Notes on Advanced Macroeconomics

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Contents

Preface

Aim of this work of mine is to provide accurate documentation of macroeconomic knowledges in language as simple as possible. Most of this note is taken from the advanced macroeconomics course of Yanfei Deng I am currently taking. This note is still a work in progress and needs tons of ammending. See the newest version at `https://github.com/xolarvill/notes-on-economics` along with my other notes.
This work was first written in markdown language on Obsidian, which provides a lightweight and highly personalized working experience. But it was later transfered into purely latex language on Sublime Text, due to poor support for rendering of long math equation blocks of Obsidian. I wrote some LaTeX snippets of Sublime Text to optimize the writing experience, and uploaded on github at `https://github.com/xolarvill/snippets-for-quick-latex-on-st`.
One can find enormous academical help on advanced macroeconomics from books listed below:

- Introduction to Modern Economic Growth, D. Acemoglu
- Economic Growth, R. Barro and S.i.M. Xavi
- The ABCs of RBCs, An Introduction to Dynamic Macroeconomic Models, G. McCandless
- Macroeconomics, A Comprehensive Textbook for First-Year Ph.D. Courses in Macroeconomics. M. Azzimonti, P. Krusell, A. McKay, and T. Mukoyama
- Adavanced Macroeconomics, D. Romer

Part I
Models of Growth

- Classical model
- Keynesian model
- Solow-Swan model
- Ramse-Cass-Koopsman model
- Overlapping generations model
- (Endogenous model)

1 Classical Model

Learning points for 23-2 Course: the goal is to understand:

- motivations for contruction of a model
- the common approach to analyze a model
- how models evolve over time
- target function and constraint

See Macroeconomic Theory (Sargint, 1987), in a perfectly competitive market, the economic system can be described as:

$$\text{AS:} \begin{cases} Y = A \cdot F(K, N) \text{ (Only N is endogenously chosen; Inada conditions)} \\ \max \pi \Rightarrow \frac{W}{P} = F_N(K, N) \\ N = N(\frac{W}{P}), N' > 0 \text{ (ad hoc, assuming labor market clearing)} \end{cases} \tag{1}$$

$$\text{AD:} \begin{cases} Y = C + I + G + \delta K \\ C = C(Y, T - \pi) \\ I = I(q - 1), I' > 0, q \equiv \frac{F_k - (r + \delta - \pi)}{r - \pi} \\ \frac{M}{P} = m(Y, r), m_1 > 0, m_2 < 0 \end{cases} \tag{2}$$

Note 1 (Exogeneous variables in this model would be)**.**

- In AS: K
- In AD: K, G, M, π, δ, T

At the equilibrium, the endogenous variables could be determined jointly by multiple exo variables. e.g.

$$\text{AS} \Rightarrow \begin{cases} Y^* = f(k) \\ N^* = g(k) \\ (\frac{W}{P})^* = h(k) \end{cases}$$

Question 1 (how to calculate N at the equilibrium?)**.**
By using total derivatives, we could have:

$$\begin{cases} dY = dY(dK) \\ d\frac{W}{P} = d\frac{W}{P}(dK) \\ dW = dW(dK) \\ dP = dP(dK) \end{cases}$$

Note 2. *All a model can contain are:*

- *behavioral function* $\begin{cases} \textit{by ad hoc} \\ \textit{by optimization} \end{cases}$
- *definitive function*
- *equilibrium conditions*

Note 3 (General way of advanced macroeconomics)**.** *By method, there are optimization, equilibrium-conducting and comparative analysis. And all of three can do both static analysis and dynamic analysis.*

Note 4. *Inada conditions making sure that profit maximization is feasible*

2 Keysian Model

In a brief way, a Keysian model can be seen as two parts:

$$\text{AS: } \begin{cases} Y = Y(F, N) \\ \frac{W}{P} = F_N(K, N) \\ N_S = N_s\left(\frac{W}{P}\right) \text{,where in most cases neglected from the equations} \end{cases} \tag{3}$$

$$\text{AD: } \begin{cases} Y = C + I + G + \delta K \\ C = C(Y, T - \pi) \\ I = I(q-1), I' > 0, q \equiv \frac{F_k - (r + \delta - \pi)}{r - \pi} \\ \frac{M}{P} = m(Y, r), m_1 > 0, m_2 < 0 \end{cases} \tag{4}$$

The Keyensian model consists of six equations, instead of seven in classical model. Because labor market is not clearing. So the wage is exogenous.

3 Solow Model

The Solow–Swan model, or exogenous growth model, or Neoclassical model is an economic model of long-run economic growth. It attempts to explain long-run economic growth by looking at capital accumulation, labor or population growth, and increases in productivity largely driven by technological progress. At its core, it is an aggregate production function, often specified to be of Cobb–Douglas type, which enables the model to make contact with microeconomics. The model was developed independently by Robert Solow and Trevor Swan in 1956, and superseded the Keynesian Harrod–Domar model.

> Mathematically, the Solow–Swan model is a nonlinear system consisting of a single ordinary differential equation that models the evolution of the per capita stock of capital.

Learning points for 23-2 Course:

- the economic environment of Solow model
- the main equations in Solow model
- Several key concepts of Solow model, i.e. golden rate, equilibrium dynamics
- how does the model converge under different conditions

3.1 Economic Environment and Specific Assumptions

Solow model seems to have simple assumptions, but in which many things are pre-assumed.
Sections

- A Solow model assumes only two sections in participation

$$Y(t) = C(t) + I(t) = C(t) + S(t) \tag{5}$$

Household

- A large number of non-optimizing homogeneous households (with ad-hoc assumptions) $\Rightarrow$ representative agent
- Constant saving rate of $s \in [0,1] \Rightarrow S(t) = sY(t) = I(t)$

Firm

- A large number of non-optimizing homogeneous firms (with ad-hoc assumptions) $\Rightarrow$ representative agent
- Since sharing a common production method, it can use an aggregate production function

Market structure

- Competitive $\Rightarrow$ Solow model is a prototypical competitive general equilibrium model

Endowment (labor and capital)

- Labor $L(t)$ provided inelastically by households at wage $w(t)$
- Households also own the capital $K(t)$ and rent it to firms at capital rate $R(t)$
- The inital capital is given as $K(0)$
- Capital exponential depreciation is denoted as δ $\Rightarrow$ real interest rate $r(t) = R(t) - \delta$
- How the capital is distributed is irrelevant to our exploring

Market clearing

- Labor market clearing condition is $L(t) = \bar{L}(t)$
- Capital market clearing condition is $K^s(t) = K^d(t)$

Technology

- Technology is free and publicly available

Production function
The aggregate production function is

- continuous
- differentiable
- positive
- diminishing MP
- CRTS

Production function in Harrold form

$$Y_t = F(K_t, A_t L_t) \tag{6}$$

- This format of production function means, in the final, the ratio of capital to production K/Y will be stable.
- It is convient to make A_t times L_t instead of other ways in Solow Model.

> **Note 5** (Production function in Hicks neutrality form)**.**
>
> $$Y_t = A_t F(K_t, L_t)$$

CRTS

$$F_t(cK_t, cL_t) = cF(K_t, L_t),\ \forall c \geqslant 0 \tag{7}$$

- One way to see CRTS as a reasonable requirement is that imagining the scale of economy large enough to cover the potiential benefits of cooperation.
- Another way is imagining all other factors are compared irrelevant than capital, labor and knowledge.
- The CRTS assumption can derive the production function to $y = f(k)$.

Inada Conditions

$$\lim_{x\to 0} y = \lim_{x\to 0} f(x) = \infty \tag{8}$$
$$\lim_{x\to \infty} y = \lim_{x\to \infty} f(x) = 0 \tag{9}$$

- Many have argued that Inada conditions are the key limits of Solow model, as well as the exogenosu tech growth rate.
- The purpose of this setting is to confine the model from wild spreading.
- Cobb-Douglas Function is very suitable in this setting. *It is rather reducted than deducted from the economic nature.*

Note 6 (No profits for firms)**.**

$$\Rightarrow \textit{PMP:} \tag{10}$$
$$\underset{L(t),K(t)}{max} \quad F - w(t)L(t) - R(t)K(t) \tag{11}$$
$$\Rightarrow \begin{cases} w(t) = F_L \\ R(t) = F_K \end{cases} \tag{12}$$
$$\Rightarrow Y(t) = w(t)L(t) + R(t)K(t) \textit{ (applying Euler's equation)} \tag{13}$$
$$\textit{Meaning the firms make no profits. They are just profit-maximizing.} \tag{14}$$

Main equations
All of these above should be considered oversimplication of the real world, but to the favor of an economist.

$$\begin{cases} y_t = F(K_t, A_t N_t) \\ I_t = S_t = sY_t \\ K_{t+1} = I_t + (1-\delta)K_t \end{cases} \tag{15}$$

3.2 Solow model in discrete time

Note 7. *The gradually smaller and smaller effect of exogenous variables on endogenous variable is called an impulse response.*

Note 8. *Even the edge or limit of a dynamic state is no way near the steady state.*

Fundamental law of motion

$$\text{Solow model's motion equations} \begin{cases} Y_t = F(K_t, A_t, L_t) \\ K_{t+1} = (1-\delta)K_t + I_t \end{cases} \tag{16}$$

Question 2 (how to solve this set of difference equations?). *See Gao Xu, 2017 and McCandless, 2008*

$$y_t = \frac{Y_t}{N_t} = A_t \cdot F(\frac{K_t}{N_t}, \frac{N_t}{N_t}) \equiv A_t f(k_t) \tag{17}$$

$$K_{t+1} = (1-\delta)K_t + I_t \tag{18}$$

$$\Rightarrow \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = \frac{(1-\delta)K_t}{N_t} + \frac{I_t}{N_t} \tag{19}$$

$$\Rightarrow k_{t+1} = \frac{(1-\delta)k_t + i_t}{1+n} \tag{20}$$

$$\Rightarrow (1+n)k_{t+1} = (1-\delta)k_t + i_t \text{ (transpose)} \tag{21}$$

$$\Rightarrow (1+n)k_{t+1} = (1-\delta)k_t + s_t \tag{22}$$

$$\Rightarrow (1+n)k_{t+1} = (1-\delta)k_t + s \cdot (1+\alpha)^t A_0 \cdot y_t \text{ (α is the growth rate for tech, in simplest case= 0)} \tag{23}$$

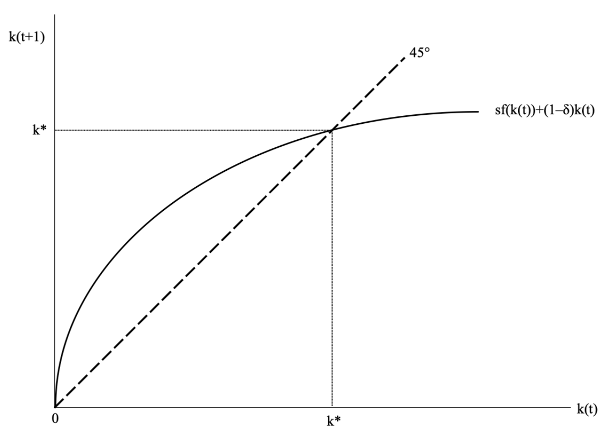
$$\Rightarrow (1+n)k_{t+1} = (1-\delta)k_t + sA_0 y_t \tag{24}$$

It is obvious the steady state of capital per worker can be found at when $k_{t+1} = k_t = k^*$

$$k_{t+1} = g(k_t) = \frac{(1-\delta)k_t + sA_0 f(k_t)}{1+n} \tag{25}$$

Equilibrium steady states in a phase diagram

With A_t and N_t being stable (not have to be constant), the solution of difference equations in discrete Solow model is $k_{t+1} = sf(k_t) + (1-\delta)k_t$, which leads to the phase diagram below



Equilibrium dynamics, solving the k^*

$$\begin{aligned}
k_{t+1} &= sf(k_t) + (1-\delta)k_t && (26)\\
&= g(k_t) && (27)\\
&\Rightarrow \text{eventually that } k^* = g(k^*) && (28)\\
&\Rightarrow \frac{k^*}{f(k^*)} = \frac{s}{n+\delta} \text{ (meaning capital-output ratio is anchored by the parameters)} && (29)\\
&\Rightarrow k^* = \cdot \text{ (solving the real final } k\text{)} && (30)
\end{aligned}$$

Growth rate
The growth rate of capital is denoted as

$$\gamma_t = \frac{k_{t+1}}{k_t} = \frac{(1-\delta)k_t + sA_0 f(k_t)}{(1+n)k_t} \tag{31}$$

Intertemporal equilibrium, convergence speed and convergence time

$$\begin{aligned}
& k_{t+1} = g(k_t) && (32)\\
& \Rightarrow k_{t+1} = k^* + g'(k^*)(k_t - k^*) && (33)\\
& \Rightarrow k_{t+1} = k^* + g'(k)(k_t - k^*) && (34)\\
& \Rightarrow \hat{k_{t+1}} \equiv \frac{k_{t+1} - k^*}{k^*} = g'(k^*)(\frac{k_t - k^*}{k^*}) = g'(k^*)\hat{k_t} \text{ (redefining)} && (35)\\
& \Rightarrow \hat{k_t} = [g'(k^*)]^t \hat{k_0} \text{ (by the method of iteration, the equil steady state of capital growth ratio)} && (36)\\
& \Rightarrow t = \frac{\log \frac{\hat{k_t}}{\hat{k_0}}}{\log g'(k^*)} \text{ (using logarithms)} && (37)\\
& \Rightarrow t = \frac{\log \frac{k_t - k^*}{k_0 - k^*}}{\log g'(k^*)} \text{ (meaning the time } t \text{ is now decided by the location of capital } k_t\text{)} && (38)
\end{aligned}$$

We have the convergence speed g$'(k^*)$ and convergence time t.

> **Note 9** (Steady State vs Intertemporal Equilibrium)**.** *Despite of seemingly different division, the two are sides of a coin.*

Balance of growth

$$\begin{aligned}
k_t &= \frac{K_t}{A_t L_t} && (39)\\
\Rightarrow \frac{dK_t}{k_t} &= \frac{dk_t}{k_t} + \frac{dA_t}{A_t} + \frac{dN_t}{N_t} && (40)
\end{aligned}$$

Convergence under different conditions:

i) first scenerio: $\begin{cases} A_t = a \\ N_t = n \end{cases}$

$$k_t \equiv \frac{K_t}{N} \tag{41}$$
$$y_t \equiv \frac{Y_t}{N} = F[\frac{K_t}{N}, A] = f(k_t) \tag{42}$$
$$k_{t+1} = sf(k_t) + (1-\delta)k_t \tag{43}$$
$$\text{The steady state equilibrium is } k(t) = k^*, \ \forall t \tag{44}$$

ii) second scenerio: $\begin{cases} A_t = a \\ \frac{N_t}{N_{t-1}} = 1 + n \end{cases}$

iii) third scenerio: $\begin{cases} \frac{A_t}{A_{t-1}} = 1 + a \\ \frac{N_t}{N_{t-1}} = 1 + n \end{cases}$

3.3 Solow model in continuous time

Let $A(t) = 1$, we have the Solow model in continous time

$$Y(t) = F[K(t), L(t)] \tag{45}$$
$$I(t) = S(t) = sY(t) \tag{46}$$
$$\dot{K(t)} = I_t - \delta K_t \tag{47}$$

Derivation of the basic function of Solow model in continuous time, which is the steady state of Solow model

$$k(t) = \frac{K(t)}{L(t)} \tag{48}$$
$$\frac{\dot{k(t)}}{k(t)} = \frac{\dot{K(t)}}{K(t)} - \frac{\dot{L(t)}}{L(t)} \tag{49}$$
$$\frac{\dot{k(t)}}{k(t)} = \frac{sF - \delta K(t)}{K(t)} - n \tag{50}$$
$$\frac{\dot{k(t)}}{k(t)} = \frac{(sF - \delta K(t)/L(t))}{K(t)/L(t)} - n \tag{51}$$
$$\Rightarrow \dot{k(t)} = sf[k(t)] - (\delta + n)k(t) \tag{52}$$

Intertemporal equilibrium, convergence speed and convergence time

$$\dot{k(t)} = sf[k(t)] - (\delta + n)k(t) \equiv g[k(t)] \tag{53}$$

$$\Rightarrow \text{eventually at steady state } g(k^*) = 0 = sf[k^*] - (\delta + n)k^* \tag{54}$$

$$\Rightarrow \frac{k^*}{f(k^*)} = \frac{s}{\delta + n} \text{ (the anchored capital-output ratio determined by parameters)} \tag{55}$$

$$\Rightarrow \dot{k(t)} \approx g(k^*) + g'(k^*)[k(t) - k^*] \text{ (linearization by using Taylor expansion)} \tag{56}$$

$$\Rightarrow \frac{dK(t)}{dt} = g'(k^*)k(t) - g'(k^*)k^* \tag{57}$$

$$\Rightarrow k(t) = [k(0) - k^*]e^{[g'(k^*)t]} + k^* \tag{58}$$

$$\Rightarrow \text{transposition} \tag{59}$$

$$\text{Denote } \hat{k(t)} \equiv \frac{k(t) - k^*}{k^*} = e^{[g'(k^*)t]}\frac{k(0) - k^*}{k^*} \tag{60}$$

$$\Rightarrow \frac{\hat{k(t)}}{\hat{k(0)}} = e^{g'(k^*)t} \tag{61}$$

$$\Rightarrow \text{Denote } t = \frac{log[\frac{\hat{k(t)}}{\hat{k(0)}}]}{g'(k^*)} \tag{62}$$

$$\Rightarrow \text{Like in discrete time, we have the convergence speed } g'(k^*) \text{ and time for convergence } t \tag{63}$$

3.4 Golden Rate

One of the philosophies upon which economic science is established is utilitism, that is achieving max happiness by max consuming. To do so in a stationary state in Solow model, the idea of golden rate is introduced.

Golden rate in discrete time, or the maximized welfare of Solow model

$$\text{Starting with two ad-hoc assumptions} \begin{cases} C_t = (1-s)Y_t \\ S_t = sY_t \end{cases} \tag{64}$$

$$\Rightarrow \text{At the steady state:} \tag{65}$$

$$c^*(s) = (1-s)f[k^*(s)] = f[(1-s)k^*(s)] = f[k^*(s)] - (n+\delta)k^*(s) \tag{66}$$

$$\Rightarrow \max c^* \tag{67}$$

Question 3 (why do we even need to introduce money in a model like Solow model?)**.**

4 Ramsey Model

See Chapter 5 to Chapter 7 from Acemoglu, 2004 for more mathematical help.

Note 10. *Ramsey model is in nature a growth model with consumer optmization.*

The Ramsey model, or the RCK model, or the Neoclassical growth model differs from the Solow model in that the choice of consumption is explicitly microfounded at a point in time and so endogenizes the savings rate. As a result, unlike in the Solow model, the saving rate may not be constant along the transition to the long run steady state.

> The Ramsey–Cass–Koopmans model aims only at explaining long-run economic growth rather than business cycle fluctuations, and does not include any sources of disturbances like market imperfections, heterogeneity among households, or exogenous shocks. Subsequent researchers therefore extended the model, allowing for government-purchases shocks, variations in employment, and other sources of disturbances, which is known as *real business cycle theory*.

4.1 Assumptions

4.1.1 Model with inherited properties from Solow Model

Production Function in the Harrold neutrality form
$Y = F(K, AL)$
Exogenous Variables
Population growth rate= n
Technology growth rate= g
Depreciation rate= 0 (for simplicity)
And most of all, the technological growth rate is assumed as a.

Note 11 (Saving rate in Ramsey model)**.** *Ramsey model endogenizes saving rate s, so s is no longer a constant anymore.*

4.1.2 Firm

- Perfect competitive firms, or homogenous firms, or representive firms
- Hire workers and capital in competitive factor markets
- Each has the production function $Y = F(K, AL)$
- Firms are owned by the families, meaning the revenue goes straight to latter

4.1.3 Household

- Homogenous families.
- Population growth of each household is assumed as n.
- Each member in a household provides one unit of labor.
- Income of a household is the sum of the labor revenue subsum, the capital revenue subsum and the firm revenue share subsum.
- A household should and must maximize the utility.

> **Note 12** (Rule of Prohibiting Ponzi Schemes)**.** *Conconstraint in Ramsey model is also called Prohibiting Ponzi Schemes Rule. It forbids the household to do a scam of rolling over debt by repeatedly borrowing. Yet in real life, many people are doing this as a harmless way to ralex debt pressure.*

4.2 Solution of Ramsey model

4.2.1 Central Planner's Problem

Originally Ramsey set out the model as a social planner's problem of maximizing levels of consumption over successive generations. Only later was a model adopted by Cass and Koopmans as a description of a decentralized dynamic economy with a representative agent. This is also why the model is called Ramsey-Cass-Koopsman model, yet abbreviated as Ramsey model.
So, chronologically the first thing we should address is central planner's problem.
Disperssive competition
UMP

$$\begin{aligned} &\max_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) \\ &s.t. C_t + K_{t+1} - (1-\delta)K_t = W_t L_t + R_t k_t + D_t \end{aligned} \tag{68}$$

Noticing that the constraint is equivalent to $C_t + I_t = Y_t + D_t$
Solving this UMP resulting Euler's equation

$$\begin{aligned} &\Rightarrow u'[F(K_t) - K_{t+1}] = u'(C_t) = \beta u'(C_{t+1}) F'(K_{t+1}) \\ &\iff \frac{u'(C_t)}{u'(C_{t+1})} = \beta R_{t+1} \end{aligned} \tag{69}$$

PMP at a perfectly competitive market

$$\underset{K_t}{max}\ \pi = F(K_t, A_t L_t) - (r+\delta)K_t - wL_t \tag{70}$$

where the $A_t L_t$ is assumed as a constant in order for model to converge

$$F.O.C. \Rightarrow r + \delta = R = 1 + r \tag{71}$$

In a conclusion, opt in the disperssive economy yields $\begin{cases} \frac{u'(C_t)}{u'(C_{t+1})} = \beta R_{t+1} \\ r + \delta = R = 1 + r \end{cases}$

Central Planner's Problem

$$\begin{aligned} &\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t) \\ &s.t.\ C_t + K_{t+1} = Y_t,\ \text{where } \delta \text{ is assumed as } 1 \end{aligned} \tag{72}$$

Using langranian method and acquire Euler's equation

$$\begin{aligned} &\Rightarrow u'(F(K_t) - K_{t+1}) = u'(C_t) = \beta u'(C_{t+1}) F'(K+1) \\ &\iff \frac{u'(C_t)}{u'(C_{t+1})} = \beta R_{t+1} \end{aligned} \tag{73}$$

Conclusion Regardless of being disperssive or central planning, we would have the same Euler's equation of consumption.

4.2.2 Method of Lagrangian

We start from scratch, then move on to methods like using Hamiltonian equation or using Bellman equation.
one-period static opt

$$\begin{aligned} &v = f(x_1, x_2) \\ &s.t.\ g(x_1, x_2) = z \end{aligned} \tag{74}$$

two-period dynamic opt
A general form is denoted as

$$\begin{aligned} &\max v = f(x_1, x_2) \\ &s.t.\ g(x_1, x_2) = z \text{ (where the subscript is denoted as time periods)} \end{aligned} \tag{75}$$

a two-period consumer maximization problem can be denoted as

$$\begin{aligned} &\max_{C_1, C_2}\ u_1 = u(C_1) + \beta u(C_2) \\ &s.t.\ \begin{cases} C_1 + S_1 \leqslant [r + (1-\delta)]S_0 + Y_1 \\ C_2 + S_2 \leqslant [r + (1-\delta)]S_1 + Y_2 \end{cases} \text{,where we assume} \begin{cases} S_0 = S_2 = 0 \\ C_1, C_2 > 0 \end{cases} \end{aligned} \tag{76}$$

$$\text{The reconstraints} \Rightarrow C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \text{ (intertemporal budget conconstraint)} \tag{77}$$

Question 4 (How to solve the intertemporal UMP?).
(1). method of elimination
$\frac{du_1}{dC_1} = 0 \Rightarrow \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r}$ (the Euler's equation of intertemporal consumption) In which the $\frac{1}{1+r}$ is the comparative price

(2). method of geomotry
On a $C_1 - C_2$ plane,

$$0 = u'(C_1)dC_1 + \beta u'(C_2)dC_2 \tag{78}$$

$$\Rightarrow \frac{dC_2}{dC_1} = -\frac{u'(C_1)}{\beta u'(C_2)} \tag{79}$$

(3). method of Lagrangian function

$$\mathscr{L} = u(C_1) + \beta u(C_2) + \lambda[C_1 + \frac{C_2}{1+r} - Y_1 - \frac{Y_2}{1+r}] \tag{80}$$

$$\Rightarrow \begin{cases} \frac{\partial L}{\partial C_1} = 0 \\ \frac{\partial L}{\partial C_2} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \tag{81}$$

$$\Rightarrow \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r} \text{ (in the difference equation lies the core of Ramsey model)} \tag{82}$$

t-period dynamic opt (finite-period)
1) The original version:

$$\max_{C_1, C_2, ..., C(T)} u_1 = u(C_1) + \beta u(C_2) + \beta^2 u(C_3) + ... + \beta^{T-1} u(C_T)$$
$$s.t. \begin{cases} C_1 + S_1 & \leqslant [1+r]S_0 + Y_1 \\ C_2 + S_2 & \leqslant [1+r]S_1 + Y_2 \\ & \dots \\ C_T + s_T & \leqslant (1+r)S_{T-1} + Y_T \end{cases} \text{,where we assume} \begin{cases} \delta = s_0 = s_T = 0 \\ C_1, C_2, ..., C_T > 0 \end{cases} \tag{83}$$

where the budget constraints can be condensed into

$$C_1 + \frac{C_2}{1+r} + ... + \frac{C_T}{(1+r)^{T-1}} = Y_1 + \frac{Y_2}{1+r} + ... + \frac{Y_T}{(1+r)^{T-1}} \tag{84}$$

2) Or the second version:

$$\max_{\{C_t\}_{t=1}^{\infty}} u_1 = \sum_{t=1}^{T} \beta^{t-1} u(C_t)$$
$$s.t. \begin{cases} C_1 + S_1 & \leqslant [1+r]S_0 + Y_1 \\ C_2 + S_2 & \leqslant [1+r]S_1 + Y_2 \\ & \dots \\ C_T + S_T & \leqslant (1+r)S_{T-1} + Y_T \end{cases} \text{,where we assume} \begin{cases} S_0 = S_T = 0, \delta = 0 \\ C_1, C_2, ..., C_T > 0 \end{cases} \tag{85}$$

To solve this t-period UMP, using Langranian method: (the λ is the shadow price)

$$\max_{\{C_t\}_{t=1}^{\infty},\lambda} \mathscr{L} = \sum_{t=1}^{T} \beta^{t-1} u(C_t) + \lambda[c_1 + \frac{c_2}{1+r} + ... + \frac{C_T}{(1+r)^{T-1}} - (Y_1 + \frac{Y_2}{1+r} + ... + \frac{Y_T}{(1+r)^{T-1}})] \tag{86}$$

$$\Rightarrow F.O.C.s \begin{cases} \frac{\partial \mathscr{L}}{\partial c_1} = 0 \\ \frac{\partial \mathscr{L}}{\partial C_2} = 0 \\ \vdots \\ \frac{\partial \mathscr{L}}{\partial C_T} = 0 \end{cases} \tag{87}$$

$$\Rightarrow \frac{\beta u'(C_{t+1})}{u'(C_t)} = \frac{1}{1+r}, t = 1, 2 \ldots, T \tag{88}$$

3) Another commonly adopted simplified version is

$$\begin{aligned} &\max_{\{C_t, S_t\}_{t=1}^{T}} u_1 = \sum_{t=1}^{T} \beta^{T-1} u(C_t) \\ &s.t. C_t + S_t \leqslant (1+r) S_{t-1} + Y_t, \text{ where} \begin{cases} C_t > 0 \\ S_0 = S_T = 0 \end{cases} \end{aligned} \tag{89}$$

Using Lagrangian function method:

$$\Rightarrow u'(C_{t+1}) = \lambda_{t+1} \tag{90}$$

$$\Rightarrow \frac{u'(C_{t+1})}{u'(C_t)} = \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{\beta(1+r)} \tag{91}$$

$$\Rightarrow \frac{\beta u'(C_{t+1})}{u'(C_t)} = \frac{1}{1+r} \tag{92}$$

infinite-period dynamic opt
Using the third version, we delve into infinity

$$\begin{aligned} &\max_{\{C_t\}_{t=1}^{\infty}} u_1 = \sum_{t=1}^{\infty} \beta^{T-1} u(C_t) \\ &s.t. C_t + S_t \leqslant (1+r) S_{t-1} + Y_t, \text{ where} \begin{cases} C_t > 0 \\ S_0 = \lim\limits_{t \to \infty} S_T = 0 \end{cases} \end{aligned} \tag{93}$$

Using Lagrangian function method:

$$\mathscr{L} \equiv \sum_{t=0}^{\infty} \{\beta^t u(c_t) + \lambda_t [(1+r) s_{t-1} + Y_t - (c_t + s_t)]\} \tag{94}$$

$$\Rightarrow F.O.C. \tag{95}$$

$$\Rightarrow u'(c_{t+1}) = \lambda_{t+1} \tag{96}$$

$$\Rightarrow \frac{u'(c_{t+1})}{u'(c_t)} = \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{\beta(1+r)} \tag{97}$$

$$\Rightarrow \frac{\beta u'(c_{t+1})}{u'(t)} = \frac{1}{1+r} \tag{98}$$

4.2.3 Method of Hamiltonian Equation (Optimal Control)

Method of Hamiltonian equation is part of the optimal control theory from mathematicals.
a) discrete time

$$\max_{\{c_t,s_t\}_{t=0}^{T}} u_1 = \sum_{t=1}^{T} \beta^{T-1} u(c_t)$$
$$s.t. c_t + s_t \leqslant (1+r)s_{t-1} + Y_t, \text{ where } \begin{cases} c_t > 0 \\ s_{-1} = s_T = 0 \\ \beta = \frac{1}{1+\rho} \end{cases} \tag{99}$$

b) continuous time without patience

$$\begin{aligned} &\max_{\{C(t),S(t)\}} u = \int_{t=0}^{T} e^{-\rho t} u[C(t)]dt = \int_{t=0}^{T} u[C(t)]dt \\ &s.t.\ C(t) + \dot{S}(t) = r \cdot S(t) + Y(t), \text{ where } S(0) = S(T) = 0 \end{aligned} \tag{100}$$

Using Lagrangian

$$\Rightarrow \mathscr{L} = \int_{t=0}^{T} \{u[C(t)] + \lambda(t)[C(t) + \dot{S}(t) - r \cdot S(t) - Y(t)]\}dt \tag{101}$$
$$= \int_{t=0}^{T} \{H(t) - \lambda(t)\dot{S}(t)\}dt\ (H \text{ is Hamiltonian}) \tag{102}$$
$$= \int_{t=0}^{T} H(t)dt - \int_{t=0}^{T} \lambda(t)\dot{S}(t)dt \text{ (Then applying fractional integral method to latter)} \tag{103}$$
$$= \int_{t=0}^{T} H(t)dt - \{\lambda(t)S(t)|_0^T - \int_{t=0}^{T} \dot{\lambda}(t)S(t)dt\} \tag{104}$$
$$= \int_{t=0}^{T} H(t)dt - \dot{\lambda}(t)S(t)dt - [\lambda(t)S(t)]|_0^T \tag{105}$$
$$= \int_{t=0}^{T} H(t)dt + \dot{\lambda}(t)dt \tag{106}$$

with this simplified version containing Hamiltonian

$$\Rightarrow \text{F.O.C. of Lagrangian function:} \begin{cases} \frac{\partial \mathscr{L}}{\partial C(t)} = 0\ (C(t) \text{ is the control variable}) \\ \frac{\partial \mathscr{L}}{\partial S(t)} = 0\ (S(t) \text{ is the state variable}) \\ \frac{\partial \mathscr{L}}{\partial \lambda(t)} = 0\ (\lambda(t) \text{ is the monodromy variable}) \end{cases} \tag{107}$$

$$\Rightarrow \text{the optimal control} \begin{cases} \frac{\partial \mathscr{H}}{\partial C(t)} = 0 \\ \frac{\partial \mathscr{H}}{\partial S(t)} = -\dot{\lambda}(t) \\ \frac{\partial \mathscr{H}}{\partial \lambda(t)} = \dot{S}(t) \end{cases} \tag{108}$$

Note 13 (control variable and state variable)**.** *If a variable is determined in the current period, it is a control variable. If it is determined period(s) before, it is a state variable.*

c) continuous time with patience
Next we consider the Lagrangian function with subjective discount rate
Right into this version's Lagrangian function

$$\mathscr{L} = \int_{t=0}^{T} \{e^{-\rho t} u[C(t)] + \lambda(t)[C(t) + \dot{S}(t) - r \cdot S(t) - Y(t)]\} dt \tag{109}$$

$$= \int_{t=0}^{T} \{e^{-\rho t} u[C(t)] + \lambda(t)[rS(t) + Y(t) - C(t)] - \lambda \dot{S}(t)\} dt \tag{110}$$

$$= \int_{t=0}^{T} \{H(0) - \lambda \dot{S}(t)\} dt \text{ } (H(0) \text{ is present value Hamiltonian}) \tag{111}$$

$$\Rightarrow H(0) \equiv e^{-\rho t} u[C(t)] + \lambda(t)[rS(t) + Y(t) - C(t)] \tag{112}$$

$$\Rightarrow \tilde{H}(t) = e^{\rho t} H(0) \text{ (the current value Hamiltonian)} \tag{113}$$

$$\Rightarrow H(t) = u[C(t)] + \eta(t)[rS(t) + Y(t) - C(t)] \tag{114}$$

Comparing the two scenarios (with and without consideration for ρ), we can see — regardless of the subjective discount rate, result would be the same.

4.2.4 Method of Bellman Equation (Dynamic Programming)

a) discrete time
UMP as below

$$\begin{aligned} &\max_{\{C_t, S_t\}_{t=0}^{\infty}} u_0 = \beta^t u(C_t) \\ &s.t. \ C_t + S_t = (1+r)S_{t-1} + Y_t, \text{ where } S_1 = \lim_{t\to\infty} s_t = 0 \end{aligned} \tag{115}$$

Using the budget constraint to rewrite utility function

$$\Rightarrow C_t = (1+r)S_{t-1} + Y_t - S_t \tag{116}$$

$$\Rightarrow u_0 = \max_{\{S_t\}_{t=0}^{\infty}} \beta^t u[(1+r)S_{t-1} + Y_t - S_t] \tag{117}$$

$$= \max_{S_0} \beta^t u[(1+r)S_{-1} + Y_0 - S_0] + \beta \max_{\{S_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u[(1+r)S_{t-1} + Y_t - S_t] \tag{118}$$

$$= \max_{s_0} u[(1+r)S_{-1} + Y_0 - S_0] + \beta u(S_0, Y_1) \tag{119}$$

Upscript o denotes optimal

$$\Rightarrow u^o(S_{t-1}, Y_t) = \max_{S_t} \{u[(1+r)S_{t-1} + Y_t - S_t] + \beta u^o(S_t, Y_{t+1})\} \text{ (Bellman equation)} \tag{120}$$

$$= \max_{S_t} \{u(C_t) + \beta u^o(S_t, Y_{t+1})\} \tag{121}$$

$$\Rightarrow \text{F.O.C.}: u_{C_t} \frac{\partial C_t}{\partial S_t} + \beta \frac{du(S_t, Y_{t+1})}{dS_t} = 0 \tag{122}$$

Let $\frac{\partial C_t}{\partial S_t}$ be -1

$$\Rightarrow \beta \frac{du(s_t, Y_{t+1})}{ds_t} = u_{c_t} \tag{123}$$

Move one period ahead and acquire Euler's equation

$$
\begin{aligned}
& \qquad (124)\\
u(s_t, Y_{t+1}) &= \max_{s_{t+1}} \{u[(1+r)s_t + Y_t - s_{t+1}] + \beta u(s_{t+1}, Y_{t+2})\} \qquad (125)\\
\Rightarrow \frac{du^o(s_t, Y_{t+1})}{ds_t} &= \frac{\partial u(\cdot)}{\partial s_t} + \frac{\partial u(\cdot)}{\partial s_{t+1}}\frac{ds_{t+1}}{ds_t} + \beta\frac{du^o(\cdot)}{ds_t}\frac{ds_{t+1}}{ds_t} \qquad (126)\\
&= \frac{\partial u(\cdot)}{\partial s_t} + [\frac{\partial u(\cdot)}{\partial s_{t+1}} + \beta\frac{du^o(\cdot)}{d_{s_{t+1}}}]\frac{ds_{t+1}}{ds_t} \text{ (using optimals' envelope therome)} \qquad (127)\\
&= u_{c,t+1}\frac{dc_{t+1}}{ds_t} + 0 \qquad (128)\\
&= u_{c,t+1} \cdot (1+r) \qquad (129)\\
\Rightarrow \beta\frac{u_{c,t+1}}{u_{c,t}} &= \frac{1}{1+r} \qquad (130)
\end{aligned}
$$

Hence the same result as before.
b) continuous time (Hamilton-Jacob-Bellman, HJB)
See in ...

4.3 Intertemporal Equilibrium of Ramsey model

4.3.1 Dynamics of c

Recalling that

$$\frac{\dot{c(t)}}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta} \qquad (131)$$

Since we know

$$r(t) = f'[k(t)] \qquad (132)$$

Combining them as

$$\frac{\dot{c(t)}}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta} = \frac{f'[k(t)] - \rho - \theta g}{\theta} \qquad (133)$$

Once we assume that

$$\exists k^* \in \mathbb{R}, f'[k^*] = \rho - \theta g \qquad (134)$$

The dynamics of c, denoted as $\frac{\dot{c(t)}}{c(t)}$ is purely dependent on the value of k according to Ramsey model.

4.3.2 Dynamics of k

Since in Ramsey model we don't assume depreciation of capital k, denoted as $\delta = 0$. The dynamics of k accordingly is $\dot{k}(t) = f[k(t)] - c(t) - (n+g)k(t)$.
This way, in order for the dynamics of k returning to 0, the model must fulfil the requirement of $c(t) = f[k(t)] - (n+g)k(t)$.

4.3.3 Phase diagram and dynamic efficiency

Combining the the two dynamics together, we would have

$$\dot{k} = 0 \tag{135}$$

$$\dot{c} = 0 \tag{136}$$

in a coordinate system of k and c.

> **Note 14** (dynamic efficiency in Ramsey model). *Beware that the equilibrium of two dynamics would always be on the left of golden rate of $\dot{k}$. The origin point $(0, 0)$ can also be considered a special case of equilibrium.*

The equilibrium in Ramsey model has given only one equilibrium path.
Recalling the Solow presented as model equation of motion

$$k_{t+1} - (1-\delta)k_t = sF(k_t) \tag{137}$$

RCK model has two equations of motion

$$\begin{cases} \frac{u'(c_t)}{u'(c_{t+1})} = \beta[F'(k_{t+1}) - (1-\delta)] \\ k_{t+1} - (1 - \delta k_t) = F(k_t) - C_t \end{cases} \tag{138}$$

Acquirring the steady state of Ramsey model

$$\begin{cases} k_{t+1} = k_t = k^*, \triangle k = 0 \\ c_{t+1} = c_t = c^*, \triangle c = 0 \end{cases} \tag{139}$$

$$\Rightarrow F'(k^*) = \rho + \delta \Rightarrow k^* \tag{140}$$

$$\Rightarrow c^* = F(k^*) - \delta k^* \tag{141}$$

$$\Rightarrow (k^*, c^*) \text{ is the steady state} \tag{142}$$

$$\Rightarrow \frac{dc^*}{dk^*} = F'(k) - \delta = R - \delta \tag{143}$$

Deriving the derivatives at steady state

$$\begin{cases} \frac{\partial \triangle c}{\partial k} < 0 \\ \frac{\partial \triangle k}{\partial c} < 0 \end{cases} \tag{144}$$

, meaning there must be seperating line for each

$$\begin{aligned}
&\triangle\, c = 0 && (145)\\
&\Rightarrow F'(k^*) = \rho + \delta \text{ (from the first equation of motion of RCK)} && (146)\\
&\Rightarrow F_{kk}dk = 0dc && (147)\\
&\Rightarrow \frac{dc}{dk} = \frac{F_{kk}}{0} \Rightarrow \text{the seperating line is vertical in C-K plane} && (148)\\
&\triangle\, k = 0 && (149)\\
&\Rightarrow C^* = F(k^*) - \delta k^* && (150)\\
&\Rightarrow \frac{dc}{dk} = F_k - \delta \Rightarrow \text{the seperating line is vertically reversed U shape} && (151)
\end{aligned}$$

4.4 Linearization

Comparing Solow and Ramsey:
Solow model's Euler equation

$$\dot{k}(t) + \delta k(t) = sF(k(t)) \tag{152}$$

Ramsey model's Euler equations

$$\dot{c}(t) = \frac{F'[k(t) - (\delta + \rho)]}{\theta}c(t), \text{ where } \theta = -\frac{cu''(c)}{u'(c)} \tag{153}$$

$$\dot{k}(t) = F[k(t)] - \delta k(t) - c(t) \tag{154}$$

Noticing the two equations of Ramsey are unlinear. This demands a process of linearization.
Linearization, method of Taylor expansion
First method we use the same technique from Solow model's linearization, Taylor's Expansion.
First order Taylor expansion of RCK' Euler equations:

$$\begin{bmatrix}\dot{c}(t)\\ \dot{k}(t)\end{bmatrix} = \begin{bmatrix}\frac{F'[k(t)-(\delta+\rho)]}{\theta}c(t)\\ F[k(t)] - \delta k(t) - c(t)\end{bmatrix} = \begin{bmatrix}\phi^1[c(t), k(t)]\\ \phi^2[c(t), k(t)]\end{bmatrix} \tag{155}$$

$$\Rightarrow \begin{bmatrix}\dot{c}(t)\\ \dot{k}(t)\end{bmatrix} = \begin{bmatrix}\phi^1[c^*, k^*]\\ \phi^2[c^*, k^*]\end{bmatrix} + \begin{bmatrix}\phi^1_c & \phi^1_k\\ \phi^2_c & \phi^2_k\end{bmatrix}|_{c^*,k^*} \begin{bmatrix}c(t) - c^*\\ k(t) - k^*\end{bmatrix} \tag{156}$$

$$= \begin{bmatrix}0\\ 0\end{bmatrix} + \begin{bmatrix}o & \frac{F''(k^*)}{\theta}c^*\\ -1 & \rho\end{bmatrix}\begin{bmatrix}c(t) - c^*\\ k(t) - k^*\end{bmatrix} \tag{157}$$

The above deduction use the following conclusions

$$\begin{cases}F'(k^*) - (\delta + \rho) = 0 \Rightarrow F'(k^*) = \rho + \delta \Rightarrow k^*\\ c^* = F(k^*) - \delta k^* \Rightarrow c^*\end{cases} \tag{158}$$

Because ...

$$
\begin{aligned}
& Ax = \lambda x && (159)\\
& \Rightarrow \lambda^2 - tr(A)\lambda + dt + \lambda = 0 && (160)\\
& \Rightarrow \lambda^2 - \rho\lambda + \frac{F'(k^*)}{\theta} \cdot c^* = 0 && (161)\\
& \Rightarrow \lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4F''(k^*)c^*}}{\theta} < 0 < \lambda_2 = \frac{\rho + \sqrt{\rho^2 - 4F''(k^*)c^*}}{\theta} && (162)
\end{aligned}
$$

$\Rightarrow$ We choose the negative one which is λ_1 so that $k(t)$ can converge over time

$$
\begin{aligned}
& \Rightarrow \dot{k}(t) = \lambda_1[k(t) - k^*] && (163)\\
& \Rightarrow k(t) = k^* + e^{\lambda_1 t}[k(0) - k^*] && (164)
\end{aligned}
$$

Linearization, method of undetermined coefficients
The second approach is method of undetermined coefficients

$$
\begin{aligned}
& \dot{c}(t) = \frac{F''(k^*)}{\theta} c^* = 0 && (165)\\
& \dot{k}(t) = \rho[k(t) - k^*] - [c(t) - c^*] && (166)\\
& \Rightarrow \ddot{k}(t) = \rho\dot{k}(t) - \dot{c}(t) && (167)\\
& = \rho\dot{k}(t) - \frac{F''(k^*)}{\theta} \cdot c^*[k(t) - k^*] && (168)
\end{aligned}
$$

We can make an assumption that $\dot{k}(t) = \lambda[k(t) - k^*]$, which is a differential equation in nature

$$
\begin{aligned}
& \Rightarrow \ddot{k}(t) = \lambda\dot{k}(t) && (169)\\
& = \lambda^2[k(t) - k^*] && (170)\\
& = \rho\dot{k}(t) - \frac{F''(k^*)c^*[k(t) - k^*]}{\theta} && (171)\\
& = \lambda^2 - \rho\lambda + \frac{F''(k^*)c^*[k(t) - k^*]}{\theta} = 0 && (172)\\
& \Rightarrow \lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4F''(k^*)c^*}}{\theta} < 0 < \lambda_2 = \frac{\rho + \sqrt{\rho^2 - 4F''(k^*)c^*}}{\theta} && (173)
\end{aligned}
$$

Samewise

$$
\Rightarrow k(t) = k^* + e^{\lambda_1 t}[k(0) - k^*] \qquad (174)
$$

Note 15 (convergence in Ramsey model)**.** *Recall Solow model's chapter of convergence speed and time, we can compare the two similar results:*

$$
\begin{cases}
\textit{Ramsey: } k(t) = k^* + e^{\lambda_1 t}[k(0) - k^*] \\
\textit{Solow: } k(t) = k^* + e^{[g'(k^*)t]}[k(0) - k^*]
\end{cases} \qquad (175)
$$

Meaning the λ we are trying to derive here is the convergence speed.

4.5 Maximizing welfare

Similiar to the concept of golden rate in Solow model, Ramsey model also provide a glimpse of maximized consumption.
At the steady state

$$c^* = F(k^*) - \delta k^* \tag{176}$$

$$\Rightarrow k_{gold} = \underset{k}{\operatorname{argmax}}\ c^* \tag{177}$$

5 OLG Model

In contrast to the Ramsey model in which individuals are infinitely-lived, in the OLG model individuals live a finite length of time, long enough to overlap with at least one period of another agent's life.

> **Note 16** (Meet only once)**.** *Two generation OLG is particularly simple because it precludes intertemporal trade (no one meets twice).*

Solow model, RCK model and OLG model

- All of them are under the assumption of perfect competitive market.
- Solow model has exogenous saving rate, whereas Ramsey model and OLG model has endogenous one.
- OLG model has heterogeneous representative agents.

5.1 Assumptions

$$C_t = L_t C_{1t}^{\text{young}} + L_{t-1} C_{2t}^{\text{old}} \text{ (where subscript 1 and 2 stands for age)} \tag{178}$$

$$L_t = (1+n)L_{t-1} \Rightarrow L_t = (1+n)^t L_0 \tag{179}$$

$$S_t = s_t L(t) \tag{180}$$

5.2 UMP of household

A general UMP would be

$$\max u(C_1) + \beta u(C_{2,t+1})$$
$$s.t. \begin{cases} C_{1t} + S_t \leqslant W_t \\ C_{2,t+1} \leqslant R_{t+1} S_t \end{cases} \tag{181}$$

Form a Lagrangian and acquire Euler's equation

$$\mathscr{L} = u(C_1) + \beta u(C_{2,t+1}) + \lambda_{1t}(c_{1t} + s_t - w_t) + \lambda_{2,t+1}(c_{2,t+1} - R_{t+1}s_t) \tag{182}$$

$$\Rightarrow \text{F.O.C.} \begin{cases} \frac{\partial \mathscr{L}}{\partial C_{1t}} = 0 \\ \frac{\partial \mathscr{L}}{\partial C_{2,t+1}} = 0 \\ \frac{\partial \mathscr{L}}{\partial S_t} = 0 \end{cases} \Rightarrow \begin{cases} u'(C_{1t}) = \lambda_{1t} \\ \beta' u'(C_{2,t+1}) = \lambda_{2,t+1} \\ \frac{\lambda_{1t}}{\lambda_{2,t+1}} = R_{t+1} \end{cases} \tag{183}$$

$$\Rightarrow u'(C_{1t}) = \beta R_{t+1} u'(C_{2,t+1}) \text{ (Euler's equation in OLG UMP)} \tag{184}$$

e.g. Given a specific utillity function

$$u(C_{gt}) = \frac{C_{gt}^{1-\theta} - 1}{1-\theta}, 0 < \theta \neq 1, g = 1, 2 \tag{185}$$

$$\text{UMP} \Rightarrow \begin{cases} C_{1t}^{-\theta} = \beta R_{t+1} C_{2,t+1}^{-\theta} \\ C_{1t} = W_t - S_t \\ C_{2,t+1} = R_{t+1} S_t \end{cases} \tag{186}$$

$$\Rightarrow (W_t - S_t)^{-\theta} = \beta R_{t+1} (R_{t+1} S_t)^{-\theta} \tag{187}$$

$$\Rightarrow (\frac{W_t - S_t}{S_t})^{-\theta} = \beta R_{t+1}^{1-\theta} \tag{188}$$

$$\Rightarrow \frac{W_t}{S_t} = (\beta R_{t+1}^{1-\theta})^{-\frac{1}{\theta}} + 1 \tag{189}$$

$$\Rightarrow S_t = \frac{w_t}{\beta^{-\frac{1}{\theta}} R_{t+1}^{-\frac{1-\theta}{\theta}} + 1} < w_t \tag{190}$$

$$\Rightarrow S'_w = \frac{1}{\beta^{-\frac{1}{\theta}} R_{t+1}^{-\frac{1-\theta}{\theta}} + 1} \in (0, 1) \tag{191}$$

$$\Rightarrow S'_R = \frac{1-\theta}{\theta} \frac{S_t}{\beta^{-\frac{1}{\theta}} R_{t+1}^{-\frac{1-\theta}{\theta}} + 1} (\beta R_{t+1})^{-\frac{1}{\theta}} \tag{192}$$

$$\Rightarrow S'_R \begin{cases} > 0, 0 < \theta < 1 \\ = 0, \theta = 1 \\ < 0, \theta > 1 \end{cases} \tag{193}$$

5.3 PMP of firm

A general PMP would be

$$\max_{k_t, L_t} \pi_t = Y_t - w_t L_t - R_t K_t \tag{194}$$

$$\text{Since } \pi_t = \frac{\Pi_t}{L_t}, y_t = \frac{Y_t}{L_t}, k_t = \frac{K_t}{L_t}, \frac{K_{t+1}}{L_t} = \frac{K_{t+1}}{L_{t+1}} \frac{L_{t+1}}{L_t}, s_t = \frac{S_t}{L_t} \tag{195}$$

Rearrange in form of per capita

$$\Rightarrow \max_{K_t} \pi_t = Y_t - W_t - R_t K_t \tag{196}$$

$$\Rightarrow \text{F.O.C.} \begin{cases} R_t = f'(K_t) \\ W_t = f(K_t) - K_t f'(K_t) \end{cases} \tag{197}$$

5.4 Dynamics

Looking at the motion equation of capital

$$K_{t+1} - (1-\delta)K_t = I_t = S_t = s_t L_t = s(w_t, R_{t+1})L_t \tag{198}$$

$$\Rightarrow K_{t+1} = \frac{s(w_t, R_{t+1})}{1+n} \tag{199}$$

$$\Rightarrow K_{t+1} = \frac{w_t}{(\beta^{-\frac{1}{\theta}} R_{t+1}^{-\frac{1-\theta}{\theta}} + 1)(1+n)} \text{ (using conditions from UMP and PMP)} \tag{200}$$

$$\Rightarrow K_{t+1} = \frac{f(K_t) - K_t f'(K_t)}{(\beta^{-\frac{1}{\theta}} R_{t+1}^{-\frac{1-\theta}{\theta}} + 1)(1+n)} \text{ (equation of motion of capital in OLG)} \tag{201}$$

The steady states (intertemporal equilibrium) would be

$$k_{t+1} = k_t = k^* \tag{202}$$

$$\Rightarrow k^* = \frac{s[f(k_t^*) - k_t^* f'(k_t^*), f'(k^*)]}{1+n} \tag{203}$$

$$= \frac{f(k_t^*) - k_t^* f'(k_t^*)}{(\beta^{-\frac{1}{\theta}} f'(^*)^{-\frac{1-\theta}{\theta}} + 1)(1+n)} \tag{204}$$

$$\Rightarrow k_{t+1} = \frac{(1-\alpha)(k^*)^\alpha}{(\beta^{-\frac{1}{\theta}}[\alpha(k^*)^{\alpha-1}]^{-\frac{1-\theta}{\theta}} + 1)(1+n)} \text{ (capital per capita at steady state in OLG)} \tag{205}$$

$$\Rightarrow k^* = \cdot \text{ (thus we have the } k \text{ at steady state)} \tag{206}$$

e.g., For a utillity function $u = \log c_{1t} + \beta \log c_{2,t+1}$, where $\theta = 1$.

$$\Rightarrow \frac{c_{2,t+1}}{c_{1t}} = \beta k_{t+1} \tag{207}$$

$$\Rightarrow s_t = \frac{w_t}{\beta^{-\frac{1}{\theta}} R_{t+1}^{-\frac{1-\theta}{\theta}} + 1} \tag{208}$$

$$= \frac{w_t}{\beta^{-1} + 1} \tag{209}$$

$$= \frac{\beta}{1+\beta} w_t \text{ (meaning the saving is a function of wage, instead of endowment)} \tag{210}$$

$$\Rightarrow k_{t+1} = \frac{s_t}{1+n} = \frac{\beta w_t}{(1+\beta)(1+n)} \tag{211}$$

$$= \frac{\beta(1-\alpha k_t^\alpha)}{(1+\beta)(1+n)} \tag{212}$$

$$\text{steady state demands } k_{t+1} = k_t = k^* \Rightarrow k^* = [\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}]^{\frac{1}{1-\alpha}} \tag{213}$$

5.5 Comparing three models so far

$$\text{Solow}\begin{cases} L_t = (1+n)L_{t-1} \\ \frac{S_t}{L_t} = s_t = sy_t = s\frac{Y_t}{L_t} \\ k_{t+1} = \frac{(1-\delta)k_t + sf(k_t)}{1+n} \end{cases} \tag{214}$$

$$\text{OLG}\begin{cases} L_t = (1+n)L_{t-1} \\ k_{t+1} = \frac{s(w_t, R_{t+1})}{1+n} = \frac{s[f(k_t) - k_t f'(k_t), f'(k_{t+1})]}{1+n} \end{cases} \tag{215}$$

$$\text{Ramsey}\begin{cases} L_t = (1+n)L_{t-1} \\ k_{t+1} = \frac{f(k_t)+c_t}{1+n} \Rightarrow 0 = F(k^*) - c^* - \rho k^* \Rightarrow F'(k^*_G) = \delta \\ \frac{u'(c_t)}{u'(c_{t+1})} = \beta[f'(k_{t+1}) + (1-\delta)] \Rightarrow 1 = \rho[F'(k^*) + (1-\delta)] \Rightarrow F'(k^*) = \rho + \delta \Rightarrow k^*_G > k^* \end{cases} \tag{216}$$

5.6 Dynamic inefficiency

Because of the micro-foundations (patience denoted as $\beta = \frac{1}{1+\rho}$) in Ramsey, there will be no over-accumulation of capital, avoiding dynamic inefficiency. Whereas in Solow model the inefficiency is inevitable. Somehow in OLG model, the dynamic inefficiency returns.

Solow

Ramsey

Ramsey has no dynamic inefficiency, here's why

$$\begin{cases} L_t = (1+n)L_{t-1} \\ k_{t+1} = \frac{f(k_t)+c_t}{1+n} \Rightarrow 0 = F(k^*) - c^* - \rho k^* \Rightarrow F'(k^*_G) = \delta \\ \frac{u'(c_t)}{u'(c_{t+1})} = \beta[f'(k_{t+1}) + (1-\delta)] \Rightarrow 1 = \rho[F'(k^*) + (1-\delta)] \Rightarrow F'(k^*) = \rho + \delta \Rightarrow k^*_G > k^* \end{cases} \tag{217}$$

OLG

OLG in central planner form:

$$\max \sum_{t=0}^{\infty} (\beta_s)^t L_t[u(c_{1t}) + \beta u(c_{2,t+1})] \text{ (β_s is the discount factor)}$$

$$s.t. \begin{cases} L_t C_{1t} + L_{t-1} C_{2,t} + K_{t+1} = C_t + K_{t+1} = F(K_t, L_t) \text{ (liquidity budget conconstraint)} \\ \Rightarrow C_{1t} + \frac{L_{t-1}C_{2,t}}{L_t} + \frac{K_{t+1}}{L_{t+1}}\frac{L_{t+1}}{L_t} = \frac{F(K_t, L_t)}{L_t} \text{ (rewrite in form of per capita)} \\ \Rightarrow c_{1t} + \frac{c_{2t}}{1+n} + (1+n)k_{t+1} = f(k_t) \end{cases} \tag{218}$$

Form a Lagrangian

$$\mathscr{L} = \sum_{t=0}^{\infty} [\beta_s^t (1+n)]^t L_t[u(c_{1t}) + \beta u(c_{2,t+1})] + \lambda_t[c_{1t} + \frac{c_{2t}}{1+n} + (1+n)k_{t+1} - f(k_t)] \tag{219}$$

Acquire two Euler's equations:

$$\begin{cases} \frac{u'(c_{1t})}{u'(c_{2,t+1})} = \beta R_{t+1} \\ \frac{u'(c_{1t})}{u'(c_{2t})} = \frac{\beta(1+n)}{\beta_s} \text{ (due to central planner)} \end{cases} \tag{220}$$

At the steady state we know

$$c^* \equiv c_1^* + \frac{c_2^*}{1+n} = f(k^*) - (1+n)k^* \tag{221}$$

Let $\frac{\partial c^*}{\partial k^*} = 0$

$$\Rightarrow f'(k_G^*) = 1+n \tag{222}$$

$$f'(k^*) = \alpha(k^*)^{\alpha-1} = \alpha\{[\frac{\beta(1-\alpha)}{(1+\beta)(1+n)}]^{\frac{1}{1-\alpha}}\}^{\alpha-1} \tag{223}$$

$$= \frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}(1+n) < 1+n = f'(k_G^*) \tag{224}$$

$$\Rightarrow k^* > k_G^* \tag{225}$$

Meaning there will be over-accumulation of capital $\iff$ existence of dynamic inefficiency in OLG model.

Part II

Models of Fluctuation

- Real Business Cycle model
- Dynamic New Keynesian model

6 RBC Model

A standard RBC model is in nature a Ramsey model with

- Endogenous (but constant in the long run, i.e. $n = 0$) labor supply
- Stochastic productivity (i.e. real shocks)

> **Note 17** (Two principles of the RBC theory of business cycles)**.** *1. Money is of little importance in business cycles.*
> *2. Business cycles are created by rational agents responding optimally to real (not nominal) shocks - mostly fluctuations in productivity growth, but also fluctuations in government purchases, import prices, or preferences.*

6.1 UMP of household

A typical household entails product-consuming for C_t and labor-providing for N_t. Only in this model, the agent value leisure $L_t = 1 - N_t$.
The UMP in RBC model is

$$\max_{C_t, N_t, B_t} E_t \sum_{t=0}^{\infty} \beta^t u(C_t, N_t) \tag{226}$$

$$s.t.\ P_t C_t + Q_t B_t \leqslant B_{t-1} + W_t N_t + D_t, \text{ where } \begin{cases} \beta \in (0,1) \\ Q_t = \frac{1}{1+i_t} \text{ is the price per bond bought today} \end{cases} \tag{227}$$

Form a Lagrangian and acquire FOCs

$$\mathscr{L} = E_0 \sum_{t=0}^{\infty} \beta^t [u(C_t, N_t) + \lambda_t (P_t C_t + Q_t B_t - B_{t-1} - W_t N_t - D_t)] \tag{228}$$

$$\Rightarrow \begin{cases} \frac{\partial \mathscr{L}}{\partial C_t} = 0 \Rightarrow \beta^t [u_{C_t} - \lambda_t P_t] = 0 \Rightarrow \frac{u_{C_t}}{E_t U_{c,t+1}} = \frac{\lambda_t}{E_t \lambda_{t+1}} \cdot \frac{P_t}{E_t P_{t+1}} \\ \frac{\partial \mathscr{L}}{\partial N_t} = 0 \Rightarrow \beta^t [u_{N_t} + \lambda_t W_t] = 0 \\ \frac{\partial \mathscr{L}}{\partial \beta_t} = 0 \Rightarrow -\lambda_t Q_t \beta^t + E_t \lambda_{t+1} \beta^{t+1} = 0 \Rightarrow \beta \frac{E_t \lambda_{t+1}}{\lambda_t} = Q_t \Rightarrow \frac{1}{Q_t} = 1 + i_t \end{cases} \tag{229}$$

using these FOCs

$$\begin{cases} \beta^t [u_{C_t} - \lambda_t P_t] = 0 \\ \beta^t [u_{N_t} + \lambda_t W_t] = 0 \end{cases} \Rightarrow \text{Labor supply equation } -\frac{u_{N_t}}{u_{C_t}} = \frac{W_t}{P_t} \tag{230}$$

$$\text{Also} \begin{cases} \frac{u_{C_t}}{E_t U_{c,t+1}} = \frac{\lambda_t}{E_t \lambda_{t+1}} \cdot \frac{P_t}{E_t P_{t+1}} \\ \beta \frac{E_t \lambda_{t+1}}{\lambda_t} = Q_t \end{cases} \Rightarrow \text{Euler's equation } \frac{Q_t}{\beta} = E_t \cdot \frac{\frac{U_{c,t+1}}{P_{t+1}}}{\frac{U_{c,t}}{P_t}} \tag{231}$$

e.g. Let utility function be specific

$$u(C_t, N_t) = \begin{cases} \frac{c_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}, \sigma \neq 1 \\ \log C_t - \frac{N_t^{1+\varphi}}{1+\varphi}, \sigma = 1 \end{cases} \tag{232}$$

$$\Rightarrow \begin{cases} \frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi} \\ Q_t = \beta E_t \cdot (\frac{C_{t+1}}{C_t})^{-\sigma} \cdot \frac{P_t}{P_{t+1}} \end{cases} \tag{233}$$

6.2 PMP of firm

$$\min_{N_t} W_t N_t \Leftrightarrow \max_{Y_t} P_t Y_t - w_t N_t \tag{234}$$

$$s.t. Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}, \alpha \in [0, 1) \tag{235}$$

$$\text{where} \begin{cases} Y_t = A_t W_t^{1-\alpha} \\ A_t = A_{t-1}^{\rho\alpha} \epsilon_t^{\alpha}, \epsilon_t^{\alpha} \overset{iid}{\sim} N(0, \sigma_{\alpha}^{\epsilon}) \text{ (technology shocks)} \end{cases} \tag{236}$$

$$\Rightarrow \text{F.O.C.: } \frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha} \text{ (labor demand equation)} \tag{237}$$

6.3 Economy described by RBC

The whole economy system is described as the balance of demand and supply

$$\text{AS} \begin{cases} Y_t = A_t N_t^{1-\alpha} \text{ (total production function)} \\ A_t = A_{t-1}^{\rho a} e^{\epsilon_t^a} \text{ (technology shocks)} \\ \frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi} \text{ (labor supply from UMP)} \\ \frac{w_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha} \text{ (labor demand from PMP)} \end{cases} \tag{238}$$

$$\text{AD} \begin{cases} Y_t = C_t + I_t \text{ (accouting equation)} \\ Q_t = \beta E_t (\frac{C_{t+1}}{C_t})^{-a} \frac{P_t}{P_{t+1}} \text{ (Euler equation from UMP)} \\ \frac{M_t}{P_t} = \frac{Y_t}{Q_t^{-n}} \text{ (demand of liquidity)} \end{cases} \tag{239}$$

6.4 Log-linearization

Two methods we can transform. 1) The first is

$$\hat{x}_t = \ln x_t - \ln x^* \tag{240}$$
$$= \ln|\frac{x_t}{x^*}| \tag{241}$$
$$= \ln|1 + \frac{x_t - x^*}{x^*}| \text{ (percentage of deviation from stable state)} \tag{242}$$
$$\approx \ln|1 + \frac{1}{x^*}(x_t - x^*)| \tag{243}$$
$$= \frac{x_t - x^*}{x^*} \tag{244}$$

2) Another approach would be

$$\ln x^* = lnx_t - \tilde{x_t} \Rightarrow \ln x_t = lnx^* + \tilde{x_t} \tag{245}$$
$$\Rightarrow e^{\ln x_t} = x_t = e^{nx^* + \tilde{x_t}} = x^* e^{\hat{x_t}} \approx x^*(1 + \hat{x}_t) \tag{246}$$

Now we turn to the linearization of RBC model

$$\begin{cases} y_t = a_t + (1-\alpha)n_t \\ y = a + (1-\alpha)n \end{cases} \Rightarrow \hat{y}_t = \hat{a}_t + (1-\alpha)\hat{n}_t \tag{247}$$
$$\Rightarrow \begin{cases} \hat{w}_t - \hat{p}_t = \hat{a}_t + (1-\alpha)\hat{n}_t \\ \hat{w}_t - \hat{p}_t = a\hat{c}_t + \rho\hat{n}_t \end{cases} \Rightarrow \hat{y}_t = \hat{c}_t,\ m_t - \hat{p}_t = \hat{y}_t - \hat{n}_{it} \tag{248}$$
$$Y_t = C_t + I_t \tag{249}$$
$$Y \cdot e^{\hat{y}_t} = C \cdot e^{\hat{c}_t} + I \cdot e^{\hat{i}_t} \tag{250}$$
$$Y(1 + \hat{y}_t) = C(1 + \hat{c}_t) + I(1 + \hat{i}_t) \tag{251}$$
$$\hat{y}_t = \frac{C}{Y}\hat{c}_t + \frac{I}{Y}\hat{i}_t \tag{252}$$
$$a_t = \rho_t a_{t-1} + \epsilon_t^a \tag{253}$$
$$1 = E_t e^{\ln\beta + + \ln\frac{1}{Q_t} - a(c_{t+1} - c_t) - (P_{t+1} - P_t)} \tag{254}$$
$$1 = E_t e^{\ln\beta + + \ln\frac{1}{Q_t} - a\triangle c_{t+1} - \pi_{t+1}} \tag{255}$$
$$1 = E_t I[1 + (i_t - i) - a(\triangle c_{t+1} - \triangle c_t] - (\pi_{t+1} - \pi_t)] \tag{256}$$

Therefore a RBC model can be rewritten in linearized log equations

$$\text{AS:}\begin{cases}\hat{y}_t = a_t + (1-\alpha)\hat{n}_t \text{ (output)}\\ \hat{w}_t - \hat{p}_t = a_t - \alpha\hat{n}_t \text{ (labor demand)}\\ \hat{w}_t - \hat{p}_t = ac_t + \varphi\hat{n}_t \text{ (labor supply)}\\ \hat{a}_t = \rho_a\hat{a}_{t-1} + \epsilon_t^a \text{ (technology)}\end{cases} \tag{257}$$

$$\text{AD:}\begin{cases}\hat{y}_t = \hat{c}_t \text{ (product market clearing)}\\ \hat{c}_t = E_t\hat{c}_{t+1} - \frac{1}{a}(\hat{i}_t - E_t\hat{\pi}_{t+1})\\ \hat{m}_t - \hat{p}_t = \hat{y}_t - \eta\cdot\hat{i}_t \text{ (money demand equation; } \eta \text{ is interest rate elasticity)}\\ \hat{r}_t = i_t - E_t\hat{\pi}_{t+1} \text{ (Fisherian equation)}\end{cases} \tag{258}$$

Note 18 (Exceptation for Expectation)**.** *Notice this method does not apply to equations involving expectations.*

Note 19 (RBC has neutrality of money)**.** *The final answer of this standard RBC model is not relevant to any price variable, inferring the model being money-neutral.*

7 DNK Model

DNK is short for Dynamic New Keynesian
This New keynesian Model contains elements being

- dispersive economy, a continuum of agents
- perfect information
- rational expectations
- perfect competitive labor market
- monopolisitc competitive product market, heterogenous products, nominal rigidities

---

7.1 UMP of household

Goal of UMP is to acquire DIS, which represents demand.
Stage I, choosing goods. Since products are heterogenous, UMP begins by choosing items $i \in I$.
One way to choose products is by method of maximization.

$$\begin{aligned} &\max_{C_{it}} (\int_0^1 C_{it}^{\frac{\epsilon-1}{\epsilon}} di)^{\frac{\epsilon}{\epsilon-1}} \\ &s.t. \int_0^1 P_{it} C_{it} di \leqslant Z_t \end{aligned} \tag{259}$$

, in which the ϵ stands for current substitute elasticity and $\frac{\epsilon-1}{\epsilon}$ is contemporaneous elasticity of substitution.
The other way to choose products is by method of minimization.

$$\begin{aligned} &\min \int_0^1 P_{it} C_{it} di \\ &s.t. (\int_0^1 C_{it}^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}} \geqslant C_t \end{aligned} \tag{260}$$

The latter is prefered as in this case Lagrangian multiplier would be shadow price.
Form a Lagrangian, we would have

$$\mathop{\mathscr{L}}_{\{C_{it}\}} = \int_0^1 P_{it} C_{it} di + P_t[(\int_0^1 C_{it}^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}} - c_t] \tag{261}$$

$$\Rightarrow P_{it} = P_t \frac{\epsilon}{\epsilon - 1}((\int_0^1 C_{it}^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}-1}) \frac{\epsilon - 1}{\epsilon} C_{it}^{\frac{\epsilon-1}{\epsilon}-1} \tag{262}$$

$$\Rightarrow P_{it} = P_t[(\int_0^1 C_{it}^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}})^{\frac{1}{\epsilon}}] C_{it}^{\frac{-1}{\epsilon}} \tag{263}$$

$$\Rightarrow \frac{P_{it}}{P_t} = (\frac{C_{it}}{C_t})^{\frac{-1}{\epsilon}} \tag{264}$$

$$\Rightarrow C_{it} = (\frac{P_{it}}{P_t})^{-\epsilon} C_t \text{ (the demand curve)} \tag{265}$$

From $\int_0^1 P_{it}C_{it}di = P_tC_t$ we have

$$\Rightarrow \int_0^1 P_{it}(\frac{P_{it}}{P_t})^{-\epsilon}C_t di = P_tC_t \tag{266}$$

$$\Rightarrow \int_0^1 P_{it}(\frac{P_{it}}{P_t})^{-\epsilon}di = P_t \tag{267}$$

$$\Rightarrow P_t = (\int_0^1 P_{it}^{1-\epsilon}di)^{\frac{1}{1-\epsilon}} \text{ (the expression of total price level)} \tag{268}$$

Apparently $\begin{cases} \max\limits_{C_{it}}(\frac{1}{I}\sum\limits_{i=1}^{I} C_{it}^{\frac{1}{1+n_t}})^{1+n_t} = C_t \\ s.t. \sum\limits_{i=1}^{I} P_{it}C_{it} \leqslant Z_t \end{cases}$ and $\begin{cases} \min\limits_{C_{it}} \sum\limits_{i=1}^{I} P_{it}C_{it} \\ s.t. (\frac{1}{I}\sum\limits_{i=1}^{I} C_{it}^{\frac{1}{1+n_t}})^{1+n_t}) \geqslant C_t \end{cases}$ are the same. But as mentioned before, the minimization approach is commonly prefered.

Using either approach, we can conduct the result that demand curve and expression of total price level being

$$C_{it} = (I \cdot \frac{P_{it}}{P_t})^{\frac{1+n_t}{-\wedge_t}} C_t \tag{269}$$

$$P_t = I(\frac{1}{I}\sum_{i=1}^{I} P_{it}^{\frac{1}{-\wedge_t}})^{-\wedge_t} \tag{270}$$

After picking products, we enter stage II, UMP at period t

$$\begin{aligned} &\max_{C_t, N_t, \frac{M_t}{P_t}} E_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, N_t, \frac{M_t}{P_t}\right) \leftarrow \text{ (MIA)} \\ &s.t. \int_0^1 P_{it}C_{it}di + M_t + B_t \leqslant M_{t-1} + (1+i_{t-1})B_{t-1} + \int_o^1 W_{it}N_{it}di + T_t \end{aligned} \tag{271}$$

constraint of this UMP can be rewritten as

$$\begin{aligned} &\max_{C_t, N_t, \frac{M_t}{P_t}} E_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, N_t, \frac{M_t}{P_t}\right) \\ &s.t. \ P_tC_t + M_t + B_t \leqslant M_{t-1} + (1+i_{t-1})B_{t-1} + W_tN_t + T_t \end{aligned} \tag{272}$$

> **Note 20** (Subscript it and t). *$C_{it} \neq C_t \Leftrightarrow$ Monopolistic competitive goods market; $N_{it} = N_t \Leftrightarrow$ Perfect competitve labor market*

Form the Lagrangian

$$\mathscr{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{u(C_t, N_t, \frac{M_t}{P_t}) + \lambda_t[P_tC_t + M_t + B_t - M_{t-1} - (1+i_{t-1})B_{t-1} - W_tN_t + T_t]\} \tag{273}$$

Solving this will deliver the F.O.C.s., resulting

$$\Rightarrow \begin{cases} Q_t = \beta E_t \cdot (\frac{u_{C,t+1}}{u_{C,t}}\frac{P_t}{P_{t+1}}) \\ \frac{W_t}{P_t} = -u_{N_t} \\ 1 - Q_t = u_{\frac{M_t}{P-t}} \end{cases} \tag{274}$$

This result is as same as in RBC model.
Next linearization, specificly focusing on the third equation of the second equation.

$$1 - \frac{1}{1+i_t} = C_t^\sigma (\frac{M_t}{P_t})^{-\nu} \text{ (using } Q_t\text{'s defintion)} \tag{275}$$
$$\Rightarrow \frac{i_t}{1+i_t} = C_t^\sigma (\frac{M_t}{P_t})^{-\nu} \tag{276}$$

Taking logs

$$\log \frac{i_t}{1+i_t} = \sigma c_t - \nu(m_t - p_t) \text{ (hereafter a lowercase letter } x \text{ denotes } \log x) \tag{277}$$
$$\log \frac{i}{1+i} = \sigma c - \nu(m - p) \tag{278}$$
$$\Rightarrow \log(\frac{i_t}{1+i_t}) - \log(\frac{i}{1+i}) = \sigma \hat{c}_t - \nu(m_t - \hat{p}_t) \tag{279}$$
$$\Rightarrow \nu(m_t - \hat{p}_t) = \sigma \hat{c}_t + \log(\frac{1+i_t}{i_t}) - \log(\frac{1+i}{i}) \tag{280}$$
$$\Rightarrow \nu(m_t - \hat{p}_t) = \sigma \hat{c}_t + [\log \frac{(\frac{1+i}{i}) + 1}{\frac{1+i}{i}} \frac{i - (1+i)}{i^2}(i_t - i) - \log(\frac{1+i}{i})] \tag{281}$$
$$\Rightarrow \nu(m_t - \hat{p}_t) = \sigma \hat{c}_t + \frac{i}{i+1} \frac{i-(1+i)}{i} \frac{i_t - i}{i} \tag{282}$$
$$\Rightarrow \nu(m_t - \hat{p}_t) = \sigma \hat{c}_t - \frac{1}{1+i} \hat{i}_t \tag{283}$$
$$\Rightarrow m_t - \hat{p}_t = \frac{\sigma}{\nu} \hat{c}_t - \frac{1}{(1+i)\nu} \hat{i}_t \tag{284}$$
$$\Rightarrow m_t - \hat{p}_t = \hat{c}_t - \eta \hat{i}_t \tag{285}$$

> **Note 21** (Must-be in exam: monopolistic competition vs perfect competition)**.** *Zero market power to finite market power, meaning infinite elasticiy to non-zero positive elasticity.*

7.2 PMP of firm

Goal here in this section is to acquire price-setting behavior equation then Philippes curve, or the AS curve.

> **Note 22** (Two features DNK firm has)**.** *Monopolistic competitive goods market and stikcy price.*

7.2.1 Price-setting behavioral equation

PMP and price-setting behavioral equation
Based on the intial period $t = 0$ for simplicity

$$
\begin{aligned}
&\max_{P_{i0}} \pi_0^n = E_0 \sum_{t=0}^{\infty} Q_{0,t} \cdot (1-\theta)^t \{P_{i0} Y_{it|0} - [TC_{it|0}^n(Y_{it|0})]\} \\
&s.t.\ Y_{it|0} = (\frac{P_{i0}}{P_t})^{-\epsilon} Y_t \text{ (since } C_t = Y_t \text{ at equilbrium, the constraint is essentially demand curve)}
\end{aligned} \tag{286}
$$

$|0$ means evaluated at prices from period 0
Y_{it} is total sales of firm i at period t
TC is a function of nominal total cost
$Q_{0,t}$ means discount rate from period 0 to period t
θ means the probability of price adjusting, therefore $1 - \theta$ means the probablity to stay put. It's sticky price. [1]

Solving the PMP by putting demand curve into profit function, it would be

$$
\pi_0^n = E_0 \sum_{t=0}^{\infty} Q_{0,t} \cdot (1-\theta)^t \{P_{i0} (\frac{P_{i0}}{P_t})^{-\epsilon} Y_t - [TC_{it|0}^n((\frac{P_{i0}}{P_t})^{-\epsilon} Y_t)]\} \tag{287}
$$

then let the F.O.C. be 0 (fig 1,2)

$$
\frac{\partial \pi_0^n}{\partial P_{i0}} = 0 \tag{288}
$$

$$
\Rightarrow E_0 \sum_{t=0}^{\infty} Q_{0,t} \cdot (1-\theta)^t \{(1-\theta)(\frac{P_{i0}}{P_t})^{-\epsilon} Y_t - [\frac{\partial TC_{it|0}^n}{\partial P_{it|0}} \frac{\partial P_{it|0}}{\partial P_{i0}}] = 0 \tag{289}
$$

$$
\Rightarrow E_0 \sum_{t=0}^{\infty} (1-\theta)^t Q_{0,t} [(1-\epsilon)(\frac{P_{i0}}{P_t})^{-\epsilon} Y_t - MC_{it|0}^n (-\epsilon)(\frac{P_{i0}}{P_t})^{-\epsilon-1} Y_t \frac{1}{P_t}] = 0 \tag{290}
$$

$$
\Rightarrow E_0 \sum_{t=0}^{\infty} (1-\theta)^t Q_{0,t} [(1-\epsilon) Y_t - MC_{it|0}^n (-\epsilon)(\frac{P_{i0}}{P_t})^{-\epsilon-1} Y_t \frac{1}{P_t}] = 0 \tag{291}
$$

$$
\Rightarrow E_0 \sum_{t=0}^{\infty} (1-\theta)^t Q_{0,t} [(1-\epsilon) Y_{it|0} + \frac{\epsilon}{P_t} MC_{it|0}^n (\frac{P_{i0}}{P_t})^{-1}] = 0 \tag{292}
$$

$$
\Rightarrow E_0 \sum_{t=0}^{\infty} (1-\theta)^t Q_{0,t} Y_{it|0} [(1-\epsilon) + \epsilon MC_{it|0}^n \cdot (P_{i0})^{-1}] = 0 \tag{293}
$$

$$
\Rightarrow E_0 \sum_{t=0}^{\infty} (1-\theta)^t \cdot Q_{0,t} \cdot Y_{it|0} \cdot (1-\epsilon) = -E_0 \sum_{t=0}^{\infty} (1-\theta)^t \cdot Q_{0,t} \cdot Y_{it|0} \cdot \epsilon \cdot MC_{it|0}^n \cdot (P_{i0})^{-1} \tag{294}
$$

$$
\Rightarrow (1-\epsilon) E_0 \sum_{t=0}^{\infty} (1-\theta)^t \cdot Q_{0,t} \cdot Y_{it|0} = -\epsilon (P_{i0})^{-1} E_0 \sum_{t=0}^{\infty} (1-\theta)^t \cdot Q_{0,t} \cdot Y_{it|0} \cdot MC_{it|0}^n \tag{295}
$$

restore Q_{0t} and $Y_{it|0}$ back to demand curve

$$
\Rightarrow (\epsilon-1) E_0 \sum_{t=0}^{\infty} (1-\theta)^t \beta^t [(\frac{Y_0}{Y_t})^{\rho} \frac{P_0}{P_t}] (\frac{P_{i0}}{P_t})^{-\epsilon} Y_t = (P_{i0})^{-1} E_0 \sum_{t=0}^{\infty} (1-\theta)^t \beta^t [(\frac{Y_0}{Y_t})^{\rho} \frac{P_0}{P_t}] (\frac{P_{i0}}{P_t})^{-\epsilon} Y_t \cdot MC_{it|0}^n \tag{296}
$$

[1] For more of price adjusting, see in Calvo, 1983 for uncertain version; Taylor, 1980 for certain version.

take P_{i0} and P_0 out since it can be treated as constants

$$\Rightarrow (\epsilon-1)\underline{P_{i0}^{-\epsilon}P_0Y_o^{\sigma}}E_0\sum_{t=0}^{\infty}(1-\theta)^t\beta^tP_t^{\epsilon-1}Y_t^{1-\rho}=\epsilon\underline{P_{i0}^{-1}P_{i0}^{-\epsilon}P_0Y_0^{\rho}}E_0\sum_{t=0}^{\infty}(1-\theta)^t\beta^tP_t^{\epsilon-1}Y_t^{1-\sigma}\cdot MC_{it|0}^n \tag{297}$$

eliminates the underlined common factors on RH and LH

$$\Rightarrow (\epsilon-1)E_0\sum_{t=0}^{\infty}(1-\theta)^t\beta^tP_t^{\epsilon-1}Y_t^{1-\rho}=\epsilon E_0\sum_{t=0}^{\infty}(1-\theta)^t\beta^tP_t^{\epsilon-1}Y_t^{1-\sigma}\cdot MC_{it|0}^n \tag{298}$$

$$\Rightarrow P_{i0}^{*}=\frac{\epsilon}{\epsilon-1}\frac{E_0\sum_{t=0}^{\infty}(1-\theta)^t\beta^tP_t^{\epsilon-1}Y_t^{1-\sigma}\cdot MC_{it|0}^n}{E_0\sum_{t=0}^{\infty}(1-\theta)^t\beta^tP_t^{\epsilon-1}Y_t^{1-\sigma}}\text{ (sticky price form price-setting equation)} \tag{299}$$

Also acquiring optimal price p^*, and $\frac{\epsilon}{\epsilon-1}$ is cost mutltiplier

$$\Rightarrow P_{i0}^{*}=P_0^{*}=\frac{\epsilon}{\epsilon-1}\frac{[(1-\epsilon)\beta]^0P_o^{\epsilon-1}Y_0^{1-\sigma}MC_{0|0}^n}{(1-\theta)^0\beta^0P_0^{\epsilon-1}Y_0^{1-\sigma}}=\frac{\epsilon}{\epsilon-1}MC_{0|0}^n\text{ (elastic form price-setting equation)} \tag{300}$$

For elastic form, the intuition is that firms are adjusting price at any time so there is no need to observe profit based on any specific period. So no summation symbol.

> **Question 5** (what if price is elastic?).
> elasticity means adjusting price at all time. it means price sticky price is 0 $\Leftrightarrow 1-\theta=0$.

> **Note 23.** *monopolistic competition and elasticity are not the same thing*

next try the period-t-based version in textbook (pic 3)

$$P_t^{*}=\frac{\epsilon}{\epsilon-1}\frac{E_t\sum_{k=0}^{\infty}(1-\theta)^k\beta^{t+k}P_{t+k}^{\epsilon-1}Y_{t+k}^{1-\sigma}\cdot MC_{t+k|t}^n}{E_t\sum_{k=0}^{\infty}(1-\theta)^k\beta^tP_{t+k}^{\epsilon-1}Y_{t+k}^{1-\sigma}} \tag{301}$$

(here the subscript i is gone because all firms are making unanimous decisions)

$$P_t^{*}=\frac{\epsilon}{\epsilon-1}MC_{t|t}^n \tag{302}$$

> **Note 24** (CRTS in DNK). *if production function is CRTS, then all firms would also be making unanimous decisions regardless of time period position*

log-linearization of sticky price form price-setting behavioral equation
the method is still first order Taylor expansion, though bit trickier (pic 3, 4, 5)
dividing P_{t-1} on both side, and try seperately

$$\underbrace{\frac{P_t^*}{P_{t-1}} E_t \sum_{k=0}^{\infty} (1-\theta)^k \beta^t P_{t+k}^{\epsilon-1} Y_{t+k}^{1-\sigma}}_{\text{LHS}} = \underbrace{\frac{1}{P_{t-1}} \frac{\epsilon}{\epsilon-1} E_t \sum_{k=0}^{\infty} (1-\theta)^k \beta^{t+k} P_{t+k}^{\epsilon-1} Y_{t+k}^{1-\sigma} \cdot MC_{t+k|t}^n}_{\text{RHS}} \tag{303}$$

LHS mutil-variate first order Taylor expansion, the constant term is LHS at steady state where all P are equal

$$\begin{aligned}
\text{LHS's 1st order Taylor expansion} = & \sum_{k=0}^{\infty} [(1-\theta)\beta]^k P^{\epsilon-1} Y^{1-\sigma} && (304) \\
& + \frac{1}{p} E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k P^{\epsilon} Y^{1-\sigma} (\frac{P_t^* - P}{P}) && (305) \\
& - \frac{P}{P^2} E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k P^{\epsilon-1} Y^{1-\sigma} (\frac{P_{t-1} - P}{P}) && (306) \\
& + (\epsilon-1) E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k P^{\epsilon-1} Y^{1-\sigma} (\frac{P_{t+k} - P}{P}) && (307) \\
& + (1-\sigma) E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k P^{\epsilon-1} Y^{1-\sigma} (\frac{Y_{t+k} - Y}{Y}) && (308)
\end{aligned}$$

Like before, we use lowercase letters denotes log version

$$\Rightarrow \text{LHS} = P^{\epsilon-1} Y^{1-\sigma} E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k P^{\epsilon-1} Y^{1-\sigma} [1 + \hat{p}_t^* - \hat{p}_{t-1} + (\epsilon-1)\hat{p}_{t+k} + (1-\sigma)\hat{y}_{t+k}] \tag{309}$$

For RHS,

$$\text{RHS} = \cdots = \frac{\epsilon}{\epsilon-1} \frac{1}{P} P^{\epsilon-1} Y^{1-\sigma} E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k P^{\epsilon-1} Y^{1-\sigma} [1 - \hat{p}_{t-1} + (\epsilon-1)\hat{p}_{t+k} + (1-\sigma)\hat{y}_{t+k} + \hat{m}c_{t+k|t}^n] \tag{310}$$

> **Note 25** (First order Taylor expansion and steady state)**.** *First term (constant term) of 1st Taylor expansion is steady state*

> **Question 6** (why do we do log-linearization)**.**
> to simulate a curve with a line, so there would be a foundable or writable solution.

Now the two sides are log-linearized, let them be in the same equation

$$\underbrace{P^{\epsilon-1} Y^{1-\sigma} E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k P^{\epsilon-1} Y^{1-\sigma} [1 + \hat{p}_t^* - \hat{p}_{t-1} + (\epsilon-1)\hat{p}_{t+k} + (1-\sigma)\hat{y}_{t+k}]}_{\text{log-linear LHS}} = \tag{311}$$

$$\underbrace{\frac{\epsilon}{\epsilon-1} \frac{1}{P} P^{\epsilon-1} Y^{1-\sigma} E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k P^{\epsilon-1} Y^{1-\sigma} [1 - \hat{p}_{t-1} + (\epsilon-1)\hat{p}_{t+k} + (1-\sigma)\hat{y}_{t+k} + \hat{m}c_{t+k|t}^n]}_{\text{log-linear RHS}} \tag{312}$$

Use sticky form price-setting equation at steady state (which because of steadiness looks like elastic form), we have

$$\frac{MC^n}{P} = \frac{\epsilon - 1}{\epsilon} \tag{313}$$

Since E_t is followed by an infinite series, it is meaningless. Eliminate it.

$$\Rightarrow \hat{p}_t^* \cancel{E_t} \sum_{k=0}^{\infty} [(1-\theta)\beta]^k = E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k \hat{m}c_{t+k|t}^n \tag{314}$$

$$\Rightarrow \frac{\hat{p}_t^*}{1-(1-\theta)\beta} = E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k MC_{t+k|t}^n \tag{315}$$

$$\Rightarrow \hat{p}_t^* = [1-(1-\theta)\beta] E_t \sum_{k=0}^{\infty} [(1-\theta)\beta]^k \hat{m}c_{t+k|t}^n \text{ (log-linear sticky price setting equation)} \tag{316}$$

Therefore we have the log-linear optimal-price-setting equation for monopolistic competition and sticky price features firms.

7.2.2 Philips curve

First way to acquire Philips curve
next we use this log-linearized price-setting equation to acquire NK Philips curve
originally PC is about inflation and employment
step I, inflation (p7,8,9)

$$P_t = I(\frac{1}{I} \sum_{i=1}^{I} P_{it}^{\frac{1}{-\wedge_t}})^{-\wedge_t} \text{ (expression of total price)} \tag{317}$$

$$= [\int_0^\theta (P_{it}^*)^{1-\epsilon} di + \int_\theta^1 P_{t-1}^{1-\epsilon} di]^{\frac{1}{1-\epsilon}} \text{ (θ portion of firms don't adjust price)} \tag{318}$$

$$= [\theta(P_{it}^*)^{1-\epsilon} + (1-\theta) P_{t-1}^{1-\epsilon}]^{\frac{1}{1-\epsilon}} \text{ (expression of total price with θ firms adjusting price)} \tag{319}$$

$$\Rightarrow P_t^{1-\epsilon} = \theta(P_{it}^*)^{1-\epsilon} + (1-\theta) P_{t-1}^{1-\epsilon} \tag{320}$$

$$\Rightarrow (\frac{P_t}{P_{t-1}})^{1-\epsilon} = \theta(\frac{P_t^*}{P_{t-1}})^{1-\epsilon} + (1-\theta) \tag{321}$$

$$\Rightarrow \Pi^{1-\epsilon} = \theta(\frac{P_t^*}{P_{t-1}})^{1-\epsilon} + (1-\theta) \text{ (the capital letter Π is for total inflation } \frac{P_t}{P_{t-1}}) \tag{322}$$

Then log-linearization by bi-variate first order Taylor expansion

$$\Rightarrow \Pi^{1-\epsilon} + (1-\epsilon)\Pi^{1-\epsilon}(\frac{\Pi_t - \Pi}{\Pi}) = \{\theta(\frac{P}{P})^{1-\epsilon} + \theta(1-\epsilon)(\frac{P}{P})^{1-\epsilon}[\frac{1}{P}(P_t^* - P) - \frac{P}{P^2}\frac{(P_{t-1}-P)}{P}]\}^{1-\theta} \tag{323}$$

Since first term of tay exp is steady state, making $\Pi^{1-\epsilon} = 1$

$$\Rightarrow 1 + (1-\epsilon)\hat{\Pi}_t = \theta + \theta(1-\epsilon)(\hat{P}_t^* - \hat{P}_{t-1}) + (1-\theta) \tag{324}$$

$$\Rightarrow \hat{\Pi}_t = \theta(\hat{P}_t^* - \hat{P}_{t-1}) \tag{325}$$

$$\Rightarrow \frac{1}{\theta}\hat{\Pi}_t = \hat{P}_t^* - \hat{P}_{t-1} \tag{326}$$

Using price-setting equation, which is related to marginal cost, then production. (making nominal price combination of real price and $\hat{p}_{t-1}$)

$$\Rightarrow \frac{1}{\theta}\hat{\Pi}_t = [1-(1-\theta)\beta]E_t\sum_{k=0}^{\infty}[(1-\theta)\beta]^k(\hat{mc}^r_{t+k|t} + \hat{p}_{t+k} - \hat{p}_{t-1}) \tag{327}$$

$$= \underbrace{[1-(1-\theta)\beta]E_t\sum_{k=0}^{\infty}[(1-\theta)\beta]^k\hat{mc}^r_{t+k|t}}_{A} + [1-(1-\theta)\beta]E_t\sum_{k=0}^{\infty}[(1-\theta)\beta]^k(\hat{p}_{t+k} - \hat{p}_{t-1}) \tag{328}$$

Let the untouched part be A for simplicity, open the sum so there will be expression of inflation in log-linearized form

$$= A + [1-(1-\theta)\beta]E_t\sum_{k=0}^{\infty}[(1-\theta)\beta]^k(\hat{p}_{t+k} - \hat{p}_{t-1}) \tag{329}$$

$$\begin{aligned}&= A + [1-(1-\theta)\beta]E_t\{(\hat{p}_t - \hat{p}_{t-1}) + [(1-\theta)\beta](\hat{p}_{t+1} - \hat{p}_t + \hat{p}_t - \hat{p}_{t-1})\\ &\quad + [(1-\theta)\beta]^2(\hat{p}_{t+2} - \hat{p}_{t+1} + \hat{p}_{t+1} - \hat{p}_t + \hat{p}_t - \hat{p}_{t-1}) + \dots\}\end{aligned} \tag{330}$$

$$= A + [1-(1-\theta)\beta]E_t\{[(\hat{\pi}_t + [(1-\theta)\beta](\hat{\pi}_{t+1} + \hat{\pi}_t) + [(1-\theta)\beta]^2(\hat{\pi}_{t+2} + \hat{\pi}_{t+1} + \hat{\pi}_t) + \dots\} \tag{331}$$

$$\begin{aligned}&= A + E_t\{[(\hat{\pi}_t + [(1-\theta)\beta](\hat{\pi}_{t+1} + \hat{\pi}_t) + [(1-\theta)\beta]^2(\hat{\pi}_{t+2} + \hat{\pi}_{t+1} + \hat{\pi}_t) + \dots\}\\ &\quad - (1-\theta)\beta E_t\{[(\hat{\pi}_t + [(1-\theta)\beta](\hat{\pi}_{t+1} + \hat{\pi}_t) + [(1-\theta)\beta]^2(\hat{\pi}_{t+2} + \hat{\pi}_{t+1} + \hat{\pi}_t) + \dots\}\\ &\text{(making elimination possible)}\end{aligned} \tag{332}$$

$$\begin{aligned}&= A + E_t\{\hat{\pi}_t + [(1-\theta)\beta](\hat{\pi}_{t+1} + \hat{\pi}_t) + [(1-\theta)\beta]^2(\hat{\pi}_{t+2}\hat{\pi}_{t+1} + \hat{\pi}_t) + \dots\}\\ &\quad - [(1-\theta)\beta]\{\hat{\pi}_t - [(1-\theta)\beta]^2E_t(\hat{\pi}_{t+1} + \hat{\pi}_t) - \dots\}\end{aligned} \tag{333}$$

Make format simple by doing iteration form, the first term is when $k = 0$, but the second still starts from k=0 so there will be some parameters changed

$$= [1-(1-\theta)\beta]E_t\sum_{k=0}^{\infty}\hat{mc}^r_{t+k|t} + E_t\sum_{k=0}^{\infty}\hat{\pi}_{t+k+1} \tag{334}$$

$$= \underbrace{[1-(1-\theta)\beta]mc^r_{t+k|t} + \pi_t}_{k=0} + [1-(1-\theta)\beta]E_t\sum_{k=0}^{\infty}[(1-\theta)\beta]^{k+1}\hat{mc}^r_{t+k+1|t} + E_t\sum_{k=0}^{\infty}[(1-\theta)\beta]^{k+1}\hat{\pi}_{t+k} \tag{335}$$

$$= [1-[(1-\theta)\beta]]\hat{mc}^r_{t|t} + \hat{\pi}_t + [(1-\theta)\beta]\{\quad\} \tag{336}$$

$$= B \text{ (back to recursive form and let it be B)} \tag{337}$$

$$\Rightarrow \frac{1}{\theta}\hat{\pi}_t = B \tag{338}$$

$$\Rightarrow \hat{\pi}_t = \underbrace{\frac{\theta[1-(1-\theta)\beta]}{1-\theta}\hat{mc}^r_t}_{\text{effect of current marginal cost}} + \underbrace{\beta E_t\hat{\pi}_{t+1}}_{\text{effect of expectation of future inflation}} \quad \text{(New Keynesian Philips Curve marginal cost version)} \tag{339}$$

Question 7 (at which variable are we making derivative of Lagrangian function ?)**.**
Theoratically we can make derivative of any variable if possible but not proper.

NKPC has another version where marginal cost is substitued with real output, thus measuring the effect of output gap.

$$\begin{aligned} &\min \frac{W_t}{P_t} N_t \\ &s.t. Y_t = A_t N_t^{1-\alpha} \end{aligned} \tag{340}$$

Form a Lagrangian

$$\Rightarrow \mathscr{L} = \frac{W_t}{P_t} N_t + MC_t^r \cdot (Y_t - A_t N_t^{1-\alpha}) \text{ (shadow price can also be marginal cost)} \tag{341}$$

Take partial derivative of N_t here since labor is decided by market, the firm only can control demand of labor

$$\Rightarrow \frac{\partial \mathscr{L}}{\partial N_t} = 0 \tag{342}$$

$$\Rightarrow \frac{W_t}{P_t} N_t = MC_t^r \cdot MPN_t \text{ , where } MPN_t = \frac{\partial Y_t}{\partial N_t} = (1-\alpha) A_t N_t^{-\alpha} \tag{343}$$

$$\Rightarrow MC_t^r = \frac{\frac{W_t}{P_t}}{MPN_t} \tag{344}$$

$$\hat{mc}_t^\sigma = \hat{w}_t - \hat{p}_t - (a_t - \alpha \hat{n}_t) \text{ (log-linearized, and using labor supply equation)} \tag{345}$$

$$= \sigma \hat{c}_t + \varphi \hat{n}_t - (a_t - \alpha \hat{n}_t) \tag{346}$$

$$= \sigma \hat{y}_t - a_t - (\varphi + \alpha) \frac{\hat{y}_t - a_t}{1-\alpha} \tag{347}$$

$$= \frac{\sigma(1-\alpha) + (\varphi+\alpha)}{1-\alpha} \hat{y}_t - \frac{1+\varphi}{1-\alpha} a_t \tag{348}$$

To acquire natural ratio of output, we use the fact that elastic marginal cost is zero.

$$\Rightarrow 0 = \hat{mc}_t^r = \frac{\sigma(1-\alpha) + (\varphi+\alpha)}{1-\alpha} \hat{y}_t^f - \frac{1+\varphi}{1-\alpha} a_t \text{ (upscript } f \text{ stands for flexible meaning elastic)} \tag{349}$$

$$\Rightarrow \frac{1+\varphi}{1-\alpha} a_t = \frac{\sigma(1-\alpha) + (\varphi+\alpha)}{1-\alpha} \hat{y}_t^f \tag{350}$$

$$\Rightarrow \hat{mc}_t^r = \frac{\sigma(1-\alpha) + (\varphi+\alpha)}{1-\alpha} (\underbrace{\hat{y}_t - \hat{y}_t^f}_{\equiv \tilde{y}_t \text{, output gap}}) \text{ (put it back, having the gap of output } \hat{y}_t - y_t^f = \tilde{t}) \tag{351}$$

If we put the last equation back to NKPC marginal cost version, we connect natural output to inflation which is NKPC real output version.

New Keynesian Philips Curve marginal cost version:

$$\hat{\pi}_t = \underbrace{\frac{\theta[1-(1-\theta)\beta]}{1-\theta} \hat{} \frac{\sigma(1-\alpha)+(\varphi+\alpha)}{1-\alpha} (\hat{y}_t - \hat{y}_t^f)}_{\text{effect of current output gap}} + \underbrace{\beta E_t \hat{\pi}_{t+1}}_{\text{effect of expectation of future inflation}} \tag{352}$$

another way to acquire the NKPC is to concern monopolistic competition, then sticky price.

$$\max_{P_{it}} \pi^r = \frac{P_{it}}{P_t} Y_{it} - \frac{W_t}{P_t} N_{it} \tag{353}$$

$$\Rightarrow \frac{P_{it}}{P_t} = \frac{\epsilon}{\epsilon - 1} MC_t^r \tag{354}$$

Note 26 (indicator of monopolistic competition)**.**
$\frac{\epsilon}{\epsilon-1}$ means monopolistic competition
integral means perfect competitive labor supply for households
ϵ is subtitute elasticity

This is real price, where ϵ is subtitute elasticity. Here MC_i comes without subscript t is because of CRTS production function.

$$\begin{aligned}
\Rightarrow \frac{P_{it}}{P_t} &= \frac{\epsilon}{\epsilon - 1} \frac{W_t/P_t}{A_t} && (355)\\
&= \frac{\epsilon}{\epsilon - 1} \frac{Y_t^\sigma N_t^\varphi}{A_t} && (356)\\
&= \frac{\epsilon}{\epsilon - 1} \frac{Y_t^\sigma}{A_t} (\int_0^1 N_{it} di)^\varphi \text{ (because of goods market p.c.)} && (357)\\
&= \frac{\epsilon}{\epsilon - 1} \frac{Y_t^\sigma}{A_t} [\int_0^1 (\frac{Y_{it}}{A_t}) di]^\varphi \text{ (based on CRTS production function)} && (358)\\
&= \frac{\epsilon}{\epsilon - 1} \frac{Y_t^\sigma}{A_t^{1+\varphi}} [\int_0^1 (\frac{P_{it}}{P_t})^{-\epsilon} Y_t di]^\varphi \text{ (using demand curve and move } A_t \text{ out)} && (359)\\
&= \frac{\epsilon}{\epsilon - 1} \frac{Y_t^\sigma}{A_t^{1+\varphi}} (\frac{P_{it}^*}{P_t})^{-\epsilon\varphi} \text{ (star means symmetric and at equil)} && (360)
\end{aligned}$$

Since at equilibrium all prices are same

$$P_{it}^* = P_t^* \Rightarrow \frac{P_{it}^*}{P_t} = \frac{\epsilon}{\epsilon - 1} \frac{Y_t^{\sigma+\varphi}}{A_t^{1+\varphi}} (\frac{P_t^*}{P_t})^{-\epsilon\varphi} \tag{361}$$

Let's consider a situation that

$$\begin{aligned}
&\text{If } P_t^* = P_t \text{ (meaning when price elastic)} && (362)\\
&1 = \frac{\epsilon}{\epsilon - 1} && (363)\\
&\Rightarrow Y_t^f \text{ or } \hat{y}_t^f \text{ (to be log-linearized)} && (364)
\end{aligned}$$

Taking logs

$$\begin{aligned}
(1 + \epsilon\varphi)(P_t^* - P_t) &= \ln \frac{\epsilon}{\epsilon - 1} - (1 + \varphi) a_t + (\sigma + \varphi) y_t && (365)\\
&= (\sigma + \phi) y_t - (1 + \phi) a_t + \ln(\frac{\epsilon}{\epsilon - 1}) \text{ (change order)} && (366)
\end{aligned}$$

Change order for furthur preparation

$$\Rightarrow P_t^* = P_t + \frac{\sigma+\varphi}{1+\epsilon\varphi}y_t - \frac{1+\varphi}{1+\epsilon\varphi}a_t + \frac{1}{1+\epsilon\varphi}\ln(\frac{\epsilon}{\epsilon-1}) \tag{367}$$

$$\Rightarrow P^* = P + \frac{\sigma+\varphi}{1+\epsilon\varphi}y_t - 0 + \frac{1}{1+\epsilon\varphi}\ln(\frac{\epsilon}{\epsilon-1}) \text{ (steady state by definition)} \tag{368}$$

Two equations above deducted, having the log-linearized result; elastic price means $p_t^* = \hat{p}_t$

$$\hat{p}_t^* = \hat{p}_t + \frac{\sigma+\varphi}{1+\epsilon\varphi}\hat{y}_t - \frac{1+\varphi}{1+\epsilon\varphi}a_t \tag{369}$$

$$P_t^* = \hat{p}_t \text{ (elastic price gives } y \text{ a } f \text{ upscript)} \tag{370}$$

$$\Rightarrow 0 = \frac{\sigma+\varphi}{1+\epsilon\varphi}\hat{y}_t - \frac{1+\varphi}{1+\epsilon\varphi}a_t \tag{371}$$

$$\Rightarrow \hat{y}_t^f = \frac{1+\varphi}{\sigma+\varphi}a_t \tag{372}$$

here notice natural output is a function of technology

$$\Rightarrow \hat{p}_t^* = \hat{p}_t + \frac{\sigma+\varphi}{1+\epsilon\varphi}\hat{y}_t - \frac{\sigma+\varphi}{1+\epsilon\varphi}\hat{y}_t^f \tag{373}$$

$$\Rightarrow \hat{p}_t^* = \hat{p}_t + \gamma(\hat{y}_t - \hat{y}_t^f) \text{ (}\gamma\text{ denotes the bundle of coefficients)} \tag{374}$$

$$\Rightarrow \hat{p}_t^* = \hat{p}_t + \gamma\tilde{y}_t \text{ (acceptable price-setting euqation with only assumption of m.c.)} \tag{375}$$

[2]

Note 27 (DNK model is perfect information rational expectation)**.** *Here E_t is not $\bar{E}_t$ because of perfect information rational expectation*

Second way to acquire NKPC

Note 28 (Price adjusting in certainty vs in randomness)**.**
Price adjusting in certainty Taylor, 1978; price adjusting in randomness, Calvo, 1983
Taylor
a, b and c denotes three firms in a monopolistic competitive market. they adjust price by turns and one person per period.
Q1: how does one choose price
$P_A^\triangle = \frac{}{3}$, the upscript $\triangle$ stands for staying put in the next three periods.
Q2: what is the total price
$P_t =$
Calvo
each firm adjust price at probablity θ^t at period t.

[2] see mankiw and Reis 2002 for more details of acceptable price-setting eq

now sticky price price-adjusting equation

$$\hat{p}_t = \underbrace{(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k E_t P^*_{t+k}}_{\text{weighted average}} = (1-\theta\beta)\hat{p}^*_t + \theta\beta E_t \hat{p}^a_{t+1} \tag{376}$$

The θ is the opposite of the one in previous PAE, here rather is to stay put. price adjusting based on acceptable price setting eq in future periods, β mean subjective discount.
With the expression of total price being

$$\hat{p}_t = (1-\theta)\sum_{k=0}^{\infty}\theta^k \hat{p}^{\triangle}_{t-k} \tag{377}$$

$$\hat{p}_t = (1-\theta)\hat{p}^a_t + \theta\hat{p}_{t-1} \text{ (expression of total price based on PAE)} \tag{378}$$

Based on Acceptable Price-setting Equation (from monopolistic competition) and Total Price (from sticky price), one can quickly conduct NKPC.

> **Note 29** (prices in DNK).
> starred price means acceptable price, triangled price means adjusting price

From the first equation

$$\hat{p}_t = (1-\theta)P^{\triangle}_t + \theta P_{t-1} \tag{379}$$

$$\Rightarrow (1-\theta)P^{\triangle}_t = \hat{p}_t - \theta P_{t-1} \tag{380}$$

$$\Rightarrow (1-\theta)E_t P^{\triangle}_{t+1} = E_t\hat{p}_{t+1} - \theta\hat{p}_t \tag{381}$$

$$\Rightarrow (1-\theta)\beta E_t P^{\triangle}_{t+1} = \beta E_t\hat{p}_{t+1} - \theta\beta\hat{p}_t \tag{382}$$

From the second equation

$$\hat{p}_t = (1-\theta\beta)\hat{p}^*_t + \theta\beta E_t\hat{p}^{\triangle}_{t+1} \tag{383}$$

$$\Rightarrow (1-\theta)\hat{p}^{\triangle}_t = (1-\theta)(1-\theta\beta)\hat{p}^*_t + (1-\theta)\theta\beta E_t P^{\triangle}_{t+1} \tag{384}$$

$$\hat{p}_t - \theta\hat{p}_{t-1} = (1-\theta)(1-\theta\beta)\hat{(}\hat{p}_t + \gamma\tilde{y}_t) + \theta\beta E_t\hat{p}_{t+1} - \theta^2\beta\hat{p}_t \text{ (replacing } y^* \text{ with APSE)} \tag{385}$$

$$\Rightarrow [1+\theta^2\beta - (1-\theta)(1-\theta\beta)]\hat{p}_t - \theta P_{t-1} = (1-\theta)(1-\theta\beta)\gamma\tilde{y}_t + \theta\beta E_t \tag{386}$$

$$\Rightarrow [1+\theta^2\beta - (1-\theta)(1-\theta\beta)]\hat{p}_t - \theta P_{t-1} - \theta\beta\hat{p}_t = (1-\theta)(1-\theta\beta)\gamma\tilde{y}_t + \theta\beta(E_t\hat{p}_{t+1} - \hat{p}_t) \tag{387}$$

$$\Rightarrow \theta(\hat{p}_t - \hat{p}_{t-1}) = (1-\theta)(1-\theta\beta)\gamma\tilde{y}_t + \theta\beta(E_t\hat{P}_{t+1} - \hat{p}_t) \text{ (match terms so there is inflation)} \tag{388}$$

$$\Rightarrow \theta\hat{\pi}_t = (1-\theta)(1-\theta\beta)\gamma\tilde{y}_t + \theta\beta E_t\hat{\pi}_{t+1} \text{ (here comes inflation)} \tag{389}$$

$$\Rightarrow \hat{\pi}_t = k\hat{y}_t + \beta E_t\hat{\pi}_{t+1} \text{ (NKPC)} \tag{390}$$

7.2.3 Dynamic IS curve

Recalling that household

$$\hat{y}_t = E_t\hat{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1}) \tag{391}$$

$$\Rightarrow \hat{y}_t^f = E_t\hat{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - 0) \text{ (inflation elastic form is assumed as 0)} \tag{392}$$

$$\Rightarrow \tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1}) \text{ (dynamic IS curve, DIS)} \tag{393}$$

7.3 Equilibrium of model

Now NKPC and DIS each represents supply and demand. The two equations have three unkown variables $\hat{\pi}_t, \tilde{y}_t, \hat{i}_t$, applying the model does not have a solution. It is so that we need to add another variable into the model. There are two ways making DNK model solvable, if not monetary supply rule then Taylor rule.

7.3.1 DNK with money supply rule

$$\text{DNK} \begin{cases} \hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + k\tilde{y}_t \text{ (NKPC)} \\ \tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1}) \text{ (DIS)} \\ \hat{m}_t = \hat{p}_t + \hat{y}_t - \eta\hat{i}_t \text{ (Money supply rule, used to be ad hoc in RBC)} \end{cases} \tag{394}$$

Seems now three equations have even more unkown variables $\hat{\pi}_t, \tilde{y}_t, \hat{i}_t, \hat{p}_t, m_t$.
But hat $\pi_t, \hat{p}_t$ are the same thing because

$$\hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1} \tag{395}$$

$$\Rightarrow \hat{p}_t = \hat{\pi}_t + \hat{p}_{t-1} \tag{396}$$

$$\Rightarrow \hat{m}_t = \hat{\epsilon}_t \tag{397}$$

this transfer works either way.
Let the m_t money supply be exogenous for simplicity.

> **Note 30** (Exogenous variables in DNK model). *Only technology and money supply are considered real exogenous variables here.*

So we are left with three equations and three unkowns, the model is now solutionable. The final result will be a function of m_t.

7.3.2 DNK with interest rate rule

$$\text{DNK} \begin{cases} \hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + k\tilde{y}_t \text{ (NKPC)} \\ \tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1}) \text{ (DIS)} \\ \hat{i}_t = \phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t + \nu_t \text{ (Interest rate rule)} \end{cases} \tag{398}$$

$$\text{where } \nu_t = \rho_v\nu_{t-1} + \epsilon_t^\nu \text{ is the exogenous shock of interest rate, with the feature of AR1} \tag{399}$$

Note 31 (Interest rate rule)**.**

$$\hat{i}_t = \phi_\pi \cdot (\hat{\pi}_t - \hat{\pi}^*) + \phi_y \cdot (\hat{y}_t - y_t^f) + \nu_t \tag{400}$$

If the target parameters $\hat{\pi}^*$ and y_t^f are given, it is Taylor Rule.
Operation parameters ϕ_π and ϕ_y are decided based on past experience.

Acquiring equilibrium and prove it is the only equilibrium
Rewrite the DIS

$$\Rightarrow \tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}[(\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t + \nu_t) - E_t\hat{\pi}_{t+1}] \text{ (apply interest rate rule to the second equation)} \tag{401}$$

$$= E_t\tilde{y}_{t+1} - \frac{1}{\sigma}\{[\phi_\pi \cdot (\beta E_t\hat{\pi}_{t+1} + k\tilde{y}_t) + \phi_y\hat{y}_t + \nu_t] - E_t\hat{\pi}_{t+1}\} \text{ (put the first in)} \tag{402}$$

$$= E_t\tilde{y}_{t+1} + \frac{1-\beta\phi_\pi}{\sigma}E_t\pi_{t+1} - \frac{\phi_\pi k + \phi_y}{\sigma}\tilde{y}_t - \frac{\nu_t}{\sigma} \tag{403}$$

$$= \frac{1}{\sigma + \phi_\pi k + \phi_y}[\sigma E_t\tilde{y}_{t+1} + (1-\beta\phi_\pi)E_t\hat{\pi}_{t+1} - \nu_t] \text{ (the relationship of } y_t \text{ and } y_{t+1} \text{ and } \pi_{t+1}\text{)} \tag{404}$$

Similarly we can change the NKPC

$$\hat{\pi}_t = \frac{1}{\sigma + \phi_\pi k + \phi_y}\{k\sigma E_t\tilde{y}_{t+1} + [k + \beta(\sigma + \phi_y)]E_t\hat{\pi}_{t+1} - \nu_t\} \tag{405}$$

Therefore three equations are two now.
Iteration in vector form

$$\begin{bmatrix}\tilde{y}_t \\ \hat{\pi}_t\end{bmatrix} = \frac{1}{\sigma + \phi_\pi k + \phi_y}\begin{bmatrix}\sigma & 1-\beta\phi_\pi \\ k\sigma & k + \beta(\sigma+\phi_y)\end{bmatrix}\begin{bmatrix}E_t\hat{y}_{t+1} \\ E_t\hat{\pi}_{t+1}\end{bmatrix} - \frac{1}{\sigma + \phi_\pi k + \phi_y}\begin{bmatrix}1 \\ k\end{bmatrix}\nu_t \tag{406}$$

$$\Rightarrow \begin{bmatrix}\tilde{y}_t \\ \hat{\pi}_t\end{bmatrix} = A\begin{bmatrix}C_{t-1} \\ K_{t-1}\end{bmatrix} - B\nu_t \text{ (at RHS one period ahead meaning the whole equation is foresight-seeing)} \tag{407}$$

$$\Rightarrow \lambda^2 - \frac{\sigma + k + \beta(\sigma + \phi_y)}{\sigma + \phi_\pi k + \phi_y}\lambda + \frac{\sigma\beta}{\sigma + \phi_\pi k + \phi_y} = 0 \tag{408}$$

Note 32 (Technique in Ramsey and DNK)**.**
From the perspective of techniques in persuiing solution, it is greatly similar to the way we did with Ramsey model, where the model is one period backward at RHS.

Note 33 (HNKPC)**.** *Hybrid New Keynesian Philips Curve*
Since NKPC lacks persistence in practical use (shocks are too slow to effects), we might as well add a 1-period-lagged inflation to the RHS as adpative expectation. This is somehow a nominal shock since it gets into the DNK model through m_t.

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + k\tilde{y}_t \text{ (NKPC)} \tag{409}$$
$$\hat{\pi}_t = \hat{\pi}_{t-1} + \beta E_t \hat{\pi}_{t+1} + k\tilde{y}_t \text{ (HNKPC)} \tag{410}$$

Note 34 (BK conditions)**.**
First brought up by Blanchard and Kahn, 1980
1)
2)
3)

[3]

Note 35.
Lasalle, 1986

$$x^2 + bx + c = 0 \tag{411}$$
$$x_1, x_2 < 1 \iff \begin{cases} |c| < 1 \\ |b| < 1 + c \end{cases} \tag{412}$$

If the equilibrium exists and it is the only one, then

$$\begin{cases} |\frac{\sigma\beta}{\sigma+\phi_\pi k+\phi_y}| < 1 \Rightarrow \sigma(\beta-1) < 0 < \phi_\pi k + \phi_y \\ |\frac{\sigma+k+\beta(\sigma+\phi_y)}{\sigma+\phi_\pi k+\phi_y}| < 1 + \frac{\sigma\beta}{\sigma+\phi_\pi k+\phi_y} \Rightarrow k(\phi_\pi - 1) + (1-\beta)\phi_y > 0 \end{cases} \tag{413}$$

Method of Undetermined Coefficients:
Let's assume

$$\begin{cases} \tilde{y}_t = \psi_{y\nu}\nu_t \\ \hat{\pi}_t = \psi_{\pi\nu}\nu_t \end{cases} \tag{414}$$

recall that

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + k\hat{y}_t \text{ (expected inflation)} \tag{415}$$

[3] Also see Nakajima 2007 for similar techniques.

$$
\begin{aligned}
\psi_{\pi\nu}\nu_t &= \beta E_t(\psi_{\pi\nu}\nu_{t+1}) + k\hat{y}_t \text{ (iterating)} && (416)\\
&= \beta E_t[\psi_{\pi\nu}(\rho_\nu\nu_t + \epsilon^\nu_{t+1})] + k\hat{y}_t \text{ (expand } \nu_{t+1} \text{ since it's AR1 shock of } i_t) && (417)\\
&= \beta\psi_{\pi\nu}\rho_\nu\nu_t + k\psi_{y\nu}\nu_t \text{ (preassuming expectation of next period noise is 0)} && (418)\\
&\Rightarrow (1-\beta\rho_\nu)\psi_{\pi\nu} = k\psi_{y\nu} && (419)\\
&\Rightarrow k\psi_{y\nu} = \frac{1-\beta\rho_\nu}{k}\psi_{\pi\nu}\text{relationship of two undetermined coefficients} && (420)
\end{aligned}
$$

Using DIS

$$
\tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}[\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t + \nu_t) - E_t\hat{\pi}_{t+1}] \tag{421}
$$

using interest rate rule

$$
\begin{aligned}
&\Rightarrow \psi_{y\nu}\nu_t = \psi_{y\nu}E_t\nu_{t+1} - \frac{1}{\sigma}(\phi_\pi\psi_{\pi\nu}\nu_t + \phi_y\psi_{y\nu}\nu_t + \nu_t - \psi_{\pi\nu}E_t\nu_{t+1}) && (422)\\
&= \psi_{y\nu}\rho_\nu\nu_t - \frac{1}{\sigma}(\phi_\pi\psi_{\pi\nu}\nu_t + \phi_y\psi_{y\nu}\nu_t + \nu_t - \psi_{\pi\nu}\rho_\nu\nu_t) && (423)\\
&= \frac{\sigma\rho_\nu - \phi_y}{\sigma}\phi_{y\nu}\nu_t - \frac{\phi_\pi - \rho_\nu}{\sigma}\psi_{\pi\nu}\nu_t - \frac{1}{\sigma}\nu_t && (424)\\
&\Rightarrow \sigma\psi_{y\nu} = (\sigma\rho_\nu - \phi_y)\psi_{y\nu} - (\phi_\pi - \rho_\nu)\psi_{\pi\nu} - 1 \text{ (elimination of } \nu_t) && (425)\\
&\Rightarrow (\sigma - (\sigma\rho_\nu - \phi_y))\psi_{y\nu} + (\phi_\pi - \rho_\nu)\psi_{\pi\nu} = -1 && (426)\\
&\Rightarrow \psi_{\pi\nu} = \frac{-k}{\dots} = -k\wedge_\nu && (427)\\
&\Rightarrow \psi_{y\nu} = -(1-\beta\rho_\nu)\wedge_\nu && (428)
\end{aligned}
$$

7.4 Analyze exogenous shocks

Now policy change will reflect in the parameter ϵ, making policy analysis possible.

$$
\left.\begin{array}{r}\text{we use } \nu \text{ to denote the exogenous shock}\\ \text{last section we have expression of } \nu\end{array}\right\} \Rightarrow \text{to analyze the shock would mean fully using it.}
$$

> **Note 36** (what is a shock?)**.**
> In a model with uncertainty, variable has the feature of causing change in variation by change itself can cause a shock.
> But it was later broadened as change of exogenous variable in a model without uncertainty.

recall output gap at equilibrium

$$
\begin{aligned}
\hat{y}_t - \hat{y}^f_t &= \hat{y}^*_t = \psi_{y\nu}\nu_t && (429)\\
&= -(1-\beta\rho_\nu)\wedge_\nu \nu_t && (430)\\
&= -(1-\beta\rho_\nu)\wedge_\nu (\rho_\nu\nu_{t-1} + \epsilon^\nu_t) && (431)\\
&= -(1-\beta\rho_\nu)\wedge_\nu \rho_\nu\nu_{t-1} - (1-\beta\rho_\nu)\wedge_\nu \epsilon^\nu_t && (432)\\
&= \rho_\nu\tilde{y}_{t-1} - (1-\beta\rho_\nu)\wedge_\nu \epsilon^\nu_t \text{ (}\tilde{y}_{t-1} \text{ backward one period)} && (433)
\end{aligned}
$$

Note 37. *we can only do analysis at equil*

similiar with inflation at equilibrium

$$\hat{\pi}_t^* = \psi_{\pi\nu}\nu_t \quad (434)$$
$$= -k \wedge_\nu \nu_t \quad (435)$$
$$= -k \wedge_\nu (\rho_\nu \nu_{t-1} + \epsilon_t^\nu) \quad (436)$$
$$= \rho_\nu \pi_{t-1} - k \wedge_\nu \epsilon_t^\nu \quad (437)$$

interest rate at equilibrium

$$\hat{r}_t^* = \hat{i}_t - E_t \hat{\pi}_{t+1} \text{ (Real interest rate at equilibrium)} \quad (438)$$
$$= \sigma(E_t \tilde{y}_{t+1} - \hat{y}_t) \leftarrow \text{DIS} \quad (439)$$
$$= \sigma(1 - \beta\rho_\nu)(1 - \rho_\nu) \wedge_\nu \nu_t \quad (440)$$
$$\Rightarrow \hat{i}_t^* = \hat{r}_t + E_t \hat{\pi}_{t+1} \text{Nominal interest rate at equilibrium} \quad (441)$$
$$= [\sigma(1 - \beta\rho_\nu)(1 - \rho_\nu) - k\rho_\nu] \wedge_\nu \nu_t \quad (442)$$

moreever we can do analysis with money supply using quantity equation of money

$$\hat{m}_t^* = \hat{p}_t^* + \hat{y}_t^* - \eta \hat{i}_t^* \text{ (LHS is solvable since all variables at RHS has been solved)} \quad (443)$$
$$= \text{ (assuming } \hat{y}_t^{f*} = 0) \quad (444)$$
$$= \quad (445)$$
$$= \text{ use solutions from above} \quad (446)$$
$$\Rightarrow \quad (447)$$
$$= \quad (448)$$
$$\frac{d\hat{m}_t^*}{d\nu_t} = \quad (449)$$
$$= \text{showing the result is decided by mutiple joint forces} \quad (450)$$

Note 38 (quantity equation of money)**.** $MV = Py$
While this theory was originally formulated by Polish mathematician Nicolaus Copernicus in 1517, it was popularized later by economists Milton Friedman and Anna Schwartz after the publication of their book, "A Monetary History of the United States, 1867-1960," in 1963.

7.5 Optimal monetary policy

A significant paper Mankiw and Reis 2002 used a new keynesian macroeconomic model to acquire the optimal monetary policy. [4]

[4] N. Gregory Mankiw, Ricardo Reis, Sticky Information versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve, The Quarterly Journal of Economics, Volume 117, Issue 4, November 2002, Pages 1295–1328, https://doi.org/10.1162/003355302320935034

Note 39 (why is there welfare loss?)**.**
monopoly and sticky price cause deviation from Pareto optimality.

Note 40 (Optimal monetary policy)**.**
No concept of optimality can be found in policy analysis in the perspective of household or firm. But the government can use welfare loss fucntion in its optimality to maximize social welfare. In the process of achieving maximized social welfare, the monetary policy is the so-called optimal monetary policy.

To acquire the optimal monetary policy, the welfare loss function must be previously introduced.
To acquire WLF, we use second order taylor expansion
first order Taylor expansion

$$\frac{X_t - X}{X} = \ln X_t - \ln X = x_t - x = \hat{x}_t \tag{451}$$

second order Taylor expansion

$$\begin{aligned} \frac{X_t - X}{X} &= \frac{1}{X} e^{x_t} - 1 \\ &= (\frac{1}{X} e^{x} - 1) + \frac{1}{X} e^{x} (x_t - x) + \frac{1}{2} \frac{1}{X} e^{x} (x_t - x)^2 \\ &= \hat{x}_t + \frac{1}{2} \hat{x}_t^2 \end{aligned} \tag{452}$$

Similarly

$$\Rightarrow \begin{cases} \frac{C_t - C}{C} \approx \hat{c}_t + \frac{1}{2} \hat{c}_t^2 \\ \frac{N_t - N}{N} \approx \hat{n}_t + \frac{1}{2} \hat{n}_t^2 \end{cases} \tag{453}$$

Bi-variate 2o taylor expansion for a utility function where people gain utilities from consuming goods and enjoying leisure.

$$u(C_t, N_t) \approx u(C, N) + u_C^C (\frac{C_t - C}{C}) + u_N^N (\frac{N_t - N}{N}) + \frac{1}{2} u_{CC}^{C^2} \frac{(C_t - C)^2}{C} + \frac{1}{2} u_{NN}^{N^2} \frac{(N_t - N)^2}{N} \tag{454}$$

$$= u(C, N) + u_C C (\hat{c}_t + \frac{1}{2} \hat{c}_t^2) + u_N N (\hat{n}_t + \frac{1}{2} \hat{n}_t^2) + \frac{1}{2} u_{CC} C^2 (\hat{c}_t + \frac{1}{2} \hat{c}_t^2) + \frac{1}{2} u_{NN} N^2 (\hat{n}_t + \frac{1}{2} \hat{n}_t^2) \tag{455}$$

$$= u(C, N) + u_C C (\hat{c}_t + \frac{1}{2} \hat{c}_t^2) + u_N N (\hat{n}_t + \frac{1}{2} \hat{n}_t^2) + \frac{1}{2} u_{CC} C^2 \hat{c}_t + \frac{1}{2} u_{NN} N^2 \hat{n}_t \text{ (reservate only to 2o)} \tag{456}$$

$$= u(C, N) + u_C C (\hat{c}_t + \frac{1 + \frac{u_{CC}}{u_C} C}{2} \hat{c}_t) + u_N N (\hat{n}_t + \frac{1 + \frac{u_{NN}}{u_N} N}{2} \hat{n}_t) \tag{457}$$

$$\Rightarrow \frac{u(C_t,N_t)-u(C,N)}{u_C C} = (\hat{c}_t + \frac{1+\frac{u_{CC}}{u_C}C}{2}\hat{c}_t^2) + \frac{u_N N}{u_C C}(\hat{n}_t + \frac{1+\frac{u_{NN}}{u_N}}{2}\hat{n}_t^2) \tag{458}$$

$$\because \hat{c}_t = \hat{y}_t \text{ (market clearing at equilibrium)} \tag{459}$$

$$u(C_t,N_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{N_t^{1+\varphi}-1}{1+\varphi} \iff \begin{cases} -\frac{u_{CC}}{u_C}C = \sigma \\ -\frac{u_{NN}}{u_N}N = \varphi \end{cases} \tag{460}$$

> **Note 41** (Coefficients of CES function)**.**
> σ is intertemporal substitute elastic, φ is frisch elasticity

continue to the last taylor expanded eq, introducing var_i as the deviation of each firm's price

$$\ldots = (\hat{y}_t + \frac{1-\sigma}{2}\hat{y}_t^2) + \frac{u_N N}{u_C C}\{\frac{1}{\alpha}(\hat{y}_t - a_t) + A\cdot var_i P_{it}\} + \frac{1+\varphi}{2}[\frac{1}{\alpha}(\hat{y}_t - a_t) + A\cdot var_i P_{it}]^2 \tag{461}$$

$$= (\hat{y}_t + \frac{1-\sigma}{2}\hat{y}_t^2) + \frac{u_N N}{u_C C}\frac{1}{\alpha}\{[(\hat{y}_t - a) + B\cdot var_i P_{it}] + \frac{1+\varphi}{2\alpha}(\hat{y}_t - a_t)^2\}, B \equiv \alpha A \tag{462}$$

, here let's ignore many fluctuation terms from taylor expansion as they are extremely small and irrelevant to monetary policy.

$$-\frac{U_N}{U_C} = MP\cdot N = \alpha A N^{\alpha-1} \tag{463}$$

$$= \alpha\frac{C}{N} \Rightarrow \frac{u_N N}{u_C C} = -\alpha \text{ (can use this to simplify equation before)} \tag{464}$$

go back to before

$$\Rightarrow \ldots = (\hat{y}_t - \frac{1-\sigma}{2}\hat{y}_t^2) - \{(\hat{y}_t - a_t) + B\cdot var_i P_{it} + \frac{1+\varphi}{2\alpha}(\hat{y}_t - a_t)^2\} \tag{465}$$

$$= \frac{1-\sigma}{2}\hat{y}_t^2 - B\cdot var_i P_{it} - \frac{1+\varphi}{2\alpha}(\hat{y}_t - a_t)^2 \tag{466}$$

$$= \frac{1-\sigma}{2}\hat{y}_t^2 - B\cdot var_i P_{it} - \frac{1+\varphi}{2\alpha}(\hat{y}_t^2 - 2\hat{y}_t a_t + a_t^2) \tag{467}$$

$$= (\frac{1-\sigma}{2} + \frac{1+\varphi}{2\alpha})\hat{y}_t^2 + \frac{\sigma+\varphi}{\alpha}\hat{y}_t a_t - B\cdot var_i P_{it} \tag{468}$$

$$= \frac{\alpha(1-\alpha)+1+\varphi}{2\alpha}\hat{y}_t^2 + \frac{\sigma+\varphi}{\alpha}\hat{y}_t\hat{y}_t^f - B\cdot var_i P_{it} \text{ (remember when we work out the } \hat{y}_t^f) \tag{469}$$

$$= \underbrace{-\frac{1}{2}[\underbrace{C(\hat{y}_t - \hat{y}_t^f)+}_{\text{deviation from output}} \underbrace{D(\hat{\pi}_t - \pi^*)^2}_{\text{inflation}}]}_{\text{welfare loss}} \text{ (welfare loss function)} \tag{470}$$

the goal is to minimize the whole term

Note 42 (paradoxy in welfare loss function).
the two goals are contradicated according to philips curve

8 Sticky wage model

Before good market is price-sticky and monopolistic competitive, now the setting goes to labor market.

$$Y_{it} = (\frac{P_{it}}{P_t})^[-\epsilon]Y_t \text{ (good demand curve, } \epsilon \text{ is substitute elasticity)} \tag{471}$$

$$P_t = (\int_0^1 P_{it}^{1-\epsilon} di)^{\frac{1}{1-\epsilon_p}} \text{ (total price)} \tag{472}$$

$$W_t = \int_0^1 W_{it} di =_{it} \tag{473}$$

$$P_{it}^* = c + \alpha \tilde{y}_t \text{ (desired price)} \tag{474}$$

labor market m.c. (as in classical model) and sticky wage (partial rigidity, in contrary to total rigidity where all wages are rigid).
1) labor demand curve and total wage price level

$$\begin{aligned} &\min \int_0^1 W_{jt} N_{ijt} dj \\ &s.t. A_t[(\int_0^1 N_{ijt}^{\frac{\epsilon_W - 1}{\epsilon_W}} dj)^{\frac{\epsilon_W}{\epsilon_W - 1}}]^{1-\alpha} \geqslant A_t N_{it}^{1-\alpha} \end{aligned} \tag{475}$$

where i indexing firm, j indexing labor, t is time
Form a Lagrangian

$$\mathscr{L} = \int_0^1 W_{jt} N_{ijt} dj + W_t \{A_t[(\int_0^1 N_{ijt}^{\frac{\epsilon_w - 1}{\epsilon_w}} dj)^{\frac{\epsilon_W}{\epsilon_W - 1}}]^{1-\alpha} - A_t N_{it}^{1-\alpha}\} \tag{476}$$

where Lagrangian multiplier is total wage level
Solving the optimization problem results

$$\Rightarrow \begin{cases} N_{ijt} = (\frac{W_{jt}}{W_t})^{-\epsilon_W} N_{it} \text{ (labor supply equation)} \\ W_t = (\int_0^1 W_{jt}^{1-\epsilon_W} dj)^{\frac{1}{1-\epsilon_W}} \text{ (expression of total wage level)} \end{cases} \tag{477}$$

2) UMP of household
in utility function N is labor demand instead of labor supply as before. this is because m.c. labor market.

$$\begin{aligned} u &= u(C_t, N_t) \\ &= u(C_t, \{N_{jt}\}) \\ &= \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \int_0^1 \frac{N_j^{1+\varphi}}{1+\varphi} dj \text{ (CES style)} \end{aligned} \tag{478}$$

the UMP can be rewritten as

$$\begin{aligned} \max_{\{C_t, W_t^*\}} \sum_{t-0}^{\infty} \beta^t u(C_t, \int_0^1 N_{jt} dj) \iff \max \sum_{t=0}^{\infty} \beta^t u(C_t, \int_0^1 (\frac{W_t^*}{W_t})^{-\epsilon_W} N_{it} dj) \\ s.t. P_t C_t + Q_t B_t \leqslant \int_0^1 W_t^* (\frac{W_t^*}{W_t})^{-\epsilon_W} N_{it} dj + B_{t-1} \end{aligned} \tag{479}$$

here W_t^* instead of original because all wages can be adjusted, eventually they will be the same
a simplified version would be

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t u(C_t, (\frac{W_t^*}{W_t})^{-\epsilon_W} N_{it}) \\ &s.t. P_t C_t + Q_t B_t \leqslant W_t^* (\frac{W_t^*}{W_t})^{-\epsilon_W} + B_{t-1} - T_t \end{aligned} \tag{480}$$

$$\begin{aligned} &\max_{W_0^*} \sum_{t=0}^{\infty} \beta^t \theta_W^t u(C_{t|0}, N_{t|0}) \\ &s.t. \begin{cases} C_{t|0} = \frac{1}{P_t}[W_0^* (\frac{W_0^*}{W_t})^{-\epsilon_W} N_{it} + B_{t-1|0} - T_{t|0} - Q_{t-1|t} B_{t|0}] \\ N_{t|0} = (\frac{W_0^*}{W_t})^{-\epsilon_W} N_{it} \text{ (with labor demand curve, left with only } W_0^*) \end{cases} \end{aligned} \tag{481}$$

sticky wage, where $\theta \equiv Prob(\text{stay put})$. θ is before utility function because the whole equation is based on period 0, entailing a prob for the whole eq to stay put.
put two constraints in the max part, now the opt problem is a unconstrainted one

$$\max_{W_0^*} \sum_{t=0}^{\infty} (\beta\theta_W)^t u(\frac{1}{P_t}[W_0^* (\frac{W_0^*}{W_t})^{-\epsilon_W} N_{it} + B_{t-1|0} - T_{t|0} - Q_{t-1|t} B_{t|0}], (\frac{W_0^*}{W_t})^{-\epsilon_W} N_{it}) \tag{482}$$

$$FOC \Rightarrow E_0 \sum_{t=0}^{\infty} (\beta\theta_W)^t [u_C(C_{t|0}, N_{t|0}) \frac{1}{P_t} (1-\epsilon_W) (\frac{W_0^*}{W_t})^{-\epsilon_W} N_{it} - u_N(C_{t|0}, N_{t|0}) \epsilon_W (\frac{W_0^*}{W_t})^{-(\epsilon_W)^{-1}} \frac{1}{W_t} N_{it})] \tag{483}$$

$$= E_0 \sum_{t=0}^{\infty} (\beta\theta_W)^t [u_C(C_{t|0}, N_{t|0}) \frac{1}{P_t} (1-\epsilon_W) (\frac{W_0^*}{W_t})^{-\epsilon_W} W_t^{\epsilon_W} N_{it} - u_N(C_{t|0}, N_{t|0}) \epsilon_W (\frac{W_0^*}{W_t})^{-(\epsilon_W)} W_t^{\epsilon_W} N_{it} \frac{1}{W_0^*})] \tag{484}$$

move some exogenous variables out since they basically can be treated as constants

$$\Rightarrow (1-\epsilon_W)(W_0^*)^{-\epsilon_W} \dots \tag{485}$$

$$= \epsilon_W (W_0^*)^{-\epsilon_W^{-1}} \tag{486}$$

$$\Rightarrow W_0^* = \frac{\epsilon_W}{1-\epsilon_W} \frac{E_0 \sum_{t=0}^{\infty} (\beta\theta_W)^t u_N(C_{t|0}, N_{t|0}) W_t^{\epsilon_W} N_{it}}{E_0 \sum_{t=0}^{\infty} (\beta\theta_W)^t u_C(C_{t|0}, N_{t|0}) \frac{1}{P_t} W_t^{\epsilon_W} N_{it}} \tag{487}$$

use marginal substitute ratio, where $MRS_{t|0} \equiv -\frac{U_N(C_{t|0})}{U_C(C_{t|0})}$, the eq is essentially an inflation eq

$$W_0^* = \frac{\epsilon_W}{\epsilon_W - 1} \frac{E_0 \sum_{t=0}^{\infty} (\beta\theta_W)^t u_N(C_{t|0}, N_{t|0}) MRS_{t|0} W_t^{\epsilon_W} N_{it}}{E_0 \sum_{t=0}^{\infty} (\beta\theta_W)^t u_C(C_{t|0}, N_{t|0}) \frac{1}{P_t} W_t^{\epsilon_W} N_{it}} \tag{488}$$

$$\Rightarrow \frac{W_0^*}{P_0} = \frac{\epsilon_W}{\epsilon_W - 1} MRS_{t|0} \text{ (if elastic wage and m.c., there will be no eq above)} \tag{489}$$

now based on the perspective at period t

$$W_t^* = \frac{\epsilon_W}{\epsilon_W - 1} \frac{E_t \sum_{k=0}^{\infty} (\beta\theta_W)^k u_C(C_{t+k|t}, N_{t+k|t}) MRS_{t+k|t} W_{t+k|t}^{\epsilon_W}}{E_t \sum_{k=0}^{\infty} (\beta\theta_W)^k u_C(C_{t+k|t}, N_{t+k|t}) \frac{1}{P_{t+k}} W_{t+k|t}^{\epsilon_W}} \text{ (at t period instead 0 period)} \tag{490}$$

$$\Rightarrow \frac{W_t^*}{P_t} = \frac{\epsilon_W}{\epsilon_W - 1} MRS \text{ (if elasticity wage and monopolistic competitive at t period)} \tag{491}$$

Note 43 (M.C. firm choice variable for opt problem)**.** *MC firm can choose price max or quantity max*

9 Final exam on June 17

> **Note 44** (exam hints).
> 1. RBC and Ramsey: endogenous variables in two models? what are the differences (growth and fluctuation) and commons (microfounded) of the two models? Lagrangian solution? Behavioral equations?
> 2. DNK and Ramsey:
> difference: one forward-looking dif with expectation; one backward-looking
> Ramsey's and DNK's convergence
> 3. Solow, Ramsey, OLG: inefficient to efficient to inefficient
> 4. monetary policies: money supply vs interest rate; what is natural output rate
> 5. sticky price and sticky wage: one more result eq for latter making more endogenous variables possible.

9.1 jian da ti, 15'x2

9.1.1 RBC model (6 eqs) vs classical model (7 eqs), match similar eqs and differ

9.1.2 ?

9.2 jisuan tuidao ti, 20'x2

9.2.1 second way to deduct NKPC, three equations, acceptable price setting eq

9.2.2 sticky price and sticky wage to labor demand eq, opt problem, with hints

9.3 lunshuti, 30'

dynamic inefficiency: solow to ramsey to olg