Kyle Cox

Professor McPartland

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Zeno's Paradox of the Dichotomy

There are essentially two schools of thought when considering the nature of space and time: Either (1) space and time are discrete or (2) space and time are continuous. Though both cannot be true, each notion of reality is perfectly reasonable, and various practices are devoted to either school of thought. A continuous spacetime, for example, is invoked in calculus and its applications, including classical mechanics, while a discrete universe is studied in quantum physics, digital machines, and computational mathematics. While there certainly exist other conceptualizations of the fundamentals of the universe—for example, the Pythagorean notion of ratio as the principle component of the cosmos—the two schools of thought in discussion are perhaps of chief importance because they are *intuitive*. We observe that the external world appears continuous in some situations and discrete in others, and from that experience our minds learn to model the world around us as continua or digital systems. From these models we design entire bodies of study, yet often neglect to consider their foundations. It is a rare occasion that in a calculus classroom the students inquire whether there exists such a thing as instantaneous velocity; it simply makes sense that it is so. Yet Zeno demonstrates that our ordinary models of space and time lead us to logical absurdities; somehow, from premises that seem self-evident, we arrive at a universe that cannot exist.

One of the consequences of continuous space is that this space is infinitely divisible. Assuming a continuity of space, Zeno presents to us his Paradox of the Dichotomy. We can reconstruct his argument from Aristotle's *Physics*. Aristotle summarizes Zeno's argument in the following way: "that which is moving must reach the midpoint before the end … It is always necessary to traverse half the distance, but these are infinite, and it is impossible to get through things that are infinite" ("Zeno of Elea" 49). It is assumed here, of course, that space can be infinitely divided. From Aristotle's recapitulation we can state Zeno's paradox formally as follows:

1. If a space can be divided into halves, it can be divided into infinitely many parts.
2. If a space can be divided into infinitely many parts, then it is infinite.
3. If a space is infinite, then it cannot be traversed.
4. All space can be divided into halves.
5. Therefore, space cannot be traversed and motion is impossible.

In the language of formal logic:

Let P_s be "a space can be divided into halves."

Let Q_s be "a space can be divided into infinitely many parts."

Let R_s be "a space is infinite."

Let S_s be "a space cannot be traversed."

Then for the arrow *a*:

$P_s \supset Q_s$

$Q_s \supset R_s$

$R_s \supset S_s$

P_s

$\therefore S_s$

Aristotle provides a very reasonable and mathematically sound response to this paradox, which rebuts Premise 2. Specifically, Aristotle seems to attack the ambiguity of the word “infinite,” as Zeno uses it in two different ways. Aristotle argues that all continuous entities are continuous in two ways: “infinite in division and infinite with respect to their extremities” (“Zeno of Elea” 50). He claims that while we cannot traverse infinities of the latter type—infinities by extension—we can traverse infinities by division. To this end, Aristotle offers time as an example: Time is infinite in divisibility, yet we observe it pass through this infinity, so it must be possible to pass through all infinities of this type, including space (“Zeno of Elea” 50).

Yet although Aristotle makes a necessary distinction between different types of infinities, in this way he only raises more questions and certainly does not solve the paradox. The time example, too, is no explanation, for we could apply the same paradox to the continuity of time. Of course, objects move through time, as do objects through space, but this appearance is only a further testament to the absurdity introduced by Zeno’s paradox.

Aristotle makes a very intriguing claim in his counterargument yet fails to investigate it: that we can traverse things that are infinite in division. Salmon codifies this argument in the language of modern mathematics describing a distance of $d = 1$ with the following series:

$$d = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \qquad (25)$$

which certainly proves what Aristotle was attempting to say: that infinitely divisible entities can still be finite in extension. And some took the convergence of this series to be the end to the paradox, but Salmon is keener, and argues that this series instead brings into question “the adequacy of the mathematical results for the description of a physical process” (26). In fact, this series simply makes Zeno’s paradox more specific. It shows that for every finite space, if we take space to be continuous, then traversing that finite space requires *an infinite number of tasks*.

Thus we reframe Zeno's paradox as a supertask: a set of infinitely many tasks that we take to be capable of completing in a finite time. Paul Benacerraf describes a famous supertask similar to Zeno's—Thomson's Lamp. He supposes a lamp that can be turned on and off by a button at its base. This button is pressed once at the end of one minute, again after the next half minute, again after the next quarter minute, and so on, until the lamp has been pressed an infinite number of times by the end of two minutes (107). Thomson, however, shows this supertask to be self-contradictory.

Thomson's Lamp has two possible states: on or off. Suppose that Person A performs the lamp series, and at the end of the series, the lamp is off. Well, this is not possible: For every time press of the button that turns the lamp off, there is a following press of the button that turns the lamp on again. Suppose that Person B performs the lamp series, and at the end of the series, the lamp is on. The same contradiction can be raised for Person B: For every press of the button that turns the lamp on, there is a press of the button that turns it off again (Benacerraf 107). Thus both possible states of the lamp are impossible at the end of the series. Such is the nature of an infinite series of tasks. For every element in the infinite task series, there is necessarily another element following it. Consequently, we cannot discuss the final state of the lamp, or any task, at the end of an infinite series.

However, Benacerraf shows this self-contradiction is the result of the misinterpretation of the infinite series. Let t_0 be the time at the beginning of the series, and let t_1 be the time at the end of the two minutes. Thomson's argument assumes that the state of the lamp at time t_1 follows the same laws that the state of the lamp did along the entire *t*-series, which we define by every on switch being followed by an off switch, and every off switch being followed by an on switch

(Benacerraf 108). The point t_1, in fact, lies outside of the *t*-series, and thus the supertask series gives us no information about the state of the task at the end of the series.

So now we ask how this pertains to Zeno's Paradox of the Dichotomy. By demonstrating that the end state of a supertask does not exist within the series of the supertask itself, Benacerraf demonstrates that the supertask is an insufficient description of Zeno's paradox. Describing Achilles' race from Point A to Point B as a supertask leaves us fundamentally incapable of commenting on Achilles' status at the end of the race. At the end of this supertask race, Thomson asks, Where is Achilles? He answers, "Nowhere" (Benacerraf 116).

Of course, one could say that the difference between Achilles' position at the end of the series and Point B, the end of the race, is merely the difference of a point. Modern mathematics shows that what Zeno's paradox really describes is an infinite series, and that the consequences of an infinite converging series are that the quantities at the end of the series become infinitesimal. We have no trouble calling these infinitesimal quantities "points" when in they exist in the context of space, but perhaps we fail to truly grasp what this means. Nevertheless, Zeno presents space as a sum, part of which includes points. Thus a new question arises: Is the sum of points a finite space?

The math is certainly consistent. But relying upon the math as a justification for the physical accuracy of this description of space may be fallacious. Salmon asks the reader, "Does a given physical situation correspond with a particular mathematical operation, in this case, the operation of summing an infinite series?" (29). The conclusion from Thomson's discussion seems to be that this is not the case, but concluding that mathematics are insufficient to describe physical processes—and very elementary ones at that—ought to leave the reader uneasy. In fact, what is problematic is not the use of math to correspond to a physical reality, but rather the type of mathematical operation invoked. Summing an infinite series of points is not an effective way to describe space, but perhaps there exist other, more effective ways to describe this physical attribute.

Having deconstructed Zeno's paradox to a question of whether space can be understood as a sum or aggregate of points, we can now analyze it in terms of a paradox of extension. Adolf Grunbaum delivers a critique of this conceptualization of measure. Grunbaum notes that if we considered space to be an aggregate of infinite points of length zero, then our result would be absurd: All space would be of zero measure (177). Further, he notes that such a description leaves both a line and a point indistinguishable (179). For this reason, Grunbaum denies the principle of countable additivity for an infinite number of infinitesimal points (195). Grunbaum redefines measure in this way: "Length, measure, or extension is defined as a property of point-sets rather than of individual points, and zero-length is assigned to the *unit set*" (179), and further, "A finite interval on a straight line is the (ordered) set of all real points between (and sometimes including one or both of) two fixed points called the 'end-points' of the interval" (182). In these definitions, Grunbaum accomplishes two things. He first redefines points as unit sets: Instead of points, we discuss sets containing only that point. Each of these sets, further, has the property of having zero measure. He then redefines measure as a set of all real points. Having defined points as unit sets, we now define measure as a set of those unit sets.

Ascribing this set-element relation to space and points resolves the problem of measure in Zeno's notion of space. Zeno's fallacy lay in attributing a part-whole relationship to space and its points. A part-whole relationship is defined by the sum of the parts being equal to the whole, a relationship that works when space is described in terms of its finite parts. This relationship, however, deteriorates when infinitesimals, or points, are introduced. In fact, part of the deceptiveness of the Zeno paradox is in its combination of finite parts and infinitesimals. For any distance Zeno describes by his paradox, it is possible to traverse half its length, or three quarters its length, or any part of its length up until the point that the finite distance is described as the sum of an infinitesimal. Upon claiming that one can traverse half the given distance, of course, the same paradox can be applied to that half-distance, and then again to the half-distance of that half-distance. The result of this recursive process is that ultimately Zeno's paradox is reduced not to an infinite series, but an infinite sum of infinitesimal points—where space is shown to be the whole of its point-parts, which Grunbaum has shown to be false.

The set-element relation succeeds in eliminating this paradox because it does not have the additivity property that a part-whole relation does. Thus we have shown that Zeno's Premise 2, "If a space can be divided into infinitely many parts, then it is infinite [in measure]" to be false, or, more accurately, space can be divided into infinitely many point-sets which have no relation to its measure, as "the length of an interval is not a function of its cardinality" (Grunbaum 193). However, despite being formally consistent, this notion of space superficially feels unintuitive, and perhaps unreasonable.

Points, however, serve a very specific purpose in mathematics and in practice. They are dimensionless units of location, and rather than having zero measure, measure is simply not a property they possess. A description of space as a set of unit-sets feels unintuitive for the same reason that Zeno's description of space arrives at a paradox: We attempt to attribute properties of finite space to entities that do not possess these properties. Points are not physically reproducible by any finite process and are not useful in aggregates, so they do not *need* to possess properties of additivity in the same way the rest of space does. Being dimensionless, they are more accurately mathematical tools than physical descriptors of the nature of space. Thus it is perfectly reasonable to isolate each point in its own unit set, united not by addition but by its union with other point-sets, because nowhere in the natural world or the mathematical world do they serve a purpose that would necessitate their being discussed as aggregates.

In this way, we prove Zeno's paradox not to be a paradox of motion, but of space, and particularly its infinitesimal parts. Though ultimately failing to disprove motion, Zeno's paradox does serve the ultimate purpose of proving that nature, and specifically, space, are not as they appear. Space may be infinitely divisible, but the result of that infinite divisibility results in objects that do not act the same as ordinary space. Perhaps it is the widespread use of the word "point" in the vernacular or our failure to accurately grasp the infinite that leads to these misconceptions, but the world of the infinite consists of very different properties than the physical world, and from our experience in a finite world it is impossible to extrapolate to both the infinitely large and the infinitesimal.

Works Cited

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