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Theory and Methodology

A stochastic dominance analysis of ranked voting systems with scoring

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Abstract: Selection procedures and elections often use a ranked voting system combined with a set of point values assigned to the various ranks. The winner is the one with the highest total points. However, the choice of the set of point values tends to be arbitrary. Stochastic dominance will be used to define a partial ordering of candidates in this situation. This shows the effect of choosing different point values and, in particular, determines all possible orderings of the candidates. Examples from sports competitions will be used.

Keywords: Decision-making; Stochastic dominance; Preference analysis; Sports

1. Introduction

Consider an election in which each voter is allowed to rank the candidates by order of preference. A scoring function is assumed which assigns point values to each rank and the candidate with the highest total number of points is declared the winner. This is a type of *ranked (or preferential) voting system*, first suggested by Borda in 1781. It is one of many voting systems that have been investigated (see Black, 1958; Fishburn, 1973; Nurmi, 1987). No voting system has all the desirable properties one could hope for and the method of Borda has had its share of criticism (see Black, 1958).

Social Choice Theory investigates voting methods and attempts to decide 'fair' ways to elect a candidate given the preference order of each voter for each candidate (Fishburn, 1973). However, in this paper we do not assume knowledge of the individual voter's rankings, only the aggregate vote by rank for each candidate. For this

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reason it is not possible, for example, to ascertain if there is a winner according to the Condorcet principle. We are not advocating the use of a ranked voting system with scoring function, but since they are frequently used we feel that it is important to investigate how changing the assignment of point values will affect the outcome of the election.

Since the point values used seem to be arbitrary, will the results of an election change if a different set of point values is used? This paper will show that first and second-order stochastic dominance can determine a partial ordering of the candidates by making some reasonable restrictions on the possible scoring functions. After the election, it can be determined if there was a unique winner or if others can declare they would be deserving of victory had different point values been used. In this sense the process may be viewed as a sensitivity analysis. Alternatively, we could have avoided specifying any point values. The result of the election is then a partial ordering of the candidates. If a unique winner must be declared then this might be unsatisfactory – but it could be used to show the need for a runoff election. In any case, we assume that a voter's ranking of the candidates is not influenced by the actual point values used.

2. Assigning point values

Consider an election with M candidates. In a ranked voting system, each voter provides a preference ordering (ranking) of the candidates. It is assumed that there are no ties. Points are awarded to a candidate as a function of the ranking assigned by a voter. In general, only ranks $1,\dots,N$ $(N \leqslant M)$ will be assigned points so that a complete ranking of all candidates may not be required from the voters. Let $v(n)$ be the points

Table 1
Votes by rank for four candidates A–D

Candidate	Rank					
	1	2	3	4	5	Score
A	27	38	15	7	4	636
B	38	15	16	14	7	614
C	21	25	30	8	5	564
D	2	8	10	23	19	214

awarded to a candidate who receives a vote for rank n and the winner is the candidate with the highest total score. $\{v(n)\}$ will be called a *scoring function*. This model includes plurality voting ($v(1) = 1$ with all other values equal to 0) and the Borda method ($v(n) = N + 1 - n$).

We shall now assume $v(1) > v(2) > \cdots > v(N) > 0$ and $v(n) = 0$ otherwise. The total score of a candidate i is then given by

$$E_i(v) \equiv \sum_{n=1}^{N} f_i(n)v(n) \tag{1}$$

where $f_i(n)$ is the frequency of rank n votes for candidate i. Tie breaking procedures between candidates with equal total scores will not be considered. It is clear that the ranking of the total scores by (1) is invariant under any increasing linear transformation of the scoring function. Therefore, it is always possible to choose $v(N) = 1$ and $v(N+1) = 0$ without loss of generality. If $N < M$, it is convenient to assume all votes for ranks other than $1,\dots,N$ will be cast for $N+1$. This has no effect on (1). A scoring function that satisfies the following conditions will be called *decreasing*:

Definition 1. A *decreasing scoring function* is any non-negative function v defined on the set $\{1,\dots,N+2\}$ and decreasing on the set $\{1,\dots, N+1\}$ with $v(N) = 1$ and $v(n) = 0$, $n > N$.

A non-increasing scoring function could be defined by replacing 'decreasing' in Definition 1 with 'non-increasing'. Scoring functions which are not strictly decreasing may seem to be unrealistic. However, consider the scoring function: 1-1-1-0. ($v(1) = v(2) = v(3) = 1$, $v(4) = 0$.) This requires voters to select their top three choices. The American Baseball Hall of Fame uses this method to select members. This is similar to approval voting (Nurmi, 1987), although in approval voting the voters may select as many or as few acceptable candidates as desired rather than precisely three.

3. An example

Consider the following ranked voting system. Each voter selects and ranks the top 5 candidates. The candidates are awarded points on a

10-7-5-3-1 basis. Thus a first place vote is worth 10 points to the candidate, a second place vote 7 points, etc. Sixth place and beyond is assigned zero points. This is an example of a decreasing scoring function with $N = 5$.

Table 1 present voting results for 4 candidates with 92 voters (additional details for the example are found in Appendix B). For each of the 4 candidates receiving a vote (for ranks 1 through 5), the distribution of votes for each rank is given along with the candidate's total score, which shows that candidate A would be the winner based on point values of 10-7-5-3-1. Notice that candidate B would have finished in first position had the point value of first place, $v(1)$, been sufficiently large. Candidate C had more third place votes than any other player and would thereby benefit from a larger point value of third place, $v(3)$. However, if we require $v(1) > v(2) > v(3)$, then the total scores for candidates A and B would also increase with such an adjustment. So it is not clear if candidate C would have won using *any* decreasing scoring function. In order to investigate further, we shall place additional restrictions on the class of point assignments we are willing to consider.

4. Convex scoring functions

We wish to go beyond a decreasing scoring function and capture the essence of a scoring function that most people would consider 'fair'. A tacit assumption usually made is that first place votes are not only more valuable than second place votes but that the difference in value between a first and a second place vote should be at least as large as between a second and a third place vote. In the example, this is shown by the fact that $v(1) - v(2) = 3$ while $v(2) - v(3) = 2$. This leads to the condition that such a function v must be *convex* on the set $\{1, \dots, N+1\}$.

Definition 2. A *convex scoring function* is any decreasing scoring function v satisfying the additional condition that

$$v(n) - v(n+1) \geqslant v(n+1) - v(n+2) \quad \text{for } n = 1, \dots, N.$$

A convex scoring function will reward the candidate with votes for both extremes of rankings, a controversial candidate, as opposed to the candidate who might have more votes at the middle ranks, a compromise candidate. Convex scoring functions include the Borda method. In addition, any convex scoring function v will always be bounded below by the linear scoring function of the Borda method:

$$v(n) \geqslant N - n + 1, \quad 1 \leqslant n \leqslant N.$$

Convex and concave scoring functions were considered in a different context by Gaertner (1983).

5. Partial orderings

Our next goal is to determine an ordering (or partial ordering) of all candidates from the results of the voting, summarized in Table 1, based on using any function from a class $\mathscr{C}$ of scoring functions. A preference relation will be defined such that candidate i is preferred to candidate k if i has at least as high a score as k for all scoring functions in $\mathscr{C}$. (If two or more candidates have the same set of votes, namely $f_i = f_k$, then they can never be distinguished so we shall place them into an equivalence class and treat them as one candidate.) Following Fishburn (1982), we now define a binary relation P on the set of candidates. For any candidates i and k:

Definition 3a. iPk if $E_i(v) \geqslant E_k(v)$ for all decreasing scoring functions v, with strict inequality for at least one v.

P will define a partial ordering on the set of candidates. If iPk, then this means i will never have a lower score for any decreasing v. If no ordering exists between i and k, then the two are essentially tied since the winner depends on the particular choice of a decreasing scoring function v. It is not possible for both iPk and kPi to occur.

Another binary relation, Q, can be defined corresponding to the set of convex scoring functions. This definition is given as follows:

Definition 3b. iQk if $E_i(v) \geqslant E_k(v)$ for all convex scoring functions v, with strict inequality for at least one v.

Note that P implies Q: If iPk then iQk since Q is based on a subset of the class of scoring functions that P is based on. In the next section, we will see how to easily determine these partial orderings.

6. Stochastic dominance

In order to analyze the partial orderings defined in the previous section, results from the theory of stochastic dominance (see Fishburn and Vickson, 1978, pp. 97–102) will be adapted. In (1), $\{f_i\}$ is the vote count for candidate i at each rank. Let the cumulative total vote be given by

$$F_i(n) = f_i(1) + \cdots + f_i(n), \quad 1 \leqslant n \leqslant N+1.$$

For each candidate i, let

$$F_i^{(2)}(n) = \sum_{j=1}^{n} F_i(j) \quad \text{for } 1 \leqslant n \leqslant N.$$

The next Theorem gives a necessary and sufficient condition for both of the relations of Definition 3.

Theorem 1. *Consider any candidates i and k with* $f_i \neq f_k$.
(a) *iPk if and only if*

$$F_i(n) \geqslant F_k(n) \quad \text{for } 1 \leqslant n \leqslant N,$$

with a strict inequality for at least one n.
(b) *iQk if and only if*

$$F_i^{(2)}(n) \geqslant F_k^{(2)}(n) \quad \text{for } 1 \leqslant n \leqslant N,$$

with a strict inequality for at least one n.

Proof. See Appendix A.

As an example of the use of Theorem 1(a), consider the 4 candidates from Table 1. The corresponding F_i are given in Table 2. For convenience, we have combined all rankings beyond N into rank $N+1$ (which assigns no points to the candidate). We see that $F_A \geqslant F_C$, $F_A \geqslant F_D$, $F_B \geqslant F_D$, and $F_C \geqslant F_D$. All other pairs of F_i intersect. The conclusion is that APC, CPD, BPD. The transitivity of P implies APD. The minimal Hasse diagram of the partial ordering is displayed in Figure 1: a line from i at a higher level to j at a lower level indicates iPk.

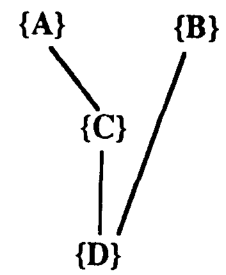

Figure 1. Partial ordering for all decreasing scoring functions based on F_i given in Table 2.

Figure 1 shows that candidates A and B are the only possible winners no matter which v is chosen. That is, they form the non-dominated set of candidates. Candidate A will finish no worse than second place but there is no dominance between candidates B and C so the only possible orderings for first, second, and third place are: A,B,C; A,C,B; or B,A,C. Candidate D could not finish in the top three positions.

Table 3 displays the result of the computations required to apply Theorem 1(b) for the four candidates from Table 1. The partial ordering for the candidates is given in Figure 2. The ordering is the same as in Figure 1 except that BQC. Since P from Figure 1 is a stronger condition than Q,

Table 2
F_i for candidates $A-D$: Computed from Table 1

Candidate	Rank					
	1	2	3	4	5	$N+1$
A	27	65	80	87	91	92
B	38	53	69	83	90	92
C	21	46	76	84	89	92
D	2	10	20	43	62	92

Table 3
$F_i^{(2)}(n) = \sum_{j=1}^{n} F_i(j)$, $1 \leqslant n \leqslant N$, for candidates $A-D$: Computed from Table 2

Candidate	Rank					
	1	2	3	4	5	$N+1$
A	27	92	172	259	350	442
B	38	91	160	243	333	425
C	21	67	143	227	316	408
D	2	12	32	75	137	229

any line in Figure 1 must be present (or is implied by transitivity) in Figure 2, but not conversely. Only the relative ordering of candidates A and B would depend on the specific choice of the scoring function from within the class of all convex scoring functions. The choice 10-7-5-3-1 leads to the selection of A while 13-7-5-3-1 leads to the selection of B.

Now suppose that in the voting it had been decided that a scoring function of the type

$$v(1) > v(2) > v(3) = v(4) = v(5) > 0,$$

such as 10-7-1-1-1 was to be used. This defines a non-increasing scoring function. Analysis of all v in this class can be easily handled by tallying all votes for ranks 3, 4, or 5 in one category and then proceed with $v(1) > v(2) > v(3) > 0$ in Theorem 1(a). (The resulting ordering for the 4 candidates in this case agrees with that given in Figure 2.)

Linear scoring functions form a subclass of convex scoring functions. Theorem 2 is an extension of Theorem 1 to the case where it is specified at which ranks linearity occurs:

Theorem 2. *Let S be any given subset of $\{1,\dots, N-1\}$. Define $\mathscr{C}$ as those convex scoring functions v with*

$$v(n) - v(n+1) = v(n+1) - v(n+2)$$
for all $n \in S$.

Define S^* as the complement of S relative to $\{1,\dots, N\}$. Then

$$E_i(v) \geqslant E_k(v) \quad \text{for all } v \in \mathscr{C}$$

if and only if

$$F_i^{(2)}(n) \geqslant F_k^{(2)}(n) \quad \text{for all } n \in S^*.$$

Proof. See Appendix A.

We will now consider three examples of the use of Theorem 2.

(a) If $N = 5$ and $\mathscr{C}$ is all the convex scoring functions then S is the empty set and $S^* = \{1, 2, 3, 4, 5\}$. So Theorem 2 now reduces to Theorem 1(b).

(b) Consider the set $\mathscr{C}$ of all linear decreasing scoring functions (the Borda method) with $N = 5$. For such functions, the set $S = \{1, 2, 3, 4\}$ and so $S^* = \{5\}$. In fact, the numerical values of $F_i^{(2)}(5)$ in Table 3 are seen to be those given by using the scoring function 5-4-3-2-1. In this case, for all $v \in \mathscr{C}$ we need only check column 5 to see that the candidate ordering will be A-B-C-D. In this election the Borda method would give the same results as the 10-7-5-3-1 function.

(c) Consider the class $\mathscr{C}$ of convex scoring functions with $N = 5$, $S = \{2, 3\}$, and $S^* = \{1, 4, 5\}$. This requires convex voting functions to have equal spacing between the second and third and the third and fourth ranks. An example is 10-7-5-3-1. By looking only at columns 1, 4, and 5 of Table 3, we see that row A exceeds C and D; row B exceeds C and D; and row C exceeds row D. This class gives the same ordering as in Figure 2 for the 4 candidates.

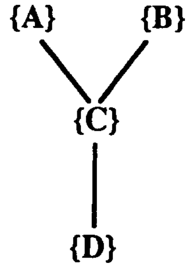

Figure 2. Partial ordering for all convex scoring functions based on $F_i^{(2)}$ given in Table 2.

7. Other examples

In the NCAA Track and Field Championship, a team champion is determined by assigning points to the top 8 contestants in each of 21 events. The team totals are then evaluated using the convex scoring function 10-8-6-5-4-3-2-1. This is similar to the voting in any of the previous examples except that a voter is replaced by an event. In each event a team may have more than one entry and therefore may be ranked more than once in that event. This was not permitted in the prior development of voting but this does not cause any complications to the theory.

Convex scoring functions are used for awards in professional baseball. Here are some examples: Rookie of the Year award (5-3-1), Cy Young award for the best pitcher (5-3-1), and the Most Valuable Player (14-9-8-7-6-5-4-3-2-1).

8. Conclusions

We have shown how stochastic dominance can be used to give a partial ordering for a large class of scoring functions in ranked voting systems. Ranked voting systems are frequently used with rather arbitrary scoring functions; however, they are virtually always convex scoring functions. Stochastic dominance will determine: (a) the existence of a candidate who would win using any decreasing or convex scoring function and (b) a partial ordering of all candidates in the election with respect to a class of scoring functions. Alternatively, if it is permissible to declare a tie in the overall ranking, as in weekly polls for American college football and basketball teams, then we could avoid specifying a specific scoring function and merely use the class of convex scoring functions. The result is then a partial ordering.

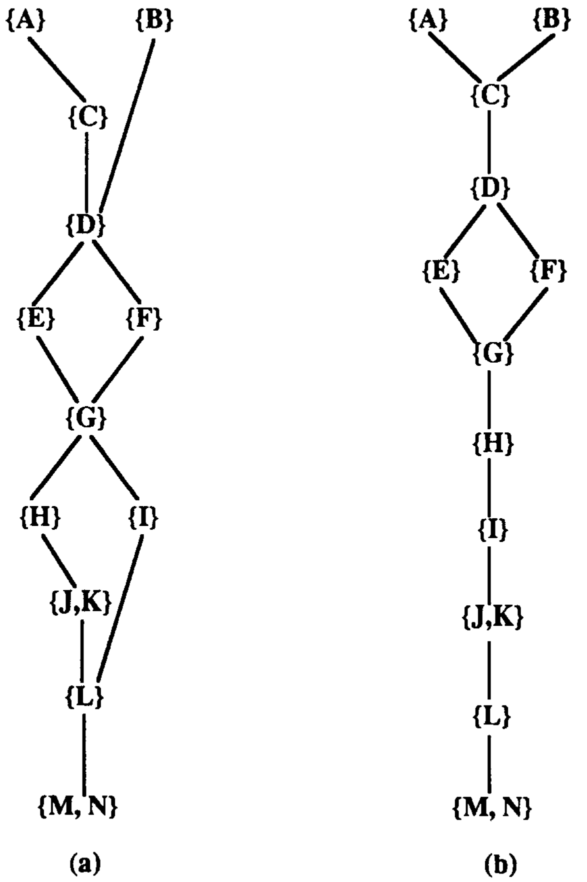

Figure 3. (a) Partial ordering for all decreasing scoring functions based on votes in Table 4. (b) Partial ordering for all convex scoring functions based on votes in Table 4.

Appendix A

This Appendix contains proofs of the Theorems. Fishburn and Vickson (1978, pp.97–102), Fishburn (1982, p.122) and Fishburn (1964, pp.202–208) use Abel's partial summation formula to prove first and second-order stochastic dominance for discrete random variables. We will adapt those results to prove Theorems 1 and 2.

Proof of Theorem 1. We shall assume for convenience that any candidates with an identical frequency distribution of votes have been combined into one equivalence class. Then the statement of the Theorem is simplified: the qualifier 'with strict inequality for at least one v' is no longer needed. From Abel's formula:

$$E_i(v) - E_k(v) = \sum_{n=1}^{N} [f_i(n) - f_k(n)]v(n)$$

$$= \sum_{n=1}^{N} [F_i(n) - F_k(n)][v(n) - v(n+1)]. \tag{2}$$

(a) Since v is a decreasing function, $F_i \geqslant F_k$ implies the right side of (2) is non-negative. For the converse: Assume

$$E_i(v) - E_k(v) \geqslant 0$$

for all decreasing v and

$$F_i(n^*) - F_k(n^*) < 0$$

for some n^*. This must be true for the particular v with

$$v(n) = 1 + (n^* - n)\epsilon,$$

for $1 \leqslant n \leqslant n^*$ and $v(n) = 0$ otherwise, for some $\varepsilon > 0$. Evaluating the right side of (2) with this v gives $F_i(n^*) - F_k(n^*)$ plus terms proportional to ε. Therefore, for small ε this quantity will be negative. This contradicts the assumption that $E_i(v) - E_k(v) \geqslant 0$. Therefore $F_i(n^*) \geqslant F_k(n^*)$ and since n^* is arbitrary this completes the proof.

(b) The proof is based on the 'second-order' version of Abel's formula (Fishburn, 1964):

$$E_i(v) - E_k(v) = \sum_{n=1}^{N} \left[F_i^{(2)}(n) - F_k^{(2)}(n)\right] \times \left[\{v(n) - v(n+1)\} - \{v(n+1) - v(n+2)\}\right]. \quad (3)$$

$F_i^{(2)}(n) \geqslant F_k^{(2)}(n)$, $1 \leqslant n \leqslant N$, and the convexity of v implies that the right side of (3) is non-negative. This proves $E_i(v) \geqslant E_k(v)$. For the converse, consider the particular convex function

$$v(n) = \max(n^* - n + 1, 0)$$

for fixed n^*, $1 \leqslant n^* \leqslant N$. With this function, the right side of (3) reduces to $F_i^{(2)}(n^*) - F_k^{(2)}(n^*)$. Since n^* is arbitrary, this completes the proof.

Proof of Theorem 2. From the right side of (3), when $n \in S$ the value of $F_i^{(2)}(n) - F_k^{(2)}(n)$ is not restricted. The result now follows as in Theorem 1(b).

To make the connection between stochastic dominance and ranked voting systems, if $F_i(N+1) = 1$ then $\{f_i\}$ can be considered the probability function of a discrete random variable X_i, F_i will be the cumulative distribution function, and $E_i(v)$ from (1) is then $E[v(X_i)]$. $F_i(N+1) = 1$ will occur if each vote in Table 1 counts 1/92 unit instead of 1. Clearly this scaling will not affect the relative standing of the candidates. It is not necessary to actually carry out this scaling; it is mentioned here to show the connection with the stochastic dominance theorem which is usually stated in terms of probability distributions.

Appendix B

The Most Valuable Player of the National Basketball Association

This Appendix further discusses the data cited in the text. The example was the voting for Most Valuable Player in 1990 of the National Basketball Association. The voters rank their choices for the top 5 players in the league. The players are then awarded points using a 10-7-5-3-1 scoring function. Sixth place and beyond is assigned zero points. This is an example of a convex scoring function with $N = 5$.

Table 4 presents the results of the Most Valuable Player voting (with 92 voters) for the 1990 season. For each of the 14 players receiving a vote (for ranks 1 through 5), the distribution of votes for each rank is given along with the player's total score. Magic Johnson (candidate A) was voted the Most Valuable Player. Notice that Charles Barkley (candidate B) would have finished in first position had the value of $v(1)$ been greater than 12.

Table 4
The complete results of the Most Valuable Player voting for the National Basketball Association (1990)

Candidate	Name	Rank					
		1	2	3	4	5	Score
A	Magic Johnson	27	38	15	7	4	636
B	Charles Barkley	38	15	16	14	7	614
C	Michael Jordan	21	25	30	8	5	564
D	Karl Malone	2	8	10	23	19	214
E	Patrick Ewing	1	1	11	22	24	162
F	David Robinson	2	4	4	7	13	102
G	Akeem Olajuwon	1	–	4	7	13	64
H	Tom Chambers	–	1	1	–	–	12
I	John Stockton	–	–	1	1	1	9
J	Larry Bird	–	–	–	1	2	5
K	Buck Williams	–	–	–	1	2	5
L	Clyde Drexler	–	–	–	1	–	3
M	Joe Dumars	–	–	–	–	1	1
N	Isiah Thomas	–	–	–	–	1	1

Figure 3a shows the partial ordering for all candidates in the 1990 Most Valuable Player voting using a decreasing scoring function and Figure 3b shows the partial ordering for all convex scoring functions. From Figure 3a, we find that the relative ordering of candidates, A, B, C, E, F, H, I, J, and K would depend on the specific decreasing scoring function chosen. From Figure 3b, only the relative ordering of candidates A, B, E, and F would depend on the specific convex scoring function chosen. For example, if the National Basketball Association had used the same scoring function (14-9-8-7-6-5-4-3-2-1) used by baseball to choose their Most Valuable Player, then Charles Barkley would have finished first by a 935–913 score (based on the votes for the top 5 ranks). In this election, declaring a first place tie between Magic Johnson and Charles Barkley would have been appropriate.

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