Notes on Advanced Microeconomics

Victor Li

Autumn semester, 2023

Contents

Preface

Aim of this work of mine is to provide clear macroeconomic knowledges in language as simple as possible. Most of this note is taken from the advanced macroeconomics course of Yanfei Deng I took. This note is still a work in progress and needs tons of ammending. See the newest version at `https://github.com/xolarvill/notes-on-economics` along with my other notes. And pull a request if you figure some part of this note is wrong.

This work was first written in markdown language on Obsidian, which provides a lightweight and highly personalized working experience. But it was later transfered into purely latex language on Sublime Text, due to poor support for rendering of long math equation blocks of Obsidian. For the tranfer I used Pandoc at `https://pandoc.org/`. I wrote some LaTeX snippets of Sublime Text to optimize the writing experience, and uploaded on github at `https://github.com/xolarvill/snippets-for-quick-latex-on-st`.

One can find enormous academical help on advanced macroeconomics from books listed below:

- Introduction to Modern Economic Growth, D. Acemoglu
- Economic Growth, R. Barro and S.i.M. Xavi
- The ABCs of RBCs, An Introduction to Dynamic Macroeconomic Models, G. McCandless
- Macroeconomics, A Comprehensive Textbook for First-Year Ph.D. Courses in Macroeconomics. M. Azzimonti, P. Krusell, A. McKay, and T. Mukoyama
- Adavanced Macroeconomics, D. Romer

Part I
Consumer and demand theory

1 Choice set

A single choice or a single consuming plan is denotes as

$$x = (x_1, x_2 ... x_n) \in \mathbb{R}^n_+$$

A choice set or a consuming set is a set of possible and mutually exclusive alternative consuming plans, denoted as

$$X = \mathbb{R}^n_+$$

It represents all plans of consuming goods, regardless of possibility.
A choice set should at least fulfil the following requirements:
1. choice set is not a null set

$$\phi \neq X \subseteq \mathbb{R}^n_+$$

2. X is closed, in this way, also continuent.
3. X is convex, denoted as

$$\forall x^1, x^2 \in X, \forall \lambda \in [0,1], \lambda x^1 + (1-\lambda)x^2 \in X$$

4. X can contain zero, meaning the choice maker can choose to not consume.

$$0 \in X$$

2 Preference relationship

Preference relationship is a binary relationship defined on X: in the choice set X, $\forall x, y, z \in X$, preference relationship (at least as good as) is denoted as

$$x \succsim y$$

> **Note 1.** *Preference relationship $\succsim$ is defined on X, so its minimum unit is a consuming plan $x = (x_1, x_2, ..., x_n)$. e.g.: $x_1 \succsim x_2 : (x^1 = 1, x^2 = 3) \succsim (x^1 = 5, x^2 = 1)$*

Restrict preference relationship is denoted as

$$x \succ y \iff x \succsim y,\ y \neg \succsim x$$

Indifference preference relationship is denoted as

$$x \sim y \iff x \succsim y, y \succsim x$$

Rational preference
Given a relationship $\succsim$ in a non-empty choice set X, we call $\succsim$ a rational preference if it possesses three properties(axioms):

$$1.(transitivity)\ x \succsim y, y \succsim z \Rightarrow x \succsim z \tag{1}$$

$$2.(completeness)\ x \succsim y \vee y \succsim x \tag{2}$$

$$3.(reflectivity)\ \forall x \in X, x \succsim x \tag{3}$$

> **Note 2.** *Many pyschological experiments ended up with results overthrowing the transitivity axiom. But for now, we assume it to hold true.*

Some propositions: if the $\succsim$ is rational:

Proposition 1 (i). *$\succ$ is not reflective($x \succ x$ does not hold true) but transitive($x \succ y, y \succ z \Rightarrow x \succ z$). This means $\succ$ is not rational.*

Proof.

$$Let\ \exists x \in X, x \succ x \tag{4}$$
$$x \succ x \Rightarrow x \succsim x \wedge x \neg \succsim x \tag{5}$$
$$\Rightarrow \text{Paradox;} \blacksquare \tag{6}$$
$$x \succ y \Rightarrow x \succsim y \wedge y \neg \succsim x \tag{7}$$
$$y \succ z \Rightarrow y \succsim z \wedge z \neg \succsim y \tag{8}$$
$$x \succsim y \succsim z(*), \tag{9}$$
$$if\ z \succsim x\ while\ x \succsim y \Rightarrow z \succsim y \Rightarrow Paradox \Rightarrow z \neg \succsim x(**), \tag{10}$$
$$(*)(**) \Rightarrow x \succ z; \blacksquare \tag{11}$$

□

Proposition 2 (ii). *$\sim$ is reflective($x \sim x$) and transitive($x \sim y, y \sim z \Rightarrow x \sim z$) and symmetric($x \sim y \Rightarrow y \sim x$). This means $\sim$ is not rational.*

Proof.

$$x \succsim x \Rightarrow x \sim x; \blacksquare \tag{12}$$
$$x \sim y \iff x \succsim y,\ y \succsim x \tag{13}$$
$$\Rightarrow x \sim y, y \sim z \Rightarrow x \sim z; \blacksquare \tag{14}$$
$$x \sim y \iff x \succsim y,\ y \succsim x \wedge y \succsim x,\ x \succsim y \blacksquare \tag{15}$$

□

Proposition 3 (iii). $x \succ y \succsim z \Rightarrow x \succ z$

Proof.

$$x \succ y : x \succsim y \wedge y \neg \succsim x \tag{16}$$
$$y \succsim z \tag{17}$$
$$x \succsim y \succsim z(*); \blacksquare \tag{18}$$
$$if\ z \succsim x, \tag{19}$$
$$y \succsim z \Rightarrow y \succsim x \Rightarrow Paradox \Rightarrow z \neg \succsim x(**); \blacksquare \tag{20}$$
$$(*)(**) \Rightarrow x \succ z; \blacksquare \tag{21}$$

□

3 Utility function

A utility function is denoted as

$$u : X \to \mathbb{R} \tag{22}$$
$$\forall x, y \in X, x \succsim y \iff u(x) \geqslant u(y) \tag{23}$$

One preference relationship can be represented by many distinct utility functions.

Question 1. *If $f : R \to R$ is strictly increasing and $u : X \to R$ is the utility function of a preference information $\succsim$, then $v(x) = f(u(x)), v : x \to R$ denotes a utility function for $\succsim$ as well. **Answer** f is monotonic function, it won't change u as a utiliy function. The $f(u(x))$ and $u(x)$ represent the same preference.*

Proposition 4. *If a preference relationship can be denoted by a utility function, it must be rational.*

Proof.

$$\begin{aligned}
\text{(Completeness) } & u : X_+^n \to R, && (24)\\
& x \to x_1, y \to x_2 && (25)\\
& \text{meaning either } x_1 \geqslant x_2 \vee x_2 \geqslant x_1; \blacksquare && (26)\\
\text{(Transitivity) } & x \succsim y \iff u(x) \geqslant u(y) && (27)\\
& y \succsim z \iff u(y) \geqslant u(z) && (28)\\
& u(x) \geqslant u(y) \geqslant u(z) \iff x \succsim y \succsim z; \blacksquare && (29)
\end{aligned}$$

□

Utility function is a order-mapping from choice set to 1-dimensional Euclidean space, of course it must be of rational preference.

If a preference is rational, does that mean there is necessarily a utility function that will represent it? The answer is no, unless in the case where X is a finite choice set:

$$X = \{x_1, x_2, ..., x_n\}, n \neq \infty$$

Question 2 (WHY?).
`https://mindyourdecisions.com/blog/2013/05/03/i-am-rational-but-you-cant-model-me-with-a-utilit`
`#:~:text=It%20can%20be%20proven%20that,defined%20by%20a%20utility%`
`20function.` this post explains why a rational preference relationship cannot be neccessarily represented by a utility function, using the special case of lexicographic preference.
`https://economics.stackexchange.com/questions/32392/`
`are-there-any-other-rational-preference-relations-without-utility-function-repre`
this gives some variation of lexicographic preference.

> http://www.econ.ucla.edu/iobara/LecturePreferenceandUtility201A.pdf this is the proving of how rational preference of finite choice can be reprensented by a utility function.

Marginal utility is denoted as

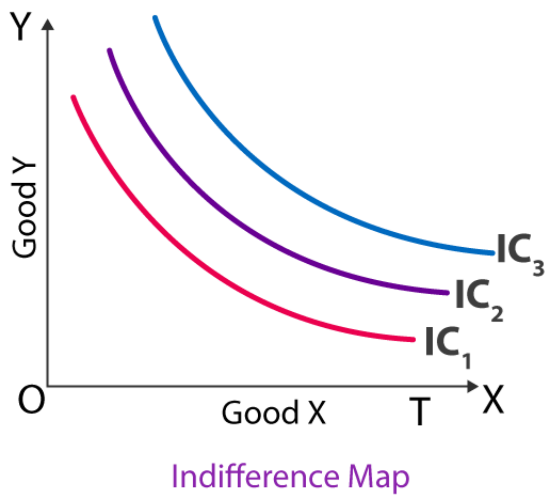
$$MU_i = \frac{\partial u}{\partial x_i}$$

Marginal rate of substitution is denoted as

$$MRS_{ij} = |\frac{dx_j}{dx_i}| = |\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}}|$$

4 Indifference curve

When a preference relationship satisfies all axioms, it can be illustrated in a 2-dimensional space, as indifference curve.



5 Utility maximization

5.1 commodity and budget

Commodity

$$L, (= 1, 2, ..., L)$$

Commodity space

$$\mathbb{R}^L$$

A commodity vector is a point in commodity space

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{pmatrix} \in \mathbb{R}^L$$

Price vector

$$p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{pmatrix} \gg 0 \in \mathbb{R}^L$$

here much greater symbol means scarcity in market economics
Consumer's wealth level

$$w \text{ or } y$$

Affordable restraint

$$p \cdot x \leqslant w \text{ or } p \cdot x \leqslant y$$

Budget set, competitive budget restraint, Walrasian budget set, budget half-space

$$B_{p,y} = \{x \in \mathbb{R}^L_+ : p \cdot x \leqslant y\} \Rightarrow B_{p,y} \text{ is convex}$$

here the restraint is caused by market
Budget hyperplane

$$\{x \in \mathbb{R}^L_+ : p \cdot x = y\}$$

properties of budget set:

Property 1 (1). *$B_{p,w}$ is a concave set*

Proof.

$$x, x' \in B \Rightarrow \alpha x + (1 - \alpha)x' \in B \tag{30}$$

$$x \in \mathbb{R} \tag{31}$$

□

Property 2 (2). *$B_{tp,tw} = B_{p,w}, \forall t > 1$. Using $t = \frac{1}{p_i}$, we can set the commodity i as denominations*

Proof.

$$B_{tp,tw} = \{x \in \mathbb{R}^L_+ : tp \cdot x = tw = p \cdot x = w\} \tag{32}$$

□

Property 3 (3). *$B(p, w) \subset B(p, w'), \forall w' > w;\ B(p, w) \supset B(p', w), \forall p' > p$*

Relationship between price and budget hyperplane

$$p \perp B_{p,y,=}$$

5.2 UMP and deduction

Method (i) The tangent point of indifference curve and budget line is where one can achieve utility maximization.
Method (ii)

$$\begin{aligned} &\max_{\{x\}} \ u(x) \\ &s.t. \ px \leqslant y \end{aligned} \tag{33}$$

Walrasian demand correspondance

$$x(p, w) = C(B_{p,w})$$

⇒ singleton walrasian demand correspondance is a demand function
Marshallian demand function or Walrasian demand function

$$x = x(p, y) = \begin{pmatrix} x(p_1, y_1) \\ x(p_2, y_2) \\ \vdots \\ x(p_L, y_L) \end{pmatrix}$$

it should be

Property 4 (homogeneous of degree zero)**.** *meaning proportional changes in price and wealth will not effect demand*

$$\forall p, w, \alpha > 0 : x(\alpha p, \alpha w) = \alpha^0 x(p, w) = x(p, w)$$

and since it is defined in a Walrasian budget set, where it is homogeneous of degree zero already, the demand function must be homogeneous of degree zero as well

Property 5 (Satisfying Walras' law)**.**

$$\forall p \gg 0[\forall i, p = (p_1, p_2, ..., p_i, ..., p_n), p_i > 0], \tag{34}$$

$$w > 0 \Rightarrow p \cdot x = w, \forall x \in x(p, w) \tag{35}$$

Proof.

$$\text{If } px < w, \exists y \in B(p, w) \tag{36}$$

$$\text{and } y \geq x \Rightarrow py > px \tag{37}$$

$$\Rightarrow y \gtrsim x \tag{38}$$

□

For convience, we can use the property of being homogeneous of degree zero to set price at 1 unit for some commodities or set the wealth level at 1 unit.
Indirect utility function

$$v(p, y) = \max_{\{x\}} \ u(x) \tag{39}$$

$$s.t. \ px \leqslant y \tag{40}$$

Expenditure function

$$e(p,u)$$

Hicks demand function

$$x^h(p,u)$$

Roy's identity

$$-\frac{\frac{\partial v(p,y)}{\partial p}}{\frac{\partial v(p,y)}{\partial y}} = x(p,y)$$

Shephard's lemma

$$\frac{\partial e(p,u)}{\partial p} = x^h(p,u)$$

6 Price change and income change

Engel function

$$If\ p = \bar{p} \in \mathbb{R}^n_+,\ x(\bar{p},w)$$

ICC, Wealth expansion path

$$E_p = \{x(\bar{p},w) : w > 0\} \in \mathbb{R}^L$$

Wealth effect of commodity

$$\frac{\partial x(p,w)}{\partial w}$$

Normal good and inferior good

$$\begin{cases} \frac{\partial x(p,y)}{\partial y} > 0 \Rightarrow \text{normal at } (p,y) \\ \\ \frac{\partial x(p,y)}{\partial y} < 0 \Rightarrow \text{inferior at } (p,y) \end{cases}$$

Total wealth effects

$$\triangledown_w x(p,w) = D_w x(p,w) = \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \frac{\partial x_2(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{pmatrix}$$

Price effect on commodity for price p_k of k

$$\frac{\partial x(p,w)}{\partial p_k}, \text{if } k = \ \text{ then it is called self price effect}$$

Giffen good
For good , it is a Giffen good if

$$\frac{\partial x(p,w)}{\partial p} < 0$$

Complementray good

For good , it is a complementary good if

$$\frac{\partial x(p,w)}{\partial p_j} < 0, \neq j$$

Substitute good
For good , it is a substitute good if

$$\frac{\partial x(p,w)}{\partial p_j} > 0, \neq j$$

Total price effects

$$D_p x(p,w) = \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ & \ddots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} \end{pmatrix}$$

PCC, offer curve

$$x(p, \bar{w})$$

Proposition 5. *If a $x(p,w)$ is homogeneous of degree zero, then CRTS, denoted as*

$$\forall p, w \tag{41}$$

$$p \cdot D_p x + w \cdot D_w x = 0 \tag{42}$$

Proof.

$$x(tp, tw) = x(p,w), \forall = 1, 2, ..., L \tag{43}$$

Take derivatives of t on both side and moves of deduction, we have when $t = 1$ (44)

$$p \cdot D_p x + w \cdot D_w x = 0 \tag{45}$$

□

This mean proportional and simultaneous change in both price and wealth will not change demand.
Recalling that if a production function is homogeneous of degree one, it satisfies both CRTS and product exhaustion theorem. CRTS:

$$f(\lambda x, \lambda y) = \lambda f(x,y), \lambda > 1$$

Product exhaustion theorem

$$x \cdot f_x + y \cdot f_y = f(x,y)$$

Proposition 6 (Cournot Aggregation)**.** *If a demand function fulfils Law of Walras, then $\forall p, w \Rightarrow p_l D_{p_k} x_l(p,w) + x_k(p,w)^T = 0$. In possible way we can imagine this as $p \times \triangle x + (\triangle p = 1) \times x = 0$. meaning the total expenditure is fixed, which is also called the Cournot Aggregation.*

Proof. As same as to prove $p \cdot x(p,w) = w$ □

Proposition 7 (Engel Aggregation)**.** *When a $x(p,w)$ fufils Walrasian Law, $\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1$ or $p \cdot D_w x(p,w) = 1$, which is also called the Engel Aggregation. Meaning the change in total expenditure is equal to the change in wealth.*

7 Elasticity and aggregation

elasticity of demand of good i with respect to wealth w

$$\eta_i = \frac{\partial lnx}{\partial lnw}$$

elasticity of demand of good i with respect to price p_i of itself

$$\epsilon_{ii} = \frac{\partial lnx_i}{\partial lnp_i}$$

elasticity of demand of good i with respect to price p_j of another good j

$$\eta_{ij} = \frac{\partial lnx_i}{\partial lnp_j}$$

percentage of cost of good i in total cost

$$S_i = \frac{px}{w} \Rightarrow \sum_{i=1}^{n} S_i = 1$$

Engel aggregation

$$\sum_{i=1}^{n} S_i \eta_i = 1$$

Cournot aggregation

$$\sum_{i=1}^{n} S_i \epsilon_{ij} = -S_j$$

Substitution matrix

$$S(p,w) = D_p x(p,w) + D_w x(p,w) \cdot X(p,w)^T \tag{46}$$

$$S_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \tag{47}$$

7.1 Duality

8 Uncertainty

9 Revealed preference

9.1 Choice structure

Budget line

Given price vector p and income vector y, a budget set B is defined as $B_{p,w} = \{x | x \in \mathbb{R}_+^n, px \leqslant w\}$. The border of a budget set is the budget line.

A family of budget sets $\mathscr{B}$ is denoted as

$$\{\mathscr{B} | B_1, B_2, ..., B_n\}$$

$\mathscr{B}$ is a set of nonempty subset, the budget set, B_i of the choice set X. It represent all possible choice sets for the choice maker, however not needed for the whole X.

> **Note 3** (Family). *Since $\mathscr{B}$ is a set of sets, it is called a family.*

Choice rule $C(\cdot)$, or a choice function is denoted as

$$C : X \to X \tag{48}$$

$$\forall B \subset \mathscr{B}, C(B) \subset B \tag{49}$$

Notice that $C(\cdot)$ is theoretically non-empty. For every B_i in $\mathscr{B}$, $C(\cdot)$ can always choose the best element(s). A choice structure is denoted as

$$(\mathscr{B}, C(\cdot))$$

Proposition 8. $x, y \in C(\mathscr{B}) \Rightarrow x \sim y$

Proof.

$$x, y \in C(\mathscr{B}) \Rightarrow x, y \in \mathscr{B} \tag{50}$$

$$x \in C(\mathscr{B}) \Rightarrow x \gtrsim y \ (*) \tag{51}$$

$$y \in C(\mathscr{B}) \Rightarrow y \gtrsim x \ (**) \tag{52}$$

$$(*)(**) \Rightarrow x \sim y \tag{53}$$

□

9.2 Weak axiom of revealed preference

WARP providing a reverse way to examine if the preference is rational

$$\exists B_1, B_2 \in \mathscr{B}, \tag{54}$$

$$\exists x, y \in B_1, \tag{55}$$

$$If \begin{cases} x \in C(B_1) \\ x, y \in B_2 \\ y \in C(B_2) \end{cases} \Rightarrow x \in C(B_2) \tag{56}$$

or another version would be

$$\exists B_1, B_2 \in \mathscr{B}, \tag{57}$$

$$\exists x, y \in B_1, \tag{58}$$

$$If \begin{cases} x \in C(B_1) \\ y \in C(B_2) \\ x \notin C(B_2) \end{cases} \Rightarrow x \notin B_2 \tag{59}$$

Either way, this axiom is trying to prove that the decision maker is *self-consistent* or not paradoxy.

A revealed preference relationship $\succsim^*$ is denoted as

$$x \succsim^* y \iff \exists \{B \in \mathscr{B} | x, y \in B \text{ } and \text{ } x \in C(B)\}$$

Noticing that this relationship is neither of completeness nor of transitivity. Meaning $\succsim^*$ need not to be complete or transitive.

> **Note 4.** *Recalling we establish the idea of completeness on X, not any B or $\mathscr{B}$.*

Lemma 1. $x, y \in C(B) \Rightarrow x \sim^* y$

Lemma 2. $x \in C(B), y \notin C(B) \Rightarrow x \succ^* y$

Lemma 3. $x \succsim^* y \Rightarrow y \neg \succ^* x$

some choice structure can be super wired. they may represent a rational preference yet fail to meet WARP. others may meet the WARP yet fail to represent a rational preference.

9.3 Relationship between rational preference and revealed preference

Now we have two ways to treat preference

$$\begin{cases} \succsim: \text{preference relationship} \leftarrow \text{rational} \\ (\mathscr{B}, C(\cdot)) : \text{choice struture} \leftarrow \text{WARP} \end{cases}$$

9.3.1 From preference to choice

Intuitively thinking, if someone is of ration (rational preference), he or she ought to be consistent (WARP). Suppose $\succsim$ is a preference relationship on X, Define

$$C^*(\cdot, \succsim) : \mathscr{B} \to B \tag{60}$$

$$\forall B \in \mathscr{B}, C^*(B, \succsim) = \{x \in B | \forall y \in B, x \succsim y\} \tag{61}$$

> **Question 3.** *what if the B is an infinite set with the property of strictly increasing? can one ever search for the most preferred x?*

From now on, suppose that $\forall B \in \mathscr{B}, C^*(B, \succsim) \neq \varnothing$

Proposition 9. *Suppose $\succsim$ is rational, then the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$ satisfies WARP.*

Proof.

$$\exists B \in \mathscr{B} \tag{62}$$
$$x, y \in B \tag{63}$$
$$x \in C^*(B, \succsim) \tag{64}$$
$$\Rightarrow x \succsim y \blacksquare \tag{65}$$
$$\exists B' \in \mathscr{B} \tag{66}$$
$$x, y \in B' \tag{67}$$
$$y \in C^*(B', \succsim) \tag{68}$$
$$y \succsim (\forall z \in B') \tag{69}$$
$$\Rightarrow x \succsim y \succsim z \text{ (transitivity axiom in rational preference)} \tag{70}$$
$$\Rightarrow x \in C^*(B', \succsim) \blacksquare \tag{71}$$

□

So we can have a final conclusion about the relationship from preference relationship to choice structure: if a rational preference is captured by a choice structure, this choice structure must satisfy WARP.

9.3.2 From choice to preference

Now from the choice structure to preference, how does it work? If someone possesses consistency, would it be enough to assure rationality? Our guts wouldn't jump to a quick decision. Turns out, it needs some kind of restrictions to be positive.

Proposition 10. *If choice structure* $(\mathscr{B}, C(\cdot))$ *satisfies*
(i) WARP
(ii) $\mathscr{B}$ *includes all subsets of* X *of up to three elements.*
then there is one and only one rational preference can rationalize this choice structure, denoted as

$$\forall B \in \mathscr{B}, C(B) = C^*(B, \succsim)$$

Proof. (i)

$$\tag{72}$$
$$\text{Meaning the preference is rational:} \tag{73}$$
$$\text{(completeness)} \tag{74}$$
$$B : \{x, y\} \in \mathscr{B} \tag{75}$$
$$[C(B) = x \vee y] \Rightarrow [x \succsim^* y] \vee [y \succsim^* x] \blacksquare \tag{76}$$
$$\text{(transitivity)} \tag{77}$$
$$\exists B : \{x, y, z\} \in \mathscr{B}, \text{ let } x \succsim^* y \text{ and } y \succsim^* z \tag{78}$$
$$\text{If } y \in C(B) \Rightarrow x \in C(B) \tag{79}$$
$$\text{If } z \in C(B), \text{ similarly} \Rightarrow x \in C(B) \blacksquare \tag{80}$$
$$\text{Thus proved} \blacksquare \tag{81}$$

□

Proof. (ii)

$$\text{Meaning } \forall B \in \mathscr{B}, C(B) = C^*(B, \succsim^*) \blacksquare \tag{82}$$
$$\text{Let } x \in C(B), \forall y \in B, x \succsim^* y \Rightarrow C(B) \subset C^*(B, \succsim^*) \blacksquare \tag{83}$$
$$\text{Let } x \in C^*(B, \succsim^*), \forall y \in B, \exists B^y \subset B, \text{ where } x, y \in B^y \tag{84}$$
$$\Rightarrow x \in C(B) \Rightarrow C^*(B, \succsim^*) \subset C(B) \blacksquare \tag{85}$$

□

Proof. (iii)
Since $\mathscr{B}$ contains all subsets of up to three elements, it assures certain binary preference relationship. □

Unfortunately, this restrict is arguably not strong enough in reality.

9.4 WARP and Law of Demand

Proposition 11. *If a walrasian demand function $x(p, w)$ fufils WARP, for two (p, w) and (p', w'): If $px(p', w') \leqslant w$ and $x(p', w') \neq x(p, w)$, then $p'x(p, w) > w'$.*

Proof.

$$px(p, w) = w, B(p, w) \tag{86}$$
$$p'x(p', w') = w', B(p', w') \tag{87}$$
$$\text{If } px(p', w') \leq w \Rightarrow x(p', w') = x' \in B(p, w) \tag{88}$$
$$\text{If } x \notin B(p', w') \Rightarrow p'x(p, w) > w' \tag{89}$$

□

Or we can prove it using WARP.

Proof.

$$\begin{cases} px(p, w) = w \\ p'x(p', w') = w' \\ px(p', w') \leq w \end{cases} \Rightarrow p'x(p, w) > w \tag{90}$$

□

Slutsky wealth compensation

$$w = px(p, w) \Rightarrow \triangle w = w' - w = \triangle p \cdot x(p, w)$$

WARP and Slutsky wealth compensation

$$x(p, w) \text{ fulfils Walrasian Law, it fulfils also the WARP } \iff \tag{91}$$
$$(p, w) \to (p', w') = (p', w' = p'x(p, w)) : (p' - p)[x'(p, w') - x(p, w)] \leqslant 0 \tag{92}$$

10 Classic Demand Theory

10.1 Preference relationship

A. desired assumption: the more goods, the better
monotonicity

$$\text{If } x \in X \text{ and } y \gg x \text{ meaning } y > x \Rightarrow \succsim \text{ is weak monotonic} \tag{93}$$

local non-saturation assumption:

$$\text{If } \exists \succsim \text{ on } X, \forall x \in X \text{ and } \forall \epsilon > 0, \text{ always } \exists y \in X \text{ that} \tag{94}$$
$$||y - x|| \leq \epsilon \text{ and } y > x \tag{95}$$
$$\Rightarrow \text{the preference relationship } \succsim \text{ is local non-satisfying.} \tag{96}$$

this is an assumption weaker than monotonicity assumption (the order in term of weak and strong is strictly monotonic > weak monotonic > local non-saturation)
Based on x and $\succsim$, three related sets are defined:
Indifference set

$$\{y \in R : y \sim x\}$$

(local non-saturation ensures the existence of indifference curves in two-dimensional space)
Upper contour set

$$\{y \in R : y \succsim x\}$$

Lower contour set

$$\{y \in R : x \succsim y\}$$

B. Convexity assumption:
diversity, diminishing marginal utility, diminishing marginal rate of substitution $\succsim$ on X is convex if

$$\forall x \in X, \text{ the upper contour set } \{y \in R : y \succsim x\} \text{ is convex}$$

That is,

$$\text{If } y \succsim x, \forall \alpha \in [0,1], \alpha y + (1-\alpha) z \succsim x$$

A real commodity set may not satisfy the convexity assumption, but for the convenience of analysis, we need to assume convexity
C. Other preference relations
Similar preferences $\succsim$ is similar if all indifference sets can be extended in equal proportion along any ray and connected together. That is, if $x \sim y$, then $ax \sim ay, \forall a \geq 0$.
Quasi-linear preference (omitted)

10.2 Utility

A. Lexicographic preference relation

$$X = \mathbb{R}_+^2, x \succsim y : \text{either } [x_1 > y_1], \text{ or } [x_1 = y_1 \text{ and } x_2 \geq y_2].$$

This is rational, monotonic, and convex, but does not satisfy the continuity assumption, and there is no utility function

B. Continuity assumption of preference If the preference relation of X is continuous, then the previous preference relation can still be retained in the limit operation

If it is continuous, then the upper and lower contour sets are closed sets If it is continuous, then there is a utility function u that can represent the continuous relation If $\succsim$ is convex, then there is a u that can represent $\succsim$, then u is quasi-concave

10.3 Utility maximization problem

Assume that consumers have rational, continuous, and locally unsaturated preference relations, and use continuous functions $u(\cdot)$ to represent these preferences

Question 4. *What is the demand function of the CD function $u = kx_1^{\alpha}x_2^{1-\alpha}, \alpha \in (0,1), k > 0$?*

$$\max_x \ u(x) \tag{97}$$

$$s.t. \ px = w \tag{98}$$

$$\text{Make some changes so that } \ln u = \ln k + \alpha \ln x_1 + (1-\alpha)\ln x_2 \tag{99}$$

$$FOC: \nabla u(x) = \lambda p \tag{100}$$

$$\Rightarrow x_1 = \frac{\alpha w}{p_1}, x_2 = \frac{(1-\alpha)w}{p_2} \tag{101}$$

Question 5. *find the demand function based on $\max\limits_x \ u(x) = [\sum\limits_{i=1}^n x_i^{\frac{\alpha-1}{\alpha}}]^{\frac{\alpha}{\alpha-1}} \ s.t. \ \sum\limits_{i=1}^n p_i x_i = w$*

$$FOC: \frac{\alpha}{\alpha-1}[\sum_{i=1}^n x_i^{\frac{\alpha-1}{\alpha}}]^{\frac{\alpha}{\alpha-1}-1}\frac{\alpha-1}{\alpha}x_j^{\frac{\alpha}{\alpha-1}-1} = \lambda p_j, \forall j = 1,2,...,n \tag{102}$$

$$\text{Multiply both sides with } x_j \tag{103}$$

$$\frac{\alpha}{\alpha-1}[\sum_{i=1}^n x_i^{\frac{\alpha-1}{\alpha}}]^{\frac{\alpha}{\alpha-1}-1}\frac{\alpha-1}{\alpha}x_j^{\frac{\alpha}{\alpha-1}} = \lambda p_j x_j, \forall j = 1,2,...,n \tag{104}$$

$$\text{Start from j=1 and add up to n} \tag{105}$$

$$[\sum_{i=1}^n x_i^{\frac{\alpha-1}{\alpha}}]^{\frac{\alpha}{\alpha-1}} = \sum_{j=1}^n \lambda p_j x_j = \lambda w \tag{106}$$

$$x_j^{-\frac{1}{\alpha}} = p_j[\sum_{i=1}^n x_i] \tag{107}$$

First-order conditions for corner solutions:

$$x_l^* = 0, \frac{\partial u_l(x^*)}{\partial x_l} \leq \lambda p_l \tag{108}$$

$$x_l^* > 0, \frac{\partial u_l(x^*)}{\partial x_l} = \lambda p_l \tag{109}$$

Indirect utility function $v(p, y)$:

Property 6 (HD0)**.**

$$v(p, w) = v(tp, tw), \forall t > 0 \tag{110}$$

$$B_{p,w} = B_{tp,tw} \tag{111}$$

$$\Rightarrow x(p, w) = x(tp, tw) \tag{112}$$

$$\Rightarrow v(p, w) = u(x(p, w)) = u(x(tp, tw)) = v(tp, tw) \tag{113}$$

Property 7. *Non-increasing for* p*, strictly increasing for* w

Proof.

$$(p1, p2) \to (p1', p2) \Rightarrow B_{p,w} \subseteq B_{p',w} \Rightarrow v(p', w) \geq v(p, w)$$

□

Property 8 (quasi-convex)**.** *That is, the set* $\{(p, w) : v(p, w) \leq D\}$ *is convex for any* D

Proof.

$$\text{Assume that } (p1, w1), (p2, w2), \tag{114}$$

$$v(p1, w1) \subseteq D, v(p2, w2) \subseteq D \tag{115}$$

$$(p3, w3) = \alpha(p1, w1) + (1 - \alpha)(p2, w2) = [\alpha p1 + (1 - \alpha)p2, aw1 + (1 - \alpha)w2], \forall \alpha > 0 \tag{116}$$

$$\tag{117}$$

□

Property 9 (Minimize expenditure EMP)**.**

Proof. Let $p \gg 0$ and $u > u(0)$, $u(\cdot)$ continuous

$$\begin{aligned} &\underset{x \geq 0}{min}\ px \\ &s.t.u(x) \geq u \end{aligned} \tag{118}$$

$$\Rightarrow x^h(p, w) \tag{119}$$

$$e(x^h(p, u)) = e(p, w) \tag{120}$$

□

10.4 Duality

EMP is the dual problem of UMP

Proposition 12 (a)**.** *x^* is the optimal consumption bundle of UMP. For a given target utility $u(x^*)$, x^* is also the most optimal in EMP, and the minimum expenditure is exactly w. $x^h(p,u) = x(p,e(p,u))$*

Proposition 13 (b)**.** *x^* is also the most optimal in EMP. When the wealth level is px^*, x^* is also the best in UMP, and the maximum utility is u. $x(p,w) = h(p,u(p,w))$*

a).

$$x^* \in x(p,w) \iff \begin{cases} px^* = w \\ u(x^*) \geq u(x'), \forall x'p \leq w \end{cases} \tag{121}$$

$$\text{If } x^* \notin x^h(p,u) \tag{122}$$

$$\begin{aligned} &\min_x px \\ &s.t. u(x) \geq u(x^*) \end{aligned} \tag{123}$$

$$\exists \bar{x}, p\bar{x} < px^*, u(\bar{x}) \geq u(x) \tag{124}$$

$$u(\bar{x}) \geq u(x^*) \geq u(x'), \forall x'p \leq w \tag{125}$$

$$u(\bar{x}) \geq u(x'), \forall x'p \leq w \tag{126}$$

$$\Rightarrow \bar{x} \text{ is the optimal option in UMP}, \bar{x} \in x(p,w) \tag{127}$$

$$\Rightarrow x^* \in x^h(p,u) \tag{128}$$

□

b).

$$\text{Assume that } x^* \notin x(p,w) \tag{129}$$

$$\exists \bar{x}, p\bar{x} \leq w, u(\bar{x}) > u(x^*), px' < w \tag{130}$$

$$u(x') \geq u(x^*), px' < w = px^* \tag{131}$$

$$x' = t\bar{x}, t \in (0,1) \tag{132}$$

$$u(x') \geq u(x^*), px' < w \tag{133}$$

$$u(x') \geq u(x^*), px' < w = px^* \tag{134}$$

$$x^* \notin x^h(p,u) \text{ and } x^* \text{ is the optimal option are paradox} \tag{135}$$

$$\Rightarrow x^* \in x(p,w) \tag{136}$$

□

Expenditure function $e(p,u)$

Property 10 (HD1 with respect to P).

$$e(p,u) = px_1, \forall x \in x^h(p,u) \tag{137}$$
$$e(tp,u) = tpx_2, \forall x_2 \in x^h(tp,u) \tag{138}$$
$$e(tp,u) = te(p,u) \tag{139}$$
$$tpx_1 = tpx_2 \Rightarrow x_1 = x_2 \tag{140}$$
$$x^h(p,u) = x^h(tp,u) \Rightarrow HD0 \tag{141}$$

Property 11. *Regarding u is strictly increasing and p is non-decreasing*

Proof.

$$u_1 > u_2 \Rightarrow e(p,u_1) > e(p,u_2) \tag{142}$$
$$\text{Assume that } u_1 > u_2, e(p,u_1) \le e(p,u_2) \text{ holds true} \tag{143}$$

□

Property 12. *Regarding the concave function of p*

Proof.

$$e[\alpha p_1 + (1-\alpha)p_2, u] \geqslant \alpha e(p_1,u) + (1-\alpha)e(p_2,u) \tag{144}$$
$$e[\alpha p_1 + (1-\alpha)p_2, u] = [\alpha p_1 + (1-\alpha)p_2]x_3 \tag{145}$$
$$x_3 \in x^h(\alpha p_1 + (1-\alpha)p_2, u) = \alpha p_1 x_3 + (1-\alpha)p_2 x_3 \tag{146}$$
$$\ge ap_1x_1 + (1-\alpha)p_2x_2 = \alpha e(p_1,u) + (1-\alpha)e(p_2,u) \tag{147}$$

□

Hicks demand function
Properties of Hicks demand function

Property 13 (HD0).

Property 14 (No excess utility). $x \in x^h(p,u), u(x) = u$

Property 15. *Hicks demand functino is the derivative of* $e(p,u)$ *with respect to* p

Proposition 14 (Hicks demand satisfies the law of compensation of demand). *For the price change of Hicks wealth compensation, the movement direction of demand and price is opposite*

$$(p'-p)[x^h(p',u) - x^h(p,u)] \leqslant 0$$

Proof.

$$x^h(p',u) \text{ is the optimal choice at price } p' \tag{148}$$
$$\Rightarrow p'x^h(p',u) \le p'x^h(p,u)\ (*) \tag{149}$$
$$\text{Same for at the price } p, \tag{150}$$
$$\Rightarrow px^h(p,u) \le px^h(p',u)\ (**) \tag{151}$$
$$p'x^h(p',u) + px^h(p,u) \le p'x^h(p,u) + px^h(p',u) \tag{152}$$
$$(p'-p)(x^h(p',u) - x^h(p,u)) \le 0 \tag{153}$$

□

Slutsky equation

$$\frac{\partial x^h(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l}{\partial w} x_k(p,w)$$

11 Choice under uncertainty

Outcome

$$X = (x_1, x_2, ..., x_n)$$

lottery

$$\text{A set of possibilities } l = (P_1, P_2, ..., P_n), \forall P_i \geq 0, \sum_{i=1} Pi = 1 \tag{154}$$

$$\Rightarrow L = (P_i x_i) \tag{155}$$

compound lottery

$$\exists \text{an amount of } k \text{ lotteries } L_k = (p_i^k) \tag{156}$$

Complex lotteries can be simplified to simple lotteries If two different complex lotteries have the same lottery after simplification, then we consider them to be the same
Simplex Geometric figure used to represent lotteries (simple or complex)

$$\triangle = \{P \in \mathbb{R}_+^n : P_1 + ... + P_n = 1\}$$

Alternative set $\mathscr{L}$
Assumption: $\succsim$ is complete and transitive Complete means $\forall L, L' \in \mathscr{L}$, which satisfies $L \succsim L'$ or $L' \succsim L$ or both Transitive means $\forall L, L', L'' \in \mathscr{L}, L \succsim L', L' \succsim L''$, then $L \succsim L''$
Continuity axiom (assumption) $\succsim$ is continuous on $\mathscr{L}$ if for $\forall L, L', L'' \in \mathscr{L}$, the set $\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succsim L''\} \subset [0,1]$ and the set $\{\alpha \in [0,1] : L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0,1]$ are both continuous
Independence axiom (assumption)

$$\forall L, L', l'' \in \mathscr{L}, \exists \alpha \in [0,1], \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L'' \iff L \succsim L' \tag{157}$$

Conclusion: a): $L \succ L'$ if and only if $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$ b): $L \sim L'$ if and only if $\alpha L + (1-\alpha)L'' \sim L' + (1-\alpha)L''$
Expected utility function $U : \mathscr{L} \to \mathbb{R}$ $U(L) = \sum_i P_i u_i$
Conclusion: a) VM utility function $\iff$ linear b) $\succsim \Rightarrow U(L), V(L) = \alpha U(L) + \beta, \alpha > 0$ in $\mathscr{L}$
Existence theorem (expected utility theorem) Preferences in lottery space are complete, transitive, continuous, and independent, so preferences can be expressed by expected utility functions.

Part II

FIRM AND SUPPLY THEORY

12 Firm

production possiblity set

$$\mathbb{R}^L$$

production vector

$$y = (y_1, y_2, ..., y_L) \in \mathbb{R}^L$$

production process

$$y^A \to y^B \Rightarrow y = y^A - y^B = y^{output} - y^{input}$$

production feasibility set

$$Y = \{y \in \mathbb{R}^L | y \text{ if feasible allowed by current technology}\}$$

transformation function

$$T : \mathbb{R}^L \to \mathbb{R} \tag{158}$$

$$\iff \tag{159}$$

$$(1).\ \forall y \in Y, T(y) \leqslant 0 \tag{160}$$

$$(2).\ T(y) < 0, \text{ if } \exists y' \in Y, y' > y \tag{161}$$

$$(3).\ T(y) = 0, \neg\exists y \in Y, y' > y \tag{162}$$

transfer function can be consider a sort of measurement of efficiency of production, where the hyperplane is set of the most efficient vectos.
input requirement set

$$\text{Let } q \geqslant 0 \in \mathbb{R}^m, x \geqslant 0 \in \mathbb{R}^{L-m}, y = (q, -x) \Rightarrow V(q) = \{x \in \mathbb{R}^L | (q, -x) \in Y\} \tag{163}$$

if $q^1 \geqslant q^2$, then $V(q^1) \subset V(q^2)$
Production function

$$\text{Define } m = 1, L - m = L - 1 \tag{164}$$

$$f : \mathbb{R}^{L-1} \to \mathbb{R} \text{ is a production function if } (f(x), -x) \in Y \text{ and } T(f(x), -x) = 0 \tag{165}$$

CD production function

$$f(x) = A \sum_{i=1}^{L-1} x_i^{a_i}$$

Leontief production function

$$f(x) = min\{a_1 x_1, ..., a_n x_n\}$$

linear production function

$$f(x) = \sum_{i=1}^{L-1} a_i x_i$$

two production factors

$$\text{CD: } f(x) = a x_i^{\alpha} x_2^{1-\alpha}, \text{Leontif: } f(x) = min\{a_1 x_1, a_2 x_2\}$$

a special case with two production factors

$$f(x_1, x_2) = [ax_1^\rho + bx_2^\rho]^{\frac{1}{\rho}}, a + b = 1 \Rightarrow \begin{cases} \rho \to 0, \text{ CD} \\ \rho = 1, \text{ linear} \\ \rho \to -\infty, \text{ Leontief} \end{cases}$$

“‘ad-abstract title: Quiz of CES to Leontief
“‘
properties of production set (i) closed, meaning the hyperplane is attained (ii) irreversable

$$\text{if } y \in Y, \text{then } -y \notin Y$$

(iii)

$$\{0\} \subset Y$$

(iv) no free lunch

$$y \in Y, y \geqslant 0 \Rightarrow y = 0$$

(v) disposal at freedom

$$\text{If } y \in Y, y' \leqslant y \Rightarrow y' \in Y$$

(vi) additivity

$$\forall y, y' \in Y, \ y + y' \in Y$$

(vii) convexity

$$\forall x, y \in Y, \exists \alpha \in [0,1], \alpha x + (1-\alpha)y \in Y$$

Marginal product

$$q = f(k,l) \Rightarrow \frac{\partial q}{\partial k} = f_l \text{ or } \frac{\partial q}{\partial l} = f_k \tag{166}$$

diminishing marginal product

$$\frac{\partial^2 f}{\partial k^2} < 0 \text{ or } \frac{\partial^2 f}{\partial l^2} < 0, \text{ but } \frac{\partial^2 f}{\partial l \partial k} \text{ can be positive.}$$

marginal rate of substitution

$$\exists f(k,l), MRST = \frac{dk}{dl} = |\frac{MP_l}{MP_k}| = -\frac{MP_l}{MP_k}$$

elasticity of substitution

$$\delta = \frac{d \ln \frac{k}{l}}{d \ln |MRST|}$$

Question 6 (Quiz of calculating elasticity of CES function).

$$q = f(k,l) = (k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}} \tag{167}$$
$$\Rightarrow MRST = -\frac{f_l}{f_k} = -(\frac{k}{l})^{1-\rho} \tag{168}$$
$$\Rightarrow \delta = \frac{dln\frac{k}{l}}{dln|MRST|} = \frac{1}{1-\rho} \tag{169}$$

13 PMP

profit maximization problem, PMP

$$\text{For a production function } f(x) = f(x_1, x_2, ..., x_n) \tag{170}$$
$$\text{The gradient is } \nabla f(x) = [\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}] \tag{171}$$
$$\text{Given production feasibility set } Y \text{ and price vector } P \gg 0 \tag{172}$$
$$\text{The profit maximization is denoted as} \tag{173}$$
$$\begin{cases} \max\limits_{y} \ py \\ s.t. \ T(y) \leqslant 0 \end{cases} \text{or} \begin{cases} \max\limits_{y} \ py \\ s.t. \ y \in Y \end{cases} \tag{174}$$
$$\Rightarrow \pi(P) = max\{py : y \in Y\} \tag{175}$$
$$y_{\text{product provided}} = \{y \in Y, py = \pi(p)\} \tag{176}$$
$$\tag{177}$$
$$\text{If transformation function T(y) is differentiable, then} \tag{178}$$
$$\text{the solution to PMP can also be denoted as} \tag{179}$$
$$\begin{cases} \text{FOC: } p = h\nabla T(y^*) \Rightarrow \frac{\partial f}{\partial x} = \frac{w_x}{p} \\ \text{SOC: } D^2T(y^*) \text{ is negative half finite} \end{cases} \tag{180}$$

properties of profit function

Property 16. *Homogeneous with degree of* 1 *with respect to price vector* p

Proof.

$$\tag{181}$$
$$\pi(tp) = tp \cdot y^* = \pi(p) = t \cdot py^* \tag{182}$$
$$\Longleftrightarrow \begin{cases} (1) \ \forall y \in Y(tp) \Rightarrow y \in Y(p) \\ (2) \ \forall y \in Y(p) \Rightarrow y \in Y(tp) \end{cases} \tag{183}$$

□

Proof.

$$
\begin{aligned}
&(1): && (184)\\
&\forall y \in Y(p), py \geqslant py', \forall y' \in Y(p) && (185)\\
&(tP)y \geqslant (tP)y', \forall y' \in Y && (186)\\
&y \in Y(P_1 = tp) && (187)\\
& && (188)\\
&(2): && (189)\\
&\forall y \in Y(tp), tpy \geqslant tpy', \forall y' \in Y(p) && (190)\\
&(tp)y \geqslant (tp)y', \forall y' \in Y && (191)\\
&y \in Y(P_2 = tp) && (192)
\end{aligned}
$$

□

2. profit function is a concave function

$$\pi[ap_1 + (1-a)p_2] \leqslant a\pi(p_1) + (1-a)\pi(p_2)$$

"‘ad-question title: Proving

$$
\begin{aligned}
&\pi(p1) = p1y1 \geq p1y3 && (193)\\
&\pi(p2) = p2y2 \geq p2y3 && (194)\\
&a\pi(p1) + (1-a)\pi(p2) \geq ap1y3 + (1-a)p2y3 = [ap1 + (1-a)p2]y3 = \pi(ap1 + (1-a)p2) && (195)
\end{aligned}
$$

"‘

3. Shephard's lemma

$$\frac{\partial \pi(p)}{\partial p} = y(p)$$

"‘ad-question title: Proving

$$
\begin{aligned}
&\pi(p) = \max_y py \; s.t. T(y) \leqslant 0 && (196)\\
&L(p, \phi) = py - \phi T(y) && (197)\\
&\frac{\partial L}{\partial p} = y - \phi \frac{\partial T}{\partial p} = 0 && (198)
\end{aligned}
$$

"‘

4. law of supply "‘ad-question title: Proving

$$
\begin{aligned}
&\pi(p_s) = y_s, \pi(p_t) = y_t && (199)\\
&\Rightarrow p_s y_s \geq p_s y_t, p_t y_t \geq p_t y_s && (200)\\
&\Rightarrow p_s y_s + p_t y_t \geq p_s y_t + p_t y_s && (201)\\
&\Rightarrow (p_s - p_t)(y_s - y_t) \geq 0 && (202)
\end{aligned}
$$

"‘

14 CMP

cost minimization From production plan $y = (q, -x)$, we see two simultaneous decisions in a single move. in the case $y = (y_1)$
goods

$$q$$

factors

$$x$$

factor price

$$w$$

conditional factor demand function

$$x(q,w) = \underset{x}{min}\ wx \tag{203}$$

$$s.t.\ f(x) \geqslant q \tag{204}$$

cost function

$$C(w,q) = x(q,w) \cdot w$$

minimization of cost function

$$\text{For a cost function } C(w,q) = x(q,w) \cdot w, \tag{205}$$

$$FOC:\ w = h\nabla f(x^*), \exists h \geq 0 \tag{206}$$

$$SOC:\ D^2 f(x^*) \text{ is negative half-finite} \tag{207}$$

"'ad-info title: Case

$$\underset{l,k}{min}\ c = wl + rk \tag{208}$$

$$s.t. f(l,k) \geq q \tag{209}$$

$$\Rightarrow \text{Using Lagrange's multiplier } L(l,k,\phi) = wl + rk - \phi[f(k,l) - q] \tag{210}$$

$$\text{Equilibrium: } \begin{cases} w = \phi f_l' \\ r = \phi f_k' \end{cases} \tag{211}$$

"'

properties of cost function

$$c(w,q) = w \cdot x^*, x^* \in x(w,q) \tag{212}$$

1. noncreasing of q and w
"'ad-question title: Proving

$$\text{As proving } q1 \geq q2 \Rightarrow c(w,q1) \geq c(w,q2) \tag{213}$$

$$c(w,q1) = wx1, x1 \in x(w,q1), x1 \in V(q1) \tag{214}$$

$$\text{If } q1 \geq q2, V(q1) \subset V(q2) \tag{215}$$

$$c(w,q2) = wx2 \leq wx1, x2 \in x(w,q2), x2 \in V(q2) \tag{216}$$

‘‘‘
2. homogeneous with degree of 1 with respect to w
‘‘‘ad-question title: Proving

$$c(aw,q) = aw \cdot x^*, x^* \in x(aw,q) \tag{217}$$
$$a \cdot c(w,q) = a \cdot wx^{**}, x^{**} \in x(w,q) \tag{218}$$
$$\Rightarrow \text{As proving} \begin{cases} \forall x \in x(w,q) \Rightarrow x \in x(aw,q) \\ \forall x \in x(aw,q) \Rightarrow x \in x(w,q) \end{cases} \tag{219}$$
$$\forall x \in x(w,q), awx \leq awx', \forall x' \in V(q) \tag{220}$$
$$\forall x \in x(aw,q), wx \leq wx', \forall x' \in V(q) \tag{221}$$

‘‘‘
3. cost function is a convex function with respect to w

$$C[aw1 + (1-a)w2] \geq aC(w1) + (1-a)C(w2), \forall a \in [0,1]$$

the convexity of cost function is to assure the slower than change of price cost thus the availablity to properly achieve optimization.
‘‘‘ad-question title: Proving

$$C[aw1 + (1-a)w2] \geq aC(w1) + (1-a)C(w2), \forall a \in [0,1] \tag{222}$$
$$C(w1) = w1x1 \leq w1x3 \tag{223}$$
$$C(w2) = w2x2 \leq w2x3 \tag{224}$$
$$aC(w1) + (1-a)C(w2) \leq aw1x3 + (1-a)w2x3 = C(aw1 + (1-a)w2) \tag{225}$$

‘‘‘

Part III

GAME THEORY

15 Game theory

Complete information stastic game theory: Nash Equilibrium Incomplete information stastic game theory: Subgame Perfection Nash Equilibrium

15.1 CISG: NE

Participants

$$I = \{1, 2, ..., n\}$$

Strategy

$$S_i = \{s_1, s_2, ..., s_n\} \text{ for the particular participant } i, \text{ whereas } S \text{ for all participants} \tag{226}$$
$$s = \{s_{ij}\}, \text{ where } i \text{ stands for participant and } j \text{ stands for each action} \tag{227}$$

Payoff structure

$$\pi(s_{ij})$$

Information

$$\text{As literally.} \begin{cases} \text{Complete information} \\ \text{Incomplete information} \end{cases}$$

Strictly inferior strategy

$$s_i, s_i' \text{ are two strategies of the participant } i, \tag{228}$$
$$\forall s_{-i} \in S_{-i}, \pi(s_i', s_{-i}) > \pi(s_i, s_{-i}) \tag{229}$$
$$\Rightarrow s_i' \text{ is strictly superior to } s_i \text{ and } s_i \text{ is strictly inferior to } s_i' \tag{230}$$

By the repetition of elimination of strictly inferior strategies, we will have the according Nash equilibrium.
Nash Equilibrium

$$\text{The strategic combination } s^* = (s_1^*, s_2^*, ..., s_n^*) \text{ is a Nash Equilibrium if and only if} \tag{231}$$
$$\forall s_i \in S_i, \tag{232}$$
$$\pi_i(s^*) \geqslant \pi_i(s_i, s_{-i}^*), i = 1, 2..., n \tag{233}$$
$$(s_i, s_{-i}^*) = (s_1^*, s_2^*, ..., s_n^*) \tag{234}$$

Mixed strategy Nash equilibrium pure strategy is a special case of mixed

$$\text{A strategic combination } p^* = (p_1^*, p_2^*, ..., p_n^*) \text{ is a MSNE if and only if} \tag{235}$$
$$\forall p_i \in P_i, \pi_i(p^*) \geqslant \pi(p_i, P_{-i}^* *), i = 1, 2..., n \tag{236}$$
$$\text{where } (p_i, p_{-i}^*) = (p_1^*, p_2^*, ..., p_n^*) \tag{237}$$