数学分析作业(4月11日交)

2019 年 5 月 3 日

习题8.4

8.

证明. $|x| \le 1$ 时，$\frac{1}{1-x} = 1+x^2+x^3+\cdots$ 绝对收敛. 故$\frac{1}{(1-x)^2} = \frac{1}{1-x}\frac{1}{1-x} = (\sum_{m=0}^{\infty} x^m)(\sum_{n=0}^{\infty} x^n) = \sum_{k=0}^{\infty}(k+1)x^k$. □

11.

证明. 考虑无穷乘积$\prod_{n=1}^{\infty} \frac{\beta+n}{\alpha+n} = \prod_{n=1}^{\infty}(1-\frac{\alpha-\beta}{\alpha+n})$. 考虑负项级数$\sum_{n=1}^{\infty} \ln(1-\frac{\alpha-\beta}{\alpha+n})$, 由于$\ln(1-\frac{\alpha-\beta}{\alpha+n}) \sim -\frac{\alpha-\beta}{\alpha+n}$, 而$\sum_{n=1}^{\infty} \frac{\alpha-\beta}{\alpha+n} = \infty$, 故$\sum_{n=1}^{\infty} \ln(1-\frac{\alpha-\beta}{\alpha+n}) \to -\infty$, 则$\lim_{n\to\infty} \frac{\beta(\beta+1)\cdots(\beta+n-1)}{\alpha(\alpha+1)\cdots(\alpha+n-1)} = 0$. □

12.

证明. 易得$\prod_{n=1}^{\infty} \frac{1}{1-\frac{1}{p_n}} = \sum_{n=1}^{\infty} \frac{1}{n}$ 发散. 则$\sum_{n=0}^{\infty} \ln(1-\frac{1}{p_n})$ 发散，而$\ln(1-\frac{1}{p_n}) < 0$, $\lim_{n\to\infty} \frac{\ln(1-\frac{1}{p_n})}{-\frac{1}{p_n}} = 1$, 则$\sum_{n=0}^{\infty} \frac{1}{p_n}$ 发散. □

习题9.1

1.

证明. $f_{n+1}(x) = x^{\frac{1}{2}} f_n^{\frac{1}{2}}(x)$, $f_2(x) = x^{\frac{1}{2}}$, $f_3(x) = x^{\frac{1}{2}} x^{\frac{1}{4}}, \cdots$, 由归纳可得$f_n(x) = x^{\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}} = x^{1-\frac{1}{2^{n-1}}}$. 则$f_n(x) \to x$逐点地，且$f_n(x)$单调递减, 故由Dini定理，$f_n$一致收敛. □

2.

证明. $|f_1(x)| \le M$, $|f_2(x)| \le \int_0^x M dx = Mx$,$|f_3(x)| \le \int_0^x Mx dx = \frac{M}{2}x^2, \cdots$, 归纳地可证，$|f_n(x)| \le \frac{M}{(n-1)!}x^{n-1}$, 则$|f_n(x)| \le \frac{Ma^{n-1}}{(n-1)!} \to 0 (n \to 0)$, 从而$f_n(x) \rightrightarrows 0$. □

3.

证明. $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall m > n > N$, 有$\sum_{k=n+1}^{m} g_k(x) \le \varepsilon, \forall x \in I$. 则$\sum_{k=n+1}^{m} f_k(x) \le \sum_{k=n+1}^{m} |f_k(x)| \le \sum_{k=n+1}^{m} g_k(x) \le \varepsilon$. □