Proposition 1. *Given a function $F : \mathbb{N} \times X \to X$ and an $x_0 \in X$ we can construct a function $f : \mathbb{N} \to X$ satisfying the recursion*

- $f(0) = x_0$
- $f(k+1) = F(k, f(k))$

Proof. We can start by constructing a familly of functions $f_n : [[n]] \to X$ where we define $[[n]] := \{0, \dots, n-1\}$. We wan to have the fact the f_n satsfies the above recursion for all $k+1 \in [[n]]$. We can construct f_n as follows.

- $f_0 : [[0]] \to X$. As $[[0]] = \emptyset$ This is the empty function so we can ommit this case
- $f_1 : [[1]] \to X$. We can define this as $f_1(0) = x_0$ and its relation is then given as follows $f_1 = \{(0, x_0)\}$ This is clearly a function. And satisfies the base case of the reccurence
- Given we already have a function f_n satisfying the reccurence we may construct f_{n+1} as follows.

  $$f_{n+1}(x) = \begin{cases} f_n(x) & x \in [[n]] \\ F(x-1, f_n(x-1)) & x = n \end{cases}$$

  . Where its relation is geiven as follows $f_{n+1} = f_n \cup \{(n, F(x-1, f_n(x-1)))\}$

  We now aim to prove that this is a function and then that it does satisfy the reccurence. To prove this is a function we need for every $x \in [[n+1]]$ There is excalt one pair $(x, f_{n+1}(x)) \in f_{n+1}(x)$ Consider two cases for x

  - Consider $x \in [[n]]$. Due to the fact f_n is a function there is exactly one pair $(x, \_) \in f_n$. And as we know $x < n$ which tells us $x \neq n$ we know that then there is exaclty one pair $(x, \_) \in f_n \cup \{(n, F(x-1, f_n(x-1)))\} = f_{n+1}$
  - Consider $x \notin [[n]]$. As x is restricted to $[[n+1]]$ we can say in this case that then $x = n$. This means that x is outside the domain of f_n so there is no pair contianing x in f_n so there then is exactly one pair $(x, \_) \in f_n \cup \{(n, F(x-1, f_n(x-1)))\} = f_{n+1}$

  As such we can say that if f_n is a function then so is f_{n+1}. We now need to show it agrees with the recursion. As $f_n \subset f_{n+1}$ we can say then that for all $x \in [[n]]$ that $f_n(x) = f_{n+1}(x)$. So as we know that f_n satsifes the recursion we can say that f_{n+1} does aswell where $x \in [[n]]$. If $x = n$ then we get from the the way we extended the function it also satisfies the recurrence

Lemma 1. *Given an $x \in \mathbb{N}$ for all $a, b > x$ we have that $f_a(x) = f_b(x)$*

Proof. By the construction we have that $f_{x+1} \subseteq f_a$ and $f_{x+1} \subseteq f_a$. As all of these are functions we then get $f_a(x) = f_{x+1}(x)$ and $f_b(x) = f_{x+1}(x)$ so $f_a(x) = f_b(x)$ $\square$

Now we must find somewhere where all these functions exist. We know that the set $\mathbb{N} \times X$ must exist as it is given as the domain of F. Any set $[[n]] \times X$ is a subset of $\mathbb{N} \times X$ as $[[n]] \subset \mathbb{N}$. For all $n, f_n \subset ([[n]] \times X) \subset (\mathbb{N} \times X)$. As such we can then say that $f_n \in \mathcal{P}(\mathbb{N} \times X)$ for all n. As such we can construct by specification a set $\hat{f} = \{x = f_n \text{ for some n } | x \in \mathcal{P}(\mathbb{N} \times X)\} = \{f_0, f_1, \dots\}$

I now propose that $f : \mathbb{N} \to X = \bigcup \hat{f}$ is a function and that it satisfies the recurion. To show its a function we shall do this in two parts, show there is at least one pair $(n, \_) \in f$ and then that there is at most one pair.

- Suppose by contradiction that there is an $n \in \mathbb{N}$ such that there is no pair $(n, \_) \in f$. As we know there is a pair $(n, \_) \in f_{n+1}$ This would imply that $f_{n+1} \not\subset f$ which contradicts the construction of f
- The other way it may fail to be a function is that for a given n there are two pairs $(n, x), (n, y) \in f$ where $x \neq y$. This means there are funtions such that $f_a(n) = x \neq y = f_b(n)$. This contradicts the above lemma

This concludes that we know that f is a function. Next we aim to prove that it satisfies the reccurence which will; be done by induction.

- Base case, $n = 0$: We have that $(0, x_0) \in f_1 \subset f$ so we can conclude $f(0) = x_0$
- Indctive case, for some n we have that $f(n)$ satisfies the reccurence : We have that $(n + 1, F(k, f(n))) \in f_{n+2} \subset f$ so we know that $f(n + 1) = F(k, f(n))$ Which satsifies the reccurence.

This concludes the proof of the existence of a function that satisfies the linear reccurence??? $\square$

Predicate for classifying f_n

$$\Phi(X) = (0, x_0) \in X \wedge ((n + 1 < |X| \wedge (n, x) \in X) \implies (n + 1, F(n, x)) \in X)$$