A Quantum Model for Sovling HCP and TSP Graph Problem Based on Quantum Z_2 Lattice Gauge Theory

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Abstract

In this paper, using the condensed closed string characteristics of Z_2 topological quantum phase transition, by mapping a graph to a lattice we develop a novel quantum model and a adiabatic algorithm to solve the shortest path, HCP and TSP graph problems, which only need N_e qubits for a given graph (the number of vertices and edges is N_v, N_e). By the simulation of a number of small random graphs we demonstrate the effectiveness of this quantum algorithm model.

I. HAMILTONIAN MODEL OF GRAPH PROBLEM BASED ON Z_2 GAUGE THEORY

The Hamiltonian of the quantum Z_2 gauge theory[1–3] is defined as

$$\begin{aligned} H &= Z + gX \\ Z &= \sum_{\Box} Z_{\Box},\ Z_{\Box} = -\prod_{l\in\Box} \sigma_l^z \\ X &= -\sum_{l} \sigma_l^x, \end{aligned} \tag{1}$$

where the $\Box$ indicates the elementary plaquette of the lattice, which is the mapping of the vertex from the graph.

For graph problem, with the dual mapping from graph to Z_2-lattice[4], the Z_2 Graph Hamiltionian can be defined as

$$H_{Z_2-G} = H_{problem} + gX \qquad or \qquad H_{Z_2-G} = \lambda H_{problem} + X \tag{2}$$

Here are three problems. (1)The Shortest Path of Two Vertex,(2)HCP, (3)TSP

A. The Shortest Path of Two Vertex

First,we need add a edge l_0 between starting vertex and end vertex with $w_{l_0} = 0$ and set $\sigma_{l_0}^z = -1$. Then the Hamiltonian of the shortest path problem can be defined as

$$\begin{aligned} H_{hcp} &= Z + W \\ Z &= \sum_{\Box} Z_{\Box}, \quad Z_{\Box} = -\prod_{l\in\Box} \sigma_l^z \\ W &= -\sum_{l\in\Box} w_l \sigma_l^z \end{aligned} \tag{3}$$

Meanings of items:

1) $\sigma_l^z = -1$ means that this edge l is selected by a path.

2) Z item : the $\Box$ indicates the elementary plaquette of the lattice, or a vertex of its mapping graph. $d_{\Box}$ represents the degre of the vertex $\Box$. Z item restricts that the paht is a loop in the graph.

3) W item reduce the number of visited vertexs and the weight sum of the path.

B. HCP

The Hamiltonian of HCP can be defined as

$$\begin{aligned} H_{hcp} &= Z + V + fF \\ Z &= \sum_{\square} Z_{\square}, \quad Z_{\square} = -\prod_{l\in\square} \sigma_l^z \\ V &= \sum_{\square} V_{\square}, \quad V_{\square} = (\sum_{l\in\square} \sigma_l^z - c_{\square})^2, \quad c_{\square} = d_{\square} - 4 \\ F &= ? \end{aligned} \tag{4}$$

Meanings of items:

1) $\sigma_l^z = -1$ means that this edge l is selected by a path.

2) the $\square$ indicates the elementary plaquette of the lattice, or a vertex of its mapping graph. $d_{\square}$ represents the degre of the vertex $\square$. Z item restricts that the paht is a loop in the graph.

3) V item restricts a single visit for each vertex in the graph(1-visit loop).

4) F facilitate the fusion of split 1-visit loops into one single loop. $f = 0 - 1$. F is in consideration, which should reduce the number of subloop.

C. TSP

The Hamiltonian of TSP can be defined as

$$\begin{aligned} H_{tsp} &= H_{hcp} + W \\ W &= -\sum_{l} w_l \sigma_l^z \end{aligned} \tag{5}$$

Meanings of items:

1) W represents the minimization of the sum of path weights. w_l represents the weight of the edge l.

Simplify Hamiltonian H_{hcp}, H_{tsp}

$$d_\square = c_\square + 4 \tag{6}$$

$$\begin{aligned}
V_\square &= (\sum_{l\in\square}\sigma_l^z - c_\square)^2 \\
&= (\sum_{l\in\square}\sigma_l^z)^2 - 2c_\square\sum_{l\in\square}\sigma_l^z + c_\square^2 \\
&= [(\sum_{l\in\square}(\sigma_l^z)^2) + (\sum_{l,m\in\square,l\neq m}\sigma_l^z\sigma_m^z)] - 2c_\square\sum_{l\in\square}\sigma_l^z + c_\square^2, \qquad (\sigma_l^z)^2 = +1 \\
&= [d_\square + (\sum_{l,m\in\square,l\neq m}\sigma_l^z\sigma_m^z)] - 2c_\square\sum_{l\in\square}\sigma_l^z + c_\square^2 \\
&= (\sum_{l,m\in\square,l\neq m}\sigma_l^z\sigma_m^z) - 2c_\square\sum_{l\in\square}\sigma_l^z + [c_\square^2 + c_\square + 4] \\
V &= \sum_\square V_\square = \sum_\square(\sum_{l,m\in\square,l\neq m}\sigma_l^z\sigma_m^z) - 2c_\square\sum_{l\in\square}\sigma_l^z + [c_\square^2 + c_\square + 4]
\end{aligned} \tag{7}$$

Remove the constant term, the Hamiltonian of HCP is

$$\begin{aligned}
V_\square &= (\sum_{l,m\in\square,l\neq m}\sigma_l^z\sigma_m^z) - 2c_\square\sum_{l\in\square}\sigma_l^z \\
V &= \sum_\square V_\square = \sum_\square[(\sum_{l,m\in\square,l\neq m}\sigma_l^z\sigma_m^z) - 2c_\square\sum_{l\in\square}\sigma_l^z] \\
H_{hcp} &= \sum_\square -\prod_{l\in\square}\sigma_l^z + \sum_\square[(\sum_{l,m\in\square,l\neq m}\sigma_l^z\sigma_m^z) - 2c_\square\sum_{l\in\square}\sigma_l^z] + fF
\end{aligned} \tag{8}$$

For TSP, we neeed add the weight of the edge w_l. So the Hamiltonian of TSP is

$$H_{tsp} = H_{hcp} - \sum_l w_l\sigma_l^z \tag{9}$$

When g is equal to 0, the ground state of the system is the optimal solution to the HCP or TSP problem. Obviously, The optimal solution can be obtained by gradually increasing λ from initial Hamiltonian $-\sum_l \sigma_l^x$ adiabatically. The adiabatic algorithm can refer to the article "Sovling Hamiltonian Cycle Problem with Quantum Z2 Lattice Gauge Theory".

II. QUANTUM SIMULATION OF HCP MODEL IN 3×3 TORUS LATTICE

Quantum simulations(the 3*3 torus lattice) verify that above model works correctly as shown in FIG.1. (1) Loop state condition: $< Z >= -N_v$; (2) 1-visit loop state condition: $< V >= 0$.

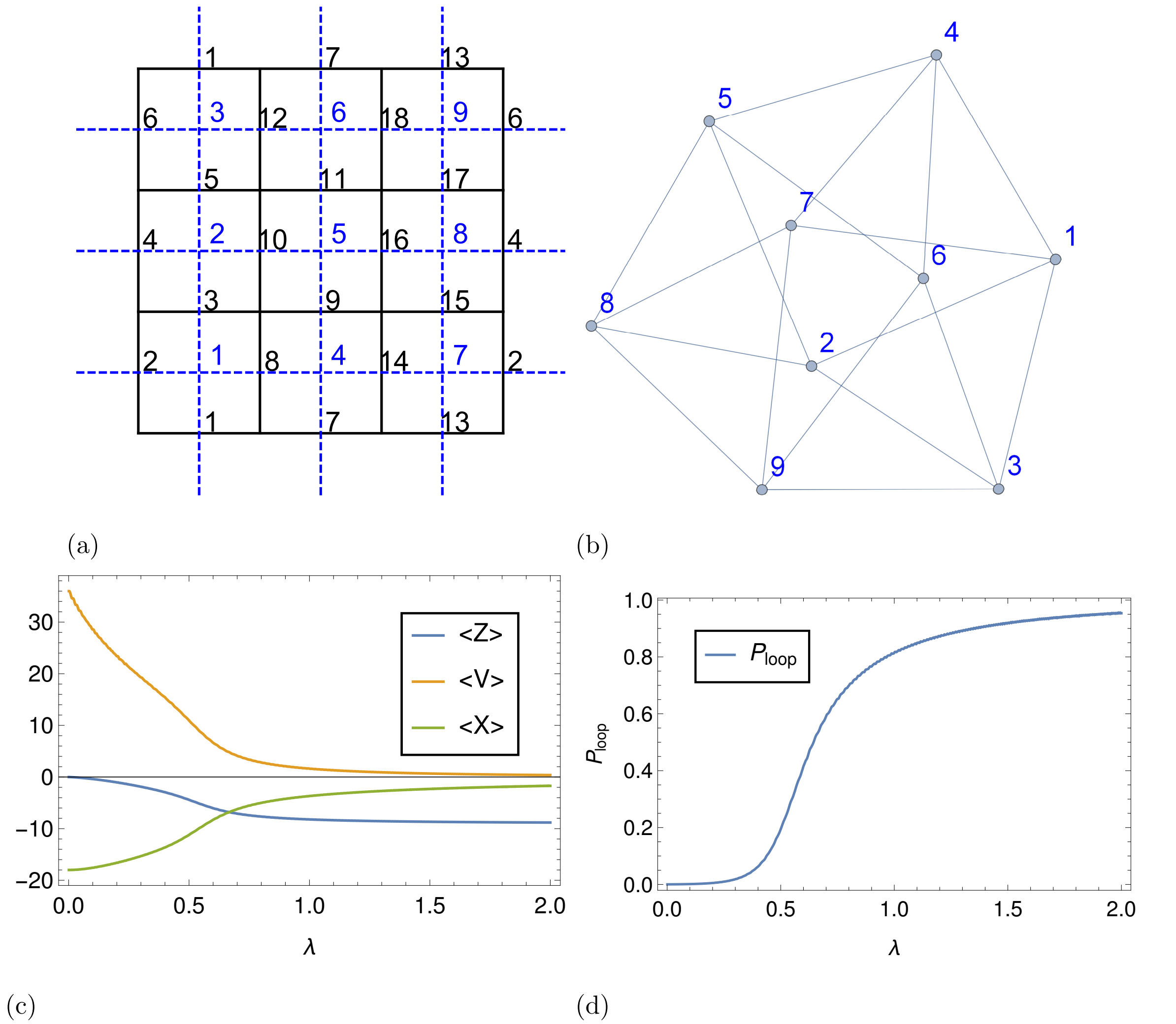

FIG. 1. Simulation results. (a) The 3*3 torus lattice. (b) The graph corresponding to the 3*3 tours lattice. (c)$f = 0$, the simulation results of H_{Z_2-hcp}. (d)$f = 0$, the probability of 1-visit loop state P_{loop} vs λ.

III. CONCLUSION AND DISCUSSION

In this paper, using the condensed close string characteristics of Z_2 topological quantum phase transition, by mapping a graph to a lattice we develop a novel quantum model and a adiabatic algorithm to solve the shortest path, HCP and TSP graph problem, which only need N_e qubits (the number of vertices and edges is N_v, N_e). Using this algorithm model, by the simulation of a number of small random graphs we demonstrate its effectiveness.

IV. ACKNOWLEDGE

[1] X. Cui, Y. Shi, and J. C. Yang, "Circuit-based digital adiabatic quantum simulation and pseudoquantum simulation as new approaches to lattice gauge theory," J. High Energy Phys **2020**, 160 (2020).

[2] S. Sachdev, "Topological order, emergent gauge fields, and fermi surface reconstruction," Rep. Prog. Phys. **82**, 014001 (2018).

[3] F. J. Wegner, "Duality in generalized ising models and phase transitions without local order parameters," J. Math. Phys. **12**, 2259 (1971).

[4] Xiaopeng Cui and Yu Shi, "Solving Hamiltonian Cycle Problem using Quantum $\mathbb{Z}_2$ Lattice Gauge Theory," arXiv e-prints , arXiv:2202.08817 (2022), arXiv:2202.08817 [quant-ph].