Sets

CS 113 Discrete Mathematics
Spring 2022

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[1] adapted from collaborative work with Carina Dreyer in Spring 2020

Outline

What is a set?

- A well-defined collection of distinct objects, considered as an object in its own right
- order has no significance, and multiplicity is generally also ignored (unlike a list)

What is a set?

- A well-defined collection of distinct objects, considered as an object in its own right
- order has no significance, and multiplicity is generally also ignored (unlike a list)

Definition

A *set* is an unordered collection of objects, called *elements* or *members* of the set.

Describing Sets

Roster Method

Listing the elements enclosed with braces

Example

- Vowels $= \{$a,e,i,o,u$\}$
- Primary colors $= \{$red, blue, yellow$\}$

Describing Sets

Set-builder Method

Defining properties or *predicates* satsified by the elements

Notation

$A = \{x \mid x \in S, P(x)\}, B = \{x \in S \mid P_1(x), P_2(x), \ldots, P_n(x)\}$

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Example

- Set-builder notation for $X = \{2, 4, 6, 8, 10\}$?

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Example

- Set-builder notation for $X = \{2, 4, 6, 8, 10\}$?
- $X = \{n \in \mathbb{Z} \mid n \text{ is even}, 2 \leq n \leq 10\}$

Describing Sets

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- Set-builder notation for $X = \{2, 4, 6, 8, 10\}$?
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- For $P = \{2, 3, 5, 7, 11, 13\}$?

Describing Sets

Set-builder Method

Defining properties or *predicates* satsified by the elements

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$A = \{x \mid x \in S, P(x)\}, B = \{x \in S \mid P_1(x), P_2(x), \dots, P_n(x)\}$

Example

- Set-builder notation for $X = \{2, 4, 6, 8, 10\}$?
- $X = \{n \in \mathbb{Z} \mid n \text{ is even}, 2 \leq n \leq 10\}$
- For $P = \{2, 3, 5, 7, 11, 13\}$?
- $P = \{n \in \mathbb{Z} \mid n \text{ is prime}, 2 \leq n \leq 13\}$

Set Membership

Notation

Set membershp is denoted by $\in$ and non-membership by $\notin$.

Example

- Vowels = {a,e,i,o,u}
- "a belongs to the set of Vowels", is written as: a $\in$ Vowels
- "j does not belong to the set of Vowels" is written as: j $\notin$ Vowels

Some Standard Sets

- $\mathbb{N}$: The set of all natural numbers (i.e. all non-negative integers)
- $\mathbb{N}^\star$: The set of all positive natural numbers
- $\mathbb{Z}$: The set of all integers
- $\mathbb{Z}^+$: The set of all positive integers
- $\mathbb{Z}^\star$: The set of all nonzero integers
- $\mathbb{Q}$: The set of all rational numbers
- $\mathbb{Q}^+$: The set of all positive rational numbers
- $\mathbb{Q}^\star$: The set of all nonzero rational numbers
- $\mathbb{R}$: The set of all real numbers
- $\mathbb{R}^+$: The set of all positive real numbers
- $\mathbb{R}^\star$: The set of all nonzero real numbers

Empty Set and Universal Set

Definition (Empty Set)

The *empty set* contains no elements.

Notation

The empty set is denoted as $\emptyset$.

Definition (Universal Set)

A *universal set*, U, is a set which contains all objects, including itself.

Example

If $X = \{1, 2, 3\}$ and $Y = \{2, 4, 5\}$, then the universal set is $U = \{1, 2, 4, 5\}$.

Universal Set and Russel's Paradox

Problem with U

Following the usual formulation of set theory, the conception of a universal set leads to *Russell's paradox* and is consequently not allowed - we will only use it to explain Venn diagrams.

Russel's Paradox (restated)

Consider a barber who shaves exactly those men who do not shave themselves. Who shaves the barber?

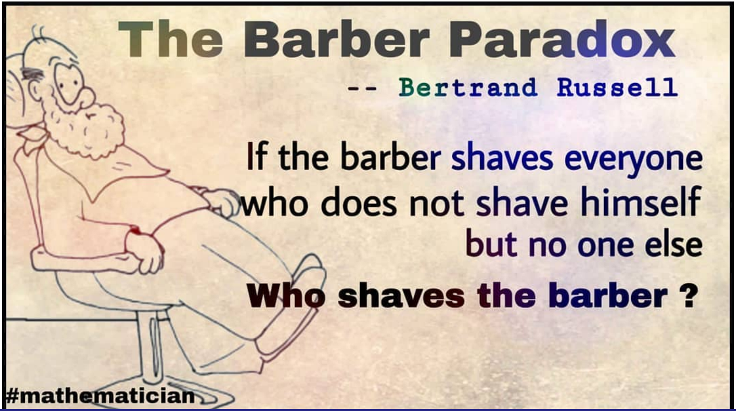
Fun Exercise

Find out how U leads to the paradox.

Russel's Paradox

Statement

What is the paradox?



Russel's Paradox

The Paradox

- If the barber shaves himself, then the barber is an example of "those men who do not shave themselves," a contradiction;
- If the barber does not shave himself, then the barber is an example of "those men who do not shave themselves," and thus the barber has to shave himself–also a contradiction.

Russel's Paradox

The Paradox

- If the barber shaves himself, then the barber is an example of "those men who do not shave themselves," a contradiction;
- If the barber does not shave himself, then the barber is an example of "those men who do not shave themselves," and thus the barber has to shave himself–also a contradiction.
- Hence the barber does not shave himself, but he also does not not shave himself, hence the paradox.

Finite and Infinite Sets

Definition (Finite Set)

X is a *finite set* iff there exists a non-negative integer n such that X has n distinct elements.

Definition (Infinite Set)

If a set is not finite, then it is an *infinite set*.

Example

- $Y = \{1, 2, 2, 3\}$
- $P = \{red, blue, yellow\}$
- $\mathbb{E}$, the set of all even integers
- $\emptyset$, the empty set

Cardinality

Definition

The *cardinality* of a set X with n distinct elements where $n \geq 0$ is n. It is denoted as $|X| = n$.

Example

- If $P = \{red, blue, yellow\}$, then $|P| = 3$
- What about $Y = \{1, 2, 2, 3\}$?
- The cardinality of $\emptyset$, the empty set, is zero.
- Some sets have infinite cardinality, e.g. $\mathbb{Z}$, the set of integers.
- Not all infinite cardinalities are equal, e.g. $|\mathbb{R}| > |\mathbb{Z}|$.
- A set with only one element is a *singleton*, e.g. $H = \{4\}, |H| = 1$.

Venn Diagram

- A diagram that shows all possible logical relations between a finite collection of different sets.
- Abstract visualization of a universal set, U, as a rectangle, with all subsets of U shown as circles.
- Used to teach elementary set theory, as well as illustrate simple set relationships in probability, logic, statistics, linguistics, and computer science

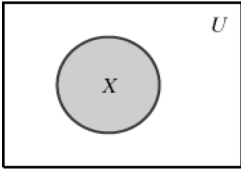


FIGURE 1.1 Set X

Subsets

Definition

A set X is a *subset* of a set Y *iff* (if and only if) every element of X is also an element of Y. That is, $\forall x \in A \colon x \in B$.

Corollary

$\emptyset$ is a subset of every set.

Notation

X is a subset of Y is denoted as $X \subseteq Y$.
X is not a subset of Y is denoted as $X \nsubseteq Y$.

Supersets

Definition

If X is a subset of Y then Y is a *superset* of X, denoted as $Y \supseteq X$.

Example

Find the sub/super-set relations among the following sets.
$X = \{a, e, i, o, u\}$, $Y = \{a, i, u\}$, and $Z = \{b, c, d, f, g\}$.

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- $Y \subseteq X$

Proof.

$\forall x \in Y : x \in X$ $\square$

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- $Y \not\subseteq Z$

Example

Find the sub/super-set relations among the following sets.
$X = \{a, e, i, o, u\}$, $Y = \{a, i, u\}$, and $Z = \{b, c, d, f, g\}$.

- $Y \subseteq X$

Proof.

$\forall x \in Y : x \in X$ $\square$

- $Y \not\subseteq Z$

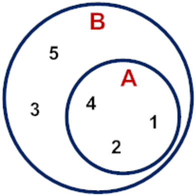
Proof.

$a \in Y \land a \notin Z$ $\square$

a is a *counterexample* to the statement: $Y \subseteq Z$.

Pictorial Representation

$A = \{1, 2, 4\}, B = \{1, 2, 3, 4, 5\}, A \subseteq B$



Terminology

B contains A. A is contained in B.

Proper Subsets

Definition

A is a *proper subset* of B iff $A \subseteq B \wedge \exists x \in B : x \notin A$. This is denoted as $A \subset B$.

Corollary

$A \subset B \implies A \subseteq B$

Proper Subsets

Example

Is there a proper subset relation among the following sets?
$X = \{a, e, i, o, u\}, Y = \{a, e, i, o, u, y\}$

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Proper Subsets

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Is there a proper subset relation among the following sets?

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$X \subset Y$

Proof.

Step 1: To prove that $X \subseteq Y$.

$\forall x \in X : x \in Y$

Proper Subsets

Example

Is there a proper subset relation among the following sets?

$X = \{a, e, i, o, u\}, Y = \{a, e, i, o, u, y\}$

$X \subset Y$

Proof.

Step 1: To prove that $X \subseteq Y$.

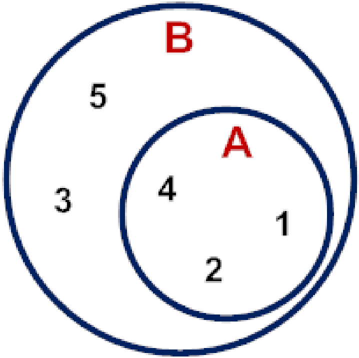
$\forall x \in X : x \in Y$

Step 2: To prove that $\exists x \in Y : x \notin X$.

$y \in Y \land y \notin X$ $\square$

Proper Subsets

Pictorial Example



Set Equality

Definition

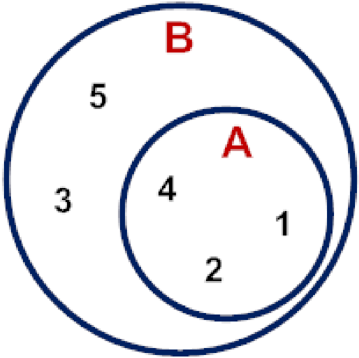
Two sets X and Y are *equal* iff every element of X is an element of Y and every element of Y is an element of X, i.e. $X = Y \iff X \subseteq Y \wedge Y \subseteq X$.

Example

- $\{1, 2, 3\} = \{2, 3, 1\}$
- $X = \{red, blue, yellow\}, Y = \{c \mid \text{c is a primary color}\}$.
- What about the sets $\{1, 2, 3\}$ and $\{2, 2, 3, 1\}$?

Set Equality

Pictorial Example



Defintion

Definition

The *power set* of a set X, written as $\mathcal{P}(X)$, is the set of all subsets of X. That is $\mathcal{P}(X) = \{A \mid A \subseteq X\}$.

Example

- If $X = \{\text{red, blue, yellow}\}$, then $\mathcal{P}(X) = \{\emptyset, \{\text{red}\}, \{\text{blue}\}, \{\text{yellow}\}, \{\text{red,blue}\}, \{\text{red, yellow}\}, \{\text{blue, yellow}\}, \{\text{red, blue, yellow}\}\}$
- What is the power set of $X = \{1, 2, 2, 3\}$ and how many elements does the power set have?

Cardinality

Definition

The *power set* of a set X, written as $\mathcal{P}(X)$, is the set of all subsets of X. That is $\mathcal{P}(X) = \{A \mid A \subseteq X\}$.

Corollary

If X is a finite set with $|X| = n$, then the number of subsets of X is $|\mathcal{P}(X)| = 2^n$.

Remark

The above is the motivation for the notation $\mathcal{P}(X) = 2^X$.

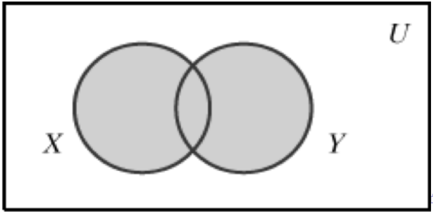
Union of Two Sets

Definition

The *union* of two sets X and Y, denoted $X \cup Y$, is defined to be the set $X \cup Y = \{x \mid x \in X \vee x \in Y\}$.

Example

If $X = \{1, 2, 3, 4, 5\}$ and $Y = \{5, 6, 7, 8, 9\}$, then $X \cup Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$



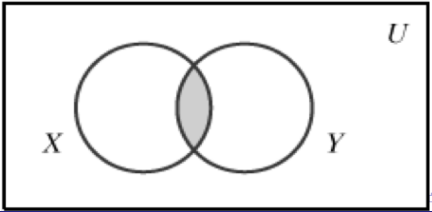
Intersection of Two Sets

Definition

The *intersection* of two sets X and Y, denoted $X \cap Y$, is defined to be the set $X \cap Y = \{x \mid x \in X \wedge x \in Y\}$.

Example

If $X = \{1, 2, 3, 4, 5\}$ and $Y = \{5, 6, 7, 8, 9\}$, then
$X \cup Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $X \cap Y = \{5\}$



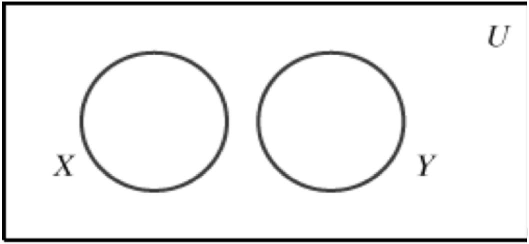
Disjoint Sets

Definition

Two sets X and Y are *disjoint* if $X \cap Y = \emptyset$.

Example

If $X = \{1, 2, 3, 4\}$ and $Y = \{5, 6, 7, 8, 9\}$, then $X \cap Y = \emptyset$. X and Y are disjoint.



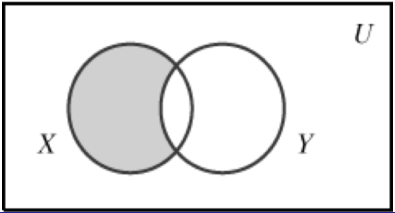
Difference of Two Sets

Definition

The *difference* of two sets X and Y (or *relative complement* of Y in X), written $X - Y$, is the set $X - Y = \{x \mid x \in X \wedge x \notin Y\}$.

Example

If $X = \{1, 2, 3, 4, 5\}$ and $Y = \{5, 6, 7, 8, 9\}$, then $X - Y = \{1, 2, 3, 4\}$ and $Y - X = \{6, 7, 8, 9\}$.



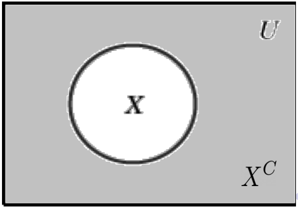
Set Complement

Definition

The complement of a set X with respect to a universal set U, denoted by X^C or $\overline{X}$ or X', is defined to be $\overline{X} = \{x | x \in U \wedge x \notin X\}$.

Example

If $X = \{1, 2, 3, 4, 5\}$ and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then
$\overline{X} = \{6, 7, 8, 9\}$



Set Identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Proving Set Identities

Example: DeMorgan's laws

Example

Prove DeMorgan's laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proving Set Identities

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Prove DeMorgan's laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof: $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Step 1: Prove that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.
Let $x \in \overline{A \cap B}$.
Then $x \notin (A \cap B)$.
That is, $\neg(x \in A \land x \in B)$
Then $x \notin A \lor x \notin B$
That is, $x \in \overline{A} \lor x \in \overline{B}$
That is, $x \in (\overline{A} \cup \overline{B})$

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Step 2: Prove that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.
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Then $x \in \overline{A} \vee x \in \overline{B}$
Then $x \notin A \vee x \notin B$
Then $x \notin (A \cap B)$
That is, $x \in \overline{A \cap B}$.
Other can be proved similarly.

Proving Set Identities

Approaches

- Prove that each is a subset of the other: $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Using known identities: $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$
- Use **Membership Tables** to verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.
- There are more ways to do it, but these are the most important ones

Proving Set Identities

Example: Membership Table

Example

Construct a membership table to show the distributive law
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1					
1	1	0					
1	0	1					
0	1	1					
1	0	0					
0	1	0					
0	0	1					
0	0	0					

Using more than one set operation

- It becomes difficult if we combine multiple operations:
- $X - (Y \cup Z) = X \cap \overline{(Y \cup Z)} = X \cap \overline{Y} \cap \overline{Z}$
- Why is that true?

Using more than one set operation

- It becomes difficult if we combine multiple operations:
- $X - (Y \cup Z) = X \cap \overline{(Y \cup Z)} = X \cap \overline{Y} \cap \overline{Z}$
- Why is that true?
- How can you rewrite $(X \cup Y) - Z$?

Using more than one set operation

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- Why is that true?
- How can you rewrite $(X \cup Y) - Z$?
- If $X = \{1, 2, 3, 4, 5\}$ and $Y = \{4, 5, 6, 7\}$ and $Z = \{3, 5, 6, 8, 9\}$, then what are those mentioned sets?

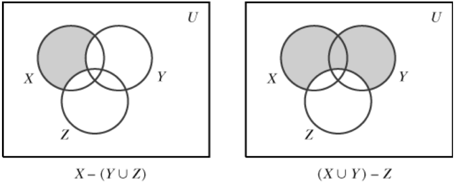


FIGURE 1.8 Venn diagrams of the sets $X - (Y \cup Z)$ and $(X \cup Y) - Z$

Generalizing Set Operations

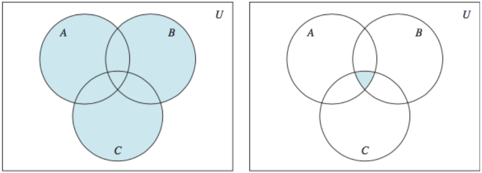
Definition

Let $A_1, A_2, ..., A_n, ...$ be an indexed collection of sets.

- $A_1 \cup A_2 \cup A_3 \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$
- $A_1 \cap A_2 \cap A_3 \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$
- $A_1 \cup A_2 \cup A_3 \ldots \cup A_n \ldots = \bigcup_{i=1}^{\infty} A_i$
- $A_1 \cap A_2 \cap A_3 \ldots \cap A_n \ldots = \bigcap_{i=1}^{\infty} A_i$

These are well defined, since union and intersection are associative.

Generalizing Set Operations

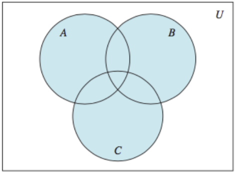


(a) $A \cup B \cup C$ is shaded. (b) $A \cap B \cap C$ is shaded.

Example

Let $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{4, 5, 6, 7, 8\}$.
What is $A \cap B \cap C$ and $A \cup B \cup C$?

Generalizing Set Operations



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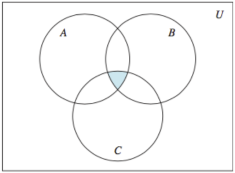

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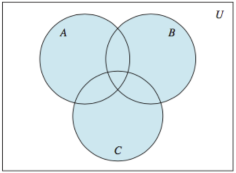
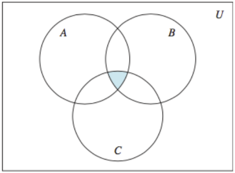
Example

Let $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{4, 5, 6, 7, 8\}$.
What is $A \cap B \cap C$ and $A \cup B \cup C$?
$A \cap B \cap C = \{5\}$

Generalizing Set Operations



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

Example

Let $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{4, 5, 6, 7, 8\}$. What is $A \cap B \cap C$ and $A \cup B \cup C$?

$A \cap B \cap C = \{5\}$

$A \cup B \cup C = \{x \in \mathbb{N}^\star | x \leq 9\}$

Ordered Pair

Definition

Given sets X and Y and $x \in X$ and $y \in Y$, an *ordered pair* is written (x, y).

Corollary

Order of elements is important! (x, y) is not necessarily equal to (y, x).

Example

- $(5, a) \neq (\circ, 7)$
- $(2, 3) \neq (3, 2)$
- $(\alpha, \gamma) = (\alpha, \gamma)$
- $(1, 1) = (1, 1)$

Cartesian Product

Definition

The *Cartesian product* of two sets X and Y, written $X \times Y$, is the set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

Corollary

For any set X, $X \times \emptyset = \emptyset = \emptyset \times X$

Example

Given $X = \{a, b\}$, $Y = \{c, d\}$

- $X \times Y = \{(a, c), (a, d), (b, c), (b, d)\}$
- $Y \times X =$

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Given $X = \{a, b\}$, $Y = \{c, d\}$

- $X \times Y = \{(a, c), (a, d), (b, c), (b, d)\}$
- $Y \times X = \{(c, a), (d, a), (c, b), (d, b)\}$
- $|X \times Y| = ?$

Cartesian Product

Example

If you have two finite sets A and B, where A has m elements and B has n elements, then $A \times B$ has how many elements?

Cartesian Product

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If you have two finite sets A and B, where A has m elements and B has n elements, then $A \times B$ has how many elements?
$|A \times B| = m \cdot n$
This rule is called the *multiplication principle*.

Definition

We can similarly define the Cartesian product of n sets $A_1, A_2, ..., A_n$ as

$$A_1 \times A_2 \times A_3 \times ... \times A_n = \{(x_1, x_2, ..., x_n) | x_1 \in A_1, x_2 \in A_2, ...x_n \in A_n\}.$$

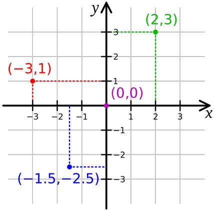
The multiplication principle states that for finite sets $A_1, A_2, ..., A_n$, if $|A_1| = m_1,\ |A_2| = m_2,\ ...|A_n| = m_n$, then
$|A_1 \times A_2 \times A_3 \times ... \times A_n| = m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_n$.

Cartesian Product

Example

An important example of sets obtained using a Cartesian product is $\mathbb{R}^n$, where n is a natural number. For $n = 2$, we have

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$$



Cardinality of the Union of Two Sets

Inclusion-Exclusion

For two sets A and B, the cardinality of $A \cup B$ is

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example

Among 10 students, 5 study mathematics, 6 study science, and 2 study both. How many of these students study neither mathematics nor science?

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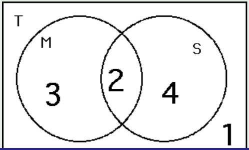

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How do you generalize that formula for bigger n?

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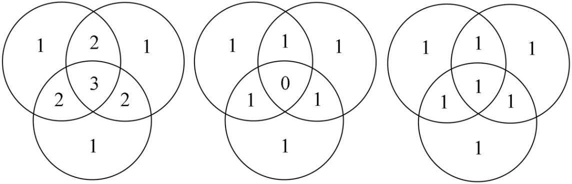

$|A| + |B| + |C|$

$|A| + |B| + |C|$
$-(|A \cap B| + |A \cap C| + |B \cap C|)$

$|A| + |B| + |C|$
$-(|A \cap B| + |A \cap C| + |B \cap C|)$
$+|A \cap B \cap C|$

Formula

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\leq i_1<i_2<\ldots<i_k\leq n}|A_{i_1}\cap\ldots\cap A_{i_k}|\right)$$

Cardinality of the Union of More Than Two Sets

Example

A large software development company employs 100 computer programmers. Of them, 45 are proficient in Java, 30 in C++, 20 in Python, six in C++ and Java, one in Java and Python, five in C++ and Python, and just one programmer is proficient in all three languages above. Determine the number of computer programmers that are not proficient in any of these three languages.

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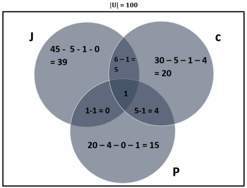
$= 100 - (45 + 30 + 20) + 6 + 1 + 5 - 1) = 100 - 84 = 16$



Partition of Sets

Definition

In general, a collection of nonempty sets $A_1, A_2, ..., A_n$ is a partition of a set A if they are all disjoint and their union is A.

Example

- If the earth's surface is our universal set, we might want to partition it to the different continents. Similarly, a country can be partitioned to different provinces.
- How could you partition the set of all natural numbers?

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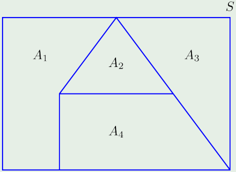
- How could you partition the set of all natural numbers?
- in the set of all odd and all even numbers.

Partition of Sets

Example

Here, the sets A_1, A_2, A_3 and A_4 form a partition of the universal set S.



Corollary

If $A_1, A_2, ..., A_n$ is a partition of a set A, then

$$|A| = |\bigcup_{i=1}^{n} A_i| = |A_1| + |A_2| + ... + |A_n|$$

Attention!

- Sets can be elements of sets.
- The empty set is different from a set containing the empty set. $\emptyset \neq \{\emptyset\}$
- Two sets are equal if and only if they have the same elements.

Example

$\{\{1, 2, 3\}, 5, \{3, 4\}\}$
$\{1, 3, 5\} = \{1, 3, 3, 3, 5, 5\}$

- We used what is called naïve set theory - we were not concerned with a formal set of axioms for set theory, which is a whole subject of itself.

Thank You!