automatically resolve conflicts, traits detect those conflicts and require programmers to explicitly resolve them. This is where the relation to disjoint intersection types comes in: the mechanism to detect incoherence of disjoint intersection types provides us with the mechanism to detect conflicts in traits. We demonstrate various trait features in a simple OO language, and then sketch a straightforward translation from that language to  $\lambda_i$  as a basic form of syntactic sugar. **GEORGE: Need to note the difference between class-based and prototype-based somewhere.**

### 3.1 Basic Traits

A trait is a collection of related methods that characterizes only a specific perspective of the features of an object. Therefore, compared with programs using inheritance, programs using traits usually have a larger number of small traits rather than fewer but larger classes. Code reuse with traits is easier than with classes, since traits are usually shorter and can be *composed*. In fact, trait composition offers a variety of possibilities: two traits can be “added” together (which is an symmetric operation); methods can be removed from a trait; and trait systems provide conflict detection, etc.

The first example shows basic trait composition. Many social networking sites allow users to “upvote” a comment and the number of upvotes that comment has received is also displayed. We would like to separate the logic for upvotes from comments so that it can be reused in other entities such as posts and sharings. The code below defines a trait, `Comment`, which contains a single method `content`.<sup>4</sup>

```
type Comment = { content: () → String } in
trait Comment(content: String) { self: Comment →
  content() = content
} in
```

Next, we create another trait, `Up`, for tracking the number of upvotes.

```
type Up = { upvotes: () → Int } in
trait Up(upvotes: Int) { self: Up →
  upvotes() = upvotes
} in
```

At this point the reader may wonder why there are duplicate declarations related to `Comment` and `Up`. In mainstream OO languages such as Java, a class declaration such as `class C { ... }` does two things at the same time:

- Declaring a *template* for creating objects;
- Declaring a new *type*.

In contrast, trait declarations in this source language only do the former. Back to our example, the purpose of declaring two types is just to use them for type annotations of the self reference. In the traits literature, a trait usually “requires” some methods and, based on that, “provides” another set of methods. In our examples, the type of `self` actually denotes what methods are required.

A trait can expect parameters, which become in scope in the entire trait body. For example, the `Comment` trait is parametrized by `content`, and the `content` method does nothing more than returning the eponymous variable.

The origin of self references is always explicit. The `Comment` trait requires `self` to be of type `Comment`, which is defined as a type synonym for a record in the first line. But the name “self” is nothing special. In fact, `self` is just another parameter after the preceding parameter list, and becomes in scope after the arrow. We could have named it `this` or even `s`.

Creating an object is via the **new** keyword, similar to many OO languages, except for one crucial novelty: we can create an object from multiple traits. More precisely, the object is created from the *composition* of those traits. Therefore, we are able to call methods from different traits on a single object. For example, we can create a single object from `Comment` and `Up` traits and test its functionality as follows:

```
let comment = new[Comment&Up] (Comment("Have fun!") &
  Up(4))
in println(comment.content(), comment.upvotes())
-- Output: "Have fun!" 4
```

### 3.2 Traits with Dependencies

The following example shows that a trait can depend on another trait. First we define the type of a point and a trait for a standard point.

```
type Point = { x: () → Int, y: () → Int } in
trait Point(x: Int, y: Int) { self: Point →
  x() = x
  y() = y
} in
```

The norm of a point can be defined as its distance to the origin. We provide two definitions of norm via two traits.

```
type Norm = { norm: () → Double } in
trait EuclideanNorm() { self: Point →
  norm() = Math.sqrt(self.x() * self.x() + self.y() *
    self.y())
} in
trait ManhattanNorm() { self: Point →
  norm() = Math.abs(self.x()) + Math.abs(self.y())
} in
```

Note how in `EuclideanNorm` and `ManhattanNorm` the type of the self-reference is `Point`! This is in contrast to a typical object-oriented language, such as Java, where the self-reference must always be of the same type as the class being defined. It is this functionality that allow us to express dependencies between traits. When the traits `EuclideanNorm` and `ManhattanNorm` are instantiated and composed with some other traits, they must be composed with an implementation of `Point`.

```
println(new[Point&Norm] (Point(3,4) &
  EuclideanNorm()).norm()) --Prints 5
println(new[Point&Norm] (Point(3,4) &
  ManhattanNorm()).norm()) --Prints 7
```

<sup>4</sup>Our source language assumes regular record type, record operations as well the unit type and unit literals.



### 3.3 Mutual Dependencies

The next example, although a little bit contrived, illustrates that when two traits are composed, any two methods in those two traits can refer to each other via the self reference, just as if they were inside the same class.

```
type EvenOdd = {
  even: Int → Bool,
  odd: Int → Bool
} in
trait Even() { self: EvenOdd →
  even(n: Int) = if n == 0 then True else self.odd(n - 1)
} in
trait Odd() { self: EvenOdd →
  odd(n: Int) = if n == 0 then False else self.even(n - 1)
} in
new[EvenOdd] (Even() & Odd()).odd(42)
```

When the two traits are composed, conceptually it is as if that a new object were being created on the fly by copying all the definitions inside those two traits. If there is any unresolved conflict, the program will be rejected by the type system.

### 3.4 Detecting and Resolving Conflicts in Trait Composition

Traits usually supports explicit conflict detection and resolution. In inheritance, one pattern to resolve conflicts is for the subclass to override methods defined in the parent. The trait-based approach analog is excluding a method from a trait. We show how the mechanism can be modeled in  $\lambda_i$ . The following example shows a counter object and how we could extend its behavior so that it supports reset. First we define a `Counter` as a type synonym for a record that contains a `val` method, which returns the current counter value. Next we define a trait `Counter` that contains two methods. The `val` method just returns the value that is bound at the parameter of the trait, and `incr` returns a new counter.

```
type Counter = { val: () → Int } in
trait Counter(val: Int) { self: Counter →
  val() = val
  incr() = new[Counter] Counter(val + 1)
} in
type Reset = { reset: () → Counter } in
trait Reset() { self: Counter →
  reset() = new[Counter] Counter(0)
} in
let counter = new[Counter&Reset] (Counter(0) & Reset())
in counter.incr()
```

In the above code, even though `counter` has a `reset` method, after we call the `incr` method, the resulting object no longer has that. Therefore, naturally we would like to override the `incr` method inside `Reset`.

```
type Counter = { val: () → Int } in
trait Counter(val: Int) { self: Counter →
  val() = val
  incr() = new[Counter] Counter(val + 1)
} in
trait Reset() { self: Counter →
  incr() = new[Counter&Reset] (Counter(val + 1) & Reset())
  reset() = new[Counter] Counter(0)
} in
let counter = new[Counter&Reset] (Counter(0) & Reset())
in counter.incr()
```

However the modified code should not typecheck according to the specification of traits, since both `Counter` and `Reset` contains a conflicting `incr` method. The code also does not typecheck since it violates the disjoint intersection typing rules of  $\lambda_i$ . The problem is that both `Counter(0)` and `Reset()` provide an implementation of

`incr`. The programmer needs to resolve the conflict by excluding the `incr` being overridden using the record exclusion operator<sup>5</sup>.

```
...
let counter = new (Counter(0) \ incr & Reset())
in counter.incr()
```

### 3.5 Dynamic Instantiation

One difference with traditional traits or classes is that in our language we are able to compose traits *dynamically* and then instantiate them. This is impossible in traditional OO languages such as Java since classes being instantiated must be known statically. Actually, since traits are just terms, traits are first-class values and can be defined inside a function, passed around or returned just as normal terms. For example, the following function takes a trait, `norm`, with unknown implementation, compose it with the statically known `Point` trait, and then instantiate it. Trait is a built-in keyword and `Trait[Norm]` is the type for traits that conform to the `Norm` type.

```
let makePoint (norm: Trait[Norm]) = new[Point & Norm]
  (Point(3,4) & norm)
in ...
```

**BRUNO:** example does not make sense (and uses no composition).

**BRUNO:** Explain the trait[Log]

### 3.6 Desugaring

Of course, this whole section will lose its point if the source language cannot be translated to  $\lambda_i$  and checked against the type system of  $\lambda_i$ . A more formal description can be found in the full version of the paper. The idea of trait translation is inspired by the functional mixin semantics [7] using open recursion, which was proposed by Cook in an untyped setting. However, our translation is done context of a statically-typed programming language, which is what provides the ability to *statically* detect conflicts in traits.

**Trait Declarations.** A trait in the source language is translated into nothing but a normal term in  $\lambda_i$ . For example,

```
trait Point(x: Int, y: Int) { self: Point →
  x() = x
  y() = self.z()
}
...
```

becomes

```
let Point (x: Int, y: Int) (self: () → Point) = {
  x = λ(_: ()) → x,
  y = λ(_: ()) → (self ()) . z()
} in
```

One difference is that the self reference becomes a thunk and all occurrences of it have been replaced by `self ()`. Moreover the position of the self reference in the parameter list is adjusted. In fact, `self` is not a special keyword. It can have any name, but `self` is a convention.

The body of the trait becomes a record whose labels are the method names. `Point` has type:

$(Int, Int) \rightarrow ((\rightarrow Point) \rightarrow Point)$

The syntax for construction such as `Point(3,4)` is just function application in  $\lambda_i$ . And note that `Point(3,4)` is of type

$(\rightarrow Point) \rightarrow Point$

Therefore it is an open recursive term: the recursive call is passed as an argument to the definition of `Point` itself.

<sup>5</sup> We use  $e \setminus 1$  to denote the term where the field 1 is removed. For example,  $\{x = 1, y = 2\} \setminus x$  gives  $\{x = 1\}$ .

**The “new” Keyword.** `new` instantiates a trait by taking the fixpoint of its corresponding open term. In fact, `new` is translated as an inlined fixpoint. For example,

```
new[Point] Point(3,4)
```

becomes

```
let rec self : () → Point = λ(_: ()) → Point (3, 4)
  self
in self ()
```

The composition of traits in the source language is desugared using the merge operator. The reason that traits built on  $\lambda_i$  have conflict detection for free is that the merge operator is enforcing that the two terms being merged are disjoint. For example,

```
new[Point3D] (Point(3,4) & Z(5))
```

is turned into

```
let rec self : () → Point3D = λ(_: ()) → (Point (3,
  4) self) ,, (Z 5 self)
in self ()
```

**The “Trait” Keyword.** The capitalized `Trait` keyword expects a type and is translated into an open type. For example,

```
Trait[Point]
```

becomes

```
((() → Point) → Point
```

## 4. The $\lambda_i$ Calculus

This section presents the syntax, subtyping, typing, as well as the (incoherent) semantics of  $\lambda_i$ : a calculus with intersection types and a merge operator. This calculus is similar to Dunfield’s calculus [13]. The differences are that, in Dunfield’s language, the coercions are source language terms and that function application allows the expression denoting the function to have an intersection type. In contrast, in  $\lambda_i$  coercions are target language terms and function applications do not allow the expression denoting the applied function to have an intersection type. The later difference will play an important role in achieving coherence. Sections 5 and 6 will present the more fundamental contributions of this paper by showing other necessary changes for supporting disjoint intersection types and ensuring coherence.

### 4.1 Syntax

Figure 1 shows the syntax. The differences to the  $\lambda$ -calculus, highlighted in gray, are intersection types  $A \& B$  at the type-level and the “merges”  $e_1, e_2$  at the term level.

**Types.** Metavariables  $A, B$  range over types. Types include function types  $A \rightarrow B$ .  $A \& B$  denotes the intersection of types  $A$  and  $B$ . We also include  $\top$  and integer types `Int`.

**Terms.** Metavariables  $e$  range over terms. Terms include standard constructs: variables  $x$ ; abstraction of terms over variables of a given type  $\lambda(x : A). e$ ; and application of terms  $e_1$  to terms  $e_2$ , written  $e_1 e_2$ . The expression  $e_1, e_2$  is the *merge* of two terms  $e_1$  and  $e_2$ . Merges of terms correspond to intersections of types  $A \& B$ . The term  $\top$  corresponds to the only inhabitant of the  $\top$  type. In addition, we also include integer literals  $i$ .

**Contexts.** Typing contexts  $\Gamma$  track bound variables  $x$  with their type  $A$ .

In order to focus on the key features that make this language interesting, we do not include other forms such as type constants and fixpoints here. However they can be included in the formalization in standard ways and we are using them in discussions and examples. The formalization of nonessential features such as records and record operations can be found in the full version of the paper.

Types	$A, B, C$	$::=$	$\text{Int}$ $A \rightarrow B$ $A \& B$ $\top$
Terms	$e$	$::=$	$i$ $x$ $\lambda(x : A). e$ $e_1 e_2$ $e_1, e_2$ $\top$
Contexts	$\Gamma$	$::=$	$\cdot \mid \Gamma, x : A$

Figure 1.  $\lambda_i$  syntax.

### 4.2 Subtyping

**BRUNO: MAJOR PROBLEM:** We are missing the well-formedness rules as well as their uses in Figure 2. These need to be added so that we can talk about coherence later!

The subtyping rules of the form  $A <: B$  are shown in the top part of Figure 2. At the moment, the reader is advised to ignore the gray-shaded part in the rules, which will be explained later. The rule (SUB-FUN) says that a function is contravariant in its parameter type and covariant in its return type. The three rules dealing with intersection types are just what one would expect when interpreting types as sets. Under this interpretation, for example, the rule (SUB-INTER) says that if  $A_1$  is both the subset of  $A_2$  and the subset of  $A_3$ , then  $A_1$  is also the subset of the intersection of  $A_2$  and  $A_3$ .

**Metatheory.** As other sane subtyping relations, we can show that subtyping defined by  $<:$  is also reflexive and transitive.

**Lemma 1** (Subtyping is reflexive). *For all type  $A$ ,  $A <: A$ .*

**Lemma 2** (Subtyping is transitive). *If  $A_1 <: A_2$  and  $A_2 <: A_3$ , then  $A_1 <: A_3$ .*

For the corresponding mechanized proofs in Coq, we refer the reader to the repository associated with the paper<sup>6</sup>.

### 4.3 Typing

The well-formedness rules are shown in the middle part of Figure 2. In addition to the standard rules, (F-WF-INTER) is also not surprising. The typing rules are shown in the bottom part of the figure. The typing judgment is of the form:

$$\Gamma \vdash e : A$$

It reads: “in the typing context  $\Gamma$ , the term  $e$  is of type  $A$ ”. The standard rules are those for variables (F-TY-VAR) and lambda abstractions (F-TY-LAM). (F-TY-MERGE) means that a merge  $e_1, e_2$ , is assigned an intersection type composed of the resulting types of  $e_1$  and  $e_2$ .

**Typing Applications** The rule that deserves more attention is the application rule (F-TY-APP). This is where the most important difference between  $\lambda_i$  and Dunfield calculus is. In Dunfield’s calculus, if  $f$  and  $g$  are both functions expecting an integer, the following program typechecks:

```
(f ,, g) 1
```

<sup>6</sup><https://github.com/zhiyuanshi/intersection/tree/master/esop-16>

$$\begin{array}{c}
\boxed{A <: B \hookrightarrow E} \\
\\
\frac{}{\alpha <: \alpha \hookrightarrow \lambda(x:|\alpha|).x} \text{SUB-VAR} \\
\\
\frac{}{\text{Int} <: \text{Int} \hookrightarrow \lambda(x:|\text{Int}|).x} \text{SUB-INT} \\
\\
\frac{B_1 <: A_1 \hookrightarrow E_1 \quad A_2 <: B_2 \hookrightarrow E_2}{A_1 \rightarrow A_2 <: B_1 \rightarrow B_2 \hookrightarrow \lambda(f:|A_1 \rightarrow A_2|).\lambda(x:|B_1|).E_2 (f (E_1 x)))} \text{SUB-FUN} \\
\\
\frac{A_1 <: A_2 \hookrightarrow E_1 \quad A_1 <: A_3 \hookrightarrow E_2}{A_1 <: A_2 \& A_3 \hookrightarrow \lambda(x:|A_1|).(E_1 x, E_2 x)} \text{SUB-INTER} \\
\\
\frac{A_1 <: A_3 \hookrightarrow E}{A_1 \& A_2 <: A_3 \hookrightarrow \lambda(x:|A_1 \& A_2|).E (\text{proj}_1 x)} \text{SUB-INTER-1} \\
\\
\frac{A_2 <: A_3 \hookrightarrow E}{A_1 \& A_2 <: A_3 \hookrightarrow \lambda(x:|A_1 \& A_2|).E (\text{proj}_2 x)} \text{SUB-INTER-2} \\
\\
\frac{}{A <: \top \hookrightarrow \lambda(\_ : |A|).()} \text{SUB-TOP} \\
\\
\boxed{\Gamma \vdash A} \\
\\
\frac{}{\Gamma \vdash \text{Int}} \text{F-WF-INT} \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \rightarrow B} \text{F-WF-FUN} \\
\\
\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \text{F-WF-INTER} \quad \frac{}{\Gamma \vdash \top} \text{F-WF-TOP} \\
\\
\boxed{\Gamma \vdash e : A \hookrightarrow E} \\
\\
\frac{x:A \in \Gamma}{\Gamma \vdash x : A \hookrightarrow x} \text{F-TY-VAR} \quad \frac{}{\Gamma \vdash i : \text{Int}} \text{F-TY-INT} \\
\\
\frac{\Gamma \vdash A \quad \Gamma, x:A \vdash e : B \hookrightarrow E}{\Gamma \vdash \lambda(x:A).e : A \rightarrow B \hookrightarrow \lambda(x:|A|).E} \text{F-TY-LAM} \\
\\
\frac{\Gamma \vdash e_1 : A_1 \hookrightarrow A_2 \hookrightarrow E_1 \quad \Gamma \vdash e_2 : A_3 \hookrightarrow E_2 \quad A_3 <: A_1 \hookrightarrow E}{\Gamma \vdash e_1 e_2 : A_2 \hookrightarrow E_1 (E E_2)} \text{F-TY-APP} \\
\\
\frac{\Gamma \vdash e_1 : A \hookrightarrow E_1 \quad \Gamma \vdash e_2 : B \hookrightarrow E_2}{\Gamma \vdash e_1, e_2 : A \& B \hookrightarrow (E_1, E_2)} \text{F-TY-MERGE} \\
\\
\frac{}{\Gamma \vdash \top : \top \hookrightarrow ()} \text{F-TY-TOP}
\end{array}$$

**Figure 2.** The type system of  $\lambda_i$ .

In our calculus, however, one needs manually extract the desired function by passing  $f$ ,  $g$  to a function. For example, if  $f : \text{Int} \rightarrow \text{Bool}$  and  $g : \text{Int} \rightarrow \text{Char}$ , we could select  $f$  from the intersection as follows:

$((\text{fun } (h : \text{Int} \rightarrow \text{Bool}) \rightarrow h) (f, g)) 1$

The reason for this design choice is that otherwise we could have another source of semantic ambiguity. Consider the following program:

$(\text{idInt}, \text{idChar}) (1, 'c')$

where  $\text{idInt}$  and  $\text{idChar}$  are, respectively, identity functions on integers and on characters. Clearly both merges in the program  $((f, g)$  and  $(1, 'c')$ ) are disjoint. Yet there are two possible outcomes for this program:  $1$  or  $'c'$ . Thus application requires a different treatment, if we wish to have coherence. Our design choice is simply to only employ subtyping in the argument of the application, but not on the expression being applied. It should be possible to design a “smarter” application rule, that still avoids the ambiguity problems that would be present in Dunfield’s calculus, while at the same time avoiding explicitly extracting functions from intersections. However such rule would necessarily be complicated, and rather ad-hoc. So we opted for simplicity. In any case there is no expressiveness loss (only loss of convinience), as it is always possibly to extract the function component from a merge and then apply it.

#### 4.4 Semantics

We define the dynamic semantics of the call-by-value  $\lambda_i$  by means of a type-directed translation to an extension of  $\lambda$ -calculus with pairs.

**Target Language.** The syntax and typing of our target language is unsurprising. The syntax of the target language is shown in Figure 3. The highlighted part shows its difference with the  $\lambda$ -calculus. The typing rules can be found in the extended version of the paper.

**Key Idea of the Translation.** This translation turns merges into usual pairs, similar to Dunfield’s elaboration approach [13]. For example,

$1, \text{"one"}$

becomes  $(1, \text{"one"})$ . In usage, the pair will be coerced according to type information. For example, consider the function application:

$(\lambda(x:\text{String}).x) (1, \text{"one"})$

This expression will be translated to

$(\lambda(x:\text{String}).x) ((\lambda(x:(\text{Int}, \text{String})).\text{proj}_2 x) (1, \text{"one"}))$

The coercion in this case is  $(\lambda(x:(\text{Int}, \text{String})).\text{proj}_2 x)$ . It extracts the second item from the pair, since the function expects a  $\text{String}$  but the translated argument is of type  $(\text{Int}, \text{String})$ .

**Type and Context Translation.** Figure 4 defines the type translation function  $|\cdot|$  from  $\lambda_i$  types  $A$  to target language types  $T$ . The notation  $|\cdot|$  is also overloaded for context translation from  $\lambda_i$  contexts  $\Gamma$  to target language contexts  $G$ .

**Coercive subtyping.** The judgment

$A_1 <: A_2 \hookrightarrow E$

extends the subtyping judgment in Figure 2 with a coercion on the right hand side of  $\hookrightarrow$ . A coercion  $E$  is just an term in the target language and is ensured to have type  $|A_1| \rightarrow |A_2|$  (by Lemma 3). For example,

$\text{Int} \& \text{Bool} <: \text{Bool} \hookrightarrow \lambda(x:|\text{Int} \& \text{Bool}|).\text{proj}_2 x$

Types	$T$	$::=$	$\text{Int}$ $()$ $T_1 \rightarrow T_2$ $(T_1, T_2)$
Terms	$E$	$::=$	$x$ $i$ $()$ $\lambda(x:T). E$ $E_1 E_2$ $(E_1, E_2)$ $\text{proj}_k E \quad k \in \{1, 2\}$
Contexts	$G$	$::=$	$\cdot \mid G, x:T$

**Figure 3.** Target language syntax.

$$|A| = T$$

$$\begin{aligned}
|\text{Int}| &= \text{Int} \\
|A_1 \rightarrow A_2| &= |A_1| \rightarrow |A_2| \\
|A_1 \& A_2| &= (|A_1|, |A_2|) \\
|T| &= ()
\end{aligned}$$

$$|\Gamma| = G$$

$$\begin{aligned}
|\cdot| &= \cdot \\
|\Gamma, \alpha:A| &= |\Gamma|, \alpha:|A|
\end{aligned}$$

**Figure 4.** Type and context translation.

generates a coercion function with type:  $\text{Int} \& \text{Bool} \rightarrow \text{Bool}$ .

In (SUB-FUN), we elaborate the subtyping of parameter and return types by  $\eta$ -expanding  $f$  to  $\lambda(x:|A_3|). f \ x$ , applying  $E_1$  to the argument and  $E_2$  to the result. Rules (SUB-INTER-1), (SUB-INTER-2), and (SUB-INTER) elaborate intersection types. (SUB-INTER) uses both coercions to form a pair. Rules (SUB-INTER-1) and (SUB-INTER-2) reuse the coercion from the premises and create new ones that cater to the changes of the argument type in the conclusions. Note that the two rules are overlapping and hence a program can be elaborated differently, depending on which rule is used. Finally, all rules produce type-correct coercions:

**Lemma 3** (Subtyping rules produce type-correct coercions). *If  $A_1 <: A_2 \hookrightarrow E$ , then  $\cdot \vdash E : |A_1| \rightarrow |A_2|$ .*

*Proof.* By a straightforward induction on the derivation<sup>7</sup>.  $\square$

**The translation judgment.** The translation judgment  $\Gamma \vdash e : A \hookrightarrow E$  extends the typing judgment with an elaborated term on the right hand side of  $\hookrightarrow$ . The translation ensures that  $E$  has type  $|A|$ . In  $\lambda_i$ , one may pass more information to a function than what is required; but not in ordinary  $\lambda$ -calculus. To account for this difference, in (F-TY-APP), the coercion  $E$  from the subtyping relation is applied to the argument. (F-TY-MERGE) straightforwardly translates merges into pairs.

<sup>7</sup>The proofs of major lemmata and theorems can be found in the full version of the paper. [BRUNO: where??](#)

$\boxed{ A }$	
$\frac{}{ T } \text{TOPLIKE-TOP}$	$\frac{ B }{ A \rightarrow B } \text{TOPLIKE-FUN}$
$\frac{ A }{ A \& B } \text{TOPLIKE-INTER-1}$	$\frac{ B }{ A \& B } \text{TOPLIKE-INTER-2}$

**Figure 5.** Top-like types.

The type-directed translation is type-safe. This property is captured by the following two theorems.

**Theorem 1** (Type preservation). *If  $\Gamma \vdash e : A \hookrightarrow E$ , then  $|\Gamma| \vdash E : |A|$ .*

*Proof.* (Sketch) By structural induction on the term and the corresponding inference rule.  $\square$

**Theorem 2** (Type safety). *If  $e$  is a well-typed  $\lambda_i$  term, then  $e$  evaluates to some  $\lambda$ -calculus value  $v$ .*

*Proof.* Since we define the dynamic semantics of  $\lambda_i$  in terms of the composition of the type-directed translation and the dynamic semantics of  $\lambda$ -calculus, type safety follows immediately.  $\square$

## 5. Disjointness and Coherence

Although the system shown in the previous section is type-safe, it is not coherent, in the sense that one source program can be elaborated into two different target programs. This section shows how to modify it so that it guarantees coherence as well as type soundness. The result is a calculus named  $\lambda_i$ . The key aspect is the notion of disjoint intersection.

### 5.1 Disjointness

Throughout the paper we already presented an intuitive definition for disjointness. Before we discuss that definition, let us introduce first the notion of a top-like type

**Definition 3** (Top-like types). One type  $A$  is a top-like type, denoted as  $|A|$ , if it has the form  $A_k \rightarrow T$ , where  $k \in 0, 1, \dots$ . That is, any type with arity  $k$  can be a top-like type, as long as  $T$  is the result type.

Thus a top-like type is a unary relation at the type level, which can be formalized according to 5. Now, we recall the definition of disjoint types

**Definition 4** (Disjoint types). Two types  $A$  and  $B$  are said to be disjoint (written  $\vdash A * B$ ) if they do not share a common supertype - excluding top-like types. That is, for all type  $C$  such that  $A <: C$  and  $B <: C$ , then  $|C|$  must hold.

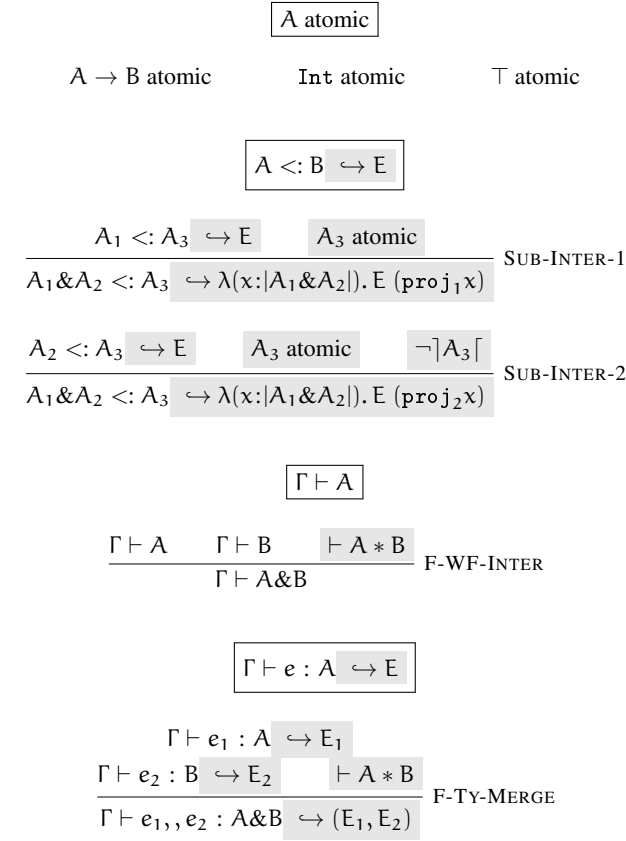
$$\vdash A * B \equiv \forall C. A <: C \wedge B <: C \Rightarrow |C|$$

For example,  $\text{Int}$  and  $\text{Char}$  are disjoint, if their only common supertype is  $T$ . On the other hand,  $\text{Int}$  is not disjoint with itself, because  $\text{Int} <: \text{Int}$ . This implies that disjointness is not reflexive as subtyping is. Two types with different “shapes” are always disjoint, unless one of them is an intersection type. For example, a function type and an intersection type may not be disjoint. Consider:

$$\text{Int} \rightarrow \text{Int} \quad \text{and} \quad (\text{Int} \rightarrow \text{Int}) \& (\text{String} \rightarrow \text{String})$$

Those two types are not disjoint since  $\text{Int} \rightarrow \text{Int}$  is their common supertype. Also,  $\text{Int} \rightarrow \text{Char}$  and  $\text{Int} \rightarrow \text{String}$  are disjoint,





**Figure 6.** Affected rules.

since their supertypes are all types with the form  $A \rightarrow T$  (where  $A$  contains Int, i.e. Int, Int&Char) and  $\top$ .

JOAO: do we want to allow this?  $f: \text{String} \rightarrow \text{Int}$   
 $g: \text{String} \rightarrow T$   
 $(\lambda h: \text{String} \rightarrow T. h \text{ ""}) (f, g)$

At last, take as an example  $(\top \& \top)$ . In a pure language (i.e. no side-effects), this type can be safely allowed since both components of the merge will evaluate to the same value. However, in an effectful language, the evaluation of either the right or left component might lead to distinct results. Following the same reasoning, we might want to allow or restrict, for instance  $\text{String} \rightarrow \text{Int} \& \text{String} \rightarrow \top$ ,

## 5.2 Typing

Figure 6 shows modifications to Figure 2 in order to support disjoint intersection types. Only new rules or rules that different are shown. Importantly, the disjointness judgment appears in the well-formedness rule for intersection types and the typing rule for merges.

**Well-formedness.** We require that the two types of an intersection must be disjoint. Under the new rules, intersection types such as  $\text{Int} \& \text{Int}$  are no longer well-formed because the two types are not disjoint.

**Meta-theory.** Since in this section we only restrict the type system in the previous section, it is easy to see that type preservation and type-safety still holds.

## 5.3 Subtyping

The subtyping rules need some adjustment. An important problem with the subtyping rules in Figure 2 is that all three rules dealing with intersection types ((SUB-INTER-1) and (SUB-INTER-2) and (SUB-INTER)) overlap. Unfortunately, this means that different coercions may be given when checking the subtyping between two types, depending on which derivation is chosen. This is the ultimate reason for incoherence. There are two important types of overlap:

1. The left decomposition rules for intersections ((SUB-INTER-1) and (SUB-INTER-2)) overlap with each other.
2. The left decomposition rules for intersections ((SUB-INTER-1) and (SUB-INTER-2)) overlap with the right decomposition rules for intersections (SUB-INTER).

Fortunately, disjoint intersections (which are enforced by well-formedness) deal with problem 1): only one of the two left decomposition rules can be chosen for a disjoint intersection type. Since the two types in the intersection are disjoint, it is impossible that both of the preconditions of the left decompositions are satisfied at the same time. More formally, with disjoint intersections, we have the following theorem:

**Lemma 4** (Unique subtype contributor). *If  $A_1 \& A_2 <: B$ , where  $A_1 \& A_2$  and  $B$  are well-formed types, then it is not possible that the following holds at the same time:*

1.  $A_1 <: B$
2.  $A_2 <: B$

Unfortunately, disjoint intersections alone are insufficient to deal with problem 2). In order to deal with problem 2), we introduce a distinction between types, and atomic types.

**Atomic Types.** Atomic types are just those which are not intersection types, and are asserted by the judgment

$$A \text{ atomic}$$

Since types in  $\lambda_i$  are simple, the only atomic types are the function type and integers. But in richer systems, it can also include, for example, record types. In the left decomposition rules for intersections we introduce a requirement that  $A_3$  is atomic. The consequence of this requirement is that when  $A_3$  is an intersection type, then the only rule that can be applied is (SUB-INTER). With the atomic constraint, one can guarantee that at any moment during the derivation of a subtyping relation, at most one rule can be used. Consequently, the coercion of a subtyping relation  $A <: B$  is uniquely determined. This fact is captured by the following lemma:

**Lemma 5** (Unique coercion). *If  $A <: B \hookrightarrow E_1$  and  $A <: B \hookrightarrow E_2$ , where  $A$  and  $B$  are well-formed types, then  $E_1 \equiv E_2$ .*

**No Loss of Expressiveness.** Interestingly, our restrictions on subtyping do not sacrifice the expressiveness of subtyping since we have the following two theorems:

**Theorem 3.** *If  $A_1 <: A_3$ , then  $A_1 \& A_2 <: A_3$ .*

**Theorem 4.** *If  $A_2 <: A_3$ , then  $A_1 \& A_2 <: A_3$ .*

The interpretation of the two theorems is that: even though the premise is made stricter by the atomic condition, we can still derive every subtyping relation which is valid in the unrestricted system.

## 5.4 Coherence of the Elaboration

Combining the previous results, we are able to show the central theorem:

**Theorem 5** (Unique elaboration). *If  $\Gamma \vdash e : A_1 \hookrightarrow E_1$  and  $\Gamma \vdash e : A_2 \hookrightarrow E_2$ , then  $E_1 \equiv E_2$ . (“ $\equiv$ ” means syntactical equality, up to  $\alpha$ -equality.)*

*Proof.* Note that the typing rules are already syntax-directed but the case of (F-TY-APP) (copied below) still needs special attention since we need to show that the generated coercion  $E$  is unique.

$$\frac{\Gamma \vdash e_1 : A_1 \rightarrow A_2 \hookrightarrow E_1 \quad \Gamma \vdash e_2 : A_3 \hookrightarrow E_2 \quad A_3 <: A_1 \hookrightarrow E}{\Gamma \vdash e_1 e_2 : A_2 \hookrightarrow E_1 (E E_2)} \text{F-TY-APP}$$

Luckily we know that typing judgments give well-formed types, and thus  $\Gamma \vdash A_1$  and  $\Gamma \vdash A_3$ . Therefore we are able to apply Lemma 5 and conclude that  $E$  is unique.  $\square$

## 6. Algorithmic Disjointness

Section 5 presented a type system with disjoint intersection types that is both type-safe and coherent. Unfortunately the type system is not algorithmic because the specification of disjointness does not lend itself to an implementation directly. This is a problem, because we need an algorithm for checking whether two types are disjoint or not in order to implement the type-system.

This section presents the set of rules for determining whether two types are disjoint. The set of rules is algorithmic and an implementation is easily derived from them. Therefore we solve the problem of finding an algorithm to compute disjointness. The derived set of rules for disjointness is proved to be sound and complete with respect to the definition of disjointness in Section 5.

### 6.1 Algorithmic Rules

$$\begin{array}{c} \boxed{\vdash A *_i B} \\[10pt] \frac{\vdash A_2 *_i B_2}{\vdash A_1 \rightarrow A_2 *_i B_1 \rightarrow B_2} \text{DIS-FUN} \\[10pt] \frac{\vdash A_1 *_i B \quad \vdash A_2 *_i B}{\vdash A_1 \& A_2 *_i B} \text{DIS-INTER-1} \\[10pt] \frac{\vdash A *_i B_1 \quad \vdash A *_i B_2}{\vdash A *_i B_1 \& B_2} \text{DIS-INTER-2} \quad \frac{A *_ax B}{\vdash A *_i B} \text{DIS-AXIOM} \\[10pt] \boxed{A *_ax B} \\[10pt] \text{Int } *_ax A \rightarrow B \text{ DISAX-INT-FUN} \quad A *_ax \top \text{ DISAX-TOP} \\[10pt] \frac{B *_ax A}{A *_ax B} \text{DISAX-SYM} \end{array}$$

**Figure 7.** Algorithmic Disjointness.

The rules for the disjointness judgment are shown in Figure 8, which consists of two judgments.

**Main Judgment.** The judgment  $\vdash A *_i B$  says two types  $A$  and  $B$  are disjoint.

The rules dealing with intersection types ((DIS-INTER-1) and (DIS-INTER-2)) are quite intuitive. The intuition is that if two

$$\begin{array}{c} \boxed{\vdash A *_i B} \\[10pt] \frac{\vdash A_2 *_i B_2 \quad \neg \vdash A_2 \mid \neg \vdash B_2 \mid}{\vdash A_1 \rightarrow A_2 *_i B_1 \rightarrow B_2} \text{DIS-FUN} \\[10pt] \frac{\vdash A_2 *_i B_2 \quad \neg \vdash A_2 \mid \mid B_2 \mid \quad A_1 *_i B_1}{\vdash A_1 \rightarrow A_2 *_i B_1 \rightarrow B_2} \text{DIS-FUN} \\[10pt] \frac{\vdash A_2 *_i B_2 \quad \mid A_2 \mid \neg \mid B_2 \mid \quad A_1 *_i B_1}{\vdash A_1 \rightarrow A_2 *_i B_1 \rightarrow B_2} \text{DIS-FUN} \\[10pt] \frac{\vdash A_1 *_i B \quad \vdash A_2 *_i B}{\vdash A_1 \& A_2 *_i B} \text{DIS-INTER-1} \\[10pt] \frac{\vdash A *_i B_1 \quad \vdash A *_i B_2}{\vdash A *_i B_1 \& B_2} \text{DIS-INTER-2} \\[10pt] \frac{A *_ax B}{\vdash A *_i B} \text{DIS-AXIOM} \end{array}$$

$$\begin{array}{c} \boxed{A *_ax B} \\[10pt] \text{Int } *_ax A \rightarrow B \text{ DISAX-INT-FUN} \quad \text{Int } *_ax \top \text{ DISAX-TOP} \\[10pt] A \rightarrow B *_ax \top \text{ DISAX-TOP} \quad \frac{B *_ax A}{A *_ax B} \text{DISAX-SYM} \end{array}$$

**Figure 8.** Impure Algorithmic Disjointness.

types  $A$  and  $B$  are disjoint to some type  $C$ , then their intersection ( $A \& B$ ) is also clearly disjoint to  $C$ . The rules capture this intuition by inductively distributing the relation itself over the intersection constructor ( $\&$ ). Although those two rules overlap, the order of applying them in an implementation does not matter as applying either of them will eventually leads to the same conclusion, that is, if two types are disjoint or not.

The rule for functions (DIS-FUN) is more interesting. It says that two function types are disjoint if and only if their return types are disjoint (regardless of their parameter types!). At first this rule may look surprising because the parameter types play no role in the definition of disjointness. To see the reason for this consider the two function types:

$$\text{Int} \rightarrow \text{String} \quad \text{Bool} \rightarrow \text{String}$$

Even though their parameter types are disjoint, we are still able to think of a type which is a supertype for both of them. For example,  $\text{Int} \& \text{Bool} \rightarrow \text{String}$ . Therefore, two function types with the same return type are not disjoint. Essentially, due to the contravariance of function types, functions of the form  $A \rightarrow C$  and  $B \rightarrow C$  always have a common supertype (for example  $A \& B \rightarrow C$ ). The lesson from this example is that the parameter types of two function types do not have any influence in determining whether those two function types are disjoint or not: only the return types matter.

**Axioms.** Up till now, the rules of  $\vdash A *_i B$  have only taken care of two types with the same language constructs. But how can be the fact that  $\text{Int}$  and  $\text{Int} \rightarrow \text{Int}$  are disjoint be decided? That is exactly the place where the judgment  $A *_ax B$  comes in handy. It provides the axioms for disjointness. What is captured by the set of

rules is that  $A *_{\text{ax}} B$  holds for all two types of different constructs unless any of them is an intersection type.

## 6.2 Metatheory

The following two theorems together say that the algorithmic disjointness judgment and the definition of disjointness are “equivalent”. For detailed proofs, we refer to the Coq code in our repository.

**Theorem 6** (Soundness of algorithmic disjointness). *For any two types  $A$  and  $B$ ,  $\vdash A *_{\text{a}} B$  implies  $\vdash A * B$ .*

*Proof.* By induction on the derivation of  $\vdash A *_{\text{a}} B$ . □

**Theorem 7** (Completeness of algorithmic disjointness). *For any two types  $A$ ,  $B$ ,  $\vdash A * B$  implies  $\vdash A *_{\text{a}} B$ .*

*Proof.* By a case analysis on the shape of  $A$  and  $B$ . □

## 7. Related Work

**Coherence.** Reynolds invented Forsythe [19] in the 1980s. Our merge operator is analogous to his operator  $p_1, p_2$ . Forsythe has a coherent semantics. The result was proved formally by Reynolds [18] in a lambda calculus with intersection types and a merge operator. However the way coherence is ensured is not general enough. He has four different typing rules for the merge operator, each accounting for various possibilities of what the types of the first and second components are. In some cases the meaning of the second component takes precedence (that is, is biased) over the first component. The set of rules is restrictive and it forbids, for instance, the merge of two functions (even when they are provably disjoint). Therefore, Forsythe treatment of coherence is rather ad-hoc. In contrast, disjointness in  $\lambda_i$  has a simple, well-defined specification and it is quite flexible.

Pierce [17] made a comprehensive review of coherence, especially on Curien and Ghelli [9] and Reynolds’ methods of proving coherence; but he was not able to prove coherence for his  $F_{\wedge}$  calculus. He introduced a primitive `glue` function as a language extension which corresponds to our merge operator. However, in his system users can “glue” two arbitrary values, which can lead to incoherence.

Our work is largely inspired by Dunfield [13]. He described a similar approach to ours: compiling a system with intersection types and a merge operator into ordinary  $\lambda$ -calculus terms with pairs. One major difference is that our system does not include unions. As acknowledged by Dunfield, his calculus lacks of coherence. He discusses the issue of coherence throughout his paper, mentioning biased choice as an option (albeit a rather unsatisfying one). He also mentioned that the notion of disjoint intersection could be a good way to address the problem, but he did not pursue this option in his work.

Recently, Castagna *et al.* [5] studied a very interesting and coherent calculus that has polymorphism and set-theoretic type connectives (such as intersections, unions, and negations). Unfortunately their calculus does not include a merge operator like ours, which is our major source of difficulty for achieving coherence.

Going in the direction of higher kinds, Compagnoni and Pierce [6] added intersection types to System  $F_{\omega}$  and used the new calculus,  $F_{\omega}^{\wedge}$ , to model multiple inheritance. In their system, types include the construct of intersection of types of the same kind  $K$ . Davies and Pfenning [11] studied the interactions between intersection types and effects in call-by-value languages. And they proposed a “value restriction” for intersection types, similar to value restriction on parametric polymorphism. None of those calculi include a merge operator.

There have been attempts to provide a foundational calculus for Scala that incorporates intersection types [1, 2]. However, the type-soundness of a minimal Scala-like calculus with intersection types and parametric polymorphism is not yet proven. Recently, some form of intersection types has been adopted in object-oriented languages such as Scala, Ceylon, and Grace. Generally speaking, the most significant difference to  $\lambda_i$  is that in all previous systems there is no explicit introduction construct like our merge operator.

**Other Type Systems with Intersection Types.** Refinement intersection [10, 12, 15] is the more conservative approach of adopting intersection types. It increases only the expressiveness of types but not terms. But without a term-level construct like “merge”, it is not possible to encode various language features. As an alternative to syntactic subtyping described in this paper, Frisch *et al.* [16] studied semantic subtyping. Semantic subtyping seems to have important advantages over syntactic subtyping. One worthy avenue for future work is to study languages with intersection types and merge operator in a semantic subtyping setting.

$\lambda\&$

“the solution seems much more powerful than simply forbidding non-overlapping intersections”

No it is not!  $\lambda\&$  rejects intersections that are accepted in our system:

`Int->Char, Int->Bool`

This example violates the well-formedness conditions of  $\lambda\&$  (see Section 3.2 of “A Calculus for Overloaded Functions with Subtyping” (1995)). So clearly the conditions imposed by  $\lambda\&$  (even for the special case of functions) do not subsume disjoint intersections.

“The problem of coherence with intersection types was studied in the 90’s by Castagna in the language  $\lambda\&$ ”

No it wasn’t! Castagna studied the overloading problem for functions and focused only on the “merge” (in our sense) of functions. Our work considers a system with **arbitrary** intersection-merges and tries to present a coherent subset of that. The well-formedness conditions presented in Section 3.2 of the  $\lambda\&$  paper assume function types only.

Moreover, although  $\lambda\&$  can encode records, **it is unclear** how to encode arbitrary merges. One failed attempt to encode arbitrary merges in  $\lambda\&$  is to consider a record with multiple labels of the same name. For example, encoding `1, 'c'` as

`l=1, l='c'`

(where label `l` is of an isolated atomic type `L`, see Section 4 of the  $\lambda\&$  paper). However, this is also rejected in  $\lambda\&$ , since it violates the well-formedness conditions.

Nevertheless  $\lambda\&$  is related work and should be discussed.

**Traits and Trait Calculi.** The seminal paper by Schärli *et al.* introduced the ideas behind traits. In their original paper, they documented an implementation of the trait mechanism in a dynamically typed version of Smalltalk. Fisher and Reppy [14] presented a statically typed calculus that models traits.  $\lambda_i$  is not dedicated to traits; but rather, it supports a source language that models traits. Compared to Fisher and Reppy’s calculus,  $\lambda_i$  is more lightweight. For example, self references (as well as other OO-specific constructs) are not built-in  $\lambda_i$ . One reason for the difference is that Fisher and Reppy’s calculus supports *classes* in addition to traits, and considers the interaction between them, whereas our object oriented source language is *prototype* (or delegation) based—the mechanism for code reuse is purely traits. Of course, there have been many other formalizations of traits, such as [22]. But most of them are heavyweight and specific to modeling traits and typical class-based models of OOP, and therefore differ from our approach.

Bettini *et al.*’s prototype language, SWRTJ [3] distinguishes, in their terminology, “records” and “traits”—the former contain

fields and the latter contain methods. Since we try to model a pure object-oriented language, we have excluded fields, which provide state reuse. In SWRTJ, traits themselves are not meant to be the generator of instances. Instead, another construct, called “classes” are, and make use of traits.

The Scala language also has a notion of “traits”. However, unlike what its name suggests, the semantics of trait composition in Scala is more similar to mixins [4]. That is, like traditional mixin semantics, when two traits are composed, Scala attempts to do *implicit resolution of conflicts*. In comparison, the traits modeled in  $\lambda_i$  are intended to model the original trait idea closely, and conflicts must be resolved explicitly. Schärli *et al.* document well the trade-offs between mixins and traits. Aside from that, Scala’s traits and our source language’s traits have four major differences:

1. Scala’s traits cannot be instantiated but only mixed into a class (which can be anonymous), whereas traits in our language can be instantiated directly.
2. Scala’s traits cannot take constructor parameters whereas ours can. As in the point example below, our trait is itself a constructor and takes the x- and y-coördinates as parameters:

```
trait Point(x: Int, y: Int) { self: Point →
  x() = x
  y() = y
} in ...
```

3. Dynamic instantiation is supported in  $\lambda_i$ , but not in Scala. In Scala instantiating an object from a class or traits requires that all classes or traits are statically known.
4. Our model of traits is purely functional, but Scala’s traits also support fields, mutable state and abstract types.

## 8. Conclusion and Future Work

This paper described  $\lambda_i$ : a language that combines intersection types and a merge operator. The language is proved to be type-safe and coherent. To ensure coherence the type system accepts only disjoint intersections. We believe that disjoint intersection types are intuitive, and at the same time expressive. We have shown the applicability of disjoint intersection types to model a simple form of traits.

We implemented the core functionalities of the  $\lambda_i$  as part of a JVM-based compiler. Based on the type system of  $\lambda_i$ , we have built an ML-like source language compiler that offers interoperability with Java (such as object creation and method calls). The source language is loosely based on the more general System  $F_\omega$  and supports a number of other features, including records, polymorphism, mutually recursive `let` bindings, type aliases, algebraic data types, pattern matching, and first-class modules that are encoded using `letrec` and records.

For the future, we intend to improve our source language and show the power of disjoint intersection types in large case studies. One pressing challenge is to address the intersection between disjoint intersection types and polymorphism. We are also interested in extending our work to systems with a  $\top$  type. This will also require an adjustment to the notion of disjoint types. A suitable notion of disjointness between two types A and B in the presence of  $\top$  would be to require that the only common supertype of A and B is  $\top$ . Finally we would like to study the addition of union types. This will also require changes in our notion of disjointness, since with union types there always exists a type  $A|B$ , which is the common supertype of two types A and B.

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