

# ST120

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## 1 Lectures 1 + 2

### 1.1 Probability Spaces

Consider a simple probability problem. I have a bag with 20 balls of three different colours 4 red balls 6 blue balls and 10 green balls. A simple question to be asked is what is  $\mathbb{P}(\text{red})$ . From intuition you know this number is  $\frac{4}{20}$  but it is important to formalise what's going on.

#### 1.1.1 Sample Space

We can say that red green and blue are three **outcomes** that can occur. It can be useful to collect these three events together into a set as such  $\{\text{red}, \text{blue}, \text{green}\}$ .

**Definition 1.** Let  $\Omega$  denote the Sample Space, The set of all outcomes.

#### 1.1.2 Event Space

Sometimes we want to discuss what we call and even instead of a specific outcome. Sometimes we may want to ask what is  $\mathbb{P}(\neg \text{red})$  we can associate  $\neg \text{red}$  with the set  $\{\text{blue}, \text{green}\}$ . We call such a thing event and events are sets of outcomes and as such every event is a subset of the Sample Space

**Definition 2.** Let  $\mathcal{F}$  denote the Event Space. The Set of all events.  $\mathcal{F} \subseteq \mathcal{P}(\Omega)$

#### 1.1.3 Probability Map

The last thing we want to do is be able to go from an event to the actual probability.

**Definition 3.** Let  $\mathbb{P}$  denote the probability map. It is a function  $\mathcal{F} \rightarrow [0, 1]$  associating an event with a probability

Such a map should fulfil some list of properties as follows

- $\mathbb{P}(\emptyset) = 0$
- $\mathbb{P}(\Omega) = 1$

- For any events  $A, B \in \mathcal{F}$  where  $A \cap B = \emptyset$  then  $\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B)$

**Definition 4.** A Probability Space is the triple  $(\Omega, \mathcal{F}, \mathbb{P})$

## 1.2 Uniform Probability Spaces

**Definition 5.** A uniform probability space is a triple  $(\Omega, \mathcal{F}, \mathbb{P})$  satisfying the normal properties as well as a uniformity where for any  $\omega, \omega' \in \Omega$  we have  $\mathbb{P}(\omega) = \mathbb{P}(\omega')$

As such we conject that  $\mathbb{P}(A) \propto |A|$

**Proposition 1.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  denote a UPS then for all  $\omega \in \Omega$   $\mathbb{P}(\{\omega\}) = \frac{1}{|\Omega|}$

*Proof.* Since for all  $\omega, \omega' \in \Omega$  we have  $\mathbb{P}(\omega) = \mathbb{P}(\omega')$  let  $p = \mathbb{P}(\omega)$

$$1 = \mathbb{P}(\Omega) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) = \sum_{\omega \in \Omega} p = p|\Omega|$$

$$p = \frac{1}{|\Omega|}$$

□

**Proposition 2.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  denote a UPS then for all  $A \in \mathcal{F}$   $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$

*Proof.*

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) = p|A| = \frac{|A|}{|\Omega|}$$

□

## 1.3 Basics of Combinatorics

Here we shall present the Fundamental rules of counting

- Correspondence Rule : If  $A$  and  $B$  are in 1 - 1 correspondence then  $|A| = |B|$
- Addition Rule : if  $A_1, A_2, \dots, A_n$  are pairwise disjoint then

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i|$$

- Fundamental Counting Principle : Suppose a finite set  $E$  can have its elements determined in  $k$  successive steps with  $n_i$  possibilities for each step  $1, \dots, k$  and different choices lead to different elements then  $|E| = \prod_{i=1}^k n_i$

## 2 Lecture 3 - Birthdays Problems

We have started to build up a theory for how to calculate probabilities. Now we will try to apply these and also see whatever definitions are needed to fill in the gaps

### 2.1 Tuples and Orderings

#### 2.1.1 Tuples

The first problem we will attempt to solve is "What is the probability, given a room of 30 people, that at least two of them share a birthday/ The first thing we will want to identify is what is the sample space. The sample space intuitively should be the space of all configurations of birthdays the group can have. Naturally this forms a uniform probability space as each outcome is equally likely. We also have to consider how to mathematically represent these outcomes so that we can find the size of the entire sample space. The way we shall represent these are as 30-tuples  $(b_1, b_2, \dots, b_{30})$  where  $b_i$  is the  $i$ th persons birthday.

Now lets discuss what a tuple is

**Definition 6.** Given a set  $A$  such that  $|A| = n \in \mathbb{N}$ . A sequence of length  $k \in \mathbb{N}$  of elements of  $A$  is an ordered  $k$ -tuple  $(a_1, \dots, a_k)$  such that  $a_i \in A$  where  $i \in \{1, \dots, k\}$  We denote these sequences  $S_{n,k}(A)$

Now we need to be able to compute the cardinality of sets of sequences

**Proposition 3.** For a set  $A$  where  $|A| = n \in \mathbb{N}$

$$|S_{n,k}(A)| = n^k$$

*Proof.* To construct an arbitrary element of  $S_{n,k}(A)$  we first choose  $a_1$  for which we have  $n$  options then we choose  $a_2$  for which we have  $n$  options and so on up till  $n_k$  where we have  $n$  options. As each option provides a different element in the final set we can use the fundamental principal of counting to compute the size.

$$|S_{n,k}(A)| = \underbrace{n \times n \times \dots \times n}_{k \text{ times}} = n^k$$

□

Using this result we can find the size of the sample space in the birthdays problem as  $|\Omega| = 365^{30}$

#### 2.1.2 Orderings

Next we wish to compute the cardinality of the event "At least two people share a birthday". Much easier than computing this is to compute the complementary event "No two people share a birthday". this means that each tuple in this event has no repeated elements. this is what we call and ordering.

**Definition 7.** Given a set  $A$  such that  $|A| = n \in \mathbb{N}$  and a  $k \leq n$  we can say an ordering of length  $k$  of elements of  $A$  is a sequence of length  $k$  of  $a$  with no repetition. We denote these  $O_{n,k}(A)$

$$O_{n,k}(A) = (a_1, \dots, a_k) : a_i \in A, \forall_{i,j} a_i \neq a_j$$

Now we would also like to be able to compute the sizes of sets of orderings.

**Proposition 4.** Given a set  $A$  such that  $|A| = n \in \mathbb{N}$  and a  $k$  such that  $k \leq n$

$$|O_{n,k}(A)| = \frac{n!}{(n-k)!}$$

*Proof.* We can determine elements of  $O_{n,k}$  elements by element. To determine  $a_1$  we have  $n$  choices. For the  $a_2$  we have  $n-1$  choices as we can't repeat the first element. For  $a_k$  we have  $n-(k-1)$  choices. As each series of choices leads to a different element we can apply the fundamental principle of counting.

$$|O_{n,k}(A)| = n \times (n-1) \times \dots \times (n-k+1) = \frac{n!}{(n-k)!}$$

□

So if we let  $B$  denote the event no one shares a birthday then  $B = O_{365,30}$  (365 days) so  $|B| = 365 \times \dots \times (365-30+1)$  so

$$\mathbb{P}(B) = \frac{|B|}{|\Omega|} \approx 0.29$$

So we then have for our original event that at least 2 people share a probability of about 0.71.

## 2.2 Another Example

Now let's consider a similar problem of there being **exactly** 2 people sharing a birthday. We can characterize each outcome in our desired event by three characteristics

1. The two people who share a birthday
2. The day that they share
3. The days everyone else has

Now to compute the size of the event by the fundamental principle of counting we need to compute the number of choices for each piece of information.

1. An intuitive guess for the first one is  $|O_{30,2}|$  but this cares about the order that the two people share a birthday are picked in which doesn't really matter so we can divide this by the ways to order the two yielding us

$$\frac{|O_{30,2}|}{2!} = \frac{30!}{28!2!}$$

2. There are 365 possibilities for this day
3. For the remaining 28 people by the Fundamental Counting Principle we have  $364 \times \cdots \times (365 - 28)$

Now via computation and the same sample space as before we get the probability is roughly 0.28

## 2.3 Combinations

**Definition 8.** Let  $A$  be a set such that  $|A| = n \in \mathbb{N}$ . We can say that a combination of  $k$  elements of  $A$  is a subset of  $A$  with  $k$  elements. Let the set of combinations be denoted  $C_{n,k}(A)$

**Proposition 5.** Let  $A$  be a set such that  $|A| = n \in \mathbb{N}$  and  $k \leq n$

$$|C_{n,k}(A)| = \binom{n}{k}$$

*Proof.* On ordering of length  $k$  can be obtained uniquely in the following steps.

1. Choose from  $C_{n,k}(A)$ . This gives  $|C_{n,k}(A)|$  choices
2. Choose a permutation of these.  $k!$  choices.

By the FPC we get

$$|O_{n,k}(A)| = k!|C_{n,k}(A)|$$

By rearranging we can get

$$|C_{n,k}(A)| = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

□

As an exemplar problem we can consider all the ways that, having rolled 8 fair dice, our outcome has three twos, three fours and two fives. We can note that each of our valid outcomes is uniquely characterised by what dice gives us the twos and fours. Initially we have  $\binom{8}{3}$  positions for the twos then  $\binom{5}{3}$  for the fours and the fives must go where is left. By the FPC this event has size being the product of these two numbers

## 3 Lecture 4 - Partitions

**Definition 9.** Let  $A$  be a set such that  $|A| = n \in \mathbb{N}$  and let  $r \leq n$ .

An ordered partition of  $A$  into  $r$  ordered subsets of cardinality  $k_1, \dots, k_r$  is a sequence of  $(A_1, \dots, A_r)$  such that  $|A_i| = k_i$ , all the elements are pairwise disjoint and  $\bigcup_{i=1}^r A_i = A$

**Proposition 6.** Let  $A$  be a set such that  $|A| = n \in \mathbb{N}$  and  $r \leq n$ . The number of partitions of  $A$  into  $r$  subsets of cardinality  $k_1, \dots, k_r$  is

$$\frac{n!}{k_1! \times \dots \times k_r!}$$

*Proof.* We can determine a partition as follows  
Choose elements as follows

$$\begin{aligned} A_1 \subseteq A \text{ such that } |A_1| = k_1 & \text{ gives } \binom{n}{k_1} \\ A_2 \subseteq A \setminus A_1 \text{ such that } |A_2| = k_2 & \text{ gives } \binom{n-k_1}{k_2} \\ & \dots \\ A_r \subseteq A \setminus \bigcup_{i=1}^{r-1} A_i \text{ such that } |A_r| = k_r & \text{ gives } \binom{n - \sum_{i=1}^{r-1} k_i}{k_r} \end{aligned}$$

By application of the fundamental principle of counting and some careful counting we get the amount of partitions is

$$\frac{n!}{k_1! \times \dots \times k_r!}$$

□

### 3.1 Samplings

Sampling is how we produce realisations of random variables.

#### 3.1.1 Example 1

Imagine a population of size  $n \in \mathbb{N}$  and there are  $n_1$  type 1 individuals and  $n_2$  type two individuals.

Suppose we draw a sample of size  $k \leq n$  from the population without replacement. The probability of a sample containing  $k_1$  type one individuals and  $k_2 = k - k_1$  of type 2 is

## 4 Lecture 5 - General Probability Spaces

We have spent a while specifically discussing uniform probability spaces. There are some more generalisations we may want to make to how our space works

- Let  $\Omega$  be an arbitrary set

- Allow  $\mathcal{F}$  to reflect partial knowledge
- Following from the previous point we want to appreciate that  $\mathcal{F} = \mathcal{P}(\Omega)$  is not always feasible.
- Finally we want to define properties of  $\mathbb{P}$  on general  $\mathcal{F}$

#### 4.1 Sample and Event Spaces

**Definition 10.**  $\Omega$  is the set of all possible outcomes for a probabilistic process.

In this course three cases are covered

- $|\Omega| = n \in \mathbb{N}$
- $|\Omega| = |\mathbb{N}|$
- $|\Omega| = |\mathbb{R}|$

**Definition 11.**  $\mathcal{F}$  is the event space and contains collections of outcomes for which you can make a judgement on whether the event happened or not

**Example 1.** Imagine rolling a dice  $\Omega = \{1, 2, 3, 4, 5, 6\}$ . The only information you are told is whether or not the outcome is odd or even. As this is the only information you are given you cannot differentiate the events  $\{2\}$  and  $\{4\}$ . As such they cannot be in the event space. Here  $\mathcal{F} = \{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}\}$ .

Now let's present a more formal definition of the event space

**Definition 12.** Let  $\Omega$  be a sample space and  $\mathcal{F} \subseteq \mathcal{P}(\Omega)$

We can say  $\mathcal{F}$  is an event space iff

1.  $\Omega \in \mathcal{F}, \emptyset \in \mathcal{F}$
2. If  $A \subseteq \Omega$  and  $A \in \mathcal{F}$  then  $A^C \in \mathcal{F}$  (closure under complements)
3. Given a set  $A = \{A_n \in \mathcal{F} | n \in \mathbb{N}\}$  then  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$  (closure under countable union)

**Proposition 7.**  $\Omega \neq \emptyset$  and  $\mathcal{F} = \mathcal{P}(\Omega)$  then  $\mathcal{F}$  is an event space

*Proof.* 1.  $\Omega \in \Omega \Rightarrow \Omega \in \mathcal{P}(\Omega) = \mathcal{F}$

2.  $A \subseteq \Omega, A \in \mathcal{F}$  then  $A^C = \Omega \setminus A \subseteq \Omega \Rightarrow A^C \in \mathcal{P}(\Omega) = \mathcal{F}$

3. Consider  $A = \{A_n \in \mathcal{F} | n \in \mathbb{N}\}$

$$\bigcup_{A_n \in A} A_n \subseteq \bigcup_{A_n \in A} \Omega = \Omega \in \mathcal{P}(\Omega) = \mathcal{F}$$

□

We can now also look at some derived properties of the event space inhering from these defined properties

**Proposition 8.** *Let  $\mathcal{F}$  be an event space then*

1.  $\mathcal{F}$  is closed under finite union
2.  $\mathcal{F}$  is closed under finite intersection
3.  $\mathcal{F}$  is closed under countable intersection

*Proof.* 1. Let  $A_1, \dots, A_n \in \mathcal{F}$ . Set  $A_j = \emptyset, j > n$ . therefore For all  $i \in \mathbb{N}^+, A_i \in \mathcal{F}$

$$\bigcup_{j=1}^n A_j = \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$$

2. Let  $A_1, \dots, A_n \in \mathcal{F}$ . By demorgans law we may state

$$\bigcap_{j=1}^n A_j = \left( \bigcup_{j=1}^n A_j^C \right)^C$$

By the closure of  $\mathcal{F}$  under complements and under finite union we can say this finite intersection is in  $\mathcal{F}$

3. For the infinite case we apply the same argument we just used only swapping finite for infinite

□

## 5 Lecture 6 - Probability measure

### 5.1 Probability measure

**Definition 13.** *Given samples and event spaces  $\Omega, \mathcal{F}$  a function  $\mathbb{P} : \mathcal{F} \rightarrow \mathbb{R}$  is a probability measure if*

1. For all  $x \in \mathcal{F}, \mathbb{P}(x) \in [0, 1]$
2.  $\mathbb{P}(\Omega) = 1$
3. For all  $m, n$  such that  $m \neq n, A_n \cap A_m = \emptyset$  then

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j)$$

**Proposition 9.** *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. We can make then the following propositions about  $\mathbb{P}$*



1.  $\mathbb{P}(\emptyset) = 0$
2. if  $A, B \in \mathcal{F}$  and  $A \subseteq B$  then  $\mathbb{P}(B - A) = \mathbb{P}(B) - \mathbb{P}(A)$
3. For all  $A \in \mathcal{F}$ ,  $\mathbb{P}(A^C) = 1 - \mathbb{P}(A)$

*Proof.* 1. For all  $n \geq 1$  let  $A_n = \emptyset$ . Therefore we can state  $\bigcup_{n=1}^{\infty} A_n = \emptyset$

$$\mathbb{P}(\emptyset) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}A_n = \sum_{n=1}^{\infty} \mathbb{P}(\emptyset)$$

This only holds if  $\mathbb{P}(\emptyset) = 0$

2. Using the same way we proved closure of finite unions in the event space we may prove a finite additivity for disjoint sets. This fact will be assumed here. Let  $B = (B - A) \cup A$ . By the definition of set difference we know that  $(B - A) \cap A = \emptyset$ . This allows us to apply finite additivity

$$\mathbb{P}(B) = \mathbb{P}(B - A) + \mathbb{P}(A)$$

3.  $\mathbb{P}(A^C) = \mathbb{P}(\Omega - A) = \mathbb{P}(\Omega) - \mathbb{P}(A) = 1 - \mathbb{P}(A)$

□

## 5.2 Principle of inclusion Exclusion

**Proposition 10.** *The Principle of Inclusion Exclusion is as follows. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Then for  $A_1, \dots, A_n \in \mathcal{F}$*

$$\mathbb{P}\left(\bigcup_{k=1}^n A_k\right) = \sum_{k=1}^n \left( (-1)^{k+1} \sum_{2 \leq i_1 < \dots < i_k \leq n} \mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k}) \right)$$

*Proof.* A proof sketch for  $n = 2$  is given here

$$A \cup B = (A - A \cap B) \cup (A \cap B) \cup (B - A \cap B)$$

By application of finite additivity PIE for  $n = 2$  may be gotten. □

**Proposition 11.** *If  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space then if  $A, B \in \mathcal{F}$  and  $A \subseteq B$  Then  $\mathbb{P}(A) \leq \mathbb{P}(B)$*

**Proposition 12.** *Booles inequality : If  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space then for a list  $A_1, \dots, A_n \in \mathcal{F}$  then*

$$\mathbb{P}\left(\bigcup_{k=1}^n A_k\right) \leq \sum_{k=1}^n \mathbb{P}(A_k)$$

*Proof.* By induction on  $n$

- Base case  $n = 2$ .  $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) \leq \mathbb{P}(A_1) + \mathbb{P}(A_2)$

- Assume its true for some  $n = k$ .

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{j=1}^{k+1} A_j\right) &= \mathbb{P}\left(\bigcup_{j=1}^k A_j \cup A_{k+1}\right) \\
&= \mathbb{P}\left(\bigcup_{j=1}^k A_j\right) + \mathbb{P}(A_{k+1}) - \mathbb{P}\left(\bigcup_{j=1}^k A_j \cap A_{k+1}\right) \\
&\leq \mathbb{P}\left(\bigcup_{j=1}^k A_j\right) + \mathbb{P}(A_{k+1}) \\
&\leq \sum_{j=1}^{k+1} \mathbb{P}(A_j)
\end{aligned}$$

□

## 6 Lecture 7 - Conditional Probability

Suppose we are doing an experiment and an we have knowledge that an event  $B$  has happened. Knowing this how can we find the probability that another event  $A$  has happened. We call theis the prbability of  $A$  given  $B$  with this event denoted as follows

$$\mathbb{P}_B(A) = \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

**Proposition 13.** *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $B \in \mathcal{F}$ . Define  $\mathbb{P}_B : \mathcal{F} \rightarrow \mathbb{R}$  as  $\mathbb{P}_B(A) = \mathbb{P}(A|B)$ .  $\mathbb{P}_B$  is a probability measure*

*Proof.* There are three propterites to be checked.

1. It is known that  $\mathbb{P}(A) \in [0, 1]$  for all  $A \in \mathcal{F}$ . We also know that for this  $A$ ,  $A \cap B \in \mathcal{F}$ . This means that  $\mathbb{P}(A \cap B)$  is defined. We know that  $\mathbb{P}(A \cap B) \leq \mathbb{P}(B)$  Therefore

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \in [0, 1]$$

- 2.

$$\mathbb{P}_B(\Omega) = \frac{\mathbb{P}(B \cap \Omega)}{\mathbb{P}(B)} = 1$$

3. The next property to check is coiuntable additivity. Let for all  $n, A_n \in \mathcal{F}$

and let them all be pairwise disjoint.

$$\begin{aligned}
\mathbb{P}_B\left(\bigcup_{n=1}^{\infty} A_n\right) &= \frac{\mathbb{P}_B\left(\bigcup_{n=1}^{\infty} A_n \cap B\right)}{\mathbb{P}(B)} \\
&= \frac{1}{\mathbb{P}(B)} \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n \cup B\right) \\
&= \sum_{n=1}^{\infty} \frac{\mathbb{P}(A_n \cup B)}{\mathbb{P}(B)} \\
&= \sum_{n=1}^{\infty} \mathbb{P}_B(A_n)
\end{aligned}$$

This assumed we were able to apply countable additivity to the original measure which requires  $A_n \cap B$  to be pairwise disjoint

$$(A_n \cap B) \cap (A_m \cap B) \subseteq A_n \cap A_m = \emptyset$$

□

This equation leads us to the multiplication rule

**Theorem 1.** *Multiplication rule* :  $\mathbb{P}(A_1 \cap \dots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2)\dots$

this is proven using an expansion of the law of conditional probability

## 7 Lecture 8 + 9 : Further applications

If we wanted to know the probability of an event  $C$  happening. We can consider all of the disjoint cases in which it could happen and take some kind of weighted mean of all of these probabilities to find  $C$ . This is the law of total probability.

**Theorem 2.**  $\mathbb{P}(C_2) = \mathbb{P}(C_2|C_1)\mathbb{P}(C_1) + \mathbb{P}(C_2|C_1^C)\mathbb{P}(C_1^C)$

*Proof.* Due to the fact  $(C_1 \cup C_2) = \Omega$  The event  $C_2 = (C_2 \cap C_1) \cup (C_2 \cap C_1^C)$ . Then by application of finite additivity and the law of conditional probability the law of total probability can be derived □

This result can be further generalised

**Theorem 3.** *Law of Total probability:* Let  $\{B_n | n \in [N]\}$  be a partition of  $\Omega$ . We can say

$$\mathbb{P}(A) = \sum_{n=1}^N \mathbb{P}(A|B_n)\mathbb{P}(B_n)$$

The proof is done in the same way by the appreciation of the union of  $B_n = \Omega$  and then applying countable additivity and the laws of conditional probability.

## 8 Lecture 10 - Random Variables

Consider having done a random experiment which is quite complex as is the sample space ( $\Omega$ ) as such the individual events ( $\mathcal{F}$ ) are also complex but these events are such that we know whether they have happened or not.

Very often we wish to abstract these elements of  $\Omega$  to something much simpler i.e something that has a numerical value given by the random process.

We can express this random variable as a function  $X : \Omega \rightarrow \mathbb{R}$ . This course will only consider random variables into the real numbers.

**Definition 14.** *Random Variable: Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. A real-valued random variable is a function  $X : \Omega \rightarrow \mathbb{R}$  such that  $\{\omega \in \Omega | X(\omega) \leq a\} \in \mathcal{F}$  for all  $a \in \mathbb{R}$  (This says that specifying conditions via  $X$  gives an event that we can determine whether it has happened or not)*

**Example 2.** *Consider rolling two dice. You win money given by the greater value amongst the dice rolled. In this case*

$$\Omega = \{1, 2, 3, 4, 5, 6\}^2$$

$$X(x, y) = \max(x, y)$$

*It can be seen that using the random variable  $X$  simplifies the outcomes in  $\Omega$*

$$\mathbb{P}(X = 5) = \mathbb{P}(\{\omega \in \Omega | X(\omega) = 5\}) = \frac{1}{5}$$

We can see at the end we use a condition on  $X$  in order to specify an event. Instead in an inequality an equality was used but this can be shown to be inside of  $\mathcal{F}$  by closure of  $\mathcal{F}$  under set difference

We can consider the a new probability measure on  $\mathbb{R}$  induced by the random variable  $X$  and if this is all we care about we can leave the old probability space behind and only consider  $\mathbb{R}$  as our sample space

**Definition 15.** *Distribution: The distribution of a random variable  $X$  is the probability measure on  $\mathbb{R}$  given by*

$$\mathbb{P}_X(B) = \mathbb{P}(\{\omega \in \Omega | X(\omega) \in B\})$$

*such that  $B \in \mathcal{B}(\mathbb{R})$*

**Example 3.** *Cont.  $\mathbb{P}_X(\{k\}) = \frac{2k-1}{36}$  for  $k = 1, 2, 3, 4, 5, 6$   
otherwise  $\mathbb{P}(\mathbb{R} \setminus \{1, 2, 3, 4, 5, 6\}) = 0$*

**Definition 16.** *Event Space: We will usually denote the event space on  $\mathbb{R}$  denoted  $\mathcal{B}(\mathbb{R})$  contains singletons  $\{x\}$  and intervals  $(a, x]$  And is closed under countable set operations as is  $\mathcal{F}$ . We have no need to be able to list elements.*

**Definition 17.** *The triplet  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$  is a simpler derived probability space.*

**Proposition 14.** *The function  $\mathbb{P}_X$  is a probability measure on  $\mathbb{R}$*

*Proof.* We must check the three conditions of probability measure

1.  $\mathbb{P}_X(B) = \mathbb{P}(\{\omega \in \Omega | X(\omega) \in B\}) \in [0, 1]$
2.  $\mathbb{P}_X(\mathbb{R}) = \mathbb{P}(\{\omega \in \Omega | X(\omega) \in \mathbb{R}\}) = \mathbb{P}(\Omega) = 1$
3. Suppose  $B_1, B_2, \dots \in \mathcal{B}(\mathcal{R})$  are disjoint.

$$\begin{aligned} \mathbb{P}_X\left(\bigcup_{n=1}^{\infty} B_n\right) &= \mathbb{P}(\{\omega \in \Omega | X(\omega) \in \bigcup_{n=1}^{\infty} B_n\}) \\ &= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{\omega \in \Omega | X(\omega) \in B_n\}\right) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(\{\omega \in \Omega | X(\omega) \in B_n\}) \\ &= \sum_{n=1}^{\infty} \mathbb{P}_X(B_n) \end{aligned}$$

This uses the facts that pre-image of the union is the union of the pre-image; that pre-image of disjoint sets are disjoint,

□

## 9 Lecture 11

**Definition 18.** Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$  be an induced probability space. We may say it is discrete if there exist a countable or finite set  $S \subseteq \mathbb{R}$  such that  $\mathbb{P}(X \in \mathbb{R} \setminus S) = 0$

**Definition 19.** Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$  be a probability space and  $X$  be a r.v. we define the probability mass function  $p_X : \mathbb{R} \rightarrow [0, 1]$  as follows

$$p_X(x) = \mathbb{P}_X(x)$$

This is a simpler function as it allows us to investigate individual probabilities instead of those for a set.

**Definition 20.** Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$  be a probability space and  $X$  be a r.v. We may define the discrete support of  $X$  or  $\mathbb{P}_X$  as follows

$$\{x \in \mathbb{R} | p_X(x) > 0\}$$

**Proposition 15.** Let  $X$  be a r.v. and  $D_X$  be its support

$$\mathbb{P}_X(B) = \sum_{x \in B \cap D_X} p_X(x)$$

*Proof.* We know there is a  $S \subseteq \mathbb{R}$  such that  $\mathbb{P}_X(S) = 0$ . From here we can determine that  $D^C$  is countable as  $D \subseteq S$ . We may say that

$$P_X(B) = P_X(B \cap D) + P_X(B \cap D^C)$$

The second term has to be zero as  $B \cap D^C \subseteq D^C$  so  $0 \leq P_X(B \cap D^C) \leq P_X(D^C) \leq 0$ . We can then evaluate as follows

$$\begin{aligned} P_X(B) &= P_X(B \cap D) \\ &= P_X\left(\bigcup_{x \in B \cap D} \{x\}\right) \\ &= \sum_{x \in B \cap D} P_X(\{x\}) \\ &= \sum_{x \in B \cap D} p_X(x) \end{aligned}$$

□

From here we have the fact that  $p_X$  contains all the information of  $\mathbb{P}_X$  as the latter can be reconstructed from the former.

## 10 Lecture 12

As said before  $D_X$  does not necessarily equal  $S$

**Example 4.** Consider rolling a die.  $\Omega = \{1, 2, 3, 4, 5, 6\}$ ,  $X : \Omega \rightarrow \mathbb{R}$ ,  $X(k) = k$ . From here we get the definition of  $D_X = \{1, 2, 3, 4, 5, 6\}$  which is a suitable candidate for  $S$ . But this is not the only choice. Any countable extension of  $D_X$  is also a candidate for example  $S = \mathbb{N}$  as there are no elements in  $\mathbb{N}^C$  that are in  $D_X$  so  $\mathbb{P}_X(\mathbb{N}^C) = 0$

**Definition 21.** A function  $f : \mathbb{R} \rightarrow [0, 1]$  is a function if the set  $D$  defined as follows

$$D = \{x \mid f(x) > 0\}$$

is countable and  $\sum_{k \in D} f(k) = 1$

This gives us a definition of a probability mass function without already having a space. We can however construct a space from here.

**Proposition 16.** Let  $f$  be a probability mass function. Then there exist a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a r.v.  $X$  such that  $p_X = f$

*Proof.* Take  $D$  to be the discrete support of  $f$ . Let  $\Omega = \mathbb{R}$ ,  $\mathcal{F} = \mathcal{B}(\mathbb{R})$ ,  $\mathbb{P}(B) = \sum_{x \in B \cap D} f(x)$

We now need to prove that  $\mathbb{P}$  is a measure

•

$$\mathbb{P}(\mathbb{R}) = \sum_{k \in \mathbb{R} \cap D} f(k) = \sum_{x \in D} f(x) = 1$$

- $\mathbb{P} \in [0, 1]$
- Suppose  $A_1, \dots, A_n \in \mathcal{F}$  are all pairwise disjoint.

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) &= \sum_{k \in (\bigcup_{n=1}^{\infty} A_n) \cap D} f(x) \\
&= \sum_{k \in (\bigcup_{n=1}^{\infty} A_n \cap D)} f(x) \\
&= \sum_{n=1}^{\infty} \sum_{x \in A_n \cap D} f(x) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(A_n)
\end{aligned}$$

Now we need to construct the DRV. Let  $X(x) = x$ .  $X$  has the right domain and codomain. and we can also say  $\mathbb{P} = \mathbb{P}_X$ . We also need to check it has an appropriate countable set  $S$  For this we may try  $D$ .

$$\mathbb{P}_X(\mathbb{R} \setminus D) = \mathbb{P}_X(D^C) = \mathbb{P}(D^C) = \sum_{x \in D^C \cap D} f(x) = 0$$

Finally we need to check  $f$  is the probability mass function for  $X$ . If  $x \in D$

$$p_X(x) = \mathbb{P}_X(\{x\}) = \mathbb{P}(\{x\}) = \sum_{x \in \{x\} \cap D} f(x) = f(x)$$

Otherwise

$$p_X(x) = \mathbb{P}_X(\{x\}) = \mathbb{P}(\{x\}) = \sum_{x \in \{x\} \cap D} f(x) = \sum_{x \in \emptyset} f(x) = 0 = f(x)$$

□

Now let's look at some experiments that are discrete spaces.

**Example 5.** Imagine tossing a  $p$ -coin which is a coin where the probability of heads is  $p$ . We may construct our space as follows.  $\Omega = \{H, T\}, \mathbb{P}(\{H\}) = p, \mathbb{P}(\{T\}) = 1 - p$ . We may define  $X : \Omega \rightarrow \mathbb{R}$  as  $X(H) = 1, X(T) = 0$ . We may then say  $X \sim \text{Bernoulli}(p)$

**Definition 22.** Where  $p \in [0, 1]$  we may say  $X \sim \text{Bernoulli}(p)$  if

$$p_X(x) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

**Example 6.** Consider another experiment where we toss the  $p$ -coin till we get heads. Let  $X$  map to the number of tosses and  $\Omega$  be the set of sequences of tails which then end in heads.

$$\mathbb{P}(\underbrace{\{T \dots T H\}}_{k \text{ times}}) = (1-p)^k p$$

$$X(\underbrace{\{T \dots T H\}}_{k \text{ times}}) = k + 1$$

**Definition 23.** We can say  $X \sim \text{Geom}(p)$  if

$$p_X(k) = \begin{cases} p(1-p)^{k-1} & k \in \mathbb{N} \\ 0 & \text{Otherwise} \end{cases}$$

**Definition 24.**  $X \sim \text{Bin}(n, p)$  where  $n \in \mathbb{N}, p \in [0, 1]$  if

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k \in \{0, \dots, n\} \\ 0 & \text{Otherwise} \end{cases}$$

## 11 Lecture 13

Expectation: Sometimes we want to consider question such as what is the average outcome of a roll of a dice. in order to do this we may calculate some form of a weighted mean.

$$\frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

We call this number the expectation of reolling a dice

**Definition 25.** Consider a discrete random variable  $X$  with a support  $D$  and a mass function  $p_X$ . We define the expectation of  $X$  denoted  $\mathbb{E}(X)$  defined as follows

$$\mathbb{E}(X) = \sum_{x \in D} x p_X(x)$$

As long as it converges absolutely (sum of absolute values converges), Otherwise the expectation is undefined

**Definition 26.** A drv  $X$  with support  $D$  and probability mass function  $p_X$  is integrable if the sum

$$\sum_{x \in D} |x| p_X(x)$$

converges



**Example 7.** Consider  $X \sim B(n, p)$

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^n n \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= n \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j+1} (1-p)^{n-j-1} \\
&= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-j-1} \\
&= np(p + (1-p))^{n-1} \\
&= np
\end{aligned}$$

Consider now fixing this value  $\lambda = np$ . This leads us to the following experiment. We may consider tossing the  $p$ -coin  $n$  times such that the expected number of heads is fixed.

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!} \frac{n!}{n^k(n-k)!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

Taking the limit of  $n \rightarrow \infty$  we get this is equal to

$$\frac{\lambda^k}{k!} e^{-\lambda}$$

**Definition 27.** Poission Distobution :  $X \sim \text{Poiss}(\lambda)$  if we may say that

$$p_X(x) = \frac{\lambda^k}{k!} e^{-\lambda}$$

**Example 8.** Let  $X$  be a drv as a indicator function of a set  $A \in \mathcal{F}$ . We can say here  $E[X] = \mathbb{P}(A)$

**Example 9.**  $X \sim \text{Geom}(p)$

$$\begin{aligned}
 E[x] &= \sum_{n=1}^{\infty} n(1-p)^{n-1}p \\
 &= p \sum_{n=1}^{\infty} n(1-p)^{n-1} \\
 &= p \sum_{n=1}^{\infty} \frac{d}{d(1-p)} (1-p)^n \\
 &= p \frac{d}{d(1-p)} \sum_{n=1}^{\infty} (1-p)^n \\
 &= p \frac{d}{d(1-p)} \frac{1}{1-(1-p)} \\
 &= p \frac{-1}{(1-(1-p)^2)} \\
 &= p \frac{1}{p^2} \\
 &= \frac{1}{p}
 \end{aligned}$$

## 12 Lecture 15

### 12.1 Expectation of functions

One interpretation of the expectation is as the centre of mass of the mean of a distribution. This allows the expectation to act as a mean.

If we have a 2 dimensional random variable  $Z = (X, Y)$  We can say that  $E[Z] = (E[X], E[Y])$  Sometimes we transform a distribution by a function  $g: \mathbb{R} \rightarrow \mathbb{R}$  and we still want to find these expectations. We have been able to do this for linear functions but we may want to do this for  $x^2$  as this is useful for variance.

**Example 10.** Consider  $X \sim \text{Geom}(p)$

$$\begin{aligned}
E[X^2] &= \sum_{n=1}^{\infty} n^2 (1-p)^{n-1} p \\
&= p \sum_{n=0}^{\infty} n x^{n-1} + p \sum_{n=0}^{\infty} n(n-1) x^{n-1}, x = 1-p \\
&= p \sum_{n=0}^{\infty} \frac{d}{dx} x^n + p x \sum_{n=0}^{\infty} n(n-1) x^{n-2} \\
&= p \frac{d}{dx} \left( \sum_{n=0}^{\infty} x^n \right) + p x \sum_{n=0}^{\infty} \frac{d^2}{dx^2} x^n \\
&= p \frac{d}{dx} \left( \frac{1}{1-x} \right) + p x \frac{d^2}{dx^2} \left( \sum_{n=0}^{\infty} x^n \right) \\
&= p \frac{1}{(1-x)^2} + p x \frac{2}{(1-x)^3} \\
&= \frac{1}{p} + 2 \frac{1-p}{p^2} \\
&= \frac{2-p}{p^2}
\end{aligned}$$

**Example 11.** Consider  $X \sim \text{Geom}(p)$

For what values of  $t$  is  $e^{tx}$  integrable (Sum for expectation exists). As such compute  $E[e^{tX}]$ .

$$\begin{aligned}
E[e^{tX}] &= \sum_{n=1}^{\infty} e^{tn} p (1-p)^{n-1} \\
&= p e^t \sum_{n=1}^{\infty} (e^t (1-p))^{n-1} \\
&= p e^t \sum_{k=0}^{\infty} (e^t (1-p))^k \\
&= \frac{p e^t}{1 - e^t (1-p)} \\
&= \frac{p}{e^{-t} + p - 1}
\end{aligned}$$

Not this is only defined where  $e^t (1-p) < 1$  so  $t < \ln \frac{1}{1-p}$

Consider for where it isnt integrable. This means that there is no expectation. Further if a random varibale does have an expectation a function of the random variable (even  $x^2$ ) may not hadve a defined expectation.

## 12.2 Variance

Variance is used to measure the dispersion of a random variable

**Definition 28.** (Square integrable r.v.) : A DRV  $X$  is square integrable if  $X^2$  is integrable.

**Definition 29.** (Variance): Consider a Square integrable random variable  $X$ . Let  $\mu := E[X]$

$$Var(x) = E[(x - \mu)^2]$$

From here we can note that

$$Var(X) = E[(X - \mu)^2] = E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - \mu^2$$

**Example 12.** Consider  $X \sim \text{Poisson}(\lambda)$

$$Var(X) = E[X^2] - E[X]^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

**Example 13.** Consider  $X \sim \text{Geom}(p)$

$$Var(X) = E[X^2] - E[X]^2 =$$