Notes on Math in Economics

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1 LOGIC

Proposition
is a declaration that can be either true or false, but not both. If withour further notice, P and Q is propositions as default.
Negation

$$\neg P$$

Conjunction

$$P \wedge Q$$

Disjunction

$$P \vee Q$$

Implication

$$P \Rightarrow Q$$

Equivalence

$$P \iff Q$$

Converse

$$P \Rightarrow Q : Q \Rightarrow P$$

Inverse

$$P \Rightarrow Q : \neg P \Rightarrow \neg Q$$

Contrapositive

$$P \Rightarrow Q : \neg Q \Rightarrow \neg P$$

Contradiction

$$(P \wedge \neg Q) \Rightarrow \neg P \text{ ,where P is a priori} \Rightarrow Q$$

Logically equivalence

$$P \equiv Q, (P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P) \equiv [(P \wedge \neg Q) \Rightarrow \neg P]$$

Truth table

P	Q	¬ P	P∨Q	P∧Q	P⇒Q	P $\iff$ Q	Q⇒P	$\neg P \Rightarrow \neg Q$	$\neg Q \Rightarrow \neg P$
T	T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	F	T	T	F
F	T	T	T	F	T	F	F	F	T
F	F	T	F	F	T	T	T	T	T

Universal quantifier

$$\forall x, P(x)$$

Existence quantifier

$$\exists x, P(x)$$

Negation of the two quantifier

$$\neg[\forall x, P(x)] \equiv [\exists x, \neg P(x)]; \tag{1}$$

$$\neg[\exists x, P(x)] \equiv [\forall x, \neg P(x)] \tag{2}$$

2 SET

Set

$$e.g.X = \{1, 2, 3\}$$

Element

$$\omega \in X$$

Family

$$\mathscr{X} = \{A, B, C\}$$

Union

$$A \cup B = \{\omega | (\omega \in A) \vee (\omega \in B)\} = A + B$$

Intersection

$$A \cap B : \{\omega | (\omega \in A) \wedge (\omega \in B)\} = AB$$

Complement

$$A \setminus B = \{\omega | (\omega \in A) \wedge (\omega \notin B)\} = A - B \tag{3}$$
$$A^c = \Omega \setminus A \tag{4}$$
$$A \setminus B = A \cap B^c \tag{5}$$

Disjoint

$$A \cap B = \varnothing \tag{6}$$
$$A \cap A^c = \varnothing \tag{7}$$

Subset

$$A \subset B : \omega \in A \Rightarrow \omega \in B$$

Superset

$$B \supset A$$

Equivalence

$$A = B \iff (A \subset B) \wedge (B \subset A)$$

Equivalent difference

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

Algorithing rules

$$A \cap B = B \cap A, A \cup B = B \cup A \tag{8}$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \tag{9}$$

De Morgan's Rule

$$(\cup A_i)^c = \cap A_i^c, (\cap A_i)^c = \cup A_i^c$$

Cartesian products

$$\underset{i \in I}{\pi} A_i = A_1 \times A_2 \times \cdots \times A_I$$

3 TOPOLOGY ON EUCLIDEAN SPACE

Euclidean space
the space of vectors of real numbers with n components, denoted as $\mathbb{R}^n$.
Metric on $\mathbb{R}^n$

$$(x, y) \in \mathbb{R}^n, (\mathbb{R}^n, d) : d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

Equal of vectors on $\mathbb{R}^n$

$$x = y \iff \forall i, x_i = y_i$$

Greater than or equal of vectors on $\mathbb{R}^n$

$$x \geqslant y \iff \forall i, x_i \geqslant y_i$$

Greater than of vectors on $\mathbb{R}^n$

$$x > y \iff \forall i, x_i \geqslant y_i; \exists j, x_j > y_j$$

Much greater than of vectors on $\mathbb{R}^n$

$$x \gg y \iff \forall i, x_i > y_i$$

ϵ-open ball/ ϵ-neighborhood of x on $\mathbb{R}^n$

$$B_\epsilon(x) = \{y \in \mathbb{R}^n | d(x, y) < \epsilon\}$$

Max on $\mathbb{R}$

$$\exists A \subset \mathbb{R}, (x \in A) \wedge (\forall a \in A, x \geqslant a) \Rightarrow x = max\ A$$

Min on $\mathbb{R}$

$$\exists A \subset \mathbb{R}, (x \in A) \wedge (\forall a \in A, x \leqslant a) \Rightarrow x = min\ A$$

Upper bound on $\mathbb{R}$

$$\exists A \subset \mathbb{R}, (x \in \mathbb{R}) \wedge (\forall a \in A, x \geqslant a) \Rightarrow x \text{ is the upper bound of } A$$

Supremum on $\mathbb{R}$ x is the least upper bound of $A \subset \mathbb{R} \iff x = sup\ A$
Lower bound on $\mathbb{R}$

$$\exists A \subset \mathbb{R}, (x \in \mathbb{R}) \wedge (\forall a \in A, x \leqslant a) \Rightarrow x \text{ is the lower bound of } A$$

Infimum on $\mathbb{R}$ x is the least lower bound of $A \subset \mathbb{R} \iff x = inf\ A$
Open set on $\mathbb{R}^n$

$$S \subset \mathbb{R}^n, \forall x \in S, \exists \epsilon > 0, B_\epsilon(x) \subset S$$

Theorem 1. *some theorems of open set:*

1. $\varnothing$ *and* $\mathbb{R}^n$ *are open* (10)
2. *the union of any collection of open sets is open* (11)
3. *the intersection of any infinite collection of open sets is open* (12)

Topology

a system $\tau = \{S_\alpha\}_{\alpha \in A}$ is a topology on space X if

$$\begin{aligned}
&1.\ \varnothing \text{ and } X \in \tau && (13)\\
&2.\ \forall\{S_\beta | S_\beta \in \tau\}, \cup S_\beta \in \tau && (14)\\
&3.\ \forall\{S_k | S_k \in \tau, k = 1, 2...n\}, \cap S_k \in \tau && (15)
\end{aligned}$$

Closed set

$$S \subset \mathbb{R}^n \text{ is closed if } S^c \text{ is open}$$

Theorem 2. *Some theorems of closed set:*

$$\begin{aligned}
&\textit{1. } \varnothing \textit{ and } \mathbb{R}^n \textit{ are open} && (16)\\
&\textit{2. the intersection of any collection of closed sets is closed} && (17)\\
&\textit{3. the union of any infinite collection of closed sets is closed} && (18)
\end{aligned}$$

Sequence in $\mathbb{R}^n$

$$\{x^i\}_{n=1}^{\infty} = \{x_1, x_2, ..., x_n, ...\}, x_i \in \mathbb{R}^n$$

Converge

if for a sequence $\{x^i\}, \forall \epsilon > 0, \exists N, i \geqslant N, d(x^i, \alpha) < \epsilon \Rightarrow x$ is the limit of the sequence$\{x^i\}$

Bounded

$$\exists M > 0, S \subset B_M(\cdot)$$

Theorem 3. *A non-decreasing and bounded sequence $\{x^i\}_{i=1}^{\infty}$ in $\mathbb{R}$ converges.*

Proof.

$$\begin{aligned}
&\text{bounded} \Rightarrow \exists M > 0, \{x_i\}_{i=1}^{\infty} \subset B_M(0) = \{y \in \mathbb{R} : \sqrt{(y-0)^2} < M\} && (19)\\
&\Rightarrow sup\{x^i\}_{i=1}^{\infty} < M && (20)\\
&\forall \epsilon > 0, x - \epsilon \text{ is not an upper bound of the sequence} && (21)\\
&\Rightarrow \exists x^N \in x_i, x^N > x - \epsilon && (22)\\
&\text{nondecreasing} \Rightarrow \forall i \geqslant N, x^i \geqslant x^N \Rightarrow x^i > x - \epsilon && (23)\\
&\forall i, x \geqslant x^i \Rightarrow x > x^i - \epsilon && (24)\\
&\Rightarrow |x - x^i| = d(x, x^i) < \epsilon \Rightarrow \text{converge} && (25)
\end{aligned}$$

□

Using convergence to define closed set

$$S \in \mathbb{R}^n, \{x^i\} \subset S, \{x^i\} \to x \in S \Rightarrow S \text{ is a closed set}$$

Non-empty closed cell in $\mathbb{R}^n$

$$\begin{aligned}
I &= \{x = (x_1, x_2, ..., x_n) : a_1 \leqslant x_1 \leqslant b_1, a_2 \leqslant x_2 \leqslant b_2, ..., a_n \leqslant x_n \leqslant b_n, \} && (26)\\
&= [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n] && (27)
\end{aligned}$$

Theorem 4 (Nested cells theorem)**.**

$$\textit{Let } \{I_i\} \textit{ be a sequence of non-empty closed cells in } \mathbb{R}^n \textit{ that are nested,} \tag{28}$$
$$i.e. I_1 \supset I_2 \supset \dots \supset I_k \supset I_{k+1} \supset \dots \tag{29}$$
$$then \bigcap_{i=1}^{\infty} I_i \neq \varnothing \tag{30}$$

Subsequence

$$\exists \{x^i\} \in \mathbb{R}^n, \{x^{i_k}\} \text{ where } i_k < i_{k+1} \tag{31}$$
$$e.g. \tag{32}$$
$$\{x^i\} = \{x_1, x_2, x_3, x_4, x_5, ...\} \tag{33}$$
$$\{x^{i_k}\} = \{x_1, x_3, x_5, ...\} \tag{34}$$

Theorem 5 (Bolzano-Weierstrass theorem)**.** *The theorem states that each infinite bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. An equivalent formulation is that a subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded. The theorem is sometimes called the sequential compactness theorem.*

Open cover
a family $\{T_\alpha\}_{\alpha \in A} \in \mathbb{R}^n$ is said to be an open cover of $S \subset \mathbb{R}^n$ if T_α is open for each $\alpha \in A$ and $S \subset \cup_\alpha T_\alpha$
Finite open subcover
given an open cover $\{T_\alpha\}_{\alpha \in A} \in \mathbb{R}^n$, $\{T_{\alpha_k}\}_{k=1}^N$ is said to be a finite subcover if for each $\alpha_k \in A, S \subset \cup_{\alpha_k} T_{\alpha_k}$.
Compact
a set $S \subset \mathbb{R}^n$ is compact if for any open cover of S, there exists a finite open subcover

Theorem 6 (Heine-Borel theorem)**.** *$S \subset \mathbb{R}^n$ is compact $\iff$ S is closed and bounded*

Function

$$f : X \to Y = \{y \in Y | \exists x \in X, f(x) \in Y\}; \tag{35}$$
$$\exists S \subset X, f(S) = \{y \in Y | \exists x \in S, f(x) = y\} \tag{36}$$

One-to-one or Injection

$$f(x_1) \neq f(x_2), x_1 \neq x_2$$

Continuent function

$$\exists f : \mathbb{R}^n \to \mathbb{R}^m, \forall x \in \mathbb{R}^n, \exists \delta > 0, \forall x' \in B_\delta(x), f(x') \in B_\delta[f(x)] \tag{37}$$
$$or \tag{38}$$
$$\forall \text{ open } T \subset \mathbb{R}^n, f^{-1}(T) \text{ is open} \tag{39}$$

Theorem 7 (Weierstrass theorem).

$$D \subset X \text{ is a nonempty compact set} \tag{40}$$
$$f: X \to Y \text{ is a continuent function} \tag{41}$$
$$\Rightarrow f(D) \subset Y \text{ is compact} \tag{42}$$

Lemma 1 (Corollary of Weierstrauss theorem on real Number space).

$$D \subset X \text{ is a nonempty compact set} \tag{43}$$
$$f: X \to \mathbb{R} \text{ is a continuent function} \tag{44}$$
$$\Rightarrow \exists x_1, x_2 \in D, \forall x \in D, f(x_1) \leqslant f(x) \leqslant f(x_2) \tag{45}$$

Distance between point and closed set
using Weierstrauss theorem to make sure there is a shortest distance between a point and a closed set
Vector/a point in Euclidean space

$$x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$$

Convex

$$C \subset \mathbb{R}^n \text{ is convex if and only if } \forall x, y \in C \text{ and } \lambda \in [0,1], \lambda x + (1-\lambda)y \in C$$

Theorem 8 (Two theorems about convexity).

$$\exists C, D \subset \mathbb{R}^n \text{ are convex} \Rightarrow \tag{46}$$
$$C + D = \{x + y | x \in C, y \in D\} \text{ is convex;} \tag{47}$$
$$\forall \alpha \in \mathbb{R}, \alpha C \text{ is also convex} \tag{48}$$

Hyperplane
$$\exists h \in \mathbb{R}^n, \lambda \in \mathbb{R}, \{x \in \mathbb{R}^n | hx = \lambda\} \text{ is a hyperplane}$$

Half-space

$$\exists h \in \mathbb{R}^n, \lambda \in \mathbb{R}, \{x \in \mathbb{R}^n | hx \leqslant \lambda\} \text{ or } \{x \in \mathbb{R}^n | hx \geqslant \lambda\} \text{ is a half-space}$$

Theorem 9 (seperating hyperplane theorem).

$$\text{Suppose that a set } C \neq \varnothing \subset \mathbb{R}^n \text{ is closed and convex, and a point } b \notin C \tag{49}$$
$$\Rightarrow \exists h \in \mathbb{R}^n, \exists \lambda \in \mathbb{R}, \forall x \in C, h \cdot b < \lambda < h \cdot x \tag{50}$$

Using hyperplane to seperate two convex sets

$$\text{Suppose that two nonempty sets } C, D \subset \mathbb{R}^n \text{ are closed and convex, and } C \cap D = \varnothing \tag{51}$$
$$\text{if at least one of the two sets is bounded, then } \exists h \neq 0 \in \mathbb{R}^n, \exists \lambda \in \mathbb{R}, \tag{52}$$
$$\forall x \in C, \forall y \in D, h \cdot x < \lambda < h \cdot y \tag{53}$$

Corrospondance

$$F : X \rightrightarrows Y \iff \forall x \in X, F(x) \subset Y$$

- $F(x)$ is assumed non-empty from now on if without furthur notice

Compact-valued correspondance

$$\forall x \in X, F(x) \subset Y \text{ is compact}$$

Convex-valued correspondance

$$\forall x \in X, F(x) \subset Y \text{ is convex}$$

Closed-valued correspondance

$$\forall x \in X, F(x) \subset Y \text{ is closed}$$

Singleton-valued correspondance

$$\forall x \in X, F(x) \subset Y \text{ is a singleton set } \iff F(x) \text{ is a function } f(x)$$

Continuous function

$$\text{A function } f \text{ is continuous if and only if for any open } \forall x \in X$$

Upper hemi-continuous correspondance

Lower hemi-continuous correspondance

Theorem 10.

F is continuous if and only if F is both upper hemi-continuous and lower hemi-continuous.

4 TOPOLOGY

a topology τ on X is a family of subsets of X satisfying

$$\varnothing, X \in \tau \tag{54}$$
$$\tau \text{ is closed under finite intersections} \tag{55}$$
$$\tau \text{ is closed under unions} \tag{56}$$

Topological space

$$(X, \tau)$$

Open set
a member of τ is an open set
Closed set
the complement of an open set is a closed set
Interior and Closure
let (x, τ) be a topological space and $A \subset X$,interior is the largest open set included by A

$$Int(A)$$

and closure is the smallest closed set including A

$$Cl(A)$$

Dense

$$If\ D \in (X, \tau), Cl(D) = X \Rightarrow D \text{ is dense}$$

Seperable

$$\exists D \in (X, \tau), Cl(D) = X \Rightarrow X \text{ is seperable}$$

Continuous function

In a topological space, $f : X \to Y$ is continuous if $f^{-1}(U) \subset X$ is open whenever $U \subset Y$ is open

Hemeomorphic

$\exists (X, \tau), (Y, \tau)$, if a one-to-one continuous function $f : X \to Y$ such that f^{-1} is continuous

Compact

$$\exists K \subset (X, \tau), \forall \text{ open cover } K \text{ includes a finite subcover } \iff K \text{ is compact}$$

Theorem 11 (Weierstrass theorem)**.** *Every continuous function from topological space to another carries compact sets to compact sets.*

5 LINEAR ALGEBRA

6 FIXED POINT

7 LATITICE AND SUPER-MODULARITY