MA147

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1 Lecture 2 + 3

1.1 Types of problem

There are 3 types of modelling problem

1. Forward Problems : Make prediction based off of a model and its parameters

2. Inverse problems : Derive parameters based off of a model and data

3. Control Problem : Enforce behaviour on a model

1.2 Dimensional Analysis

1.2.1 Variables

Before discussing any dimensions variables may have it is important to discuss the two kinds of variables. The first is the independent variables. These are things that exist independently or are parameters of our model. So time is a independent variable and an infection rate parameter is also independent.
A dependant variable evolves on a function of the dependant variables. So the amount infected or position may both be dependant variables with respect to time. We can express this relation between dependant and independent variables as follows.

$$\vec{d} = u(\vec{i}), d \in \mathbb{R}^n, i \in \mathbb{R}^m, u : \mathbb{R}^n \to \mathbb{R}^m$$

Here n and m are the number of dependant and independent variables respectively. Its important to note if x is a variable its derivative is the same variable for the purposes of counting as the derivative is an operator applied to it so to count this separately would be to count x^2 separately.

1.2.2 Introducing Dimensions

Dimensional analysis as a tool allows us to simplify models,check models and generalize. It is built on 2 key premises.

1. All quantities have dimensions (this includes dimensionless)

2. Laws relating quantities do not depend on units

Its work noting units are not dimensions. The relationship between these is shown below

Notation	Dimension	Unit
L	length	metre,foot
T	time	seconds,hours
M	Mass	grams,kg
A	Amount	mol
Θ	temp	K
Q	Charge	coulomb,e

Definition 1. *Given a variable v let $[v]$ denote the dimension of v*

For example $[t] = T$.

1.2.3 Dimensional Manipulation

We have 4 rules for the dimensional manipulation of variables.

1. $[v_1 v_2] = [v_1][v_2]$
2. $v_1 + v_2$ iff $[v_1] = [v_2]$
3. $\left[\frac{dx}{dy}\right] = \left[\frac{x}{y}\right]$ We also get a converse rule for integration by the fundamental theorem of algebra.
4. An argument x to a complex function such as sin or e^x must satisfy $[x] = 1$.

We can use dimensional analysis to reduce mathematical dependencies between variables. Suppose $d = u(i_1, i_2, \ldots, i_n)$ we may take the following steps to rewrite u to change its dependencies.

1. Write the dimensions of all variables
2. Express fundamental dimensions in terms of these variables (these are scalings)
3. Create a dimensionless version of all quantities by dividing by scaling
4. Make a change of variables to make d in terms of the new variables
5. Use the scaling to sub back in original values

This is quite a lot to do so consider the basic model given by

$$\frac{dP}{dt} = \alpha P(t), t > 0$$

$$P(0) = p_0$$

Then let

$$P = u(t, \alpha, p_0)$$

Expressing P in terms of its dependant variables. Now lets go through the steps

1. $[P] = A, [t] = T, [p_0] = A, [\alpha] = T^{-1}$
2. $A = [p_0], T = [\alpha^{-1}]$ these are the only applicable dimensions to the question.
3. $\tilde{p} = \frac{P}{p_0}, \tilde{t} = \alpha t, \tilde{\alpha} = \frac{\alpha}{\alpha}, \tilde{p_0} = \frac{p_0}{p_0}$ its important to note that for all of these $[\tilde{p}] = 1$
4. Now we can re-express d as a function of each these new variables. $\tilde{p} = \tilde{u}(\tilde{t}, \tilde{\alpha}, \tilde{p_0})$
5. And now subbing back we get $\frac{P}{p_0} = \tilde{u}(\alpha t, 1, 1)$ Eliminating constant dependencies and rearranging for P gives $P = p_0 \tilde{u}(\alpha t)$.

This process reveals that P only really depends on αt together and not separately then depends on p_0 only as a final scaling factor. This is represented in the solution to the differential equation of $P(t) = p_0 e^{\alpha t}$

2 Lecture 4 - Buckingham Π

Theorem 1. *Buckingham* Π *theorem : In a problem*