Ex-Ante versus Ex-Post Compromise

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Introducing the problem

Setting: Several voters express their preferences over a set of alternatives

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Goal: Find a procedure determining a collective choice that promote a notion of compromise

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- **MC**: MVR and ties are broken according to the quantity of support these receive
- **FB**: bargainers fall back to less and less preferred alternatives until they reach a unanimous agreement
- **q-approval FB**: picks the alternatives which receive the support of q voters at the highest possible quality breaking ties according to the quantity of support

Example 1

$|N| = 100, A = \{a, b, c\}$

51	a	b	c
49	c	b	a

- Plurality: $\{a\}$

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- Plurality: $\{a\}$
- MVR: $\{a\}$
- MC: $\{a\}$
- FB: $\{b\}$
- q-approval FB $q \in \left\{1, ..., \frac{n}{2} + 1\right\}$: $\{a\}$

Example 2

$|N| = 100, A = \{a, b, c, d\}, z \in \{64, ..., 99\}$

26	a	b	c	d
25	c	b	a	d
$\mathbf{z-51}$	d	b	a	c
$\mathbf{100-z}$	d	a	c	b

- Plurality: $\{d\}$

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- Plurality: $\{d\}$
- MVR: for $z < 76$ $\{a, b\}$

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	1°	2°
a	26	$126 - z$
b	0	z
c	25	25
d	49	49

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- Plurality: $\{d\}$
- MVR: for $z < 76$ $\{a, b\}$

	1°	2°
a	26	51
b	0	75
c	25	25
d	49	49

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$\mathbf{z - 51}$	d	b	a	c
$\mathbf{100 - z}$	d	a	c	b

- Plurality: $\{d\}$
- MVR: for $z < 76$ $\{a, b\}$, for $z \geq 76$ $\{b\}$

	1°	2°
a	26	50
b	0	76
c	25	25
d	49	49

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$\mathbf{z - 51}$	d	b	a	c
$\mathbf{100 - z}$	d	a	c	b

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- MVR: for $z < 76$ $\{a, b\}$, for $z \geq 76$ $\{b\}$
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- q-approval FB $q \in \{\frac{n}{2}, ..., z\}$: $\{b\}$

Motivation

$|N| = 2, |A| = 2k + 2$

$\mathbf{i_1}$	$\mathbf{i_2}$
x	b_1
a_1	$\cdot$
$\cdot$	$\cdot$
$\cdot$	b_{k-1}
$\cdot$	y
a_k	x
y	b_k
b_1	a_1
$\cdot$	$\cdot$
$\cdot$	$\cdot$
b_k	a_k

Ex-Ante versus Ex-Post Perspective

ex-ante compromise

imposes over individuals a willingness to compromise but it does not ensure an outcome where everyone has effectively compromised

ex-post compromise

favors an outcome where every voter gives up her most preferred positions if this increases equality

Cardinal Compromise

Setting

A alternatives

N voters

$u_i \in U(A) \subseteq \mathbb{R}^A$ utility function defined over A by the rank of voter i

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$\lambda_i^u(x) = \max_{a \in A} u_i(a) - u_i(x)$ represents the loss of utility for the voter i if the alternative x is elected instead of her favorite one; and $\lambda^u(x)$ represents the vector of these losses

Spread Measure

$\sigma : \mathbb{R}_+^N \longrightarrow \mathbb{R}_+$

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Pure Equality Recognition

$$r_i = r_j \ \forall i, j \in N \Rightarrow \sigma(r) = 0 \quad r \in \mathbb{R}_+^N$$

Spread Measure

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Pure Equality Recognition

$$r_i = r_j \ \forall i, j \in N \Rightarrow \sigma(r) = 0 \quad r \in \mathbb{R}_+^N$$

Pairwise Pareto Dominance

$$[|r_i - r_j| \leq |s_i - s_j| \ \forall i, j \in N] \Rightarrow \sigma(r) \leq \sigma(s) \quad r, s \in \mathbb{R}_+^N$$

Spread Measure

Example

$$\bar{r} = \frac{\sum_{i=1}^{n} r_i}{n}$$

$$\sigma_{avg}(r) = \sum_{i=1}^{n} |\bar{r} - r_i|$$

Spread Measure

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Examples:

$$s = (3, 3, 3, 3) \qquad \sigma_{avg}(s) = \textstyle\sum_{i=1}^{4} (3-3) = 0$$

$$t = (1, 2, 3, 4) \qquad \sigma_{avg}(t) = |2.5-1| + |2.5-2| + |2.5-3| + |2.5-4| = 4$$

$$w = (1, 3, 5, 7) \qquad \sigma_{avg}(w) = |4-1| + |4-3| + |4-5| + |4-7| = 8$$

Cardinal Compromise

$U(A)^N$ set of injective utility functions defined over A

$PO(u)$ set of Pareto optimal alternatives at $u \in U(A)^N$

$\lambda^u(x)$ losses vector when electing the alternative x

σ spread measure

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σ spread measure

$$C^\sigma(u) = \arg\min_{x \in PO(u)} (\sigma \circ \lambda^u)(x) = \{x \in PO(u) : \sigma(\lambda^u(x)) \leq \sigma(\lambda^u(y)), \forall y \in A\}$$

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	i_1	...	i_{51}	i_{52}	...	i_{100}
a	2	...	2	0	...	0
b	1	...	1	1	...	1
c	0	...	0	2	...	2

Cardinal Compromise

Example

$|N| = 100, A = \{a, b, c\}$

51	a	b	c
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$$\begin{array}{rcccccc} & i_1 & \dots & i_{51} & i_{52} & \dots & i_{100} \\ \lambda(a) = (& 0, & \dots & 0, & 2, & \dots & 2) \\ \lambda(b) = (& 1, & \dots & 1, & 1, & \dots & 1) \\ \lambda(c) = (& 2, & \dots & 2, & 0, & \dots & 0) \end{array}$$

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\lambda(c) = (& 2, & \dots & 2, & 0, & \dots & 0)
\end{array}
$$

$\sigma_{avg}(\lambda(a)) = 99.96, \quad \sigma_{avg}(\lambda(b)) = 0, \quad \sigma_{avg}(\lambda(c)) = 100.04$

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\lambda(b) = (& 1, & \dots & 1, & 1, & \dots & 1) \\
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\end{array}
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$$\sigma_{avg}(\lambda(a)) = 99.96, \quad \sigma_{avg}(\lambda(b)) = 0, \quad \sigma_{avg}(\lambda(c)) = 100.04$$

$$C^{\sigma_{avg}}(u) = \arg\min_{x \in PO(u)} (\sigma \circ \lambda^u)(x) = b$$

Ordinal Compromise

$P_i \in L(A)$	linear order over A which represents the preference of $i \in N$
$v : \{1, ..., m\} \to \mathbb{R}$	utility assignment
$v_{P_i} \in U(A)$	utility function for $P_i \in L(A)$ induced by v
$\mathbf{v} : L(A)^N \to U(A)^N$	function mapping a profile of ordinal preferences to a utility profile

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$$(C^\sigma \circ \mathbf{v})(\{P_i, i \in N\}) = C^\sigma(\{v_{P_i}, i \in N\})$$

Ordinal Compromise

$\Sigma^{\mathsf{All}} = \mathbb{R}_+^{\mathbb{R}_+^N}$ the set of all spread measures

UA-independence

A class of spread measure $\Sigma \subseteq \Sigma^{\mathsf{All}}$ is UA-independent iff, given any $\sigma \in \Sigma$ and any two UAs v and v', there exists a $\sigma' \in \Sigma$ such that $C^{\sigma} \circ \mathbf{v} = C^{\sigma'} \circ \mathbf{v}'$

Ordinal Compromise

UA-independence

$\Sigma^{\text{PPd}} \subseteq \Sigma^{\text{All}}$ the class of spread measures that satisfy PPd

Proposition 1:

Σ^{PPd} is not UA-independent

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$$
\begin{array}{lcccccccc}
\mathbf{i_1} & x & a_1 & a_2 & a_3 & y & b_1 & b_2 & b_3 \\
\mathbf{i_2} & b_1 & b_2 & y & x & b_3 & a_1 & a_2 & a_3
\end{array}
$$

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$\mathbf{i_1}$	x	a_1	a_2	a_3	y	b_1	b_2	b_3
$\mathbf{i_2}$	b_1	b_2	y	x	b_3	a_1	a_2	a_3

$k =$	1	2	3	4	5	6	7	8
$v(k)$	7	6	5	4	3	2	1	0
$v'(k)$	1000	999	998	997	3	2	1	0

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Proposition 1:

Σ^{PPd} is not UA-independent

	$v_{P_1}(\cdot)$	$v_{P_2}(\cdot)$	$\lambda^P(\cdot)$
x	7	4	$(0,3)$
y	3	5	$(4,2)$
a_1	6	2	$(1,5)$
a_2	5	1	$(2,6)$
a_3	4	0	$(3,7)$
b_1	2	7	$(5,0)$
b_2	1	6	$(6,1)$
b_3	0	3	$(7,4)$

$C^\sigma(v) \in \{y\}$

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	$v'_{P_1}(\cdot)$	$v'_{P_2}(\cdot)$	$\lambda'^{P}(\cdot)$
x	1000	997	$(0, 3)$
y	3	998	$(997, 2)$
a_1	999	2	$(1, 998)$
a_2	998	1	$(2, 999)$
a_3	997	0	$(3, 1000)$
b_1	2	1000	$(998, 0)$
b_2	1	999	$(999, 1)$
b_3	0	3	$(1000, 997)$

$C^{\sigma}(v') \in \{x, b_3\}$

Ordinal Compromise

UA-independence

$\Sigma_{\text{threshold}} = \{\sigma^k, k \in \mathbb{R}\}$ where $\sigma^k(\lambda) = \#\{i \in N \mid \lambda_i \geq k\}$

Proposition 2:

$\Sigma_{\text{threshold}}$ is UA-independent

Ordinal Compromise

UA-independence

Proposition 3:

$\sigma^k \in \Sigma_{\text{threshold}}$ fails Pure Equality Recognition and Pairwise Pareto Dominance

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$$\lambda(x) = (4, 4, 4, 4) \qquad \sigma^3(\lambda(x)) = 4$$

Ordinal Compromise

UA-independence

Proposition 3:

$\sigma^k \in \Sigma_{\text{threshold}}$ fails Pure Equality Recognition and Pairwise Pareto Dominance

$\mathbf{i_1}$	a	d	b	c
$\mathbf{i_2}$	b	c	d	a

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v assigns the utility values $10, 2, 1, 0$ respectively to the ranks $1, 2, 3, 4$

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v assigns the utility values $10, 2, 1, 0$ respectively to the ranks $1, 2, 3, 4$

$$
\begin{array}{lcl c c}
 & & & & \sigma^1(\lambda^P(\cdot)) \\
\lambda^P(a) & = & (\ 0 \quad 10\ ) & & 1 \\
\lambda^P(b) & = & (\ 9 \quad 0\ ) & & 1 \\
\lambda^P(c) & = & (\ 10 \quad 8\ ) & & 2 \\
\lambda^P(d) & = & (\ 8 \quad 9\ ) & & 2
\end{array}
$$

Questions

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- ...

Appendix I
Minimal Liberty

A social planner must choose between a world x where individuals may sell their organs, and a world y where they do not

	u_1	u_2
x	1	100
y	0	0

Even though y is Pareto dominated, the social planner might prefer y to x

Thank You!

S. J. Brams and D. M. Kilgour.
Fallback bargaining.
Group Decision and Negotiation, 10(4):287–316, Jul 2001.

J. F. Nash.
The bargaining problem.
18:155, 1950.

Y. Chun and H. Peters.
The lexicographic equal-loss solution.
22:151–161, 1991.

V. Merlin, İpek Özkal Sanver, and M. R. Sanver.
Compromise rules revisited.
28:63–78, 2019.

M. R. Sertel and B. Yılmaz.
The majoritarian compromise is majoritarian-optimal and subgame-perfect implementable.
16:615–627, 1999.

W. Thomson.
Cooperative models of bargaining.
Handbook of game theory with economic applications, 2:1237–1284, 1994.

A. Sen.
Rationality and Freedom.
12:382–407, 2004.