Bonus 3: $\int_{-1}^{1} P_n(x) P_n(x) = \dfrac{2}{2n+1}$

use generating function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

multiply by generating function again (but in sigma notation in R.H.S)

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_n(x) P_m(x) t^{n+m}$$

now integrate both sides from -1 to 1 with respect to x

$$\int_{-1}^{1} \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} t^{n+m} \int_{-1}^{1} P_n(x) P_m(x) \, dx$$

use orthogonality: $\int_{-1}^{1} P_n(x) P_m(x) = 0$ where $m \neq n$

$$\int_{-1}^{1} \frac{1}{1-2xt+t^2} dt = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} \left[P_n(x) \right]^2$$

$$L.H.S = \int_{-1}^{1} \frac{1}{1-2xt+t^2} \, dx \doteq \int_{-1}^{1} \frac{1}{(1+t^2)-2xt} = \frac{-1}{2t} \int_{-1}^{1} \frac{-2t}{(1+t^2)-2xt}$$

$$= \frac{-1}{2t} \left[\ln(1-2xt+t^2) \right]_{-1}^{1} \doteq \frac{-1}{2t} \left[\ln(1-t+t^2) - \ln((1+t)^2) \right]$$

$$= \frac{-1}{2t} \left[2\ln(1-t) - 2\ln(1+t) \right] \doteq \frac{-1}{t} \left[\ln(1-t) - \ln(1+t) \right]$$