

MA-138

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1 Lecture 1 - Sets

1.1 What are Sets

Definition 1. *A set is a collection of elements*

Commonly these are denoted by a listing of elements within braces. For example $\{1, 2, 3\}$ is the set containing 1, 2, and 3. Sometimes ellipses may be used inside the set

Example 1. $\{0, 1, 2, 3, \dots\}$ denotes the set of the natural numbers \mathbb{N}

When ellipses are seen a natural continuation of elements is assumed in this case the rest of the natural numbers. Note here \mathbb{N} contains 0.

Definition 2. *If x is an element of a set X we may write $x \in X$*

Example 2. $1 \in \mathbb{N}$

We can also demonstrate the converse

Example 3. $-1 \notin \mathbb{N}$

1.2 Set Relations

1.2.1 Equality

The first thing we want to be able to tell about pairs of sets is whether two are the same.

Definition 3. *Sets X and Y are equal if for every $x \in X$ we have $x \in Y$ and for every $y \in Y$ we have $y \in X$*

From this definition falls out two interesting things

- Sets do not care about the order of their elements
- Sets do not care how many times their elements occur

This makes them fundamentally different from lists.

Example 4. $\{1, 2, 3\} = \{2, 2, 3, 1, 1, 3, 3\}$

1.2.2 Empty Set

Before the next relation it is useful to introduce a special set known as the empty set.

Definition 4. *There exists a set \emptyset such that there does not exist an $x \in \emptyset$ called the empty set*

An interesting result is given two empty sets \emptyset_1 and \emptyset_2 these two are always equal. This is because any element in the first is necessarily in the second and vice versa as there are no elements in either. This tells us that there is only one empty set.

1.2.3 Numeric Sets

Another useful set to know is as follows before the next relation

Definition 5. *The set denoted $[n] = \{0, 1, 2, \dots, n-1\}$*

This is the set of all the natural numbers less than n and is non-standard notation.

1.2.4 Subsets

Definition 6. *We can say a set X is a subset of a set Y if for every $x \in X$ we also know that $x \in Y$ this is denoted $X \subseteq Y$*

Example 5. $\{0, 1\} \subseteq \{0, 1, 2\}$

Example 6. $[n] \subseteq [n+1]$

From here we can derive the fact that if $X \subseteq Y$ and $Y \subseteq X$ we can say $X = Y$. Each direction of the subset satisfies half the condition for equality so having both directions of the subset we can claim equality. this is similar to how when $x \leq y$ and $y \leq x$ we know that $x = y$.

For every set X we can also say two things

1. $\emptyset \subseteq X$
2. $X \subseteq X$

Which is also similar to what we can say for \leq

1.3 Set Function

1.4 Power Set

Every set can have many power sets so it is useful to be able to easily reference this collection of subsets

Definition 7. *For a set X let $\mathcal{P}(X)$ denote its power set such that $x \in \mathcal{P}(X)$ if and only if $x \subseteq X$*

Example 7. $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Example 8. $\mathcal{P}(\emptyset) = \{\emptyset\}$

It is important to remember that $\{\emptyset\}$ is a distinct set from \emptyset as the first contains \emptyset as an element $\emptyset \in \{\emptyset\}$. We also have a useful result that

$$|\mathbb{P}([n])| = 2^n$$

As for each element in $[n]$ for each subset of $[n]$ it can either be in or out of the subset.

2 Lecture 2 + 3 - Functions

2.1 Specification

Definition 8. *Specification* : Let X be a set and $S(x)$ be a property for $x \in X$. We can form a set

$$\{x \in X | S(x)\}$$

This gives the subset of X satisfying $S(x)$ for all x in the subset

An example of this is $\{k \in \mathbb{N} | k < n\}$ being the set of natural numbers less than n . This is the same as the set $[n]$ as defined earlier.

It is important to make sure you keep track of what set you are specifying against. For example the sets $\{k \in \mathbb{N} | x^2 - 1 = 0\}$ and $\{k \in \mathbb{Z} | x^2 - 1 = 0\}$ are different sets as the latter also contains -1 . It's also important to remember you are taking a subset of a set in this invocation. Not doing so is unrestricted comprehension and leads to Russell's paradox.

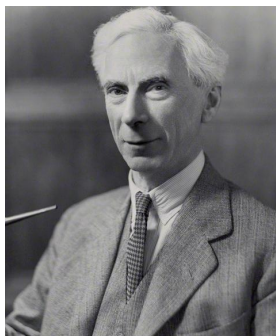


Figure 1: Russell angered by your use of unrestricted comprehension

2.2 Functions

2.2.1 Basics

Here we can start with a slightly informal definition of a function.

Definition 9. Let X and Y be sets. A function f from X to Y (denoted $f : X \rightarrow Y$) contains three pieces of information

1. A domain X
2. A codomain Y
3. A rule mapping every $x \in X$ to one $f(x) \in Y$

Two examples of functions are given as $f_1 : \mathbb{N} \rightarrow \mathbb{N}$ with $f_1(x) = x^2$ and another example is $f_2 : \mathbb{N} \rightarrow \mathbb{Z}$ with $f_2(x) = x^2$. These both are different functions as they have different codomains. Two functions are only equal if all three pieces of information agree.

It is important to note that the rule portion of a function need not be given by a formula and can instead just tell you where each element goes. For example $g : \{0, 1\} \rightarrow \{2, 3\}$ can have its rule expressed as $g(0) = 2$ and $g(1) = 3$. We can also get the useful property

$$|[[n]] \rightarrow [[m]]| = m^n$$

If a function f is $X \rightarrow X$ we say it is a function on X . A special function is the identity function on X $id_X : X \rightarrow X$ given by rule $id_X(x) = x$

2.2.2 Types of functions

Definition 10. A function $f : X \rightarrow Y$ is called injective if $f(x) = f(x') \Rightarrow x = x'$ for all $x, x' \in X$

Definition 11. A function $f : X \rightarrow Y$ is called surjective if for any $y \in Y$ there is an $x \in X$ such that $y = f(x)$

And if a function satisfies both of these we can call it bijective.

For finite sets we can say a lot of things about injections and surjections between them. For a pair of sets X, Y if $|X| = |Y|$ then any injective function is surjective. this is due to the fact to be injective each element in the domain needs a separate element in the codomain. So there are as many elements in the image as the domain which is the same as the size of the codomain so every element in the codomain is hit. And for the reverse argument each element in the codomain binds a unique element in the domain which is every element in the domain. It can't bind 2 elements to one element in the codomain otherwise the image would not be the codomain and as such would not be surjective.

We also get the fact we have no injections $f : [[n]] \rightarrow [[m]]$ if $m < n$ and no surjections $f : [[n]] \rightarrow [[m]]$ if $n < m$

2.2.3 Cardinality

Definition 12. Given any sets X and Y , X and Y have the same cardinality iff there exists a bijection $f : X \rightarrow Y$. This is denoted $|X| = |Y|$

Definition 13. For a finite set X if there is a bijection $f : [[n]] \rightarrow X$ then we may say that $|X| = n$

This makes sense for finite sets but has some interesting implications for infinite sets. Consider and $f : \mathbb{Z} \rightarrow \mathbb{N}$ defined as follows

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$$

This function is a bijection between the two sets mapping even naturals to positives and odds to negatives. As such we have $|\mathbb{N}| = |\mathbb{Z}|$ despite $\mathbb{N} \subseteq \mathbb{Z}$. We can construct a similar argument between \mathbb{N} and \mathbb{Q} by use of **The Cantor Pairing Function** which constructs the bijection between pairs of natural numbers and naturals which gets you 99% of the way to a bijection and infact shows that $|\mathbb{N}| \geq |\mathbb{Q}|$ as there are multiple pairs of naturals that can give a single rational number.

This is however not possible between \mathbb{R} and \mathbb{N} . It can be shown there is no such bijection. this is done using Cantor's diagonalization argument to prove that there is no surjection.

Proof. Assume that theres is a surjection $f : \mathbb{N} \rightarrow \mathbb{R}$ This would allow us to list elements of \mathbb{R} on the output. Here we will represent elements of \mathbb{R} in binary expansions

$$\begin{aligned} f(1) &= 0.010110101111\dots \\ f(2) &= 0.101100110001\dots \\ f(\dots) &= \dots \end{aligned}$$

Now we can construct a new number x . We make x by making the n th digit of x the opposite of the n th digit of $f(n)$. Now the question is : is x on our list. If x is in the image of f then there is some k such that $f(k) = x$ but this means that x will differ from $f(k)$ in the k th digit so x cannot be $f(k)$ so x cannot be in the image so f cannot be a surjection \square

We can also very similarly consider cantor theorem

Proposition 1. There is no surjection $X \rightarrow \mathcal{P}(X)$

Proof. By contradiction assume there is a function $f : X \rightarrow \mathcal{P}(X)$ such that f is a surjection. The means for an $A \in \mathcal{P}(X)$ we know there is an $a \in X$ so that $f(a) = A$ We can define a set $C \subseteq X$ as follows

$$C = \{x \in X | x \notin f(x)\} \in \mathcal{P}(X)$$

As f is a surjection there exists a d such that $f(d) = C$. We can consider two cases.

1. Consider $d \in C$. this gives that $d \in f(d)$ so by definition $d \notin f(d)$ leading to a contradiction
2. Consider $d \notin C$ this gives that $d \notin f(d)$ so by definition $d \in C$ leading to a contradiction

Both paths lead to a contradiction so we can assume our premise was wrong meaning that there is no such surjection. \square

This is somewhat related to how russels pradox works.

3 Lecture 4 + 5 + 6- More sets

3.1 Products

Definition 14. Given $x \in X, y \in Y$ We can construct an ordered pair (x, y) . We may say for two pairs $(x, y) = (x', y')$ if and only if $x = x', y = y'$

Definition 15. The cartesian product $X \times Y$ is the set of all pairs (x, y) such that $x \in X, y \in Y$

$$X \times Y = \{(x, y) | x \in X, y \in Y\}$$

We can from here get a few algebraic properties of the cartesian product.

Proposition 2. $X \times \emptyset = \emptyset$

Proof. Suppose by contradiction $X \times \emptyset \neq \emptyset$. this means there is a pair $(a, b) \in X \times \emptyset$ this means there is a $b \in \emptyset$ which is false yeilding a contradiction. \square

We also know that $[[m]] \times [[n]] = mn$ and that $X \times Y$ does not generally equal $Y \times X$.

Definition 16. $X^2 = X \times X$

3.2 Relations

Definition 17. A relation R from $X \rightarrow Y$ consists of three parts

1. A set X as the domain
2. A set Y as the codomain
3. A set $R \subseteq X \times Y$

You may write xRy to say that x is related to y is $(x, y) \in R$

Definition 18. A relation R from $X \rightarrow Y$ may be called graphical if the every $x \in X$ there is only one pair $(x, y) \in R$

3.2.1 Functions

Definition 19. A function $f : X \rightarrow Y$ is a graphical relation F from $X \rightarrow Y$ such that

$$f(x) = y \Leftrightarrow (x, y) \in F$$

Example 9. 1. $\{(0, 1), (1, 2), (2, 3)\} \subseteq [[3]] \times [[5]]$ is a graphical relation and associates to $f(x) = x + 1$

2. $\{(0, 0), (0, 1)\} \subseteq [[1]] \times [[2]]$ is not graphical as there is more than one pair for zero

Definition 20. A function f with graphical relation F can be called injective if for every $y \in Y$ there is at most one $(x, y) \in F$

3.3 Unions

Definition 21. Given two sets X, Y we can define their union as the set

$$\{z | z \in X \text{ or } z \in Y\}$$

And given a set of sets \mathbb{X}

$$\bigcup_{X \in \mathbb{X}} X = \{x | x \in X \text{ for some } X \in \mathbb{X}\}$$

For example let $\mathbb{X} = \{[[n]] | n \in \mathbb{N}\}$ then $\bigcup_{X \in \mathbb{X}} X = \mathbb{N}$

3.4 Intersections

Definition 22. The intersection of two sets X, Y is

$$X \cap Y = \{z | z \in X \text{ and } z \in Y\}$$

Let \mathbb{X} be a set of sets

$$\bigcap_{x \in \mathbb{X}} X = \{z | z \in X \text{ for all } X \in \mathbb{X}\}$$

For example let $\mathbb{X} = \{[[n]] | n \in \mathbb{N}\}$ then $\bigcap_{X \in \mathbb{X}} X = \emptyset$

3.5 Set difference

Definition 23. The set difference of X, Y is

$$X - Y = \{x \in X | x \notin Y\}$$

For example $\mathbb{Z} - \mathbb{N} = \mathbb{Z}_{>0}$

3.6 Algebra of Sets

3.6.1 Difference

The following identities hold for the set difference

- $X - \emptyset = X$
- $\emptyset - X = \emptyset$
- $X - X = \emptyset$

3.6.2 Union

The union is associative and commutative. The following identities also hold

- $X \subseteq X \cup Y$
- $X \cup \emptyset = X$

3.6.3 Intersection

The intersection is also associative and commutative. The following identities also hold

- $X \cap Y \subseteq X$
- $X \cap \emptyset = \emptyset$

3.6.4 Misc

Both the intersection and the union can distribute over each other. The following identity also holds.

$$X - (Y \cup Z) = (X - Y) \cap (X - Z)$$

The identity also holds if you reverse the unions and intersections

4 Lecture 6 - Composition

Definition 24. Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both functions then we can compose these to get the new function $(g \circ f) : X \rightarrow Z$ where $(g \circ f)(x) = g(f(x))$

The associated graphical relationship is as follows

$$\{(x, g(f(x))) | x \in X\} \subseteq X \times Z$$

Theorem. $(f \circ g) \circ h = f \circ (g \circ h)$

Definition 25. Given an $f : X \rightarrow Y$ we can say a function $g : Y \rightarrow X$ is a

- *left inverse of f if $(g \circ f) = id_X$*
- *right inverse of f if $(f \circ g) = id_Y$*

Theorem. *Given $f : X \rightarrow Y$ we can say*

- *If f has a left inverse it is injective*
- *If f has a right inverse it is surjective*

5 Lecture 7 - Relations

5.1 Functions on Functions

Definition 26. *Given a function $f : X \rightarrow Y$ we say show the image of f as*