数学分析作业 3 月 21 日交

May 5, 2019

习题 6.3

14.

证明. $\int_0^1 \frac{nf(x)}{1+n^2x^2}dx \int_0^n \frac{f(\frac{y}{n})}{1+y^2}dy = \int_0^{\sqrt{n}} \frac{f(\frac{y}{n})}{1+y^2}dy + \int_{\sqrt{n}}^n \frac{f(\frac{y}{n})}{1+y^2}dy = \int_0^{\sqrt{n}} \frac{f(0)}{1+y^2}dy + \int_0^{\sqrt{n}} \frac{f(\frac{y}{n})-f(0)}{1+y^2}dy + \int_{\sqrt{n}}^n \frac{f(\frac{y}{n})}{1+y^2}dy = I_1 + I_2 + I_3$. $I_1 = f(0)\arctan\sqrt{n} \to f(o)\frac{\pi}{2}$. $|I_2| \le \frac{\epsilon}{2}$ 当 n 充分大时（用 f 在 0 连续）。$|I_3| \le K(\arctan n - \arctan\sqrt{n}) \to 0$. □

习题 7.1

7.

证明. 不妨设 $t > 0$ 时，$\sigma(t) \neq \sigma(0)$. 令 $F(t) = |\sigma(t) \neq \sigma(0)|$, 则 $F(0) = 0$. F 可导，且 $F'(t) = (\sqrt{(x(t)-x(0))^2 + (y(t)-y(0))^2})' = \frac{x'(t)(x(t)-x(0))+y(t)(y(t)-y(0))}{\sigma(t)\neq\sigma(0)} = \frac{\sigma^{'}(t)(\sigma(t)\neq\sigma(0))}{\sigma(t)\neq\sigma(0)}$ $\Rightarrow |F'(t)| \le \|\sigma^{'}(t)\|$. $\Rightarrow |\sigma(1)-\sigma(0)| = F(1)-F(0) = \int_0^1 F'(t)dt \le \int_0^1 |F'(t)|dt \le \int_0^1 \|\sigma'(t)\|dt = L(\sigma)$. □

习题 7.2

2. (1) $\int_2^{+\infty} \frac{dx}{x(lnx)^p} = \int_{ln2}^{+\infty} \frac{dt}{t^p} = \frac{1}{1-p}t^{1-p}|_{ln2}^{+\infty} = \frac{(ln2)^{1-p}}{p-1}, when\ p > 1;\ +\infty, when\ p \le 1.$

(3) $\int_{e^2}^{+\infty} \frac{dx}{xlnxln^2(lnx)} = \int_{e^2}^{+\infty} \frac{dlnx}{lnxln^2(lnx)} = \int_{e^2}^{+\infty} \frac{dlnlnx}{ln^2(lnx)} = -\frac{1}{ln(lnx)}|_{e^2}^{+\infty} = \frac{1}{ln2}$.

(5) $\int_0^{+\infty} xe^{-x}dx = 1$.

(7) $\int_1^{+\infty} \frac{dx}{x(1+x)} = \int_1^{+\infty}(\frac{1}{x} - \frac{1}{x+1})dx = ln\frac{x}{x+1}|_1^{+\infty} = -ln\frac{1}{2} = ln2$.

(9) $\int_0^{+\infty} \frac{dx}{x^2+2x+2)} = \int_0^{+\infty} \frac{dx}{(1+x)^2+1} = arctan(x+1)|_0^{+\infty} = \frac{\pi}{4}$.

3. (1) $\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx = frac12\int_0^1 \frac{x^2dx^2}{\sqrt{1-x^2}} = frac12\int_0^1 \frac{tdt}{\sqrt{1-t}} = frac12\int_0^1(\frac{1}{\sqrt{1-t}} - \sqrt{1-t})dt = frac12\int_0^1(\frac{1}{\sqrt{s}} - \sqrt{s})ds = (\sqrt{s} - \frac{1}{3}s^{\frac{3}{2}})|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$.

(3) $\int_{-1}^1 \frac{arcsinxdx}{\sqrt{1-x^2}} = 0$.

(5) $\int_\alpha^\beta \frac{xdx}{\sqrt{(x-\alpha)(x-\beta)}} \overset{t=\frac{x-\alpha}{\beta-\alpha}}{=} \int_0^1 \frac{(\alpha+(\beta-\alpha)t)(\beta-\alpha)dt}{\sqrt{(\beta-\alpha)^2t(1-t)}} = \int_0^1 \frac{\alpha+(\beta-\alpha)t}{\sqrt{t(t-1)}}dt = \int_0^1 \frac{2(\alpha+(\beta-\alpha)t)dt}{\sqrt{1-(2t-1)^2)}} = \frac{\beta+\alpha}{2}arcsin(2t-1)|_0^1 + \int_0^1 \frac{(\beta-\alpha)(2t-1))dt}{\sqrt{1-(2t-1)^2)}} = \frac{\beta+\alpha}{2}(\frac{\pi}{2}) + \frac{\pi}{2}) - \frac{\beta-\alpha}{2}\sqrt{1-(2t-1)^2}|_0^1 = \frac{\beta+\alpha}{2}\pi$.

4. (2) 收敛. $\int_0^1 \frac{dx}{e^x\sqrt{x}}$ 在 0 处有瑕点。$\lim_{x\to0}(\frac{1}{e^x\sqrt{x}}/\frac{1}{\sqrt{x}}) = 1$. 而 $\int_0^1 \frac{dx}{\sqrt{x}}$ 收敛，故 $\int_0^1 \frac{dx}{e^x\sqrt{x}}$ 收敛。而 $\lim_{x\to\infty}(\frac{1}{e^x\sqrt{x}})/\frac{1}{x^{2+\frac{1}{2}}} = \lim_{x\to\infty}\frac{x^2}{e^x} = 0$. 而 $\int_1^{+\infty} \frac{dx}{x^{2+\frac{1}{2}}}$ 收敛，故 $\int_1^{+\infty} \frac{dx}{e^x\sqrt{x}}$ 收敛。

5. (1) $\int_0^1 \frac{dx}{lnx}$ 发散，0 是好的，$\lim_{x\to1-}(\frac{1}{lnx})/\frac{1}{x-1} = \lim_{x\to1-}\frac{x-1}{lnx} = 1$. 故发散。

(3) 发散，1 是瑕点，$\int_0^1 \frac{dx}{(x-1)^2} = +\infty = \int_1^2 \frac{dx}{(x-1)^2}$.

8.

证明. 如果不然,$\exists\epsilon_0 > 0\ and\ x_n \to +\infty\ s.t.\ |f(x_n)| \ge \epsilon_0$. 由一致连续性,$\exists\delta > 0,\ s.t.\ when\ |x-y| < \delta,\ we\ have\ |f(x) - f(y)| < \frac{\epsilon_0}{2}$. 于是 $|\int_{x_n}^{x_n+\delta} f(x)dx| = |\int_{x_n}^{x_n+\delta} f(x_n)dx| + |\int_{x_n}^{x_n+\delta}(f(x) - f(x_n))dx| \ge \epsilon_0\delta - \frac{\epsilon_0}{2}\delta = \frac{1}{2}\epsilon_0\delta$. 与 Cauchy 准则矛盾。 □

9.

证明. f' 有界 $\Rightarrow$ Lip $\Rightarrow$ 一致连续，再用上题。 □