数学分析作业 4 月 18 日交

May 5, 2019

习题 9.1

4. 解

(1) $\sum\limits_{i=1}^{\infty} ne^{-nx}$, $x \in (0,+\infty)$ since $\sqrt[n]{ne^{-nx}} = \sqrt[n]{n}e^{-x} \to e^{-x} < 1$ 故它收敛，但不一致收敛：如果一致收敛，则必有 $ne^{-nx} \rightrightarrows 0$ 但 $x = \frac{1}{n}$ 时，它取值为 $\frac{n}{e} > 1$, 于是 ne^{-nx} 不一致收敛于 0.

(3) $\sum\limits_{n=1}^{\infty} \frac{1}{x^2+n^2}$ 一致收敛：$\frac{1}{x^2+n^2} < \frac{1}{n^2}$. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} < +\infty$.

(5) $\sum\limits_{n=1}^{\infty} \frac{1}{(1+nx)^2}$, $x \in (0,+\infty)$ 不一致收敛：假设一致收敛，则 $\forall \epsilon > 0, \exists N, s.t. n > N$ 时，$|\sum\limits_{k=n+1}^{2n} \frac{kx}{1+k^3x^2}| < \epsilon, \forall x \in (-\infty,+\infty)$, but $|\sum\limits_{k=n+1}^{2n} \frac{kx}{1+k^3x^2}| = \sum\limits_{k=n+1}^{2n} \frac{k|x|}{1+k^3x^2} = |x| \sum\limits_{k=n+1}^{2n} \frac{1}{1/k+k^2x^2} \geq |x| \sum\limits_{k=n+1}^{2n} \frac{1}{1/n+4n^2x^2} = \frac{n^2|x|}{1+4n^3x^2}$ but when $x = n^{-3/2}$, $|\sum\limits_{k=n+1}^{2n} \frac{kx}{1+k^3x^2}| \geq \frac{n^2n^{-3/2}}{1+4} = \frac{\sqrt{n}}{5}$.

$\Rightarrow \sup\limits_{x\in(-\infty,+\infty)} |\sum\limits_{k=n+1}^{2n} \frac{kx}{1+k^3x^2}| \geq \frac{\sqrt{n}}{5} \to \infty$.

5.

证明. $b_n(x) := \frac{1}{n^x}$ for $\forall x \in [0,+\infty), b_n(x)$ 对 n 单调且一直有界：$|b_n(x)| \leq 1$. 由 Abel 判别法，$\sum\limits_{n=1}^{\infty} \frac{a_n}{n^x}$ 在 $[0,+\infty)$ 一致收敛。 □

7.

证明. $\forall \epsilon > 0$. since $f_n \rightrightarrows f, \exists N, s.t. \forall n > N$ 时, 有 $|f_n(x) - f(x)| < \frac{\epsilon}{3} \forall x \in I$. 由 f_N 的一致收敛性，$\exists \delta > 0$, s.t. $\forall x, y \in I$ *and* $|x-y| < \delta$, 有 $|f_N(x) - f_N(y)| < \frac{\epsilon}{3}$. 于是 $|f(x)-f(y)| \leq |f(x)-f_N(x)| + |f_N(x)-f_N(y)| + |f_N(y)-f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. □

8.

证明. let $|g(x)| \leq K, K > 0$. $\forall \epsilon > 0, \exists N$, *s.t.* $\forall n > N$ 时, 有 $|f_n(x) - f(x)| < \frac{\epsilon}{K}, \forall x \in I$. so $|f_n(x)g(x) - f(x)g(x)| \leq K|f_n(x) - f(x)| < \epsilon, \forall x \in I$. therefore, $f_ng \rightrightarrows fg$. □

10.

证明. (换成 (a,b) 不对，例：$f_n(x) = x - \frac{1}{n}$ *on* $(0,1)$, $f_n \rightrightarrows x$ *on* $(0,1)$ but 每个 f_n 都有负的部分). $f \in C^0[a,b]$ *and* $f > 0 \Rightarrow \exists \delta > 0$ *s.t.* $f(x) \geq \delta$, $\forall x \in [a,b]$. since $f_n \rightrightarrows f, \exists N$, *s.t.* $\forall n > N, |f_n(x)-f(x)| < \frac{\delta}{2} \Rightarrow f_n(x) \geq f(x) - \frac{\delta}{2} \geq \frac{\delta}{2} > 0$. so $|\frac{1}{f_n(x)} - \frac{1}{f(x)}| = \frac{|f_n(x)-f(x)|}{|f_n(x)f(x)|} \leq \frac{|f_n(x)-f(x)|}{\delta\frac{\delta}{2}} = \frac{2}{\delta^2}|f_n(x)-f(x)|$. $\forall \epsilon > 0$, *take* N_1 *s.t.* $\forall n > N_1, |f_n(x)-f(x)| < \frac{\delta^2}{2}\epsilon$. so *when* $n > max\{N, N_1\}$, $|\frac{1}{f_n(x)} - \frac{1}{f(x)}| < \epsilon, \forall x \in [a,b] \Rightarrow \frac{1}{f_n(x)} \rightrightarrows \frac{1}{f(x)}$ *on* $[a,b]$. □

12.

证明. let $|f_n(x)| \leq K_n$. since $f_n \rightrightarrows f$, $\exists N$, *s.t.* $\forall n > N$ 时, 有 $|f_n(x) - f_N(x)| \leq 1, \forall x \in I$. $\Rightarrow |f_n(x)| \leq |f_N(x)| + 1 \leq K_N + 1 \Rightarrow |f(x)| \leq K_N + 1$. 同理，$g$ 有界，设 $|f_n|, |g_n| \leq K, \forall n, K > 0, \forall \epsilon > 0$, $\exists N$, *s.t.* $\forall n > N, |f_n(x) - f(x)| < \frac{\epsilon}{2K}, |g_n(x) - g(x)| < \frac{\epsilon}{2K}, \forall x \in I$. $\Rightarrow |f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| < \frac{\epsilon}{2K}K + \frac{\epsilon}{2K}K = \epsilon$. $\Rightarrow f_ng_n \rightrightarrows fg$. □

13.

证明. 由 Cauchy 准则，对 $\epsilon = 1$, $\exists N,\ s.t.\ \forall m, n > N, |P_m(x) - P_n(x)| \leq 1$ 但 $P_m - P_n$ 是多项式，在 $\mathbb{R}$ 上有界，当且仅当 $deg(P_m - P_n) = 0$. 所以 $n > N$ 时，$P_n = P(x) + c_n$, $P(x)$ 为一固定多项式，$c_n \in \mathbb{R}$. $\Rightarrow c_n \to c \in \mathbb{R}$. $\Rightarrow f(x) = P(x) + c$ 为多项式。 □

习题 9.2

1.

证明. 首先 f 绝对一致收敛，故 f 连续。而 $\sum\limits_n \frac{cosnx}{n^2}$ 在 $\mathbb{R}$ 上一致收敛且极限连续。于是 $f'(x) = \sum\limits_{n=1}^{\infty} \frac{(sinnx)'}{n^3} = \sum\limits_{n=1}^{\infty} \frac{cosnx}{n^2}$. □

3.

证明. $(\frac{1}{n^x})' = (n^{-x})' = (e^{-lnnx})' = (-lnn)\frac{1}{n^x}$. $\Rightarrow (\frac{1}{n^x})^k = \frac{(-lnn)^k}{n^x}$. $\forall \delta > 0$, 在 $[1+\delta, +\infty)$ 上有 $\sum\limits_{n=1}^{\infty} \frac{(lnn)^k}{n^x} \leq \sum\limits_{n=1}^{\infty} \frac{(lnn)^k}{n^{1+\delta}} < +\infty$. 故可逐项求导。

□

5.(1). (一致收敛的验证：$0 < ne^{-nx} \leq ne^{-nln2} = \frac{n}{2^n}$. $\sqrt[n]{\frac{n}{2^n}} \to \frac{1}{2} < 1 \Rightarrow \sum\limits_n \frac{n}{2^n} < +\infty \Rightarrow \sum\limits_n ne^{-nx}$ 在 $[ln2, ln5]$ 上一致收敛。)

$\int_{ln2}^{ln5} \sum\limits_{n=1}^{\infty} ne^{-nx}dx = \sum\limits_{n=1}^{\infty} \int_{ln2}^{ln5} ne^{-nx}dx = \sum\limits_{n=1}^{\infty} \int_{nln2}^{nln5} e^{-y}dy = \sum\limits_{n=1}^{\infty} (\frac{1}{2^n} - \frac{1}{5^n}) = \frac{1/2}{1-1/2} - \frac{1/5}{1-1/5} = 1 - 1/4 = \frac{3}{4}$.

8.

证明. 设 $F_n' = f_n$. 改造 $G_n(x) = F_n(x) - F_n(a)$. 则 $G_n' = f_n$ 且 $G_n(a) = 0$. $\Rightarrow \{G_n(a)\}_{n=1}^{\infty}$ 收敛。$\Rightarrow G_n(x) \rightrightarrows G(x)\ on\ [a, b]$ 且 $G'(x) = f$. □