Stima & Filtraggio: Lab 2

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Today's Lab

Wiener Filtering & Applications

1. Stochastic processes in **MATLAB®**
2. **MATLAB®** tools for Wiener filtering

Today's Lab

Wiener Filtering & Applications

1. Stochastic processes in **MATLAB**®

- Generating white noise
- Generating filtered noise
- Useful **MATLAB**® commands

(weakly stationary)
white noise processes

$$\{e(t)\}_{t\in\mathbb{Z}} \text{ s.t. } \mathbb{E}[e(t)] = \mu,\ \mathbb{E}[e(t)e(s)] = \sigma^2\delta(t-s) + \mu^2$$

(weakly stationary)
white noise processes

$$\{e(t)\}_{t\in\mathbb{Z}} \text{ s.t. } \mathbb{E}[e(t)] = \mu,\ \mathbb{E}[e(t)e(s)] = \sigma^2\delta(t-s) + \mu^2$$

$$\bar{e}(t) := e(t) - \mu,\ \bar{e}(t) \perp \bar{e}(s),\ s \neq t$$

(weakly stationary)
white noise processes

$$\{e(t)\}_{t\in\mathbb{Z}} \text{ s.t. } \mathbb{E}[e(t)] = \mu,\ \mathbb{E}[e(t)e(s)] = \sigma^2\delta(t-s) + \mu^2$$

100 samples of Gaussian white noise with mean μ and var. σ^2:

```
>> rvE = dMu + dSigma*randn(1,100)
```

μ σ

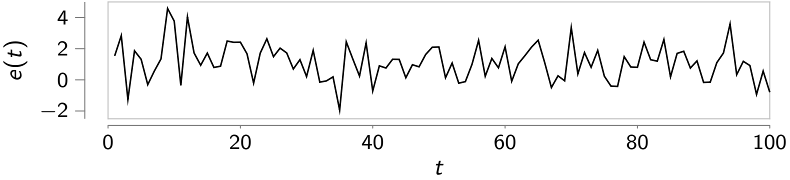
(weakly stationary)
white noise processes

$$\{e(t)\}_{t\in\mathbb{Z}} \text{ s.t. } \mathbb{E}[e(t)] = \mu,\ \mathbb{E}[e(t)e(s)] = \sigma^2\delta(t-s) + \mu^2$$

100 samples of Gaussian white noise with mean μ and var. σ^2:

```
>> rvE = dMu + dSigma*randn(1,100)
```

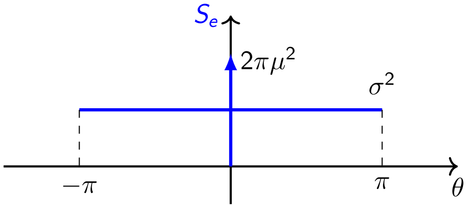
$\mu = 1$ $\sigma = 1$



(weakly stationary) **white noise processes**

$$\{e(t)\}_{t\in\mathbb{Z}} \text{ s.t. } \mathbb{E}[e(t)] = \mu,\ \mathbb{E}[e(t)e(s)] = \sigma^2\delta(t-s) + \mu^2$$

Spectral density: $S_e(e^{j\theta}) = 2\pi\mu^2\,\delta(\theta) + \sigma^2,\ \theta \in [-\pi, \pi]$



Spectral density: related to the *power of a signal*; obtained from spectrum $S_e(z)$ evaluated on the unit circle

(weakly stationary)
white noise processes

– multivariate case –

$$\{\boldsymbol{e}(t)\}_{t\in\mathbb{Z}} \text{ s.t. } \mathbb{E}[\boldsymbol{e}(t)] = \boldsymbol{\mu},\ \mathbb{E}[\boldsymbol{e}(t)\boldsymbol{e}^\top(s)] = \Sigma\,\delta(t-s) + \boldsymbol{\mu}\boldsymbol{\mu}^\top$$

$$\Sigma = \Sigma^\top \geq 0$$

(weakly stationary)
white noise processes

– multivariate case –

$$\{\boldsymbol{e}(t)\}_{t\in\mathbb{Z}} \text{ s.t. } \mathbb{E}[\boldsymbol{e}(t)] = \boldsymbol{\mu},\ \mathbb{E}[\boldsymbol{e}(t)\boldsymbol{e}^\top(s)] = \Sigma\,\delta(t-s) + \boldsymbol{\mu}\boldsymbol{\mu}^\top$$

100 samples of 2-dim. Gaussian white noise with mean $\boldsymbol{\mu} \in \mathbb{R}^2$ and cov. $\Sigma = \Sigma^\top \in \mathbb{R}^{2\times 2}$, $\Sigma \geq 0$:

```
>> mE = rvMu + sqrtm(dSigma)*randn(2,100)
```

$\boldsymbol{\mu}$ $\Sigma^{\frac{1}{2}}$

symmetric matrix

square root

(weakly stationary) **white noise processes**

– multivariate case –

$$\{\boldsymbol{e}(t)\}_{t\in\mathbb{Z}} \text{ s.t. } \mathbb{E}[\boldsymbol{e}(t)] = \boldsymbol{\mu},\ \mathbb{E}[\boldsymbol{e}(t)\boldsymbol{e}^\top(s)] = \Sigma\,\delta(t-s) + \boldsymbol{\mu}\boldsymbol{\mu}^\top$$

100 samples of 2-dim. Gaussian white noise with mean $\boldsymbol{\mu} \in \mathbb{R}^2$ and cov. $\Sigma = \Sigma^\top \in \mathbb{R}^{2\times 2}$, $\Sigma \geq 0$:

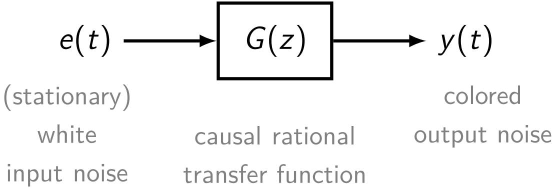
```
>> mE = rvMu + chol(dSigma,'lower')*randn(2,100)
```

$\boldsymbol{\mu}$

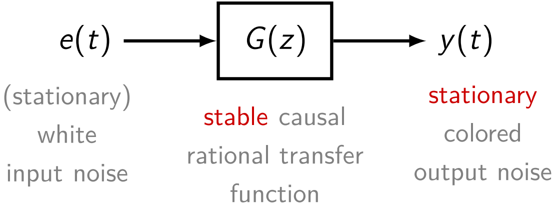
$L\,:\,\Sigma = LL^\top$

Cholesky factor (numerically convenient)

Filtered noise processes



Filtered noise processes



Filtered noise processes

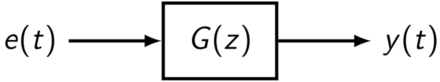
$$e(t) \longrightarrow \boxed{G(z)} \longrightarrow y(t)$$

$$G(z) = \frac{N(z)}{D(z)}, \qquad N(z), D(z) \text{ polynomials}$$

$$D^*(z^{-1})y(t) = N^*(z^{-1})e(t) \qquad \left(P^*(z^{-1}) = z^{-n}P(z),\ n = \deg(P(z))\right)$$

ARMA representation,
z^{-1} = delay operator

Filtered noise processes



Task. Generate 100 samples of the process

$$y(t) = 0.1y(t-1) + e(t) - 0.2e(t-2)$$

where $\{e(t)\}_{t \in \mathbb{Z}}$ is a Gaussian WN process ($\mu = 0$, $\sigma = 1$)

Filtered noise processes

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where $\{e(t)\}_{t \in \mathbb{Z}}$ is a Gaussian WN process ($\mu = 0$, $\sigma = 1$)

Note that $N^*(z^{-1}) = 1 - 0.2z^{-2}$, $D^*(z^{-1}) = 1 - 0.1z^{-1}$

$$\implies \; G(z) = \frac{z^2 - 0.2}{z^2 - 0.1z} \quad \text{(stable!)}$$

Filtered noise processes

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```
>> rvN = [1 0 -0.2]
>> rvD = [1 -0.1 0]
```

define the numerator and denominator polynomials in z^{-1} (does something change in z?)

Filtered noise processes

$$e(t) \longrightarrow \boxed{G(z)} \longrightarrow y(t)$$

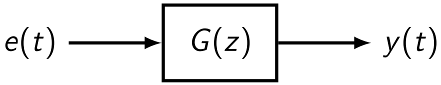
Task. Generate 100 samples of the process

$$y(t) = 0.1y(t-1) + e(t) - 0.2e(t-2)$$

where $\{e(t)\}_{t\in\mathbb{Z}}$ is a Gaussian WN process ($\mu = 0$, $\sigma = 1$)

`>> rvE = randn(1,100)` create white noise sequence

Filtered noise processes



Task. Generate 100 samples of the process

$$y(t) = 0.1y(t-1) + e(t) - 0.2e(t-2)$$

where $\{e(t)\}_{t \in \mathbb{Z}}$ is a Gaussian WN process ($\mu = 0$, $\sigma = 1$)

```
>> rvY = filter(rvN,rvD,rvE)
```
generate the process!

N.B. Initial conditions set to 0

Filtered noise processes

$$e(t) \longrightarrow \boxed{G(z)} \longrightarrow y(t)$$

Task. Generate 100 samples of the process

$$y(t) = 0.1y(t-1) + e(t) - 0.2e(t-2)$$

where $\{e(t)\}_{t \in \mathbb{Z}}$ is a Gaussian WN process ($\mu = 0$, $\sigma = 1$)

---

an alternative procedure...

Filtered noise processes

$$e(t) \longrightarrow \boxed{G(z)} \longrightarrow y(t)$$

Task. Generate 100 samples of the process

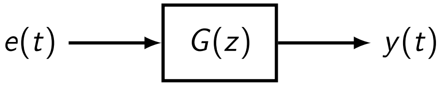
$$y(t) = 0.1y(t-1) + e(t) - 0.2e(t-2)$$

where $\{e(t)\}_{t\in\mathbb{Z}}$ is a Gaussian WN process ($\mu = 0$, $\sigma = 1$)

```
>> z = tf('z')
```
define the transfer function

```
>> tfG = (z^2-0.2)/(z^2-0.1*z)
```

Filtered noise processes



Task. Generate 100 samples of the process

$$y(t) = 0.1y(t-1) + e(t) - 0.2e(t-2)$$

where $\{e(t)\}_{t \in \mathbb{Z}}$ is a Gaussian WN process ($\mu = 0$, $\sigma = 1$)

```
>> cvY = lsim(tfG,rvE)
```

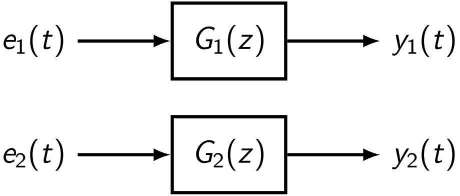
generate the process!

N.B. Initial conditions set to 0

Filtered noise processes

– an important remark –

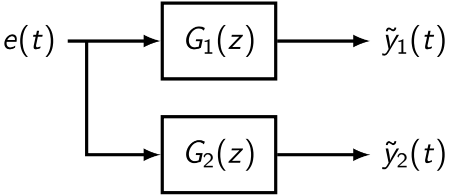
$\{e_1\}$, $\{e_2\}$ WNs, $\{e_1\} \perp \{e_2\}$



$\boldsymbol{y} := [y_1, y_2]^\top$

$\{e_1\}$, $\{e_2\}$, $\{e\}$ same mean and variance

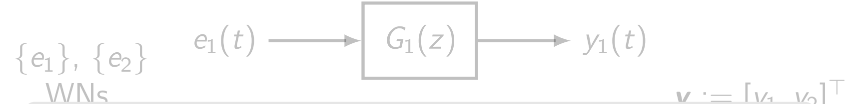
$\{e\}$ WN



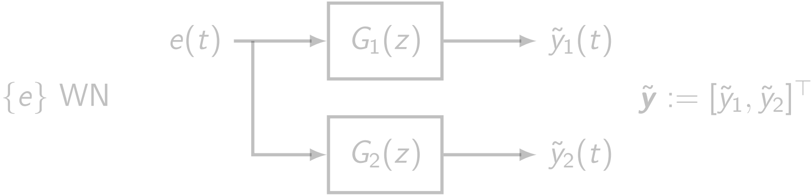
$\tilde{\boldsymbol{y}} := [\tilde{y}_1, \tilde{y}_2]^\top$

Filtered noise processes

– an important remark –



Question: Do $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$ define the same process?



Filtered noise processes

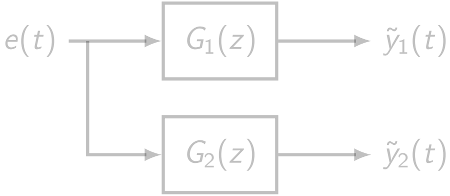
– an important remark –

$\{e_1\}$, $\{e_2\}$
WNs,
$\{e_1\} \perp \{e$

$\mathbf{y} := [y_1, y_2]^\top$

Answer: No, since $\{e_1\} \perp \{e_2\}$!

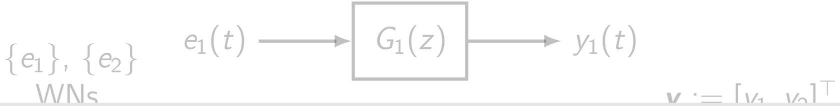
$\{e\}$ WN



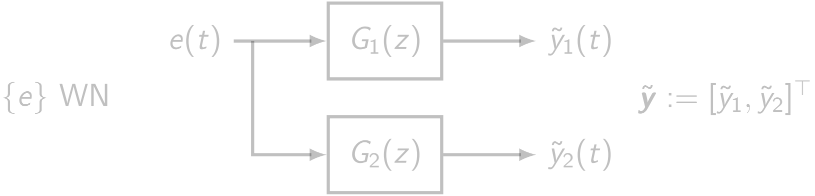
$\tilde{\mathbf{y}} := [\tilde{y}_1, \tilde{y}_2]^\top$

Filtered noise processes

– an important remark –



Exercise: compute and compare the spectra of $\mathbf{y}$ and $\tilde{\mathbf{y}}$



Useful MATLAB® commands

(Pay attention on z and z^{-1}!)

Create TF $\frac{N(z)}{D(z)}$	`>> tfG = tf(rvN,rvD,-1)`
Filter $\{e(t)\}_t$ by $\frac{N^*(z^{-1})}{D^*(z^{-1})}$	`>> rvY = filter(rvN,rvD,rvE)`
Filter $\{e(t)\}_t$ by $G(z)$	`>> cvY = lsim(tfG,rvE)`
Recover $N(z)$, $D(z)$	`>> [rvN,rvD] = tfdata(tfG,'v')`
From TF to SS	`>> ssG = ss(tfG)`
From TF to ZPK	`>> zpkG = zpk(tfG)`

✎ Practice time 1!

Ex 1.1. Create a function

```
[bS,bMP] = checkTFStability(tfG)
```

that has as input a causal scalar discrete-time transfer function object `tfG`. The function returns

- boolean `bS = true` if any coprime representation of `tfG` is (strictly) ***stable***, and `bS = false` otherwise,
- boolean `bMP = true` if any coprime representation of `tfG` is ***minimum phase***, i.e. it is stable with marginally stable inverse, and `bMP = false` otherwise.

Then, test the function on the following TFs:

$$G_1(z) = \frac{z+2}{z^2+0.4z-0.45}, \; G_2(z) = \frac{z^2-0.7z+1}{z^2+0.4z-0.45}, \; G_3(z) = \frac{z(z+1)}{z^2+2.5z+1}.$$

Practice time 1!

Ex 1.2. Create a function

`plotSpectrum(tfG)`

that has as input a causal scalar discrete-time transfer function `tfG`. The function plots the ***spectral density*** in the frequency interval $\theta \in [-\pi, \pi]$ of the process generated by filtering a Gaussian WN process ($\mu = 0$, $\sigma = 1$) through `tfG`. If `tfG` has poles on the unit circle, the function throws an error and displays an error message.

$$e(t) \longrightarrow \boxed{G(z)} \longrightarrow y(t)$$

Then, test the function on the TFs of Ex. 1.1.

Extra question. What happens if $\sigma \neq 1$?

Today's Lab

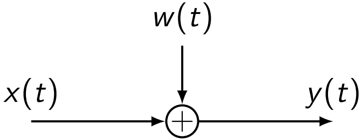
Wiener Filtering & Applications

2. Wiener filtering in **MATLAB®**

- Quick recap
- Computing integrals via residues
- Computing causal and anticausal parts
- Perform spectral factorization

Wiener filtering

Setup



$\{x(t)\}_{t\in\mathbb{Z}}$: stationary input process

$\{w(t)\}_{t\in\mathbb{Z}}$: stationary external noise

$\{y(t)\}_{t\in\mathbb{Z}}$: stationary output process

$$\begin{bmatrix} S_x & S_{xy} \\ S_{xy}^* & S_y \end{bmatrix}$$

Joint spectrum

Wiener filtering

Wiener estimate

$$H(z) := \left[\frac{S_{xy}(z)}{L(z^{-1})}\right]_+ \frac{1}{L(z)}$$

$$S_y(e^{j\theta}) = L(e^{j\theta})L(e^{-j\theta})$$

$$y(t) \longrightarrow \boxed{H(z)} \longrightarrow \hat{x}(t|t)$$

Wiener filtering

Wiener estimate

$$H(z) := [S_{x\epsilon}(z)]_+ \frac{1}{L(z)}$$

$$S_y(e^{j\theta}) = L(e^{j\theta})L(e^{-j\theta})$$

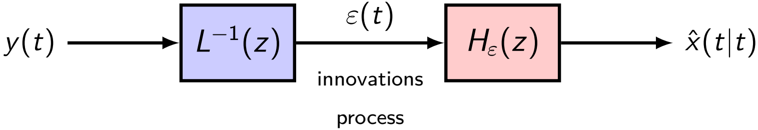
$$y(t) \longrightarrow \boxed{H(z)} \longrightarrow \hat{x}(t|t)$$

Wiener filtering

Wiener estimate

$$H(z) := \left[\frac{S_{xy}(z)}{L(z^{-1})}\right]_+ \frac{1}{L(z)}$$

$$S_y(e^{j\theta}) = L(e^{j\theta})L(e^{-j\theta})$$



Wiener filtering

Wiener estimate

$$H(z) := \left[\frac{S_{xy}(z)}{L(z^{-1})}\right]_+ \frac{1}{L(z)}$$

$$S_y(e^{j\theta}) = L(e^{j\theta})L(e^{-j\theta})$$

$$\mathrm{Var}\,\tilde{x}(t|t) = \int_{-\pi}^{\pi} \left(S_x(e^{j\theta}) - H_\varepsilon(e^{j\theta})H_\varepsilon(e^{-j\theta})\right) \frac{\mathrm{d}\theta}{2\pi}$$

estimation error variance

Wiener filtering *in practice*

How to compute the integral of a rational function upon the unit circle?

How to compute the causal and (strictly) anticausal part of a rational function?

How to perform spectral factorization?

Wiener filtering *in practice*

How to compute the integral of a rational function upon the unit circle?

How to compute the causal and (strictly) anticausal part of a rational function?

How to perform spectral factorization?

Computing integrals via residues

Let $G(z) = N(z)/D(z)$ be a scalar rational function. We can decompose $G(z)$ using partial fraction expansion

$$G(z) = \sum_{i=1}^{n_p} \sum_{j=1}^{\mu_i} \frac{R_{ij}}{(z-p_i)^j} + k_{n_\infty} z^{n_\infty} + \cdots + k_1 z + k_0$$

- $\{p_i\}_{i=1}^{n_p}$ are the poles of $G(z)$ and $\{\mu_i\}_{i=1}^{n_p}$ the corresponding multiplicities
- R_{ij} are (in general, complex) coefficients corresponding to the term of multiplicity μ_j of the pole p_i
- $n_\infty = \max\{\deg N(z) - \deg D(z), 0\}$

Computing integrals via residues

The coefficients R_{ij} are called ***residues**** and are of crucial importance in complex analysis.

If p_i has multiplicity $\mu_i = 1$, then

$$R_{i1} = \lim_{z \to p_i} G(z)(z - p_i)$$

If p_i has multiplicity $\mu_i > 1$, then

$$R_{ij} = \frac{1}{(\mu_i - j)!} \lim_{z \to p_i} \left[\frac{\mathrm{d}^{\mu_i - j}}{\mathrm{d}z^{\mu_i - j}} G(z)(z - p_i)^{\mu_i} \right], \; j = 1, \ldots, \mu_i$$

*Formally, the residues are only the coefficients R_{i1}'s.

Computing integrals via residues

Let $F(z)$ be a rational function analytic on the unit circle. The computation of the integral of $F(z)$ upon the unit circle boils down to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\theta})\,\mathrm{d}\theta = \frac{1}{2\pi j}\oint_{|z|=1} \frac{F(z)}{z}\,\mathrm{d}z = \sum_{i\,:\,|p_i|<1} R_{i1}$$

Hence, it suffices to

1. Compute the partial fraction expansion of $G(z) = F(z)/z$
2. Sum the (one-multiplicity) residues R_{i1} of poles p_i s.t. $|p_i| < 1$

Computing integrals via residues

...in MATLAB®

```
>> [cvR,cvP,rvK] = residue(rvN,rvD)
```

- `cvR` $= [R_{11}\ R_{12}\ \ldots\ R_{1\mu_1}\ R_{21}\ \ldots]^\top$
- `cvP` $= [p_1\ p_1\ \ldots\ p_1\ p_2\ \ldots]^\top$ *(every p_i is repeated μ_i times!)*
- `rvK` $= [k_{n_\infty}\ \ldots\ k_1\ k_0]$

Computing integrals via residues

...in MATLAB®

```
>> [cvR,cvP,rvK] = residue(rvN,rvD)
>> [rvN,rvD] = residue(cvR,cvP,rvK)
```

Inverse operation!

- $\texttt{cvR} = [R_{11}\ R_{12}\ \dots\ R_{1\mu_1}\ R_{21}\ \dots]^\top$
- $\texttt{cvP} = [p_1\ p_1\ \dots\ p_1\ p_2\ \dots]^\top$ *(every p_i is repeated μ_i times!)*
- $\texttt{rvK} = [k_{n_\infty}\ \dots\ k_1\ k_0]$

Wiener filtering *in practice*

How to compute the integral of a rational function upon the unit circle?

How to compute the causal and (strictly) anticausal part of a rational function?

How to perform spectral factorization?

Computing causal and (strictly) anticausal parts

Let $G(z) = N(z)/D(z)$ be a scalar rational function analytic on the unit circle. We can decompose $G(z)$ as

$$
\begin{aligned}
G(z) &= \sum_{i=1}^{n_p} \sum_{j=1}^{\mu_i} \frac{R_{ij}}{(z-p_i)^j} + k_{n_\infty} z^{n_\infty} + \cdots + k_1 z + k_0 \\
&= \sum_{i\,:\,|p_i|<1} \mathrm{Res}_i(z) + k_0^+ + k_0^- + \sum_{i\,:\,|p_i|>1} \mathrm{Res}_i(z) + \sum_{i=1}^{n_\infty} k_i z^i
\end{aligned}
$$

where $\mathrm{Res}_i(z) := \sum_{j=1}^{\mu_i} \frac{R_{ij}}{(z-p_i)^j}$, $k_0 = k_0^+ + k_0^-$ with

$$
k_0^- := -\sum_{|p_i|>1} \mathrm{Res}_i(0)
$$

Computing causal and (strictly) anticausal parts

Let $G(z) = N(z)/D(z)$ be a scalar rational function analytic on the unit circle. We can decompose $G(z)$ as

$$G(z) = \sum_{i=1}^{n_p} \sum_{j=1}^{\mu_i} \frac{R_{ij}}{(z-p_i)^j} + k_{n_\infty} z^{n_\infty} + \cdots + k_1 z + k_0$$

$$= \sum_{i\,:\,|p_i|<1} \mathrm{Res}_i(z) + k_0^+ + k_0^- + \sum_{i\,:\,|p_i|>1} \mathrm{Res}_i(z) + \sum_{i=1}^{n_\infty} k_i z^i$$

where $\mathrm{Res}_i(z) := \sum_{j=1}^{\mu_i} \frac{R_{ij}}{(z-p_i)^j}$, $k_0 = k_0^+ + k_0^-$ with

$$k_0^- := -\sum_{|p_i|>1} \mathrm{Res}_i(0)$$

Why $\mathbf{k_0^-}$? At $z = 0$, the strictly anticausal part must be 0

Computing causal and (strictly) anticausal parts

Let $G(z) = N(z)/D(z)$ be a scalar rational function analytic on the unit circle. We can decompose $G(z)$ as

$$G(z) = \sum_{i=1}^{n_p} \sum_{j=1}^{\mu_i} \frac{R_{ij}}{(z-p_i)^j} + k_{n_\infty} z^{n_\infty} + \cdots + k_1 z + k_0$$

$$= \underbrace{\sum_{i\,:\,|p_i|<1} \mathrm{Res}_i(z) + k_0^+}_{[G(z)]_+ \text{ causal part}} + \underbrace{k_0^- + \sum_{i\,:\,|p_i|>1} \mathrm{Res}_i(z) + \sum_{i=1}^{n_\infty} k_i z^i}_{[[G(z)]]_- \text{ strictly anticausal part}}$$

$[G(z)]_+$
causal part

$[[G(z)]]_-$
strictly anticausal part

Computing causal and (strictly) anticausal parts

Let $G(z) = N(z)/D(z)$ be a scalar rational function analytic on the unit circle. We can decompose $G(z)$ as

$$G(z) = \sum_{i=1}^{n_p} \sum_{j=1}^{\mu_i} \frac{R_{ij}}{(z-p_i)^j} + k_{n_\infty} z^{n_\infty} + \cdots + k_1 z + k_0$$

$$= \underbrace{\sum_{i\,:\,|p_i|<1} \mathrm{Res}_i(z) + k_0^+}_{[G(z)]_+} + \underbrace{k_0^- + \sum_{i\,:\,|p_i|>1} \mathrm{Res}_i(z) + \sum_{i=1}^{n_\infty} k_i z^i}_{[[G(z)]]_-}$$

$[G(z)]_+$
causal part

$[[G(z)]]_-$
strictly anticausal part

Moral: $[G(z)]_+$ and $[[G(z)]]_-$ can be computed using residues!

✎ *Practice time 2!*

Ex 2.1. Create a function

```
dInt = resIntegral(tfF)
```

that has as input a rational transfer function object `tfF`. The function returns the integral `dInt` upon the unit circle of `tfF` computed via the residue method.

Recall: $$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\theta})\,\mathrm{d}\theta = \frac{1}{2\pi j}\oint_{|z|=1} \frac{F(z)}{z}\,\mathrm{d}z = \sum_{i\,:\,|p_i|<1} R_{i1}$$

✎ Practice time 2!

Ex 2.2. Create a function

```
[tfCaus,tfACaus] = causalPart(tfG)
```

that has as input a rational transfer function object `tfG`. The function returns the causal part `tfCaus` and the strictly anticausal part `tfACaus` of `tfG`.

Recall: $$G(z) = \underbrace{\sum_{i\,:\,|p_i|<1} \mathrm{Res}_i(z) + k_0^+}_{[G(z)]_+} + \underbrace{k_0^- + \sum_{i\,:\,|p_i|>1} \mathrm{Res}_i(z) + \sum_{i=1}^{n_\infty} k_i z^i}_{[[G(z)]]_-}$$

Wiener filtering *in practice*

How to compute the integral of a rational function upon the unit circle?

How to compute the causal and (strictly) anticausal part of a rational function?

How to perform spectral factorization?

Scalar rational spectral factorization

the procedure

Let $\{p_i\}_{i=1}^{n}$ be the *strictly stable* (finite) poles of $S(z)$ different from zero, counted with multiplicity. Let $\{z_i\}_{i=1}^{m}$ be the *marginally stable* (finite) zeros of $S(z)$ different from zero, counted with multiplicity for the strictly stable zeros and with *half-multiplicity* for the marginally stable zeros.

Scalar rational spectral factorization

the procedure

Let $\{p_i\}_{i=1}^n$ be the strictly stable (finite) poles of $S(z)$ different from zero, counted with multiplicity. Let $\{z_i\}_{i=1}^m$ be the marginally stable (finite) zeros of $S(z)$ different from zero, counted with multiplicity for the strictly stable zeros and with half-multiplicity for the marginally stable zeros.

1. $S(z) = cz^r \dfrac{\prod_{i=1}^m (z - z_i)(z - 1/\bar{z}_i)}{\prod_{i=1}^n (z - p_i)(z - 1/\bar{p}_i)}$, $r = n - m$ (by symmetry)

(zero-pole-gain decomposition)

Scalar rational spectral factorization

the procedure

Let $\{p_i\}_{i=1}^n$ be the strictly stable (finite) poles of $S(z)$ different from zero, counted with multiplicity. Let $\{z_i\}_{i=1}^m$ be the marginally stable (finite) zeros of $S(z)$ different from zero, counted with multiplicity for the strictly stable zeros and with half-multiplicity for the marginally stable zeros.

1. $S(z) = cz^r \dfrac{\prod_{i=1}^m (z - z_i)(z - 1/\bar{z}_i)}{\prod_{i=1}^n (z - p_i)(z - 1/\bar{p}_i)}, \; r = n - m$

2. $S(z) = \lambda^2 \dfrac{\prod_{i=1}^m (z - z_i)(z^{-1} - \bar{z}_i)}{\prod_{i=1}^n (z - p_i)(z^{-1} - \bar{p}_i)}, \; \lambda^2 := c \dfrac{\prod_{i=1}^m -\frac{1}{\bar{z}_i}}{\prod_{i=1}^n -\frac{1}{\bar{p}_i}} > 0$

("symmetrizing" the equation)

Scalar rational spectral factorization

the procedure

Let $\{p_i\}_{i=1}^n$ be the strictly stable (finite) poles of $S(z)$ different from zero, counted with multiplicity. Let $\{z_i\}_{i=1}^m$ be the marginally stable (finite) zeros of $S(z)$ different from zero, counted with multiplicity for the strictly stable zeros and with half-multiplicity for the marginally stable zeros.

1. $S(z) = cz^r \dfrac{\prod_{i=1}^m (z - z_i)(z - 1/\bar{z}_i)}{\prod_{i=1}^n (z - p_i)(z - 1/\bar{p}_i)}, \; r = n - m$

2. $S(z) = \lambda^2 \dfrac{\prod_{i=1}^m (z - z_i)(z^{-1} - \bar{z}_i)}{\prod_{i=1}^n (z - p_i)(z^{-1} - \bar{p}_i)}, \; \lambda^2 := c \dfrac{\prod_{i=1}^m -\frac{1}{\bar{z}_i}}{\prod_{i=1}^n -\frac{1}{\bar{p}_i}} > 0$

3. $L(z) = z^{n-m} \lambda \dfrac{\prod_{i=1}^m (z - z_i)}{\prod_{i=1}^n (z - p_i)}$ *("balancing" relative degree)*

Scalar rational spectral factorization

the procedure

Let $\{p_i\}_{i=1}^n$ be the strictly stable (finite) poles of $S(z)$ different from zero, counted with multiplicity. Let $\{z_i\}_{i=1}^m$ be the marginally stable (finite) zeros of $S(z)$ different from zero, counted with multiplicity for the strictly stable zeros and with half-multiplicity for the marginally stable zeros.

1. $S(z) = cz^r \dfrac{\prod_{i=1}^m (z - z_i)(z - 1/\bar{z}_i)}{\prod_{i=1}^n (z - p_i)(z - 1/\bar{p}_i)}, \; r = n - m$

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3. $L(z) = z^{n-m} \lambda \dfrac{\prod_{i=1}^m (z - z_i)}{\prod_{i=1}^n (z - p_i)}$ $\qquad (\lambda > 0 \Rightarrow \textit{canonical } L(z))$

Scalar rational spectral factorization

...in MATLAB®

```
>> [cvZ,cvP,dK] = zpkdata(tfS,'v');
```

- `cvZ`: column vector containing the zeros of $S(z)$, counted with multiplicity
- `cvP`: column containing the poles of $S(z)$, counted with multiplicity
- `dK` $= c$, gain of $S(z)$

Pay attention on the term z^{n-m}!

Application: the Wiener predictor

Wiener prediction

k-step ahead Wiener predictor

$$H_k(z) := \left[z^k \frac{S_{xy}(z)}{L(z^{-1})} \right]_+ \frac{1}{L(z)}$$

$$S_y(e^{j\theta}) = L(e^{j\theta})L(e^{-j\theta})$$

$$y(t) \longrightarrow \boxed{H_k(z)} \longrightarrow \hat{x}(t+k|t)$$

Wiener prediction

k-step ahead Wiener predictor

Case $\{x(t)\}_{t\in\mathbb{Z}} = \{y(t)\}_{t\in\mathbb{Z}}$

$$H_k(z) := \left[z^k L(z)\right]_+ \frac{1}{L(z)}$$

$$S_y(e^{j\theta}) = L(e^{j\theta})L(e^{-j\theta})$$

$$y(t) \longrightarrow \boxed{H_k(z)} \longrightarrow \hat{y}(t+k|t)$$

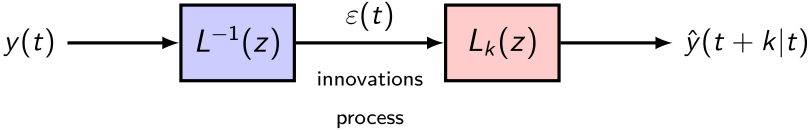
Wiener prediction

k-step ahead Wiener predictor

Case $\{x(t)\}_{t\in\mathbb{Z}} = \{y(t)\}_{t\in\mathbb{Z}}$

$$H_k(z) := \left[z^k L(z)\right]_+ \frac{1}{L(z)}$$

$$S_y(e^{j\theta}) = L(e^{j\theta})L(e^{-j\theta})$$



How to compute $\left[z^k L(z)\right]_+$?

Suppose $S_y(z)$ rational:

$$L(z) = \frac{C(z)}{A(z)} = \ell_0 + \ell_1 z^{-1} + \ell_2 z^{-2} + \dots$$

$$z^k L(z) = \left[z^k L(z)\right]_+ + \left[\left[z^k L(z)\right]\right]_-$$

How to compute $\left[z^k L(z)\right]_+$?

Suppose $S_y(z)$ rational:

$$L(z) = \frac{C(z)}{A(z)} = \ell_0 + \ell_1 z^{-1} + \ell_2 z^{-2} + \dots$$

$$z^k L(z) = \left[z^k L(z)\right]_+ + \left[\left[z^k L(z)\right]\right]_-$$

$$\boxed{\frac{C_k(z)}{A(z)}} \qquad \boxed{\ell_0 z^k + \ell_1 z^{k-1} + \dots + \ell_{k-1} z}$$

How to compute $\left[z^k L(z)\right]_+$?

Suppose $S_y(z)$ rational:

$$L(z) = \frac{C(z)}{A(z)} = \ell_0 + \ell_1 z^{-1} + \ell_2 z^{-2} + \dots$$

$$z^k \frac{C(z)}{A(z)} = \frac{C_k(z)}{A(z)} + \sum_{i=0}^{k-1} \ell_i z^{k-i}$$

How to compute $\left[z^k L(z)\right]_+$?

Suppose $S_y(z)$ rational:

$$L(z) = \frac{C(z)}{A(z)} = \ell_0 + \ell_1 z^{-1} + \ell_2 z^{-2} + \dots$$

$$z^k C(z) = A(z) \sum_{i=0}^{k-1} \ell_i z^{k-i} + C_k(z)$$

$z^k C(z)/zA(z) \rightarrow$ polynomial division

$Q(z) := \sum_{i=0}^{k-1} \ell_i z^{k-1-i} \rightarrow$ quotient

$R(z) := C_k(z) \rightarrow$ remainder

How to compute $\left[z^k L(z)\right]_+$?

Suppose $S_y(z)$ rational:

$$L(z) = \frac{C(z)}{A(z)} = \ell_0 + \ell_1 z^{-1} + \ell_2 z^{-2} + \ldots$$

$$z^k C(z) = zA(z)Q(z) + R(z)$$

...in MATLAB®

```
>> [rvQ,rvR] = deconv([rvC 0 .. 0], [rvA 0])
```

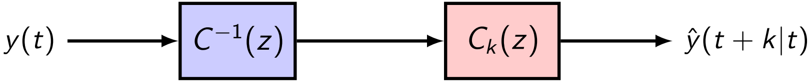
k zeros

Wiener prediction: rational case

$$H(z) := \left[z^k L(z)\right]_+ \frac{1}{L(z)} = \frac{C_k(z)}{C(z)}$$

$$\operatorname{Var} \tilde{y}(t+k|t) = \sum_{i=0}^{k-1} \ell_i^2$$

prediction error variance



✐ *Practice time 3!*

Ex 3.1. Create a function

```
tfW = spectralFactor(tfS)
```

that has as input a transfer function object `tfS` corresponding to a rational scalar discrete-time spectral density. The function returns the canonical minimum-phase spectral factor of `tfS`.

Ex 3.2. Create a function

```
[cvYhat,dVar] = WienerPredictor(tfS,cvY,iK)
```

that has as input a rational scalar spectral density `tfS` of the process $\{y(t)\}_{t\in\mathbb{Z}}$, a trajectory `cvY` of the latter process, and a positive integer $\texttt{iK} = k$. The function returns the Wiener predictions $\hat{y}(t+k|t)$ in the vector `cvYhat` and the prediction error variance $\mathrm{Var}\,\tilde{y}(t+k|t)$ in the variable `dVar`.