Ex-Ante versus Ex-Post Compromise

Submission ID

November 23, 2020

A classical social choice setting is composed of a group of individuals, or voters, that express their preferences over a set of alternatives. The social choice problem consists in defining a procedure able to determine a collective choice for this group of voters, starting from their individual preferences. Such procedure is called social choice rule and it can be defined as a function mapping preference profiles to alternatives. Depending on the properties that this function satisfies, very different outcomes can be produced starting from the same initial profile. The plurality rule is one of the most common social choice rule and it consists in selecting, as a winner, the alternative that is considered the best by the largest number of voters forming the society. Yet, this rule can pick, as a winner, an alternative that is considered the worst by a strict majority of voters. Such outcome may be undesirable. Several procedures, the so-called compromise rules, have been proposed in the literature that aim to find a compromise. Nevertheless, all those rules can be defined as *ex-ante compromises* or *procedural compromises*, i.e., they impose over individuals a willingness to compromise but they do not ensure an outcome where everyone has effectively compromised. In this work, we approach the problem of compromise from an *ex-post* perspective, favoring an outcome where every voter gives up her most preferred positions if this increases equality. We propose a new notion of compromise in the social choice context, considering ordinal utilities.

1. Introduction

In a classical social choice scenario, several individuals express their preferences over a set of alternatives and there is no unique procedure for selecting a common agreement between them. Nevertheless, there is an accepted understanding that collective choices must reflect compromises. One of the first to explicitly refer to a Social Choice Rule (SCR) as a compromise is Sertel [1986] introducing the *majoritarian compromise*. This SCR, further analyzed by Sertel and Yılmaz [1999], is a rediscovery of a method suggested by James W. Bucklin at the beginning of the 20th century (for more details see Erdélyi et al. [2015]). It falls back from considering everyone's ideal alternative to considering the voters' second, third and more generally k-*th* best until one of the alternatives is among the first k best for a majority. Brams and Kilgour [2001] generalize this concept and introduce a class of SCRs called $q-$*approval fall-back bargaining* where q is the level of required support which can vary from a single voter up to unanimity. Naturally, different choices of q lead to different SCRs, such as $q = 1$ giving the plurality rule; q being majority giving the majoritarian compromise and q being unanimity giving a bargaining procedure called *fall-back bargaining* which has been further analyzed by Kibris and Sertel [2007] and Congar and Merlin [2012].

As Özkal-Sanver and Sanver [2004] discuss, the concept of compromising is mostly understood as the trade off between the number of voters supporting an alternative (i.e., the quantity of support) and the distance of that alternative from the supporters' ideal alternative (i.e., the quality of support). This trade off, which is explicit for $q-$approval fall-back bargaining, is also the basis for several other SCRs such as the *median voting rule* proposed by Bassett and Persky [1999] and further analyzed by Gehrlein and Lepelley [2003] or the *Condorcet practical method* described by Nurmi [1999].

Merlin et al. [2019] identify and analyze a large class of *compromise rules* which are based on balancing the trade off between the quality and the quantity of support. On the other hand, as conflicting individual preferences over the available alternatives makes impossible to ensure the best outcome for every member of the group, one can argue that making a collective choice *per se* implies compromising. In that sense, every SCR incorporates some understanding of what a compromise means. To be sure, there are instances where this understanding may contradict common sense, such as dictatorships where one voter always ensures his best outcome whatever the others prefer. Nevertheless, interesting SCRs base the collective choice on the principle that all voters may have to fall back from their ideal position. Whether, at the end of the day, all voters do effectively fall back or not is another issue which is the subject matter of this paper.

Sometimes, indeed, they do not. This observation was the basis for an objection made by Jean-François Laslier to the nomenclature on compromising.[1] Consider the following example.

Example 1 Let N be a set of $n \geq 3$ voters and A a set of alternatives. $\mathcal{L}(A)$ represents the set of linear orders over A. Consider the following preference profile $P \in \mathcal{L}(A)^N$:

$$\begin{array}{ccccc} \mathbf{1} & & c & b & a \\ \mathbf{n-1} & & a & b & c \end{array},$$

which represents one individual who prefers c to b, b to a, hence c to a; and $n-1$ individuals who prefer a to b, b to c, hence a to c. At P, all BK-compromises, except fall-back bargaining i.e. when $q = n$, will ignore the single voter and will pick a as the collective outcome. □

As a matter of fact, almost every interesting SCR will ignore this "marginal minority" and choose a in this situation. While this choice is defensible on the grounds of qualified majoritarianism, the presence of b, which receives unanimous support when each voter falls back one step from his ideal point, renders questionable whether a can be qualified as a compromise. The question becomes even more acute for majoritarian SCRs, including the majoritarian compromise, where a would remain the collective choice even when the ignored group is much larger.

Example 2 Consider the following preference profile with $n = 100$:

$$\begin{array}{ccccc} \mathbf{49} & & c & b & a \\ \mathbf{51} & & a & b & c \end{array}.$$

When $q \in [\![1, \frac{n}{2}+1]\!]$, all BK-compromises pick a, and, again, it does not appear as a compromise as 51 voters reach their best alternative while the remaining 49 voters have to be contented with their worst one. Note that for $q \in [\![1, \frac{n}{2}-1]\!]$ the set of possible common agreements determined by the fall-back bargaining procedure is $\{a, c\}$. Nevertheless, a receives the highest support, thus it is elected. □

It is important to observe that all these SCRs impose to voters a willingness to compromise which does not mean that under the collective choice that will

[1] This happened at a CNRS workshop on compromising hosted by Istanbul Bilgi University at Buyukada, Istanbul on fall 2018.

be made, all voters will be effectively compromising. In other words, the term "compromise" in this literature refers to procedural or *ex-ante* compromises, which is different than outcome oriented or *ex-post* compromises, a conceptual distinction that seems to be overlooked in the literature.

To define an ex-post compromise, we adopt to our framework a concept of equal losses that is prevalent in the literature that considers the allocation of continuous utilities. This principle is used for bargaining problems [Chun, 1988], [Chun and Peters, 1991] as well as for bankruptcy problems [Herrero and Villar, 2001]. We introduce two definitions for being a compromise. In both of them, we pick a spread measure that determines how equally a given vector of real numbers is distributed and make a collective choice where voters give up from their ideal points "as equal as possible" (see Section 4.4 for a literature review). However, one of them, called *egalitarian compromise*, insists on equality at the expense of Pareto efficiency while the other, called *Paretian compromise*, is constrained to pick among the Pareto efficient alternatives. The two concepts are logically incompatible. As a result, Pareto efficient SCRs cannot ensure egalitarian compromises and this is valid under any spread measure. Moreover, several well-known SCRs of the literature such as Condorcet extensions, scoring rules, $q-$approval fall-back bargaining, all fail to be Paretian compromises under any spread measure. In fact, we are able to observe the existence of instances where being a Paretian compromise necessitates to pick an alternative that is, although Pareto optimal, ranked so low by all voters that this alternative wouldn't be picked by any of the popular SCRs of the literature. All these observations make the equal-loss principle appear quite inadequate for collective choice problems, unless envy-freeness is a major concern.

Collective choice models with two individuals present instances where envy-freeness matters. Here, the model is interpreted as a bargaining problem and bargaining procedures replace voting rules. As prominent examples, we have fallback bargaining proposed by Brams and Kilgour [2001]; the unanimity compromise and the rational compromise introduced by Kibris and Sertel [2007]; the veto-rank and short listing procedures analyzed by de Clippel et al. [2014] and the Pareto-and-veto rules analyzed by Laslier et al. [2020]. As is ours, these are all models with discrete alternatives which are not contained by the classical Nash [1950] bargaining environment with convex utilities. However, as Mariotti [1998] and Nagahisa and Tanaka [2002] illustrate, the two worlds can be interconnected, as we do for the equal-loss principle of Chun [1988] and Chun and Peters [1991]. As a rather surprising finding, several interesting SCRs used as a bargaining solution also fail to be Paretian compromises.

Section 2 presents the basic notions and notation. Section 3 introduces egalitarian compromises and Paretian compromises, two concepts that turn out to be logically incompatible. Section 4 shows that with at least three individuals, several SCRs fail to pick a compromise. Section 5 considers the two-individual case, again showing that several SCRs of the literature fail to pick compromises. Section 6 makes some concluding remarks.

2. Basic notions and notation

Consider a finite set N of individuals with $\#N = n \geq 2$ and a finite set A of alternatives with $\#A = m \geq 3$. We write $\mathcal{L}(A)$ for the set of linear orders over A. A generic element $\succ_i$ of $\mathcal{L}(A)$ stands for a preference of $i \in N$. This implies that, given any $x \neq y \in A$, precisely one of $x \succ_i y$ and $y \succ_i x$ holds while $x \succ_i x$ holds for no $x \in A$. Moreover, $x \succ_i y$ and $y \succ_i z$ implies $x \succ_i z$ $\forall x, y, z \in A$.

A *profile* $P : N \to \mathcal{L}(A)$ associates with each individual $i \in N$ a preference order $P(i) = \succ_i$. A *Social Choice Rule* (SCR) is a mapping $f : \mathcal{L}(A)^N \to 2^A \setminus \{\emptyset\}$.

We write $r_{\succ_i}(x) = \#\{y \in A \mid y \succ_i x\} + 1$ for the *rank* of $x \in A$ at $\succ_i \in \mathcal{L}(A)$. We denote by $\lambda_{\succ_i}(x) = r_{\succ_i}(x) - 1$ the loss in terms of ranks for $i \in N$ with preference $\succ_i$, when x is elected instead of the best alternative for i. The mapping $\lambda_P : A \to [\![0, m-1]\!]^N$ assigns to each $x \in A$ the loss vector $\lambda_P(x) = (\lambda_{\succ_i}(x))_{i \in N}$ induced by the election of x. The use of double brackets denotes intervals in the integers.

We are interested in measuring the spread of loss vectors. To this end, we adopt a *spread measure* $\sigma : [\![0, m-1]\!]^N \to \mathbb{R}_+$ that associates a spread value to every possible loss vector. We write Σ for the set of spread measures σ that satisfy for every $l \in [\![0, m-1]\!]^N$, $\sigma(l) = 0 \iff l_i = l_j \ \forall i, j \in N$. Thus, the spread of l gets its lowest value 0 in case of perfect equality and only in this case.

Given any distinct $x, y \in A$, we say that x *Pareto dominates* y at $P \in \mathcal{L}(A)^N$ (or equivalenty y is *Pareto dominated* by x at P) iff $x \succ_i y, \forall i \in N$. We denote $\mathrm{PO}(P) = \{x \in A \mid \forall y \neq x \in A, \exists i \in N \mid x \succ_i y\}$ the set of *Pareto optimal* alternatives at P. A SCR f is *Paretian* iff $f(P) \subseteq \mathrm{PO}(P)$ $\forall P \in \mathcal{L}(A)^N$.

3. Egalitarian versus Paretian compromises

3.1. Egalitarian compromises

We denote the minimal elements of $X \in 2^A \setminus \{\emptyset\}$ according to $(\sigma \circ \lambda_P)$ with $\min_{\sigma\circ\lambda_P}(X) = \{x \in X \mid \forall y \in X : \sigma(\lambda_P(x)) \leq \sigma(\lambda_P(y))\}$. Thus, $\min_{\sigma\circ\lambda_P}(X)$ denotes the alternatives in X whose loss vectors are the most equally distributed according to the spread measure σ.

In what follows, we define some classes of SCRs that we are interested in analyzing.

Definition 1 A SCR f is an *Egalitarian Compromise* (EC) iff

$$\exists \sigma \in \Sigma \mid \forall P \in \mathcal{L}(A)^N \text{ we have } f(P) \subseteq \min_{\sigma\circ\lambda_P}(A). \qquad \square$$

Definition 2 A SCR f is *Egalitarian Compromise Compatible* (ECC) iff

$$\exists \sigma \in \Sigma \mid \forall P \in \mathcal{L}(A)^N \text{ we have } f(P) \cap \min_{\sigma\circ\lambda_P}(A) \neq \emptyset. \qquad \square$$

Under a SCR that is EC (resp., ECC), *all* (resp., *some*) winners are among the alternatives with most equally distributed losses. Clearly, EC is a subclass of ECC. Perhaps less obviously, being ECC (or EC) is incompatible with being Paretian. This will be deduced from the following proposition, which will also be useful to prove other theorems.

Proposition 1 *For $n \geq 2, m \geq 3$, there exists a profile $P \in \mathcal{L}(A)^N$ and an alternative a_m such that $\forall i \in N$: $r_{\succ_i}(a_m) = m$, and such that $\forall \sigma \in \Sigma$: $\min_{\sigma\circ\lambda_P}(A) = \{a_m\}$; hence, $\min_{\sigma\circ\lambda_P}(A) \cap PO(P) = \emptyset$.* $\square$

Proof Consider the following profile P:

$$\begin{array}{cccccc} \mathbf{1} & a_1 & a_2 & \dots & a_{m-1} & a_m \\ \mathbf{n-1} & a_{\pi(1)} & a_{\pi(2)} & \dots & a_{\pi(m-1)} & a_m \end{array},$$

where π is the following permutation over $[\![1, m-1]\!]$:

$$\pi(i) = \begin{cases} i+1 & \text{if } i \in [\![1, m-2]\!] \\ 1 & \text{if } i = m-1 \end{cases}.$$

In P, a_m is the only alternative such that $r_{\succ_i}(a_m) = m$, $\forall i \in N$; hence, $\sigma(\lambda_P(a) > 0, \forall a \in A \setminus \{a_m\}, \forall \sigma \in \Sigma$. Thus, the set $\min_{\sigma\circ\lambda_P}(A)$ consists of the sole element a_m, and, because a_m is Pareto dominated, $\min_{\sigma\circ\lambda_P}(A) \cap \mathrm{PO}(P) = \emptyset$. ■

Our main result for Section 3.1 follows easily.

Theorem 1 *For $n \geq 2,\ m \geq 3$, no Paretian SCR is ECC.* □

Proof Proving this amounts to show that $\forall \sigma \in \Sigma, \exists P \in \mathcal{L}(A)^N \mid \mathrm{PO}(P) \cap \min_{\sigma\circ\lambda_P}(A) = \emptyset$. Suffices to use Proposition 1, which asserts that there exists a profile P such that $\forall \sigma \in \Sigma : \min_{\sigma\circ\lambda_P}(A) \cap \mathrm{PO}(P) = \emptyset$. ■

3.2. Paretian compromises

Having seen the tension for a SCR being Paretian and ECC, we investigate the consequences of inverting the order of priorities by insisting that at least some of the winning alternatives are Pareto optimal, and considering the most equally distributed loss vectors among those.

We consider two classes of SCRs. Observe that $\min_{\sigma\circ\lambda_P}(\mathrm{PO}(P))$ denotes the set of Pareto optimal alternatives whose loss vectors are the most equally distributed according to the spread measure σ.

Definition 3 A SCR f is a *Paretian Compromise* PC iff

$$\exists \sigma \in \Sigma \mid \forall P \in \mathcal{L}(A)^N \text{ we have } f(P) \subseteq \min_{\sigma\circ\lambda_P}(\mathrm{PO}(P)).$$ □

Definition 4 A SCR f is *Paretian Compromise Compatible* PCC iff

$$\exists \sigma \in \Sigma \mid \forall P \in \mathcal{L}(A)^N \text{ we have } f(P) \cap \min_{\sigma\circ\lambda_P}(\mathrm{PO}(P)) \neq \emptyset.$$ □

Again, it is clear that PC is a subclass of PCC. It will also probably come with no surprise that for a SCR, being PC is incompatible with being ECC, as being PC requires to be Paretian, which permits to use Theorem 1. On the other hand, it is less immediate that being EC is incompatible with being PCC, because being PCC does not require to be Paretian. This is however true.

Theorem 2 *For $n \geq 2, m \geq 3$, no SCR is both EC and PCC.* □

Proof Letting $\bar{P}$ denote the profile of Proposition 1, with a_m the alternative mentioned there, and considering any EC f and any $\sigma \in \Sigma$, suffices to prove that $f(\bar{P}) \cap \min_{\sigma\circ\lambda_{\bar{P}}}(\mathrm{PO}(\bar{P})) = \emptyset$.

First, from Proposition 1, $\{a_m\} \cap \mathrm{PO}(\bar{P}) = \emptyset$, hence $\{a_m\} \cap \min_{\sigma\circ\lambda_{\bar{P}}}(\mathrm{PO}(\bar{P})) = \emptyset$.

Second, because f is an EC, for some $\bar{\sigma}$, $f(\bar{P}) \subseteq \min_{\bar{\sigma}\circ\lambda_{\bar{P}}}(A)$. Using Proposition 1 again, we see that $\min_{\bar{\sigma}\circ\lambda_{\bar{P}}}(A) = \{a_m\}$, hence $f(\bar{P}) = \{a_m\}$.

That $f(\bar{P}) \cap \min_{\sigma\circ\lambda_{\bar{P}}}(\mathrm{PO}(\bar{P})) = \emptyset$ follow from these two facts. ■

It is interesting to note that the incompatibility is not complete, however.

Remark 1 For $n \geq 2$, $m \geq 3$, there exist SCRs that are both ECC and PCC, such as the SCR that selects the whole set of alternatives at every profile. However, this SCR fails to be Paretian, as it must be for every SCR that is ECC. $\square$

4. Which SCRs are compromises?

In this section we assume $n \geq 3$ and leave the analysis of $n = 2$ to the next section.

4.1. Condorcet consistent rules

An alternative $x \in A$ is a *Condorcet winner* at $P \in L(A)^N$ iff for all $y \in A \backslash \{x\}$, $\#\{i \in N \mid x \succ_i y\} > \#\{i \in N \mid y \succ_i x\}$. So each profile admits either no or a unique Condorcet winner. An SCR f is *Condorcet consistent* iff $f(P) = \{x\}$ at each $P \in L(A)^N$ that admits x as the unique Condorcet winner.

Theorem 3 *Let $n \geq 3$ and $m \geq 3$. A Condorcet consistent SCR f is neither ECC nor PCC.* $\square$

Proof Consider the following profile P, where the dots represent the sequence a_4 to a_m:

$$\begin{array}{ccccccc} \mathbf{n-1} & & a_1 & a_2 & a_3 & \dots & \\ \mathbf{1} & & a_3 & a_2 & \dots & a_1 & \end{array} .$$

Consider any Condorcet consistent SCR f, then $f(P) = \{a_1\}$. However, $\min_{\sigma \circ \lambda_P}(A) = \min_{\sigma \circ \lambda_P}(\mathrm{PO}(P)) = \{a_2\}$ $\forall \sigma \in \Sigma$, so there exists a profile P such that both $f(P) \cap \min_{\sigma \circ \lambda_P}(A)$ and $f(P) \cap \min_{\sigma \circ \lambda_P}(\mathrm{PO}(P))$ are empty. $\blacksquare$

Note that Condorcet consistent rules need not be Paretian so the fact that they all fail ECC does not follow from Theorem 1.

4.2. Scoring rules

A *score vector* is an $m-$tuple $w = (w_1, \dots, w_m) \in [\![0, 1]\!]^m$ with $w_1 = 1$, $w_m = 0$ and $w_i \geq w_{i+1}$ $\forall i \in [\![1, m-1]\!]$. Given a score vector w, we write $s^w(x, P) = \sum_{i \in N} w_{r_{\succ_i}(x)}$ for the score of $x \in A$ at $P \in L(A)^N$.

Every score vector w identifies a *scoring rule* f_n^w defined as $f_n^w(P) = \{x \in A : s^w(x,P) \geq s^w(y,P)\ \forall y \in A\}$ for every $P \in L(A)^N$.

We first show that no scoring rule is ECC, for any value of n and m at least 3.

Theorem 4 *Let $n \geq 3$ and $m \geq 3$. No score vector w induces a scoring rule f_n^w that is ECC.* □

Proof Take any score vector w. Consider the profile P of Proposition 1. Observe that $\min_{\sigma\circ\lambda_P}(A) = \{a_m\}\ \forall \sigma \in \Sigma$. However, as $w_1 > w_m$, we have $s^w(a_1,P) > s^w(a_m,P)$ which implies $a_m \notin f^w(P)$. ■

We call antiplurality score vector the score vector w formed such that $w_i = 1, \forall i \in [\![1, m-1]\!]$ and $w_m = 0$.

Theorem 5 *Let $m \geq 3$ and let w be the antiplurality score vector. The SCR f_n^w satisfies PCC for all $n \geq 3$.* □

Proof Define $\bar{\sigma} \in \Sigma$ as, $\forall l \in [\![0, m-1]\!]^N$: $\bar{\sigma}(l) = 1$ iff $\exists i,j \in N \mid l_i \neq l_j$; $\bar{\sigma}(l) = 0$ otherwise. We show the non-emptyness of $f_n^w(P) \cap \min_{\bar{\sigma}\circ\lambda_P}(\mathrm{PO}(P))$ for any profile P.

Let $k = \min_{x\in \mathrm{PO}(P)}\{(\bar{\sigma}\circ\lambda_P)(x)\}$ be the minimal value attained by $\bar{\sigma}\circ\lambda_P$ over $\mathrm{PO}(P)$. By construction of $\bar{\sigma}$, k equals either 0 or 1.

For $k = 1$, take any $x \in f_n^w(P) \cap \mathrm{PO}(P)$, which exists because the antiplurality rule, although not Paretian, never picks only non-Pareto optimal alternatives. By definition of $\bar{\sigma}$, $\bar{\sigma}(x) \leq 1$, hence, $x \in \min_{\bar{\sigma}\circ\lambda_P}(\mathrm{PO}(P))$.

If $k = 0$, take any $x \in \min_{\bar{\sigma}\circ\lambda_P}(\mathrm{PO}(P))$. As $\bar{\sigma}(\lambda_P(x)) = 0$, we have, $\forall i,j \in N$: $\lambda_i^P(x) = \lambda_j^P(x)$, hence, $\forall i,j \in N$: $r_{\succ_i}(x) = r_{\succ_j}(x)$. The case $r_{\succ_i}(x) = m, \forall i \in N$ is ruled out by $x \in \mathrm{PO}(P)$. Hence, $r_{\succ_i}(x) \leq m-1, \forall i \in N$, hence, $x \in f_n^w(P)$. ■

It is worth noting that the antiplurality rule f_n^w is not Paretian, hence fails PC for all $n \geq 3$. This, can be seen by picking a unanimous profile $P \in \mathcal{L}(A)^N$ with $a_1 \succ_i a_2 \succ_i \ldots \succ_i a_m\ \forall i \in N$, where $\min_{\sigma\circ\lambda_P}(\mathrm{PO}(P)) = \{a_1\}\,\forall\sigma \in \Sigma$ while $f_n^w(P) = A \setminus \{a_m\}$.

Theorem 6 *Let $m \geq 3$. Take any score vector w which is not the antiplurality score vector. The SCR f_n^w fails PCC for some $n \geq 3$.* □

Proof Take any $m \geq 3$ and a score vector w such that it is not the antiplurality score vector. Therefore, $w_{m-1} < 1$. Observe the existence of two natural numbers n_1, $n_2 \geq 3$ with $n_1 \geq m-1$ and $\frac{n_2-1}{n_2} > w_{m-1}$. Let $n = \max\{n_1, n_2\}$ and let $A = \{a_1, \ldots, a_m\}$. Take some $P \in L(A)^N$ with

$$\begin{array}{cccccc} i=1 & a_2 & \dots & a_m & a_1 & \\ 2 \leq i \leq m-2 & a_1 & \dots & a_m & a_i & , \\ m-1 \leq i \leq n & a_1 & \dots & a_m & a_{m-1} & \end{array}$$

where all alternatives except a_m appear at least once in the last rank. Thus, for every $\sigma \in \Sigma$, we have $\sigma(\lambda_P(x)) > 0\ \forall x \in A \setminus \{a_m\}$ while $\sigma(\lambda_P(a_m)) = 0$. Moreover, $a_m \in \mathrm{PO}(P)$. Thus, $\min_{\sigma\circ\lambda_P}(\mathrm{PO}(P)) = \{a_m\}\ \forall \sigma \in \Sigma$. On the other hand, $s^w(a_1;P) = n-1$, $s^w(a_m;P) = n \cdot w_{m-1}$ and as $\frac{n-1}{n} > w_{m-1}$, we have $s^w(a_1;P) > s^w(a_m;P)$, establishing $a_m \notin f^w(P)$, thus $f^w(P) \cap \min_{\sigma\circ\lambda_P}(\mathrm{PO}(P)) = \emptyset\ \forall \sigma \in \Sigma$. ■

4.3. BK-compromises

Given any $k \in [\![1,m]\!]$, we write $n_k(x,P) = \#\{i \in N \mid r_{\succ_i}(x) \leq k\}$ for the *k-support* that x gets at P, that is, the number of individuals for whom the rank of alternative $x \in A$ is lower than or equal to k in the profile $P \in L(A)^N$. Note that $n_k(x,P) \in [\![1,n]\!]$ is non-decreasing on k and $n_m(x,P) = n$. For each $q \in [\![1,n]\!]$, we define $\rho_q(x,P) = \min\{k \in [\![1,m]\!] \mid n_k(x,P) \geq q\}$ as the minimal rank k at which the k-support that x gets at P is at least q. We write $\rho_q(P) = \min_{x\in A}\{\rho_q(x,P)\}$ for the minimal rank k at which the k-support that some alternative gets at P is at least q. *A Brams and Kilgour (BK) compromise with threshold* q is the SCR f_q defined for each $P \in \mathcal{L}(A)^N$ as $f_q(P) = \{x \in A | n_{\rho_q(P)}(x,P) \geq n_{\rho_q(P)}(y,P)\ \forall y \in A\}$.

Theorem 7 *Let $n \geq 3$ and $m \geq 3$. The BK compromise f_n satisfies PC.* □

Proof Define $\bar\sigma \in \Sigma$ as, $\forall l \in [\![0,m-1]\!]^N$: $\bar\sigma(l) = 1$ iff $\exists i,j \in N \mid l_i \neq l_j$; $\bar\sigma(l) = 0$ otherwise. Considering any $x \in f_n(P)$, let us show that $x \in \min_{\bar\sigma\circ\lambda_P}(\mathrm{PO}(P))$. Because $x \in f_n(P)$, $x \in \mathrm{PO}(P)$, and therefore, suffices to show that $\forall y \in \mathrm{PO}(P)$, $\bar\sigma(\lambda_P(y)) \geq \bar\sigma(\lambda_P(x))$. Given the choice of $\bar\sigma$, picking any $y \in \mathrm{PO}(P)$ with $y \neq x$, suffices to show that $\bar\sigma(\lambda_P(y)) = 1$, equivalently, that $\exists i,j \in N \mid r_{\succ_i}(y) \neq r_{\succ_j}(y)$. Because $x \in f_n(P)$, $\rho_n(P) = \rho_n(x,P) = \max_{i\in N} r_{\succ_i}(x)$. It follows from $\rho_n(P) = \min_{z\in A}\{\rho_n(z,P)\}$ that $\rho_n(y,P) \geq \rho_n(x,P)$, thus, $\exists i \in N \mid r_{\succ_i}(y) \geq \rho_n(P)$. Also, $y \in \mathrm{PO}(P)$ implies that $\exists j \in N \mid r_{\succ_j}(y) < r_{\succ_j}(x)$, thus $\exists j \in N \mid r_{\succ_j}(y) < \rho_n(P)$. Therefore, $r_{\succ_i}(y) \neq r_{\succ_j}(y)$. ■

Theorem 8 *Let $n \geq 3$ and $m \geq 3$. The BK compromise f_n fails ECC.* □

Proof As f_n is Paretian, the proof comes straightforward from Theorem 1. ■

Theorem 9 *Let $n \geq 3$ and $m \geq 3$. A BK compromise f_q with threshold $q \in [\![1, n-1]\!]$ is neither ECC nor PCC.* □

PROOF Consider the following profile P (also used in the proof of Theorem 3), where the dots represent the sequence a_4 to a_m:

$$\begin{array}{cccccc} \mathbf{n-1} & & a_1 & a_2 & a_3 & \dots \\ \mathbf{1} & & a_3 & a_2 & \dots & a_1 \end{array}.$$

We have that $f_q(P) = \{a_1\}$, and, because $\sigma(\lambda_P(a_2)) = 0$ and $\sigma(\lambda_P(a_1)) > 0$, neither $\min_{\sigma\circ\lambda_P}(A)$ nor $\min_{\sigma\circ\lambda_P}(\mathrm{PO}(P))$ contain a_1 for any $\sigma \in \Sigma$. ■

4.4. Restrictions on sigma

The perfect equality recognition condition we adopt for spread measures, i.e., that the spread gets its lowest value 0 in case of perfect equality and only in this case, is very basic. Unless this condition is violated, Σ is the largest set of spread measures we could conceive. On the other hand, it is possible to let Σ shrink by imposing additional conditions over spread measures. Nevertheless, as the satisfaction of PC, PCC, EC, or ECC requires the existence of a spread measure, all of our negative results, namely, those expressed by Theorems 1, 2, 3, 4, 6, 8 and 9 prevail when Σ is restricted. In a similar vein, the positive results in Theorems 5 and 7 risk to be lost with additional conditions over spread measures.

Definition 5 Given any $m \geq 3$ and $n \geq \max\{3, m-1\}$, we say that a spread measure σ satisfies condition $C_{m,n}$ iff we have $\sigma(m-3, m-1, m-2, \dots, m-2) < \sigma(m-2, m-3, \dots, 1, 0, \dots, 0)$. □

As both vectors are n dimensional, the term $m-2$ repeats $n-2$ times in the first vector and the term 0 repeats $n-m+2$ times in the second vector.

The condition is more convincing for larger values of m and n. In fact, asking for $\sigma(0, 2, 1)$ to be smaller than $\sigma(1, 0, 0)$ is very demanding while asking for $\sigma(3, 5, 4, 4, 4, 4, 4)$ to be smaller than $\sigma(6, 5, 4, 3, 2, 1, 0)$ reflects a mild assumption. In any case, as we state below, several well-known spread measures of the literature (see Allison [1978] for a comprehensive account) satisfy Definition 5 for reasonably small values of m and n. Letting $\bar{l} = \frac{\sum_{i=1}^{n} l_i}{n}$ denote the arithmetic mean of the values of $l = (l_1, \dots, l_n)$, we consider the following measures:

- the mean absolute difference $\sigma_{mad}(l) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |l_i - l_j|$;

- the average absolute deviation $\sigma_{ad}(l) = \frac{\sum_{i=1}^{n}|l_i - \bar{l}|}{n}$;
- the standard deviation $\sigma_{sd}(l) = \sqrt{\frac{\sum_{i=1}^{n}(l_i - \bar{l})^2}{n}}$;
- the Gini coefficient $\sigma_G(l) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}|l_i - l_j|}{2 \cdot n \cdot \sum_{i=1}^{n} l_i}$.

Proposition 2 *σ_{mad}, σ_{ad}, σ_{sd} and σ_G all satisfy condition C $\forall m \geq 4$ and $n \geq \max\{4, m-1\}$.* □

The proof of Proposition 2 is given in Appendix A.

We write $\Sigma^{C_{m,n}} \subseteq \Sigma$ for the set of spread measures that satisfy condition $C_{m,n}$.

Theorem 10 *For all $m \geq 3$, $n \geq \max\{3, m-1\}$, under $\Sigma^{C_{m,n}}$,*

1) f_n^w fails PCC when w is the antiplurality score vector;

2) the BK compromise f_n fails PCC. □

Proof Take any $m \geq 3$ and any $n \geq \max\{3, m-1\}$ and consider $\Sigma^{C_{m,n}} \subseteq \Sigma$, the set of spread measures that satisfy condition $C_{m,n}$. Take some $x, y \in A$ and some $P \in \mathcal{L}(A)^N$ with $r_{\succ_1}(x) = m-2$, $r_{\succ_2}(x) = m$, $r_{\succ_i}(x) = m-1$ $\forall i \in N \setminus \{1,2\}$, and $r_{\succ_i}(y) = m - i$ $\forall i \in [\![1, m-1]\!]$, $r_{\succ_n}(y) = 1$ $\forall i \in [\![m, n]\!]$. Moreover, for each $z \in A \setminus \{x, y\}$, we have $r_{\succ_1}(z) = m$ for some $i \in N$. Note that both f_n^w and f_n pick only y at P. On the other hand, $\lambda^P(x) = (m-3, m-1, m-2, \ldots, m-2)$ and $\lambda_P(y) = (m-2, m-3, \ldots, 1, 0, \ldots, 0)$. As all the spread measure $\sigma \in \Sigma^{C_{m,n}}$ satisfy the condition $C_{m,n}$, then $\sigma(\lambda_P(x)) < \sigma(\lambda_P(y))$ $\forall \sigma \in \Sigma^{C_{m,n}}$, implying $y \notin \min_{\bar{\sigma} \circ \lambda_P}(\mathrm{PO}(P))$ $\forall \sigma \in \Sigma^{C_{m,n}}$. ■

5. Two voters case

In Section 4 we focused on the analysis of voting rules when the number of voters involved into the decision process is greater than two. Keeping the notation introduced in Section 2, we consider here the case $n = 2$. Two individuals express their preference over a set of alternatives A, and the goal is to find a common agreement on the alternative to select. This class of problems is often referred to as bargaining problems. In addition to *fallback bargaining (FB)* [Brams and Kilgour, 2001] (defined in Section 4.3) we consider three prominent solutions of the literature.

Pareto-and-Veto rules (PV) [Laslier et al., 2020] distribute a veto power of v_1 and v_2 alternatives to voters 1 and 2, respectively, with $v_1 + v_2 = m - 1$. So, every voter $i = 1, 2$ (simultaneously) vetoes his worst v_i alternatives. The SCR picks all non-vetoed and Pareto optimal alternatives.

The *Veto-Rank mechanism (VR)* is commonly used in the selection of arbitrators [de Clippel et al., 2014]. Given a list of m (odd) alternatives (that are candidates to be arbitrators), each of the two voters (that are the two parties that must agree on an arbitrator) simultaneously vetoes his worst $\frac{m-1}{2}$ alternatives. The selected alternatives are the ones with the highest Borda score among the non-vetoed alternatives.

Again within the context of selecting arbitrators, de Clippel et al. [2014] propose and analyze *Shortlisting (SL)* where one of the two parties starts by vetoing her worst $\frac{m-1}{2}$ alternatives (m being odd), and then the second party chooses her best alternative out of the remaining ones. As the outcome of the procedure depends on the party that starts, symmetry among players is ensured by defining the solution as the union of the two outcomes where one and the other party starts.

Definition 6 Given any $m \geq 4$, a spread measure $\sigma \in \Sigma$ satisfies condition D_m iff $\sigma(\lceil \frac{m}{2} \rceil, \lfloor \frac{m}{2} \rfloor - 1) < \sigma(0, \lfloor \frac{m}{2} \rfloor)$ and $\sigma(\lfloor \frac{m}{2} \rfloor - 1, \lceil \frac{m}{2} \rceil) < \sigma(\lfloor \frac{m}{2} \rfloor, 0)$. $\square$

For $m = 4$ the condition requires $\sigma(2, 1) < \sigma(0, 2)$ and $\sigma(0, 2) < \sigma(2, 1)$ which is reasonable in our context. When the value of m is larger, the condition appears even more convincing. As m grows, the distance between 0 and $\lfloor \frac{m}{2} \rfloor$ grows, while the distance between $\lceil \frac{m}{2} \rceil$ and $\lfloor \frac{m}{2} \rfloor - 1$ remains constant (1 if m is even, 2 otherwise). Requiring, for example, the spread of $(15, 14)$ to be smaller than the spread of $(0, 15)$ is very reasonable.

We write $\Sigma^{D_m} \subseteq \Sigma$ for the set of spread measures that satisfy condition D_m.

Theorem 11 *Let $m \geq 5$. Under Σ^{D_m}, FB and PV fail PCC. Furthermore, when m is odd, VR and SL also fail PCC.* $\square$

Proof Take any $m \geq 5$ and any $\sigma \in \Sigma^{D_m}$. Define $\alpha = \lceil \frac{m}{2} \rceil - 1$ and $\beta = \lfloor \frac{m}{2} \rfloor - 1$. Note that $\sigma(\alpha+1, \beta) < \sigma(0, \beta+1)$ and $\sigma(\beta, \alpha+1) < \sigma(\beta+1, 0)$, and $\alpha + \beta + 2 = m$. Consider the profile P where voter i_1 has the preference order $x \succ a_1 \succ \ldots \succ a_\alpha \succ y \succ b_1 \succ \ldots \succ b_\beta$ and voter i_2 has the preference order $b_1 \succ \ldots \succ b_\beta \succ y \succ x \succ a_1 \succ \ldots \succ a_\alpha$. Note that $\sigma(\lambda_P(y)) = \sigma(\alpha + 1, \beta)$ and that $\sigma(\lambda_P(x)) = \sigma(0, \beta + 1)$. Therefore, $\sigma(\lambda_P(y)) < \sigma(\lambda_P(x))$. As y is not Pareto-dominated, an SCR that uniquely picks x at P cannot be PCC.

In a similar vein, at the profile P' which is obtained by the inversion of the preferences of i_1 and i_2 at P, an SCR that is PCC cannot pick x uniquely.

The proof will be concluded by showing that FB, PV, and (when m is odd) VR and SL all pick only x at P or at P'.

We readily see that FB picks only x at P (and at P') since x is the first alternative which reaches the unanimous consent.

For PV, let $v_{i_1} \geq v_{i_2}$ (thus $v_{i_1} \geq \lceil \frac{m-1}{2} \rceil$ and $v_{i_2} \leq \lfloor \frac{m-1}{2} \rfloor$, or equivalently, $v_{i_1} \geq \lfloor \frac{m}{2} \rfloor = \beta + 1$ and $v_{i_2} \leq \lceil \frac{m-2}{2} \rceil = \alpha$), and consider the profile P. Observe that the first voter vetoes at least y and every b_j $(1 \leq j \leq \beta)$ while no voter vetoes x. As x Pareto-dominates every a_j $(1 \leq j \leq \alpha)$, PV picks only x at P. When $v_{i_2} \geq v_{i_1}$, a similar reasoning yields that PV picks only x at P'.

Now let m be odd.

For VR, a reasoning similar to the one applied to PV yields x as the unique choice at P: each voter vetoes her worst $\frac{m-1}{2}$ alternatives, thus i_1 vetoes y and every b_j $(1 \leq j \leq \beta)$ and i_2 vetoes every a_j $(1 \leq j \leq \alpha)$. The alternative x is the only non-vetoed alternative, so it is selected as the sole winner.

Finally, SL also picks x, as it is the unique winner no matter which voter starts the veto phase. If i_1 starts, y and every b_j $(1 \leq j \leq \beta)$ get vetoed, then i_2 chooses her best alternative out of the remaining ones which is x. If i_2 starts, every a_j $(1 \leq j \leq \alpha)$ get vetoed, then i_1 chooses her best alternative which is x. ■

5.1. More on the two voters case

We say that an alternative x is ranked among the first half of a preference ranking $\succ_i$ if $r_{\succ_i}(x) \leq \lceil \frac{m+1}{2} \rceil$.

Definition 7 A SCR f satisfies the condition *First Half* if it always picks one alternative which is ranked among the first half of both voters's preferences. □

Please note that if m is odd such an alternative always exists.

Proposition 3 *FB, PV (when the veto powers are equally distributed), SL and VR all satisfy First Half.* □

Proof For FB, it is straightforward to see that, even if the two voters have opposite preferences, when descending level to reach unanimity we reach at most rank $\lceil \frac{m+1}{2} \rceil$. Consider now PV, if the vetoes are equally distributed then by definition PV picks all Pareto alternatives that are ranked among the first half of both voters' preferences. The same reasoning can be applied for VR, where both voters exclude their least preferred $\frac{m-1}{2}$ alternatives, and for SL where one of the voter vetoes the alternatives appearing in the second half of her preferences. In this latter case, as m is considered to be odd, the second voter has at least one alternative appearing both in the first half of her preferences and among the non-vetoed alternatives. ■

Thus a SCR f is first half and Pareto iff f is a sub correspondence of PV. FB, SL and VR are first half and Pareto, thus being (proper) sub correspondences of PV.

Proposition 4 *Under Σ^{D_m} a SCR f which is PCC does not satisfy first half.* □

Consider, as an example, the profile P we described in the proof of Theorem 11. We said that any PCC rule would pick y as a winner. But for every $m \geq 5$, the alternative y is not in the first half of the voter i_1 preferences.

Remark 2 (Consideration) Now, suppose, under a given sigma, we pick among the PV alternatives the one with the smallest loss spread. For example, at the profile

$$\begin{array}{llllll} \mathbf{i_1} & a & b & c & d & e \\ \mathbf{i_2} & d & c & a & b & e \end{array}$$

PV would pick a-c and the approach suggest, under a reasonable sigma, would refine this to the singleton c (which is also a refinement of FB).

Question: what is this new rule? It refines PV. It is not FB. At the profile

$$\begin{array}{llllll} \mathbf{i_1} & a & b & c & d & e \\ \mathbf{i_2} & c & b & a & d & e \end{array}$$

PV picks a-b-c, this rule refines it to b (which is the FB outcome) while SL picks a-c. □

6. Concluding remarks

We define an ex-post compromise as an outcome where individuals give up as equally as possible from their ideal points. With three or more individuals, several interesting SCRs of the literature fail to pick ex-post compromises, under any reasonable meaning attributed to “giving up equally”. This failure is valid whether Pareto optimality is adopted or not. Our findings cover Condorcet extensions and scoring rules but also BK-compromises which impose a willingness to compromise over individuals but may eventually pick an outcome so that some individuals do not effectively compromise at all. This failure prevails for social choice problems with two individuals: all well-known two-person SCRs of the literature, namely, fallback bargaining, Pareto and veto rules, short listing and veto rank, fail to pick ex-post compromises. The exclusion of the equal-loss principle by almost all SCRs of the literature questions whether the principle is uninteresting in a discrete social choice context. We think so, when there are three or more individuals which present voting contexts where the number of voters usually exceeds the number of candidates by far and every candidate is ranked last by at least one voter. In such situations, picking an alternative whose "average rank" is highest is more of a concern than every voter giving up "equally". On the other hand, this latter concern appears to be much more valid in two-person collective choice problems which are typically interpreted as bargaining situations which require mutual consent, a fact which is strongly justified by the vast literature on the ultimatum game (https://www.sciencedirect.com/science/article/abs/pii/S0167268114001759). Thus, our analysis raises the question of designing new discrete bargaining solutions compatible with the equal-loss principle.

References

P. D. Allison. Measures of inequality. *American Sociological Review*, 43:865, 1978. ISSN 0003-1224. doi:10.2307/2094626.

G. W. Bassett and J. Persky. Robust voting. *Public Choice*, 99(3-4):299–310, 1999. doi:10.1023/A:1018324807861.

S. J. Brams and D. M. Kilgour. Fallback bargaining. *Group Decision and Negotiation*, 10(4):287–316, Jul 2001. ISSN 1572-9907. doi:10.1023/A:1011252808608.

Y. Chun. The equal-loss principle for bargaining problems. *Economics Letters*, 26(2):103–106, 1988. doi:10.1016/0165-1765(88)90022-5.

Y. Chun and H. Peters. The lexicographic equal-loss solution. *Mathematical Social Sciences*, 22:151–161, 1991. ISSN 0165-4896. doi:10.1016/0165-4896(91)90004-b.

R. Congar and V. Merlin. A characterization of the maximin rule in the context of voting. *Theory and Decision*, 72, 01 2012. doi:10.1007/s11238-010-9229-0.

G. de Clippel, K. Eliaz, and B. Knight. On the selection of arbitrators. *American Economic Review*, 104:3434–3458, 2014. ISSN 0002-8282. doi:10.1257/aer.104.11.3434.

G. Erdélyi, M. R. Fellows, J. Rothe, and L. Schend. Control complexity in bucklin and fallback voting: A theoretical analysis. *Journal of Computer and System Sciences*, 81:632–660, 2015. ISSN 0022-0000. doi:10.1016/j.jcss.2014.11.002.

W. V. Gehrlein and D. Lepelley. On some limitations of the median voting rule. *Public Choice*, 117(1/2):177–190, 2003. ISSN 00485829, 15737101. doi:10.1023/A:1026147319249. URL `http://www.jstor.org/stable/30025893`.

C. Herrero and A. Villar. The three musketeers: four classical solutions to bankruptcy problems. *Mathematical Social Sciences*, 42(3):307–328, 2001. doi:10.1016/S0165-4896(01)00075-0.

Ö. Kibris and M. R. Sertel. Bargaining over a finite set of alternatives. *Social Choice and Welfare*, 28(3):421–437, 2007. doi:10.1007/s00355-006-0178-z.

J.-F. Laslier, M. Nunez, and M. Remzi Sanver. A solution to the two-person implementation problem. working paper or preprint, May 2020. URL `https://hal.archives-ouvertes.fr/halshs-02173504`.

M. Mariotti. Nash bargaining theory when the number of alternatives can be finite. *Social Choice and Welfare*, 15:413–421, 1998. ISSN 0176-1714. doi:10.1007/s003550050114.

V. Merlin, İ. Özkal Sanver, and M. R. Sanver. Compromise rules revisited. *Group Decision and Negotiation*, 28:63–78, 2019. ISSN 0926-2644. doi:10.1007/s10726-018-9598-2.

R. Nagahisa and M. Tanaka. An axiomatization of the kalai-smorodinsky solution when the feasible sets can be finite. *Social Choice and Welfare*, 19: 751–761, 2002. ISSN 0176-1714. doi:10.1007/s003550100150.

J. F. Nash. The bargaining problem. *Econometrica: Journal of the Econometric Society*, 18:155, 1950. ISSN 0012-9682. doi:10.2307/1907266.

H. Nurmi. *Voting paradoxes and how to deal with them*. Springer, 1999. ISBN 978-3-540-66236-5. doi:10.1007/978-3-662-03782-9.

İ. Özkal-Sanver and M. R. Sanver. Efficiency in the degree of compromise: A new axiom for social choice. *Group Decision and Negotiation*, 13:375–380, 2004. ISSN 0926-2644. doi:10.1023/b:grup.0000042925.01972.ad.

M. R. Sertel. Lecture notes on microeconomics (unpublished). *Boğaziçi University, İstanbul*, 1986.

M. R. Sertel and B. Yılmaz. The majoritarian compromise is majoritarian-optimal and subgame-perfect implementable. *Social Choice and Welfare*, 16:615–627, 1999. ISSN 0176-1714. doi:10.1007/s003550050164.

A. Spread Measures

In what follows are showed the proofs for Proposition 2.

A.1. Mean Absolute Difference

PROOF for σ_{mad}.
Recall that

$$\sigma_{mad}(l) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |l_i - l_j|.$$

Let us define $f_l(i) = \sum_j |l_i - l_j|$ and $s(l) = n^2 \sigma_{\text{mad}}(l)$. Thus $n^2 \sigma_{mad}(l) = s(l) = \sum_i f_l(i)$.

Consider the two vectors $l_1 = (m-3, m-1, m-2, \ldots, m-2)$ (where $n-2$ terms are equal to $m-2$) and $l_2 = \sigma(m-2, m-3, \ldots, 1, 0, \ldots, 0)$ (where $m-2$ terms go from $m-2$ to 1, and $m-n+2$ terms are equal to 0). The thesis is now that $s(l_1) < s(l_2)$.

Let us consider l_1 first:

$$\begin{aligned} f_{l_1}(1) &= 2 + (n-2) = n \\ f_{l_1}(2) &= 2 + (n-2) = n \\ f_{l_1}(i) &= 2 \quad \forall\ 3 \leq i \leq n \end{aligned}$$

In fact, considering the first term of l_1 $(m-3)$, the difference between itself and the second term of the vector $(m-1)$ is 2; the difference between itself and any of the other $n-2$ terms of the vector $(m-2)$ is 1. The sum of these differences is represented by $f_{l_1}(1)$. The same argument holds for the second term $(m-1)$. Any of the remaining $n-2$ terms $(m-2)$ has a difference 1 with the first and the second term and a difference 0 with the rest. Therefore

$$s(l_1) = 2(2 + (n-2)) + (n-2) \cdot 2) = 4n - 4$$

To compute $s(l_2)$, let us distinguish the cases $1 \leq i \leq m-2$ and $m-1 \leq i$. For $1 \leq i \leq m-2$,

$$\begin{aligned}
f_{l_2}(i) &= \sum_{1 \leq j \leq i} |l_i - l_j| + \sum_{i < j \leq m-2} |l_i - l_j| + \sum_{m-2 < j \leq n} |l_i - l_j| \\
&= [(i-1) + (i-2) + \ldots + 0] + [1 + \ldots + (m-2-i)] \\
&\quad + (m-1-i)(n-m+2) \\
&= \frac{(i-1)i}{2} + \frac{(m-2-i)(m-1-i)}{2} + (m-1-i)(n-m+2) \\
&= i^2/2 - i/2 + \frac{(m-2)(m-1) - i(m-2+m-1) + i^2}{2} \\
&\quad + (m-1)(n-m+2) - i(n-m+2) \\
&= i^2 - i\frac{1 + 2m - 3 + 2(n-m+2)}{2} + (m-1)\frac{m-2+2(n-m+2)}{2} \\
&= i^2 - i(n+1) + (m-1)\frac{-m+2n+2}{2}.
\end{aligned}$$

For $m-1 \leq i$, $f_{l_2}(i) = m-2+m-3+\ldots+1 = \frac{(m-2)(m-1)}{2}$.
Thus,

$$\begin{aligned}
s(l_2) &= \sum_{1 \leq i \leq m-2} [i^2 - i(n+1) + (m-1)\frac{-m+2n+2}{2}] \\
&\quad + (n-(m-2))(m-2)(m-1)/2 \\
&= (m-2)(m-1)(2m-3)/6 - (n+1)(m-2)(m-1)/2 \\
&\quad + (m-2)(m-1)(-m+2n+2)/2 \\
&\quad + (n-m+2)(m-2)(m-1)/2 \\
&= (m-2)(m-1)\left(\frac{2m-3}{6} - \frac{n+1}{2} + \frac{-m+2n+2}{2} + \frac{n-m+2}{2}\right) \\
&= (m-2)(m-1)\left(\frac{-2m+3}{3} + n\right).
\end{aligned}$$

Our thesis is now that

$$4n - 4 < \frac{-2m+3}{3}(m-2)(m-1) + n(m-2)(m-1)$$

or equivalently, that

$$\frac{(2m-3)(m-2)(m-1)}{3} - 4 < n[(m-2)(m-1)-4].$$

When $m = 4$, using the fact that $4 \le n$, the inequality holds: $5(2)(3)/3-4 < 4[2] \le n[2]$.

Now assume that $m \ge 5$. Using the fact that $m-1 \le n$, suffices to show that $(2m-3)(m-2)(m-1)/3-4 < (m-1)[(m-2)(m-1)-4]$, or equivalently, that $-12 < (m-1)[3(m-2)(m-1)-12-(2m-3)(m-2)]$. Note that the right hand side equals $(m-1)[(m-2)(3(m-1)-(2m-3))-12] = (m-1)(m^2-2m-12) = (m-1)(m-1+\sqrt{13})(m-1-\sqrt{13})$. As all multiplicands are positive when $m \ge 5$, the inequality is true when $m \ge 5$. ∎

A.2. Average Absolute Deviation

Proof for σ_{ad}.
Recall that

$$\sigma_{ad}(l) = \frac{\sum_{i=1}^{n} |l_i - \bar{l}|}{n},$$

where $\bar{l} = \frac{\sum_{i=1}^{n} l_i}{n}$. Let us define $s(l) = \sum_i |l_i - \bar{l}|$, so that $s(l) = n\sigma_{ad}(l)$. Consider the two vectors $l_1 = (m-3, m-1, m-2, \ldots, m-2)$ and $l_2 = (m-2, m-3, \ldots, 1, 0, \ldots, 0)$. The thesis is now that $s(l_1) < s(l_2)$.

For l_1, the arithmetic mean of its values is $m-2$, so

$$s(l_1) = [|m-3-m+2| + |m-1-m+2| + 0 + \cdots + 0] = 2.$$

For l_2, the arithmetic mean is

$$\bar{l_2} = \frac{1}{n}\sum_{i=1}^{m-2} i = \frac{(m-2)(m-1)}{2n}.$$

Recall that

$$\sum_{i=k}^{m} x = (m-k+1)x$$
$$\sum_{i=k}^{m} i = -\frac{(k-m-1)(k+m)}{2}$$
$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$$

We can write $s(l_2)$ as:

$$
\begin{aligned}
s(l_2) &= \sum_{i=1}^{m-2} |i-\bar{l_2}| + (n-(m-2))|0-\bar{l_2}| = \\
&= \sum_{i=1}^{m-2} |i-\bar{l_2}| + (n-m+2)\bar{l_2} = \\
&= \sum_{i=\lfloor\bar{l_2}\rfloor+1}^{m-2} (i-\bar{l_2}) - \sum_{i=1}^{\lfloor\bar{l_2}\rfloor} (i-\bar{l_2}) + (n-m+2)\bar{l_2} = \\
&= \sum_{i=\lfloor\bar{l_2}\rfloor+1}^{m-2} i - \sum_{i=\lfloor\bar{l_2}\rfloor+1}^{m-2} \bar{l_2} - \sum_{i=1}^{\lfloor\bar{l_2}\rfloor} i + \sum_{i=1}^{\lfloor\bar{l_2}\rfloor} \bar{l_2} + (n-m+2)\bar{l_2} = \\
&= -\frac{(\lfloor\bar{l_2}\rfloor+1-m+2-1)(\lfloor\bar{l_2}\rfloor+1+m-2)}{\lfloor\bar{l_2}\rfloor} - (m-2-\lfloor\bar{l_2}\rfloor-1+1)\bar{l_2} \\
&- \frac{\lfloor\bar{l_2}\rfloor(\lfloor\bar{l_2}\rfloor+1)}{2} + \lfloor\bar{l_2}\rfloor\bar{l_2} + (n-m+2)\bar{l_2} = \\
&= -\frac{(\lfloor\bar{l_2}\rfloor-m+2)(\lfloor\bar{l_2}\rfloor+m-1)}{2} - (m-2-\lfloor\bar{l_2}\rfloor)\bar{l_2} - \frac{\lfloor\bar{l_2}\rfloor(\lfloor\bar{l_2}\rfloor+1)}{2} \\
&+ \lfloor\bar{l_2}\rfloor\bar{l_2} + (n-m+2)\bar{l_2} = \\
&= -\frac{(\lfloor\bar{l_2}\rfloor)(\lfloor\bar{l_2}\rfloor+m-1)}{2} + \frac{(m-2)(\lfloor\bar{l_2}\rfloor+m-1)}{2} - (m-2)\bar{l_2} + (\lfloor\bar{l_2}\rfloor)\bar{l_2} \\
&- \frac{\lfloor\bar{l_2}\rfloor(\lfloor\bar{l_2}\rfloor+1)}{2} + \lfloor\bar{l_2}\rfloor\bar{l_2} + (n-m+2)\bar{l_2} = \\
&= -\frac{(\lfloor\bar{l_2}\rfloor)(\lfloor\bar{l_2}\rfloor+m-1)}{2} + \frac{(m-2)\lfloor\bar{l_2}\rfloor}{2} + \underbrace{\frac{(m-2)(m-1)}{2}}_{=n\bar{l_2}} - (m-2)\bar{l_2} \\
&- \frac{\lfloor\bar{l_2}\rfloor(\lfloor\bar{l_2}\rfloor+1)}{2} + 2\lfloor\bar{l_2}\rfloor\bar{l_2} + (n-m+2)\bar{l_2} = \\
&= \lfloor\bar{l_2}\rfloor(-\frac{\lfloor\bar{l_2}\rfloor+m-1}{2} + \frac{m-2}{2} - \frac{\lfloor\bar{l_2}\rfloor+1}{2} + 2\bar{l_2}) + 2\bar{l_2}(n-m+2) = \\
&= \lfloor\bar{l_2}\rfloor(\frac{-\lfloor\bar{l_2}\rfloor-m+1+m-2+4\bar{l_2}-\lfloor\bar{l_2}\rfloor-1}{2}) + 2\bar{l_2}(n-m+2) = \\
&= \lfloor\bar{l_2}\rfloor(\frac{4\bar{l_2}-2\lfloor\bar{l_2}\rfloor-2}{2}) + 2\bar{l_2}(n-m+2) = \\
&= \lfloor\bar{l_2}\rfloor(2\bar{l_2}-\lfloor\bar{l_2}\rfloor-1) + 2\bar{l_2}(n-m+2) \qquad (1)
\end{aligned}
$$

If m is even then $\lfloor \bar{l_2} \rfloor = \bar{l_2}$ and we can write $s(l_{2ev})$

$$\begin{aligned}
&= \bar{l_2}(2\bar{l_2} - \bar{l_2} - 1) + 2\bar{l_2}(n - m + 2) = \\
&= \bar{l_2}(\bar{l_2} - 1 + 2(n - m + 2)) = \qquad (2)\\
&= \frac{(m-2)(m-1)}{2n}\left(\frac{(m-2)(m-1)}{2n} + 2n - 2m + 3\right)
\end{aligned}$$

Recall that now our thesis is $s(l_1) < s(l_2)$, or in this case that

$$\begin{aligned}
&2 < \frac{(m-2)(m-1)}{2n}\left(\frac{(m-2)(m-1)}{2n} + 2n - 2m + 3\right) \\
&4n < (m-2)(m-1)\left(\frac{(m-2)(m-1)}{2n} + 2n - 2m + 3\right) \\
&4n < \frac{(m-2)^2(m-1)^2}{2n} + (m-2)(m-1)(2n - 2m + 3) \\
&8n^2 < (m-2)^2(m-1)^2 + 2n(m-2)(m-1)(2n - 2m + 3)
\end{aligned}$$

If $m = 4$ then we have:

$$\begin{aligned}
&8n^2 < (4-2)^2(4-1)^2 + 2n(4-2)(4-1)(2n - 2(4) + 3) \\
&8n^2 < 4 \cdot 9 + 2n \cdot 2 \cdot 3(2n - 8 + 3) \\
&8n^2 < 36 + 12n(2n - 5) \\
&8n^2 < 36 + 24n^2 - 60n \\
&8n^2 - 24n^2 + 60n - 36 < 0 \\
&-16n^2 + 60n - 36 < 0 \\
&4n^2 - 15n + 9 > 0
\end{aligned}$$

When $m = 4$, $n \geq 4$ and this inequality holds. If $m > 4$ then $n \geq m + 1$, let us consider $n = m + 1$ first:

$$\begin{aligned}
&8(m+1)^2 < (m-2)^2(m-1)^2 + 2(m+1)(m-2)(m-1)(2(m+1) - 2m + 3) \\
&8(m+1)^2 < (m-2)^2(m-1)^2 + 2(m+1)(m-2)(m-1)(2m + 2 - 2m + 3) \\
&8(m+1)^2 < (m-2)^2(m-1)^2 + 10(m+1)(m-2)(m-1) \\
&0 < \frac{(m-2)^2(m-1)^2}{8(m+1)^2} + \frac{10(m+1)(m-2)(m-1)}{8(m+1)^2} \\
&0 < \frac{(m-2)^2(m-1)^2}{8(m+1)^2} + \frac{10(m-2)(m-1)}{8(m+1)}
\end{aligned}$$

Since all the terms in left-hand side are positive the inequality holds. We still need to consider the case where m is odd. In this case we can say that $\lfloor \bar{l_2} \rfloor > \bar{l_2} - 1$, so we can continue from Equation (1)

$$\begin{aligned}
(1) &> (\bar{l_2} - 1)(2\bar{l_2} - (\bar{l_2} - 1) - 1) + 2\bar{l_2}(n - m + 2) \\
(1) &> (\bar{l_2} - 1)(2\bar{l_2} - \bar{l_2} + 1 - 1) + 2\bar{l_2}(n - m + 2) \\
(1) &> \bar{l_2}(\bar{l_2} - 1) + 2\bar{l_2}(n - m + 2)
\end{aligned}$$

The right side of this inequality is the same equation (2) obtained for the case m even. Thus, since we proved that $s(l_1) < s(l_{1ev})$ we can conclude that $s(l_1) < s(l_{2ev}) < s(l_{2odd})$. ■

A.3. Standard Deviation

Proof for σ_{sd}.
Recall that

$$\sigma_{sd}(l) = \sqrt{\frac{\sum_{i=1}^{n}(l_i - \bar{l})^2}{n}}$$

where $\bar{l} = \frac{\sum_{i=1}^{n} l_i}{n}$. Let us define $s(l) = \sum_i |l_i - \bar{l}|$, so that $s(l) = n\sigma_{sd}(l)^2$. Consider the two vectors $l_1 = (m-3, m-1, m-2, \ldots, m-2)$ and $l_2 = (m-2, m-3, \ldots, 1, 0, \ldots, 0)$. The thesis is now that $s(l_1) < s(l_2)$.

For l_1, the arithmetic mean of its values is $m-2$ so $s(l_1)$:

$$s(l_1) = (m - 3 - m + 2)^2 + (m - 1 - m + 2)^2 + 0 + \cdots + 0 = 2$$

For l_2, the arithmetic mean is

$$\bar{l_2} = \frac{1}{n}\sum_{i=2}^{m-2} i = \frac{(m-2)(m-1)}{2n}.$$

Recall that

$$\begin{aligned}
\sum_{i=k}^{m} x &= (m - k + 1)x \\
\sum_{i=1}^{m} i &= \frac{m(m+1)}{2} \\
\sum_{i=1}^{m} i^2 &= \frac{m(m+1)(2m+1)}{6}
\end{aligned}$$

We can write $s(l_2)$ as:

$$
\begin{aligned}
s(l_2) &= \sum_{i=1}^{m-2}(i-\bar{l_2})^2 + (n-(m-2))(0-\bar{l_2})^2 = \\
&= \sum_{i=1}^{m-2}(i^2 - 2i\bar{l_2} + \bar{l_2}^2) + (n-m+2)\bar{l_2}^2 = \\
&= \sum_{i=1}^{m-2} i^2 - \sum_{i=1}^{m-2} 2i\bar{l_2} + \sum_{i=1}^{m-2}\bar{l_2}^2 + (n-m+2)\bar{l_2}^2 = \\
&= \frac{(m-2)(m-2+1)(2m-4+1)}{6} - 2\bar{l_2}\frac{(m-2)(m-2+1)}{2} \\
&+ (m-2)\bar{l_2}^2 + (n-m+2)\bar{l_2}^2 =
\end{aligned}
$$

■

A.4. Gini coefficient

Proof for σ_G.
Consider the two vectors $l_1 = (m-3, m-1, m-2, \ldots, m-2)$ and $l_2 = \sigma(m-2, m-3, \ldots, 1, 0, \ldots, 0)$; their spread using the Gini Coefficient is:

$$\sigma_G(l_1) = \frac{4(n-1)}{n^2 \cdot 2 \cdot (m-2)} = \frac{2(n-2)}{(m-2)n^2}$$

■

$$\sigma_G(l_2) = \frac{\frac{1}{3}\cdot(m-2)(m-1)(3n-2m+3)}{n^2 \cdot 2 \cdot \frac{m^2-3m+2}{2n}} = \frac{3n-2m+3}{3 \cdot n}$$

Definition 8 (OLD Definition: Pairwise Pareto dominance) For all r, $s \in \mathbb{R}_+^N$:

$$[|r_i - r_j| \leq |s_i - s_j| \, \forall i, j \in N] \Rightarrow \sigma(r) \leq \sigma(s).$$ □

We write $\Sigma^{\text{PPd}} \subseteq \Sigma^{\text{All}}$ for the class of spread measures that satisfy also PPd. Given a vector $r \in \mathbb{R}^N$ of n elements, some examples of spread measures are the following.