

Notes on Math in Economics

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Contents

1	LOGIC	2
2	SET	3
3	TOPOLOGY ON EUCLIDEAN SPACE	4
4	TOPOLOGY	9
5	LINEAR ALGEBRA	10
6	FIXED POINT	11
7	LATITICE AND SUPER-MODULARITY	12
8	STATIC OPTIMIZATION	13
9	DYNAMIC OPTIMIZATION	14

1 LOGIC

Proposition

is a declaration that can be either true or false, but not both. If without further notice, P and Q is propositions as default.

Negation

$$\neg P$$

Conjunction

$$P \wedge Q$$

Disjunction

$$P \vee Q$$

Implication

$$P \Rightarrow Q$$

Equivalence

$$P \Longleftrightarrow Q$$

Converse

$$P \Rightarrow Q : Q \Rightarrow P$$

Inverse

$$P \Rightarrow Q : \neg P \Rightarrow \neg Q$$

Contrapositive

$$P \Rightarrow Q : \neg Q \Rightarrow \neg P$$

Contradiction

$$(P \wedge \neg Q) \Rightarrow \neg P, \text{ where } P \text{ is a priori } \Rightarrow Q$$

Logically equivalence

$$P \equiv Q, (P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P) \equiv [(P \wedge \neg Q) \Rightarrow \neg P]$$

Truth table

P	Q	$\neg P$	$P \vee Q$	$P \wedge Q$	$P \Rightarrow Q$	$P \Longleftrightarrow Q$	$Q \Rightarrow P$	$\neg P \Rightarrow \neg Q$	$\neg Q \Rightarrow \neg P$
T	T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	F	T	T	F
F	T	T	T	F	T	F	F	F	T
F	F	T	F	F	T	T	T	T	T

Universal quantifier

$$\forall x, P(x)$$

Existence quantifier

$$\exists x, P(x)$$

Negation of the two quantifier

$$\neg[\forall x, P(x)] \equiv [\exists x, \neg P(x)]; \quad (1)$$

$$\neg[\exists x, P(x)] \equiv [\forall x, \neg P(x)] \quad (2)$$

2 SET

Set

$$e.g. X = \{1, 2, 3\}$$

Element

$$\omega \in X$$

Family

$$\mathcal{X} = \{A, B, C\}$$

Union

$$A \cup B = \{\omega | (\omega \in A) \vee (\omega \in B)\} = A + B$$

Intersection

$$A \cap B = \{\omega | (\omega \in A) \wedge (\omega \in B)\} = AB$$

Complement

$$A \setminus B = \{\omega | (\omega \in A) \wedge (\omega \notin B)\} = A - B \quad (3)$$

$$A^c = \Omega \setminus A \quad (4)$$

$$A \setminus B = A \cap B^c \quad (5)$$

Disjoint

$$A \cap B = \emptyset \quad (6)$$

$$A \cap A^c = \emptyset \quad (7)$$

Subset

$$A \subset B : \omega \in A \Rightarrow \omega \in B$$

Superset

$$B \supset A$$

Equivalence

$$A = B \iff (A \subset B) \wedge (B \subset A)$$

Equivalent difference

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

Algorithning rules

$$A \cap B = B \cap A, A \cup B = B \cup A \quad (8)$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \quad (9)$$

De Morgan's Rule

$$(\cup A_i)^c = \cap A_i^c, (\cap A_i)^c = \cup A_i^c$$

Cartesian products

$$\prod_{i \in I} A_i = A_1 \times A_2 \times \cdots \times A_I$$

3 TOPOLOGY ON EUCLIDEAN SPACE

Euclidean space

the space of vectors of real numbers with n components, denoted as \mathbb{R}^n .

Metric on \mathbb{R}^n

$$(x, y) \in \mathbb{R}^n, (\mathbb{R}^n, d) : d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

Equal of vectors on \mathbb{R}^n

$$x = y \iff \forall i, x_i = y_i$$

Greater than or equal of vectors on \mathbb{R}^n

$$x \geq y \iff \forall i, x_i \geq y_i$$

Greater than of vectors on \mathbb{R}^n

$$x > y \iff \forall i, x_i \geq y_i; \exists j, x_j > y_j$$

Much greater than of vectors on \mathbb{R}^n

$$x \gg y \iff \forall i, x_i > y_i$$

ϵ -open ball/ ϵ -neighborhood of x on \mathbb{R}^n

$$B_\epsilon(x) = \{y \in \mathbb{R}^n | d(x, y) < \epsilon\}$$

Max on \mathbb{R}

$$\exists A \subset \mathbb{R}, (x \in A) \wedge (\forall a \in A, x \geq a) \Rightarrow x = \max A$$

Min on \mathbb{R}

$$\exists A \subset \mathbb{R}, (x \in A) \wedge (\forall a \in A, x \leq a) \Rightarrow x = \min A$$

Upper bound on \mathbb{R}

$$\exists A \subset \mathbb{R}, (x \in \mathbb{R}) \wedge (\forall a \in A, x \geq a) \Rightarrow x \text{ is the upper bound of } A$$

Supremum on \mathbb{R} x is the least upper bound of $A \subset \mathbb{R} \iff x = \sup A$

Lower bound on \mathbb{R}

$$\exists A \subset \mathbb{R}, (x \in \mathbb{R}) \wedge (\forall a \in A, x \leq a) \Rightarrow x \text{ is the lower bound of } A$$

Infimum on \mathbb{R} x is the least lower bound of $A \subset \mathbb{R} \iff x = \inf A$

Open set on \mathbb{R}^n

$$S \subset \mathbb{R}^n, \forall x \in S, \exists \epsilon > 0, B_\epsilon(x) \subset S$$

Theorem 1. *some theorems of open set:*

$$1. \emptyset \text{ and } \mathbb{R}^n \text{ are open} \tag{10}$$

$$2. \text{ the union of any collection of open sets is open} \tag{11}$$

$$3. \text{ the intersection of any infinite collection of open sets is open} \tag{12}$$

Topology

a system $\tau = \{S_\alpha\}_{\alpha \in A}$ is a topology on space X if

$$1. \emptyset \text{ and } X \in \tau \quad (13)$$

$$2. \forall \{S_\beta | S_\beta \in \tau\}, \cup S_\beta \in \tau \quad (14)$$

$$3. \forall \{S_k | S_k \in \tau, k = 1, 2, \dots, n\}, \cap S_k \in \tau \quad (15)$$

Closed set

$S \subset \mathbb{R}^n$ is closed if S^c is open

Theorem 2. *Some theorems of closed set:*

$$1. \emptyset \text{ and } \mathbb{R}^n \text{ are open} \quad (16)$$

$$2. \text{ the intersection of any collection of closed sets is closed} \quad (17)$$

$$3. \text{ the union of any infinite collection of closed sets is closed} \quad (18)$$

Sequence in \mathbb{R}^n

$$\{x^i\}_{i=1}^\infty = \{x_1, x_2, \dots, x_n, \dots\}, x_i \in \mathbb{R}^n$$

Converge

if for a sequence $\{x^i\}$, $\forall \epsilon > 0, \exists N, i \geq N, d(x^i, \alpha) < \epsilon \Rightarrow x$ is the limit of the sequence $\{x^i\}$

Bounded

$$\exists M > 0, S \subset B_M(\cdot)$$

Theorem 3. *A non-decreasing and bounded sequence $\{x^i\}_{i=1}^\infty$ in \mathbb{R} converges.*

Proof.

$$\text{bounded} \Rightarrow \exists M > 0, \{x_i\}_{i=1}^\infty \subset B_M(0) = \{y \in \mathbb{R} : \sqrt{(y-0)^2} < M\} \quad (19)$$

$$\Rightarrow \sup\{x^i\}_{i=1}^\infty < M \quad (20)$$

$$\forall \epsilon > 0, x - \epsilon \text{ is not an upper bound of the sequence} \quad (21)$$

$$\Rightarrow \exists x^N \in x_i, x^N > x - \epsilon \quad (22)$$

$$\text{nondecreasing} \Rightarrow \forall i \geq N, x^i \geq x^N \Rightarrow x^i > x - \epsilon \quad (23)$$

$$\forall i, x \geq x^i \Rightarrow x > x^i - \epsilon \quad (24)$$

$$\Rightarrow |x - x^i| = d(x, x^i) < \epsilon \Rightarrow \text{converge} \quad (25)$$

□

Using convergence to define closed set

$$S \in \mathbb{R}^n, \{x^i\} \subset S, \{x^i\} \rightarrow x \in S \Rightarrow S \text{ is a closed set}$$

Non-empty closed cell in \mathbb{R}^n

$$I = \{x = (x_1, x_2, \dots, x_n) : a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, \dots, a_n \leq x_n \leq b_n, \} \quad (26)$$

$$= [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] \quad (27)$$

Theorem 4 (Nested cells theorem).

Let $\{I_i\}$ be a sequence of non-empty closed cells in \mathbb{R}^n that are nested, (28)

i.e. $I_1 \supset I_2 \supset \dots \supset I_k \supset I_{k+1} \supset \dots$ (29)

then $\bigcap_{i=1}^{\infty} I_i \neq \emptyset$ (30)

Subsequence

$\exists \{x^i\} \in \mathbb{R}^n, \{x^{i_k}\}$ where $i_k < i_{k+1}$ (31)

e.g. (32)

$\{x^i\} = \{x_1, x_2, x_3, x_4, x_5, \dots\}$ (33)

$\{x^{i_k}\} = \{x_1, x_3, x_5, \dots\}$ (34)

Theorem 5 (Bolzano-Weierstrass theorem). *The theorem states that each infinite bounded sequence in \mathbb{R}^n has a convergent subsequence. An equivalent formulation is that a subset of \mathbb{R}^n is sequentially compact if and only if it is closed and bounded. The theorem is sometimes called the sequential compactness theorem.*

Open cover

a family $\{T_\alpha\}_{\alpha \in A} \in \mathbb{R}^n$ is said to be an open cover of $S \subset \mathbb{R}^n$ if T_α is open for each $\alpha \in A$ and $S \subset \cup_\alpha T_\alpha$

Finite open subcover

given an open cover $\{T_\alpha\}_{\alpha \in A} \in \mathbb{R}^n$, $\{T_{\alpha_k}\}_{k=1}^N$ is said to be a finite subcover if for each $\alpha_k \in A$, $S \subset \cup_{\alpha_k} T_{\alpha_k}$.

Compact

a set $S \subset \mathbb{R}^n$ is compact if for any open cover of S , there exists a finite open subcover

Theorem 6 (Heine-Borel theorem). $S \subset \mathbb{R}^n$ is compact $\iff S$ is closed and bounded

Function

$f : X \rightarrow Y = \{y \in Y | \exists x \in X, f(x) = y\};$ (35)

$\exists S \subset X, f(S) = \{y \in Y | \exists x \in S, f(x) = y\}$ (36)

One-to-one or Injection

$$f(x_1) \neq f(x_2), x_1 \neq x_2$$

Continuent function

$\exists f : \mathbb{R}^n \rightarrow \mathbb{R}^m, \forall x \in \mathbb{R}^n, \exists \delta > 0, \forall x' \in B_\delta(x), f(x') \in B_\delta[f(x)]$ (37)

or (38)

\forall open $T \subset \mathbb{R}^n, f^{-1}(T)$ is open (39)

Theorem 7 (Weierstrass theorem).

$$D \subset X \text{ is a nonempty compact set} \quad (40)$$

$$f : X \rightarrow Y \text{ is a continuous function} \quad (41)$$

$$\Rightarrow f(D) \subset Y \text{ is compact} \quad (42)$$

Lemma 1 (Corollary of Weierstrauss theorem on real Number space).

$$D \subset X \text{ is a nonempty compact set} \quad (43)$$

$$f : X \rightarrow \mathbb{R} \text{ is a continuous function} \quad (44)$$

$$\Rightarrow \exists x_1, x_2 \in D, \forall x \in D, f(x_1) \leq f(x) \leq f(x_2) \quad (45)$$

Distance between point and closed set

using Weierstrauss theorem to make sure there is a shortest distance between a point and a closed set

Vector/a point in Euclidean space

$$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

Convex

$$C \subset \mathbb{R}^n \text{ is convex if and only if } \forall x, y \in C \text{ and } \lambda \in [0, 1], \lambda x + (1 - \lambda)y \in C$$

Theorem 8 (Two theorems about convexity).

$$\exists C, D \subset \mathbb{R}^n \text{ are convex} \Rightarrow \quad (46)$$

$$C + D = \{x + y | x \in C, y \in D\} \text{ is convex}; \quad (47)$$

$$\forall \alpha \in \mathbb{R}, \alpha C \text{ is also convex} \quad (48)$$

Hyperplane

$$\exists h \in \mathbb{R}^n, \lambda \in \mathbb{R}, \{x \in \mathbb{R}^n | hx = \lambda\} \text{ is a hyperplane}$$

Half-space

$$\exists h \in \mathbb{R}^n, \lambda \in \mathbb{R}, \{x \in \mathbb{R}^n | hx \leq \lambda\} \text{ or } \{x \in \mathbb{R}^n | hx \geq \lambda\} \text{ is a half-space}$$

Theorem 9 (seperating hyperplane theorem).

$$\text{Suppose that a set } C \neq \emptyset \subset \mathbb{R}^n \text{ is closed and convex, and a point } b \notin C \quad (49)$$

$$\Rightarrow \exists h \in \mathbb{R}^n, \exists \lambda \in \mathbb{R}, \forall x \in C, h \cdot b < \lambda < h \cdot x \quad (50)$$

Using hyperplane to separate two convex sets

$$\text{Suppose that two nonempty sets } C, D \subset \mathbb{R}^n \text{ are closed and convex, and } C \cap D = \emptyset \quad (51)$$

$$\text{if at least one of the two sets is bounded, then } \exists h \neq 0 \in \mathbb{R}^n, \exists \lambda \in \mathbb{R}, \quad (52)$$

$$\forall x \in C, \forall y \in D, h \cdot x < \lambda < h \cdot y \quad (53)$$

Correspondance

$$F : X \rightrightarrows Y \iff \forall x \in X, F(x) \subset Y$$

- $F(x)$ is assumed non-empty from now on if without further notice

Compact-valued correspondance

$$\forall x \in X, F(x) \subset Y \text{ is compact}$$

Convex-valued correspondance

$$\forall x \in X, F(x) \subset Y \text{ is convex}$$

Closed-valued correspondance

$$\forall x \in X, F(x) \subset Y \text{ is closed}$$

Singleton-valued correspondance

$$\forall x \in X, F(x) \subset Y \text{ is a singleton set} \iff F(x) \text{ is a function } f(x)$$

Continuous function

A function f is continuous if and only if for any open $V \subset Y$

Upper hemi-continuous correspondance

Lower hemi-continuous correspondance

Theorem 10.

F is continuous if and only if F is both upper hemi-continuous and lower hemi-continuous.

4 TOPOLOGY

a topology τ on X is a family of subsets of X satisfying

$$\begin{cases} \emptyset, X \in \tau \\ \tau \text{ is closed under finite intersections} \\ \tau \text{ is closed under unions} \end{cases}$$

Topological space

$$(X, \tau)$$

Open set

a member of τ is an open set

Closed set

the complement of an open set is a closed set

Interior and Closure

let (X, τ) be a topological space and $A \subset X$, interior is the largest open set included by A

$$Int(A)$$

and closure is the smallest closed set including A

$$Cl(A)$$

Dense

$$\text{If } D \in (X, \tau), Cl(D) = X \Rightarrow D \text{ is dense}$$

Seperable

$$\exists D \in (X, \tau), Cl(D) = X \Rightarrow X \text{ is seperable}$$

Continuous function

In (X, τ) , $f : X \rightarrow Y$ is continuous if $f^{-1}(U) \subset X$ is open whenever $U \subset Y$ is open

Hemeomorphic

$\exists (X, \tau), (Y, \tau)$, if a one-to-one continuous function $f : X \rightarrow Y$ such that f^{-1} is continuous

Compact

$$\exists K \subset (X, \tau), \forall \text{ open cover } K \text{ includes a finite subcover} \iff K \text{ is compact}$$

Theorem 11 (Weierstrass theorem). *Every continuous function from topological space to another carries compact sets to compact sets.*

5 LINEAR ALGEBRA

6 FIXED POINT

7 LATITICE AND SUPER-MODULARITY

8 STATIC OPTIMIZATION

Lagrangian method

KKT method

9 DYNAMIC OPTIMIZATION

Hamiltonian method

Bellman method

Discrete time optimization