A Quantum Model for Sovling HCP and TSP Graph Problem Based on Quantum Z_2 Lattice Gauge Theory

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Abstract

In this paper, using the condensed closed string characteristics of Z_2 topological quantum phase transition, by mapping a graph to a lattice we develop a novel quantum model and a adiabatic algorithm to solve the shortest path, HCP and TSP graph problems, which only need N_e qubits for a given graph (the number of vertices and edges is N_v, N_e). By the simulation of a number of small random graphs we demonstrate the effectiveness of this quantum algorithm model.

I. HAMILTONIAN MODEL OF GRAPH PROBLEM BASED ON Z_2 GAUGE THEORY

The Hamiltonian of the quantum Z_2 gauge theory[1–3] is defined as

$$\begin{aligned} H &= Z + gX \\ Z &= \sum_{\square} Z_{\square},\ Z_{\square} = -\prod_{l\in\square}\sigma_l^z \\ X &= -\sum_{l}\sigma_l^x, \end{aligned} \tag{1}$$

where the $\square$ indicates the elementary plaquette of the lattice, which is the mapping of the vertex from the graph.

For Graph problem, the Z_2 Graph Hamiltonian can be defined as

$$H_{Z_2-G} = H_{problem} + gX \qquad or \qquad H_{Z_2-G} = \lambda H_{problem} + X \tag{2}$$

Here are three problems. (1)The Shortest Path of Two Vertex,(2)HCP, (3)TSP

A. The Shortest Path of Two Vertex

First,we need add a edge l_0 between starting vertex and end vertex with $w_{l_0} = 0$ and set $\sigma_{l_0}^z = -1$. Then the Hamiltonian of the shortest path problem can be defined as

$$\begin{aligned} H_{hcp} &= Z + W \\ Z &= \sum_{\square} Z_{\square}, \quad Z_{\square} = -\prod_{l\in\square}\sigma_l^z \\ W &= -\sum_{l\in\square} w_l\sigma_l^z \end{aligned} \tag{3}$$

Meanings of items:

1) $\sigma_l^z = -1$ means that this edge l is selected by a path.

2) Z item : the $\square$ indicates the elementary plaquette of the lattice, or a vertex of its mapping graph. $d_{\square}$ represents the degre of the vertex $\square$. Z item restricts that the paht is a loop in the graph.

3) W item reduce the number of visited vertexs and the weight sum of the path.

B. HCP

The Hamiltonian of HCP can be defined as

$$
\begin{aligned}
H_{hcp} &= Z + V + fF \\
Z &= \sum_{\square} Z_{\square}, \quad Z_{\square} = -\prod_{l\in\square} \sigma_l^z \\
V &= \sum_{\square} V_{\square}, \quad V_{\square} = (\sum_{l\in\square} \sigma_l^z - c_{\square})^2, \quad c_{\square} = d_{\square} - 4 \\
F &= ?
\end{aligned}
\tag{4}
$$

Meanings of items:

1) $\sigma_l^z = -1$ means that this edge l is selected by a path.

2) the $\square$ indicates the elementary plaquette of the lattice, or a vertex of its mapping graph. $d_{\square}$ represents the degre of the vertex $\square$. Z item restricts that the paht is a loop in the graph.

3) V item restricts a single visit for each vertex in the graph(1-visit loop).

4) F facilitate the fusion of split 1-visit loops into one single loop. $f = 0 - 1$. F is in consideration, which should reduce the number of subloop.

C. TSP

The Hamiltonian of TSP can be defined as

$$
\begin{aligned}
H_{tsp} &= H_{hcp} + W \\
W &= -\sum_{l} w_l \sigma_l^z
\end{aligned}
\tag{5}
$$

Meanings of items:

1) W represents the minimization of the sum of path weights. w_l represents the weight of the edge l.

Simplify Hamiltonian H_{hcp}, H_{tsp}

$$d_\Box = c_\Box + 4 \tag{6}$$

$$\begin{aligned}
V_\Box &= (\sum_{l\in\Box} \sigma_l^z - c_\Box)^2 \\
&= (\sum_{l\in\Box} \sigma_l^z)^2 - 2c_\Box \sum_{l\in\Box} \sigma_l^z + c_\Box^2 \\
&= [(\sum_{l\in\Box} (\sigma_l^z)^2) + (\sum_{l,m\in\Box,l\neq m} \sigma_l^z \sigma_m^z)] - 2c_\Box \sum_{l\in\Box} \sigma_l^z + c_\Box^2, \qquad (\sigma_l^z)^2 = +1 \\
&= [d_\Box + (\sum_{l,m\in\Box,l\neq m} \sigma_l^z \sigma_m^z)] - 2c_\Box \sum_{l\in\Box} \sigma_l^z + c_\Box^2 \\
&= (\sum_{l,m\in\Box,l\neq m} \sigma_l^z \sigma_m^z) - 2c_\Box \sum_{l\in\Box} \sigma_l^z + [c_\Box^2 + c_\Box + 4] \\
V &= \sum_\Box V_\Box = \sum_\Box (\sum_{l,m\in\Box,l\neq m} \sigma_l^z \sigma_m^z) - 2c_\Box \sum_{l\in\Box} \sigma_l^z + [c_\Box^2 + c_\Box + 4]
\end{aligned} \tag{7}$$

Remove the constant term, the Hamiltonian of HCP is

$$\begin{aligned}
V_\Box &= (\sum_{l,m\in\Box,l\neq m} \sigma_l^z \sigma_m^z) - 2c_\Box \sum_{l\in\Box} \sigma_l^z \\
V &= \sum_\Box V_\Box = \sum_\Box [(\sum_{l,m\in\Box,l\neq m} \sigma_l^z \sigma_m^z) - 2c_\Box \sum_{l\in\Box} \sigma_l^z] \\
H_{hcp} &= \sum_\Box - \prod_{l\in\Box} \sigma_l^z + \sum_\Box [(\sum_{l,m\in\Box,l\neq m} \sigma_l^z \sigma_m^z) - 2c_\Box \sum_{l\in\Box} \sigma_l^z] + fF
\end{aligned} \tag{8}$$

For TSP, we neeed add the weight of the edge w_l. So the Hamiltonian of TSP is

$$H_{tsp} = H_{hcp} - \sum_l w_l \sigma_l^z \tag{9}$$

When g is equal to 0, the ground state of the system is the optimal solution to the HCP or TSP problem. Obviously, The optimal solution can be obtained by gradually increasing λ from initial Hamiltonian $-\sum_l \sigma_l^x$ adiabatically. The adiabatic algorithm can refer to the article ”Sovling Hamiltonian Cycle Problem with Quantum Z2 Lattice Gauge Theory”.

II. QUANTUM SIMULATION OF HCP MODEL IN 3×3 TORUS LATTICE

Quantum simulations(the 3*3 torus lattice) verify that above model works correctly as shown in FIG.1. (1) Loop state condition: $<Z>=-N_v$; (2) 1-visit loop state condition: $<V>=0$.

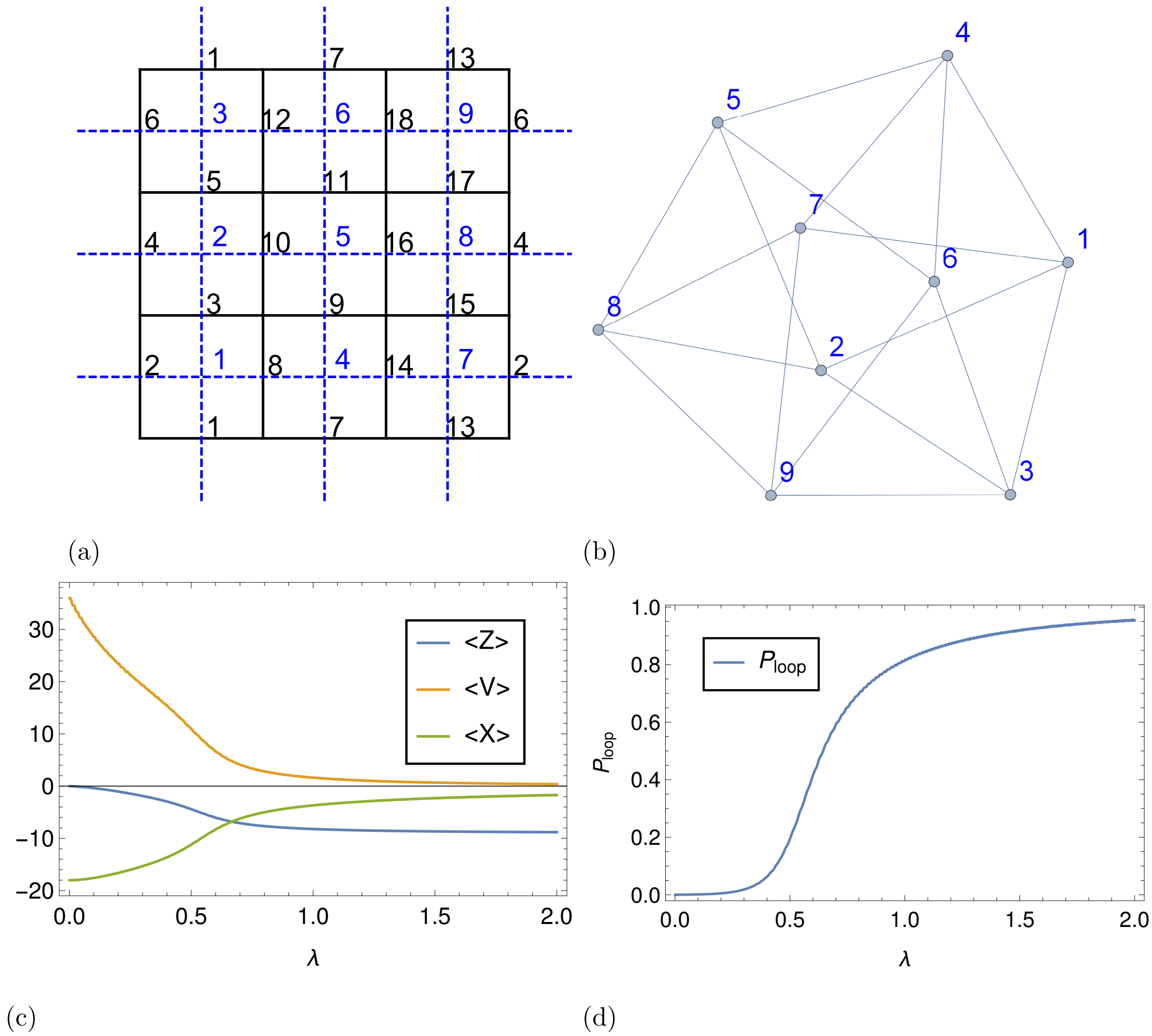

FIG. 1. Simulation results. (a) The 3*3 torus lattice. (b) The graph corresponding to the 3*3 tours lattice. (c)$f=0$, the simulation results of H_{Z_2-hcp}. (d)$f=0$, the probability of 1-visit loop state P_{loop} vs λ.

III. CONCLUSION AND DISCUSSION

In this paper, using the condensed close string characteristics of Z_2 topological quantum phase transition, by mapping a graph to a lattice we develop a novel quantum model and a adiabatic algorithm to solve the shortest path, HCP and TSP graph problem, which only need N_e qubits (the number of vertices and edges is N_v, N_e). Using this algorithm model, by the simulation of a number of small random graphs we demonstrate its effectiveness.

IV. ACKNOWLEDGE

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