An Introduction to Ergodic Dynamics and Benford's Law

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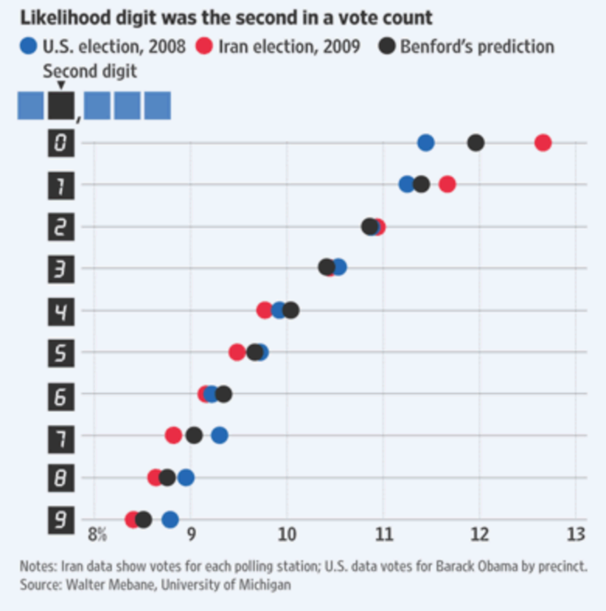
Benford's Law



Dynamical Systems

Consider the transformation $T : [0, 1) \to [0, 1)$ defined by

$$T(x) = 2x \pmod 1$$

Value of x	.1	.11	Difference
$f(x)$	.2	.22	+.02
$f^2(x)$	.4	.44	+.04
$f^3(x)$	.8	.88	+.08
$f^4(x)$	.6	.76	+.16
$f^5(x)$	.2	.52	+.32
$f^6(x)$	.4	.04	−.36

Ergodicity

Definition

Let $(X, \mathbb{B}, \mu, f)$ be a probability-measure-preserving dynamical system with μ a σ-finite measure.
We say a transformation f is **ergodic** if $f^{-1}(A) = A(\mu \mod 0)$ implies either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

Circle Rotations

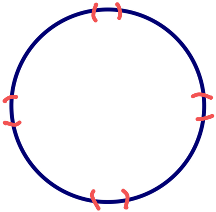
Let $X = \mathbb{R}/\mathbb{Z}$ with Lebesgue measure, and for $\theta \in [0, 1)$, consider

$$T(x) = x + \theta \mod 1.$$

We have two classifications of circle rotations: rational and irrational. Consider the rational circle rotation by $\theta = \frac{1}{4}$.

Is this transformation ergodic?
Another way of asking this is, does there exist an A with $0 < \mu(A) < 1$ which is invariant under this rotation?

Consider the set $A = \bigcup_{i=0}^{3} B_\epsilon(\frac{i}{4})$ for $\epsilon > 0$.



Circle Rotations

Lemma

A probability-measure-preserving dynamical system $(X, \mathbb{B}, \mu, f)$, *with* $\mu(X) = 1$ *is ergodic if and only if for every* $\phi \in L^2, \phi \circ f(x) = \phi(x)$ *for* μ *almost everywhere implies* ϕ *is constant for* μ *almost everywhere.*

Lemma

Irrational rotation on the circle given by the system $(\mathbb{T}, \mathbb{B}, m, R_\alpha)$ *is ergodic with respect to Lebesgue measure* m.

Irrational Circle Rotations

Proof.

Let $\phi \in L^2(X, \mathbb{B}, m)$. We can write ϕ in its Fourier expansion

$$\phi(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$$

Suppose $\phi \circ R_\alpha = \phi$ almost everywhere. Then

$$\phi(R_\alpha(x)) = \phi(x + \alpha) = \phi(x)$$

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n(x+\alpha)} = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x} e^{2\pi i n \alpha} = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$$

By uniqueness of Fourier coefficients, $a_n = a_n e^{2\pi i n \alpha}$ for all $n \in \mathbb{Z}$. Then it must be that $a_n = 0$ for all integers n except for 0.
So $\phi(x) = a_0$, and thus ϕ is constant almost everywhere and R_α is ergodic. □

Unique Ergodicity Theorem

Theorem (Space average = time average)

Let $(X, \mathbb{B}, f)$ be a continuous dynamical system on a compact metric space X. Then the following are equivalent.

1. *f is uniquely ergodic.*
2. *There exists an invariant measure μ such that for every continuous function $\phi \in C(X)$ and for every $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) = \int_X \phi \mathrm{d}\mu.$$

Benford Data Sets

We call a data set $\{y_n\} \subset \mathbb{R}$ *Benford* if the frequency of $k \in \{1, 2, \ldots, 9\}$ occuring as the leading digit in base 10 expansion is $\log \frac{k+1}{k}$.

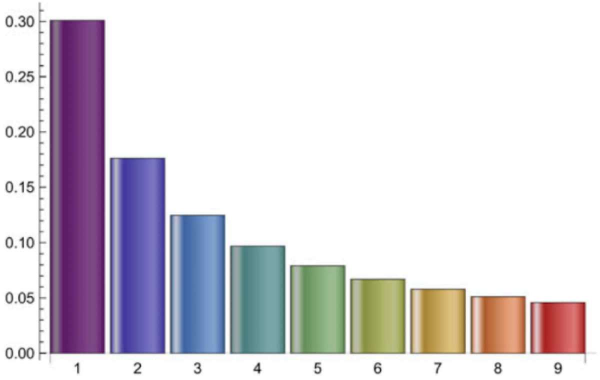


Figure: Distribution of leading digits according to Benford's Law

Benford Data Sets

More generally, we call a data set in base $b \geq 2$ Benford if the frequency of the leading digit $k \in \{1, \ldots, b\}$ is $\log_b \frac{k+1}{k}$

Note that the rule does define a probability distribution as the sum of the probabilities is 1.

$$\sum_{k=1}^{b-1} \log_b \frac{k+1}{k}$$

$$\begin{aligned} &= \log_b b - \log_b (b-1) + \log_b (b-1) - \log_b (k-2) + \\ &\quad + \cdots - \log_b 2 + \log_b 2 - \log_b 1 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Proving a Data Set is Benford

Theorem

The set $\{2^n\}_{n\in\mathbb{N}}$ *is Benford.*

First we note that an integer $a_n = 2^n$ has leading digit r if and only if, for some $m \in \mathbb{N}$,

$$r10^m \leq 2^n < (r+1)10^m$$

Applying $\log_{10}$, this holds if and only if

$$\log r + m \leq n\log 2 < \log(r+1) + m$$

where m is an integer, so taking the inequality mod 1 we have

$$n\log 2 \in [\log r, \log(r+1))$$

Proving a Data Set is Benford

Recall that the irrational rotations on the circle are uniquely ergodic with respect to Lebesque measure. $\log 2$ is irrational, so we consider the transformation on $X = \mathbb{R}/\mathbb{Z}$ given by

$$T(x) = x + \log 2 \mod 1.$$

Consider the interval $I_r = [\log r, \log(r+1))$. Then using Lebesgue measure,

$$\int_X \mathbb{1}_{I_r} dm = m(I_r) = \log\left(\frac{r+1}{r}\right)$$

Then by the unique ergodicity theorem, for every $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_r}(T^j x) = \int_X \mathbb{1}_{I_r} dm = \log\left(\frac{r+1}{r}\right).$$

Taking $x = 0$, this proves the theorem.

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