数学分析作业(3月14日交)

2019 年 5 月 3 日

习题6.2

1.

证明. $\max\{f,g\}=\frac{f+g}{2}+\frac{|f-g|}{2}$,$\min\{f,g\}=\frac{f+g}{2}-\frac{|f-g|}{2}$, 显然，两函数均是可积的. □

2.

证明. 根据$\sqrt{x}$在$[0,\infty)$的连续性知，$D_{|f|}\subset D_{f^2}$, 因此$f^2\in R[a,b], D_{f^2}$零测集，则$D_{|f|}$零测集，$|f|\in R[a,b]$. □

3.

证明. $sgn(x)=\begin{cases}1,x>0\\0,x=0\\-1,x<0\end{cases}$, $R(x)=\begin{cases}\frac{1}{q},x=\frac{p}{q}\\1,x=0,1\\0,x\in Q^c\end{cases}$ 则$sgn\circ R(x)=\begin{cases}1,\in Q\\0,x\in Q^c\end{cases}=D(x)$ 不可积. □

4.

证明. 设$F(x)=\int_a^x f(t)dt-\frac{1}{2}\int_a^b f(t)dt$, 由于$f(x)$在$[a,b]$ 上可积，则$F(x)$ 为$[a,b]$上的连续函数，且$F(a)=-\frac{1}{2}\int_a^b f(t)dt, F(b)=\frac{1}{2}\int_a^b f(t)dt$, 则$F(a)F(b)\le 0$, 由介值定理，$\exists\xi\in[a,b], F(\xi)=0$, 故$\int_a^\xi f(t)dt=\frac{1}{2}\int_a^b f(t)dt$, 从而$\int_a^\xi f(t)dt=\int_\xi^b f(t)dt$. □

5.

证明. $\int_a^b(tf+g)^2dx\ge 0$, i.e. $t^2(\int_a^b f^2dx)+2t(\int_a^b fgdx)+\int_a^b g^2dx\ge 0,\forall t\in R$. 若$\int_a^b f^2dx\ge 0$, 则由判别式$\Delta\ge 0$,则立得所证不等式.若$\int_a^b f^2dx=0$, 必有$\int_a^b f\cdot gdx=0$. □

6.

证明. 由Cauchy－Schwarz不等式：

$$\begin{aligned}(\int_a^b f(x)\cos\lambda xdx)^2+(\int_a^b f(x)\sin\lambda xdx)^2 &\le (\int_a^b\sqrt{f(x)}\sqrt{f(x)}\cos\lambda xdx)^2+(\int_a^b\sqrt{f(x)}\sqrt{f(x)}\sin\lambda xdx)^2\\ &\le \int_a^b f(x)dx\cdot\int_a^b f(x)\cos^2\lambda xdx+\int_a^b f(x)dx\cdot\int_a^b f(x)\sin^2\lambda xdx\\ &= (\int_a^b f(x)dx)^2\end{aligned}$$

□

8.

证明. 由于$f(x)\in R[0,1]$, 则$f(x)$ 有界，设$M>0,|f(x)|\le M$. 则$|\int_0^1 x^nf(x)dx|\le\int_0^1 x^n|f(x)dx\le M\int_0^1 x^ndx=\frac{M}{n+1}\to 0$, 当$n\to\infty$. □

9.

证明. 令$g(x)=f(x)-f(1)\in C[0,1]$, $g(1)=0$, 且$\exists M>0,|g(x)|\le M$. $\forall\varepsilon>0,\exists\delta>0$, 当$x\in(1-\delta,1),g(x)\le\frac{\varepsilon}{2}$. 则$\int_0^1 nx^n g(x)dx=\int_0^{1-\delta}nx^n g(x)dx+\int_{1-\delta}^1 nx^n g(x)dx\le M\int_0^{1-\delta}nx^n dx+\frac{\varepsilon}{2}\int_{1-\delta}^1 nx^n dx\le M\cdot(1-\delta)^{n+1}+\frac{\varepsilon}{2}$. 由于$0<1-\delta<1,\exists N>0,\forall n>N,M\cdot(1-\delta)^{n+1}\le\frac{\varepsilon}{2}$. 由此$\int_0^1 nx^n g(x)dx<\varepsilon,\lim_{n\to\infty}\int_0^1 nx^n g(x)dx=0$. 则$\lim_{n\to\infty}\int_0^1 nx^n f(x)dx=\lim_{n\to\infty}\int_0^1 nx^n[g(x)+f(1)]dx=\lim_{n\to\infty}\int_0^1 nx^n f(1)dx=f(1)$. □

10.

证明. (1) $\forall\varepsilon>0$, 取$\delta=\frac{\varepsilon}{2}$,$|\int_0^{\frac{\pi}{2}}\sin^n xdx|\le|\int_0^{\frac{\pi}{2}-\delta}\sin^n xdx|+|\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}}\sin^n xdx|\le(\frac{\pi}{2}-\delta)\sin^n(\frac{\pi}{2}-\delta)+\delta$, 由$0<\sin^n(\frac{\pi}{2}-\delta)<1$, 故对于上述$\varepsilon,\exists N>0,\forall n>N,\sin^n(\frac{\pi}{2}-\delta)\le\frac{\varepsilon}{\pi}$, 则$|\int_0^{\frac{\pi}{2}}\sin^n xdx|\le\varepsilon$, 即$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\sin^n xdx=0$. □

12.

证明. 由于$f(x)\in R[c,b]$,则$\forall h>0$,存在分割$\Pi: c=x_0<x_1<x_2...<x_t=a<...<x_s=b<...x_n=d$, 且$\sum_{i=0}^n w_i\Delta x_i<h$. 定义$g(x)$, 当$x\in[x_{i-1},x_i]$ 时，$g(x)=f(x_{i-1})\frac{x_i-x}{\Delta x_i}+f(x_i)\frac{x-x_{i-1}}{\Delta x_i}$, 则$g(c)=f(c),g(d)=f(d)$, g 是逐段线性连续函数，则$\int_a^b|f(x)-g(x)|dx\le\sum_{i=t}^s w_i\Delta x_i<h$. 此外，$\int_a^b|g(x+h)-g(x)|dx=\sum_{i=t}^{s-1}\int_{x_i}^{x_{i+1}}|g(x+h)-g(x)|dx\le h(f(b)-f(a))$, 则$\int_a^b|f(x+h)-f(x)|dx\le\int_a^b|f(x+h)-g(x+h)|dx+\int_a^b|g(x+h)-g(x)|dx+\int_a^b|g(x)-f(x)|dx\le 2h+h(f(b)-f(a))\to 0$, 当$h\to 0$. □

13.

证明. 在$[-a,b]$上，$(x+a)(b-x)\ge 0$, 则$\int_{-a}^b(x+a)(b-x)dx\ge 0$, 即$ab\int_{-a}^b f(x)dx-\int_{-a}^b x^2 f(x)dx\ge 0$ 移项即可得. □

14.

证明. 原不等式等价于$\int_a^b(f(x)-\frac{1}{b-a}\int_a^b f(x)dx)g(x)\ge 0$.令$\tilde f(x)=f(x)-\frac{1}{b-a}\int_a^b f(x)dx$, 则$\tilde f(x)$单减，且$\int_a^b\tilde f(x)dx=0$. 则$\exists c\in(a,b),s.t.\tilde f(x)\ge 0$ on $[a,c]$, $\tilde f(x)\le 0$ on $[c,b]$. 于是，$\int_a^b\tilde f(x)g(x)dx=\int_a^c\tilde f(x)g(x)dx+\int_c^b\tilde f(x)g(x)dx\ge g(c)\int_a^c\tilde f(x)dx+g(c)\int_c^b\tilde f(x)dx=g(c)\int_a^b\tilde f(x)dx=0$. □

习题6.3

1.(2) 解：$\frac{d}{dx}\int_{\sin x}^x\frac{dt}{\sqrt{1+\sin^2 t}}=\frac{1}{\sqrt{1+\sin^2 t}}-\frac{\cos x}{\sqrt{1+\sin^2(\sin^x)}}$.

3.解：(1)$\int_0^1\frac{(1_x^2)x^4}{1+x^2}dx=\int_0^1(-x^4+2x^2-2+\frac{2}{1+x^2})dx=-\frac{23}{15}+\frac{\pi}{2}$.

(2)$\int_0^1\frac{(1_x)^4x^4}{1+x^2}dx=\int_0^1(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2})dx=\frac{22}{7}-\pi$. 由$\int_0^1\frac{(1_x^2)x^4}{1+x^2}dx>0,\int_0^1\frac{(1_x)^4x^4}{1+x^2}dx>0$, 则$-\frac{23}{15}+\frac{\pi}{2}>0,\frac{22}{7}-\pi>0$, 即$3+\frac{1}{15}<\pi<3+\frac{1}{7}$.

4.解(2) $\lim_{n\to\infty}\frac{1}{n}(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+...\sin\frac{(n-1)\pi}{n})=\lim_{n\to\infty}\sum_{i=1}^{n-1}\sin\frac{i\pi}{n}\frac{1}{n}=\lim_{n\to\infty}\sum_{i=0}^{n-1}\sin\frac{i\pi}{n}\frac{1}{n}=\int_0^1\sin\pi xdx=\frac{2}{\pi}$.

10.

证明.

$$\begin{aligned}
&= \frac{1}{\sqrt{\alpha\beta}}\int_{-1}^1\frac{dx}{\sqrt{4x^2-2(\alpha+\beta+\alpha^{-1}+\beta^{-1})x+\alpha\beta+\alpha\beta^{-1}+\beta\alpha^{-1}+\alpha\beta^{-1}}}\\
&= \frac{1}{\sqrt{\alpha\beta}}\int_{-1}^1\frac{dx}{\sqrt{4x^2-2(\alpha+\beta)(\alpha\beta^{-1}+1)x+\alpha\beta+\alpha\beta^{-1}+\beta\alpha^{-1}+\alpha\beta^{-1}}}\\
&= \frac{1}{\sqrt{\alpha\beta}}\int_{-1}^1\frac{dx}{\sqrt{[2x-\frac{1}{2}(\alpha+\beta)(\alpha\beta^{-1}+1)]^2+\alpha\beta+\alpha\beta^{-1}+\beta\alpha^{-1}-\frac{(\alpha+\beta)^2}{4}(1+\frac{1}{\alpha\beta})^2}}\\
&= \frac{1}{\sqrt{\alpha\beta}}(\ln(1+\sqrt{\alpha\beta})-\ln(1-\sqrt{\alpha\beta}))=\frac{1}{\sqrt{\alpha\beta}}\ln\frac{1+\sqrt{\alpha\beta}}{1-\sqrt{\alpha\beta}}
\end{aligned}$$

□

13.

证明. 设$nT \le \lambda \le (n+1)T$, 且$|f(x)| \le M$.

$$
\begin{aligned}
\frac{1}{\lambda}\int_0^1 f(x)dx &\le \frac{1}{nT}\int_0^\lambda f(x)dx \\
&\le \frac{1}{nT}(n\int_0^T f(x)dx + T\cdot M) = \frac{1}{T}\int_0^T f(x)dx + \frac{M}{n} \\
&\to \frac{1}{T}\int_0^T f(x)dx \quad n\to\infty
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\lambda}\int_0^1 f(x)dx &\ge \frac{1}{(n+1)T}\int_0^\lambda f(x)dx \\
&\ge \frac{1}{(n+1)T}(n\int_0^T f(x)dx - T\cdot M) = \frac{n}{(n+1)T}\int_0^T f(x)dx - \frac{M}{n+1} \\
&\to \frac{1}{T}\int_0^T f(x)dx \quad n\to\infty
\end{aligned}
$$

令$\lambda\to\infty$,则$n\to\infty$. 故$\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^1 f(x)dx = \frac{1}{T}\int_0^T f(x)dx$. □

14.

证明.

$$
\begin{aligned}
\int_0^1 \frac{nf(x)}{1+n^2x^2}dx &= \int_0^n \frac{f(\frac{y}{n})}{1+y^2}dy \\
&= \int_0^{\sqrt{n}} \frac{f(0)}{1+y^2} + \int_0^{\sqrt{n}} \frac{f(\frac{y}{n})-f(0)}{1+y^2} + \int_{\sqrt{n}}^n \frac{f(\frac{y}{n})}{1+y^2}dy \\
&= I_1 + I_2 + I_3
\end{aligned}
$$

其中$I_1 = f(0)\arctan\sqrt{n} \to \frac{\pi}{2}f(0)$, as $n\to\infty$. 由于$f(x)$ 在0处连续，则$\forall\varepsilon \ge 0$, 当n 充分大，$|\frac{y}{n}| \le \delta$, 有$|f(\frac{y}{n}) - f(0)| \le \frac{2\varepsilon}{\pi}$，则$|I_2| \le \varepsilon$; $|I_3| \le M(\arctan n - \arctan\sqrt{n}) \to 0$, as $n\to\infty$, where $|f(x)| \le M, x\in[0,1]$. 因此，原积分极限为$\frac{\pi}{2}f(0)$. □

15.

证明. $\int_{-1}^1 \frac{hf(x)}{h^2+x^2}dx = \int_{-\frac{1}{h}}^{\frac{1}{h}} \frac{f(hy)}{1+y^2}dx$ 此积分形式与上题类似，可用类似方法证明. □

习题7.1

1.解：

(1)

$$
\begin{aligned}
L &= \int_0^4 \sqrt{1+\frac{9t^2}{4}}dt \\
&= \frac{8}{27}(\sqrt[3]{100}-1).
\end{aligned}
$$

(2)

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2}dt \\
&= \int_0^{2\pi} \sqrt{2}e^t dt = \sqrt{2}(e^{2\pi}-1).
\end{aligned}
$$

(3)

$$\begin{aligned}
L &= \int_0^{\frac{\pi}{2}} 4a\sin t\cos t\sqrt{1-\frac{1}{2}\sin^2 2t}dt + \int_{\frac{\pi}{2}}^{\pi} 4a\sin t(-\cos t)\sqrt{1-\frac{1}{2}\sin^2 2t}dt \\
&= \int_0^{\frac{\pi}{2}} 2a\sin 2t\sqrt{1-\frac{1}{2}\sin^2 2t}dt - \int_{\frac{\pi}{2}}^{\pi} 2a\sin 2t\sqrt{1-\frac{1}{2}\sin^2 2t}dt \\
&= 2a\int_0^{\pi}\sin x\sqrt{\frac{1}{2}+\frac{1}{2}\cos^2 x}dx \\
&= \sqrt{2}a\int_{-1}^{1}\sqrt{1+t^2}dt = 2\sqrt{2}a\int_0^1\sqrt{1+t^2}dt
\end{aligned}$$

有

$$\begin{aligned}
I=\int_0^1\sqrt{1+t^2}dt &= \sqrt{2}-\int_0^1\sqrt{1+t^2}dt+\int_0^1\frac{1}{\sqrt{1+t^2}}dt \\
&= \sqrt{2}-\int_0^1\sqrt{1+t^2}dt+\ln(\sqrt{2}+1)
\end{aligned}$$

则$I=\frac{\sqrt{2}}{2}+\frac{1}{2}\ln(\sqrt{2}+1), L=2a+\sqrt{2}a\ln(1+\sqrt{2})$.

(4)

$$\begin{aligned}
L &= 2\int_0^{\sqrt{2}a}\sqrt{(\frac{t}{a})^2+1dt} \\
&= 2a\int_0^{\sqrt{2}}\sqrt{1+t^2}dt
\end{aligned}$$

又

$$\begin{aligned}
\int_0^{\sqrt{2}}\sqrt{1+t^2}dt &= =\sqrt{6}-\int_0^{\sqrt{2}}\sqrt{1+t^2}dt+\int_0^{\sqrt{2}}\frac{1}{\sqrt{1+t^2}}dt \\
&= \sqrt{6}-\int_0^{\sqrt{2}}\sqrt{1+t^2}dt+\ln(\sqrt{2}+\sqrt{3})dt
\end{aligned}$$

则$L=(\sqrt{6}+\ln(\sqrt{2}+\sqrt{3}))a$.

2.解:

(1)$S=\int_0^a(\sqrt{ax}-\frac{x^2}{a})dx=\frac{a^2}{3}$.

(2)$S=\int_0^1(-x^2)-(x^2-2x)dx=\int_0^1 2x-2x^2dx=\frac{1}{3}$.

(3)$S=\int_0^1 x(x-1)(x-2)dx-\int_1^2 x(x-1)(x-2)dx=\frac{1}{2}$.

(4)$S=2\int_0^a 2x\sqrt{a^2-x^2}dx=2\int_0^{a^2}\sqrt{a^2-t}dt=\frac{4a^3}{3}$.

(5)$S=\int_0^{2\pi}\frac{1}{2}a^2(1-\cos\theta)^2d\theta=\frac{3\pi}{2}a^2$.

(6)$S=3\int_0^{\frac{\pi}{3}}\frac{1}{2}r^2d\theta=\frac{3}{2}\int_0^{\frac{\pi}{3}}a^2\sin^2 3\theta d\theta=\frac{a^2\pi}{4}$.

3.解:

(1)$S=2\pi\int_0^{\frac{\pi}{4}}\tan x\sqrt{1+\frac{1}{\cos^4 x}}dx=2\pi\int_1^{\frac{\sqrt{2}}{2}}\frac{\sqrt{1+t^4}}{t^3}dt=\pi\int_0^{\frac{1}{2}}\frac{\sqrt{1+s^2}}{s^2}ds=\pi(\sqrt{5}-\sqrt{2})+\pi\ln\frac{(\sqrt{5}-1)(\sqrt{2}+1)}{2}$.

(2)$S=2\pi\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}(-a\cos t)\sqrt{(a-\cos t)^2+(a\sin t)^2}dt=\frac{16}{3}\sqrt{2}\pi a^2$.

(3)$S=4\pi\int_0^a\sqrt{a^2-x^2}\sqrt{1+\frac{x^2}{a^2-x^2}}dx=4\pi a\int_0^a dx=4\pi a^2$.

(4)$S=2\int_0^{\frac{\pi}{2}}2\pi|y|\sqrt{x'(\theta)^2+y'(\theta)^2}d\theta=4\pi\int_0^{\frac{pi}{2}}a^2\sqrt{\sin 2\theta}\sin\theta\sqrt{\frac{1}{\sin 2\theta}}d\theta=4\pi a^2$.

4.解:

(1)$V=\int_{-1}^1\frac{1}{2}(1-z^2)^2dz=\frac{8}{15}$.

(2)$V=4\int_0^a\sqrt{b(a-x)}\cdot\sqrt{x(a-x)}dx=4\sqrt{b}\int_0^a\sqrt{a}(a-x)dx=\frac{16}{15}a^2\sqrt{ab}$.

5.解:

(1)$V=\int_0^{2\pi}\pi a^2(1-\cos t)^2d(a(t-\sin t))=\pi a^3\int_0^{2\pi}(1-\cos t)^3dt=5\pi^2a^3$.

(2)$V=2\int_0^{\sqrt{2ab}}\pi(b-x)^2dy=2\int_0^{\sqrt{2ab}}\pi(b-\frac{y^2}{2a})^2dy=\frac{16}{15}\pi b^2\sqrt{2ab}$.