On-line size Ramsey number for monotone k-uniform ordered paths with uniform looseness.

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joint work with

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and

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Notation: H, G (k-uniform) hypergraphs

$H \xrightarrow[t]{} G$ (H forces G)

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Ramsey numbers:

$R_t(G) = \min\{n : \exists H \text{ on } n \text{ vertices}, H \xrightarrow[t]{} G\}$ (vertex Ramsey)

$\hat{R}_t(G) = \min\{m : \exists H \text{ on } m \text{ edges}, H \xrightarrow[t]{} G\}$ (size Ramsey)

(Similar for other parameters)

Ordered version:

Vertices of H, G are ordered.

Only consider monochromatic copies of G that respect the order.

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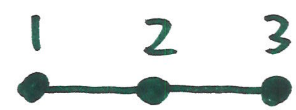
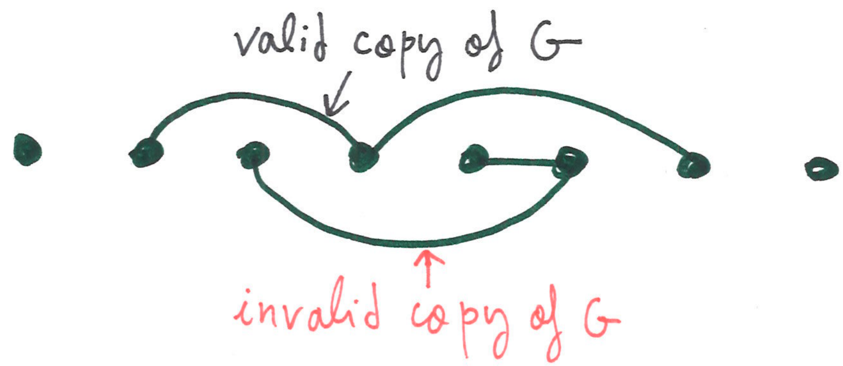
E.g.

$G = P_2$ (monotone path)



H



Game interpretation (Off-line setting)

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  - Builder: Presents a graph H
  - Painter: Colours $E(H)$ with t colours trying to avoid monochromatic G.
- Builder wins if she can force monochromatic G
- Painter wins otherwise.
- Ramsey number: smallest parameter of H with which Builder wins.

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Note:

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- Infinite (or suff. large) board (i.e. there are always available new vertices)

On-line (size) Ramsey number:

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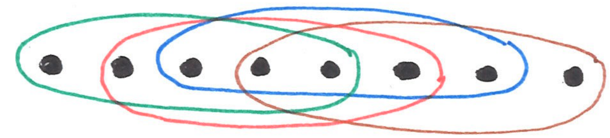
Easy claim:

$$\tilde{R}_t(G) \leqslant \hat{R}_t(G) \leqslant \binom{R_t(G)}{k}$$

k-uniform monotone tight path:

$P_m^{(k)}$ m k-edges

$n = m + k - 1$ vertices



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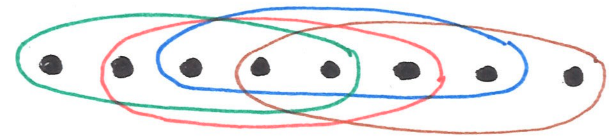

Thm (Moshkovitz, Shapira 2014):

$$R_t(P_m^{(k)}) = |Q_k| + 1, \quad \text{where}$$

$$\underbrace{2^{\cdot^{\cdot^{2}}}}_{k-2}{}^{(m^{t-1}/2\sqrt{t})} \le |Q_k| \le \underbrace{2^{\cdot^{\cdot^{2}}}}_{k-2}{}^{(2m^{t-1})}$$

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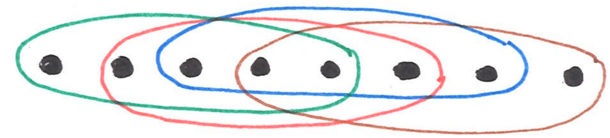

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(They use high-dimensional integer partitions)

Interpretation of Q_k

$Q_1, \ldots, Q_k$ are posets. $Q_{i+1} = \left\{ \begin{array}{l} \text{downsets of } Q_i \\ \text{ordered by } \subseteq \end{array} \right\}$

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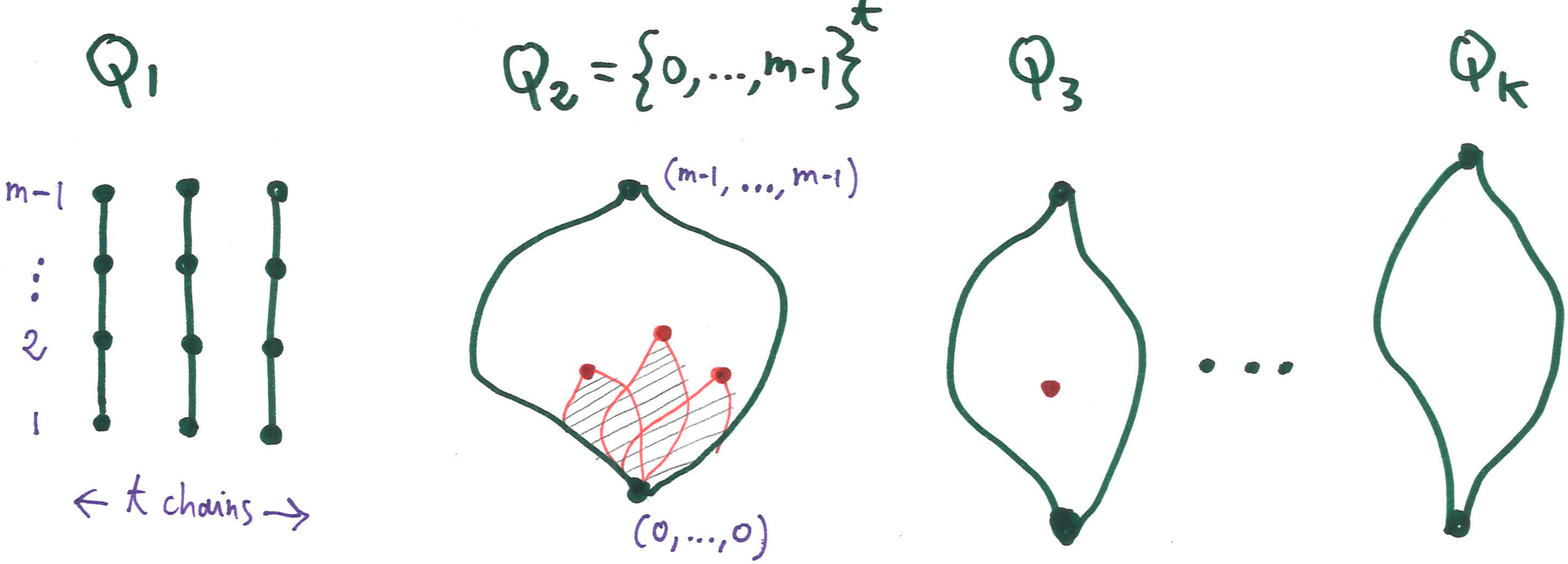

<u>Thm</u> (PG, Prałat, West 2017+):

$$\frac{|Q_k|}{k \lg |Q_k|} \leq \tilde{R}_t\left(P_m^{(k)}\right) \leq |Q_k| \left(\lg |Q_k|\right)^{2+\varepsilon}$$

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Extensions:

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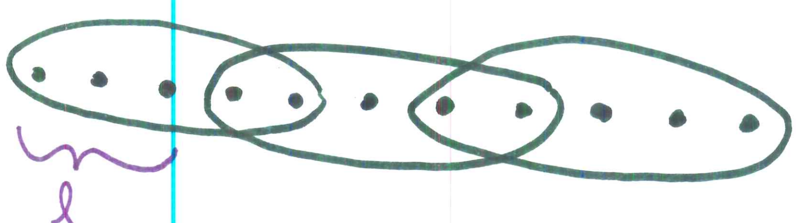
- ℓ-loose monotone paths ($1 \leq \ell \leq k$)



3-loose path

Graph case $(k=2)$

$$\frac{|A|}{2} \leq \tilde{R}_t(P_m^{(2)}) \leq |Q_2|\left((m-1)t+1\right)+2,$$

where $Q_2 = \{0,\ldots,m-1\}^t$, A largest antichain in Q_2

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In particular,

$$\frac{m^{t-1}}{3\sqrt{t}} \leq \tilde{R}_t(P_m^{(2)}) < t\,m^{t+1}$$

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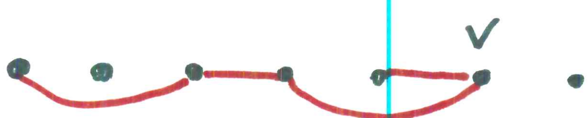
$\ell(v) = (\ell_1, \ell_2, \dots, \ell_t)$ if longest path in color i



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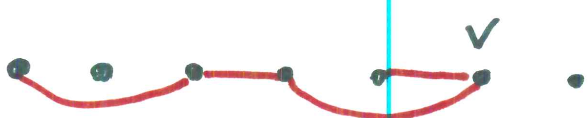

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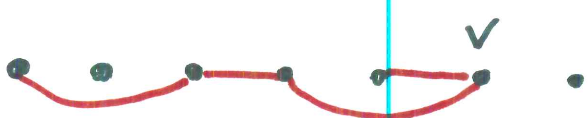

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Labels updated as: new edges are added by Builder & coloured by Painter.

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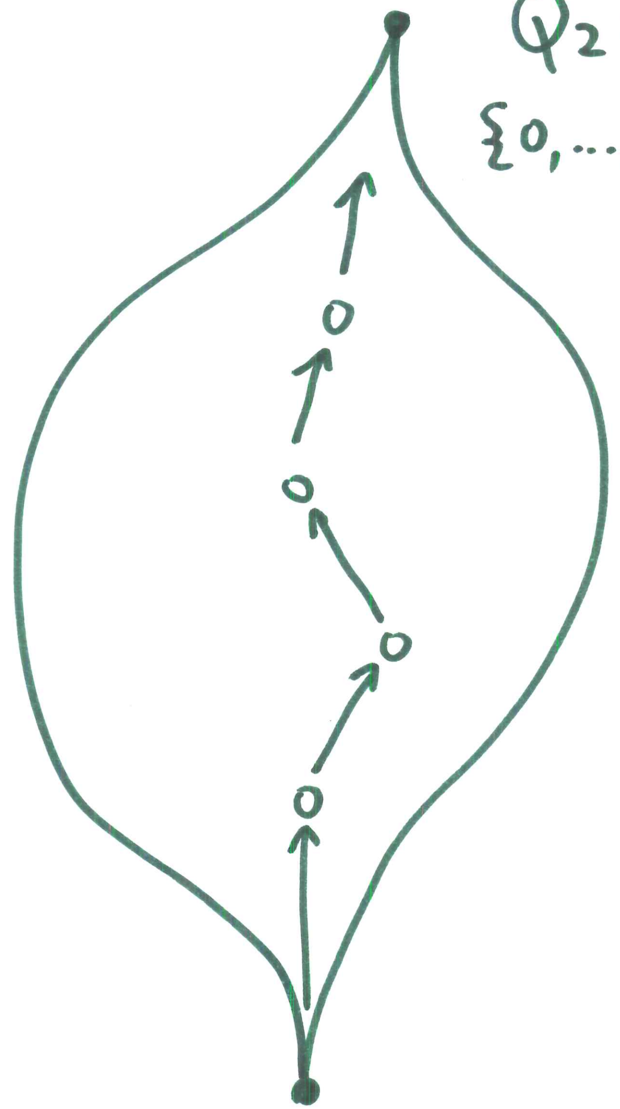

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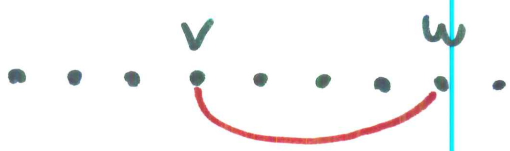

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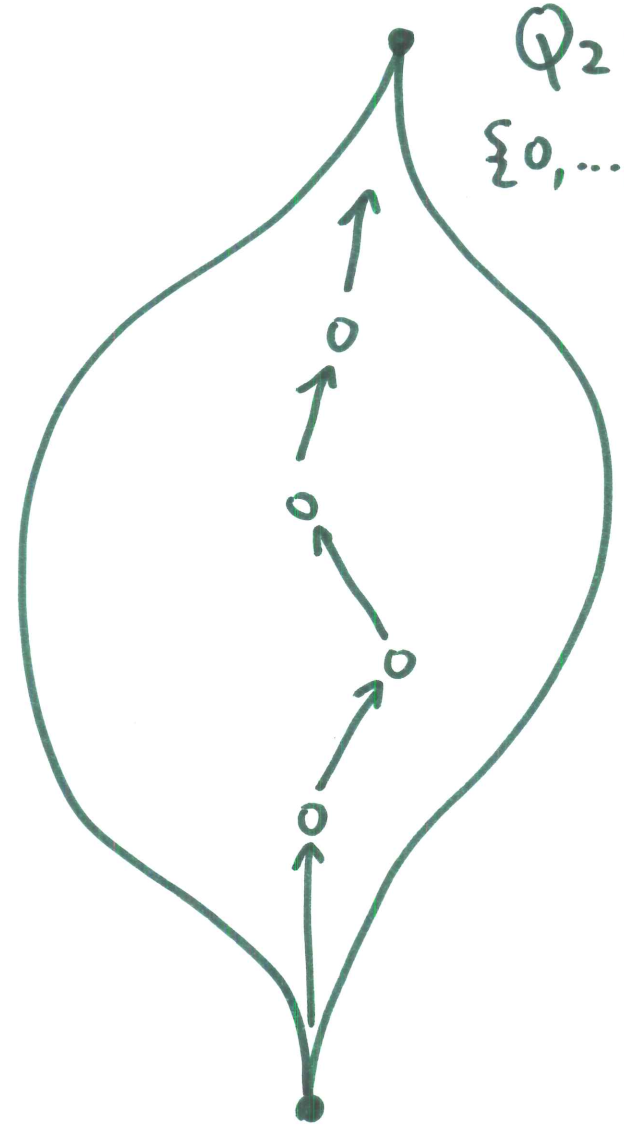

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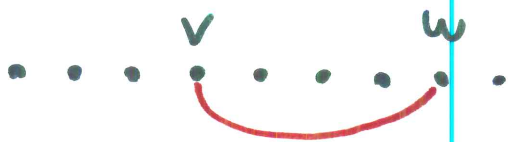

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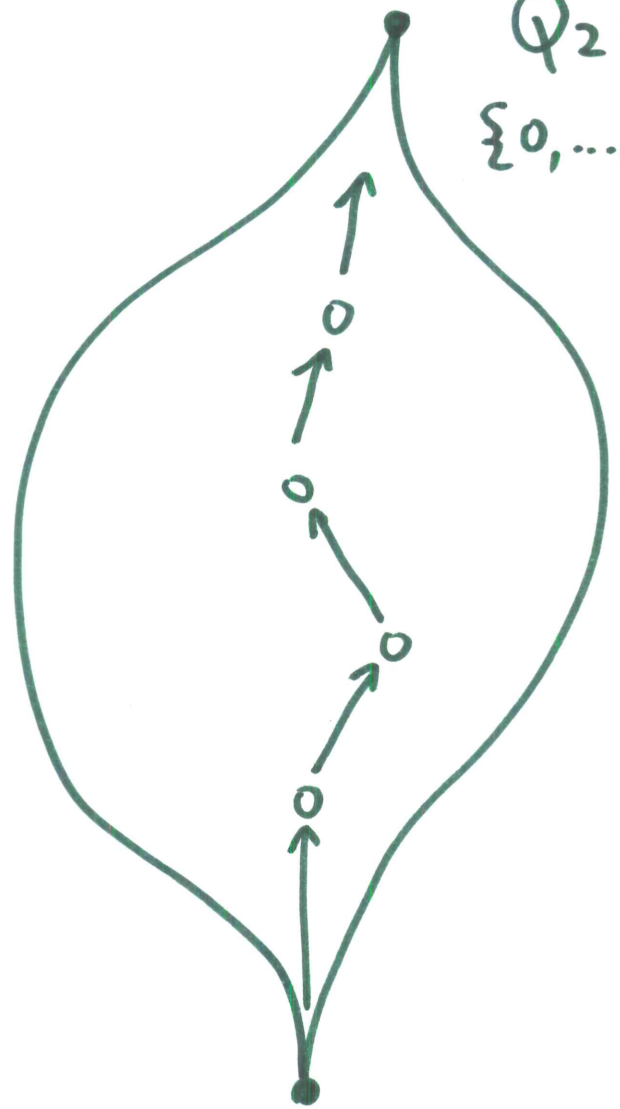

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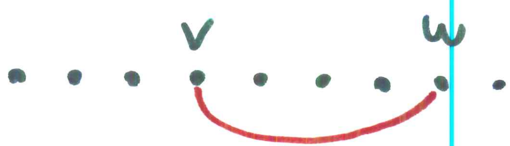

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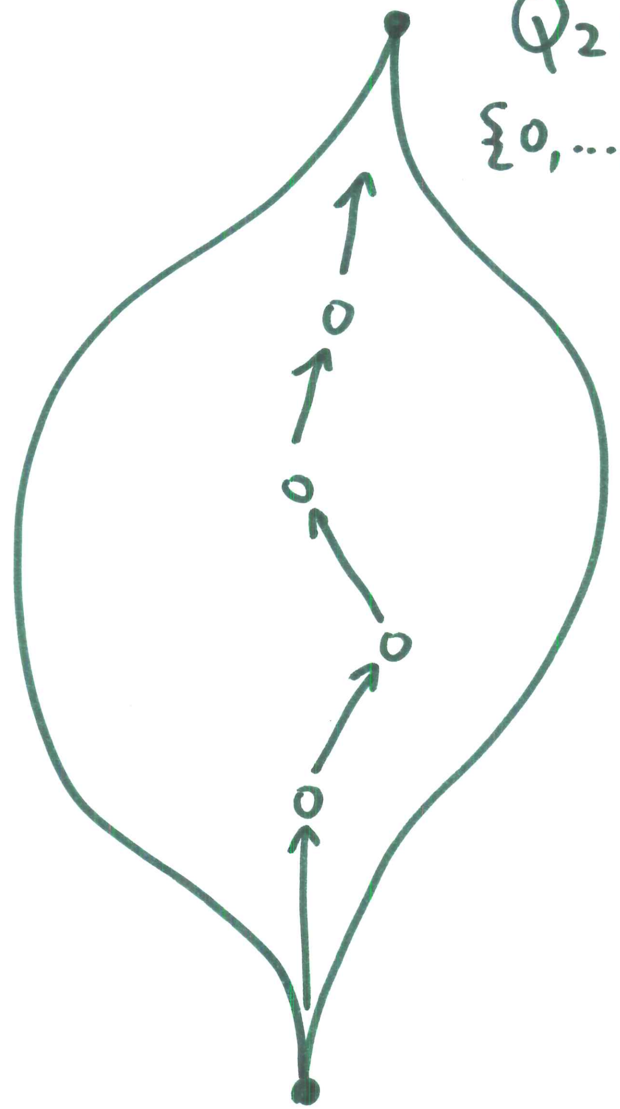

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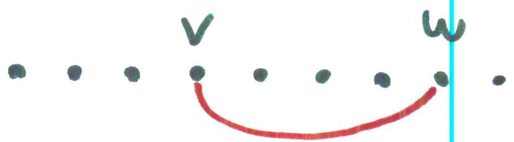

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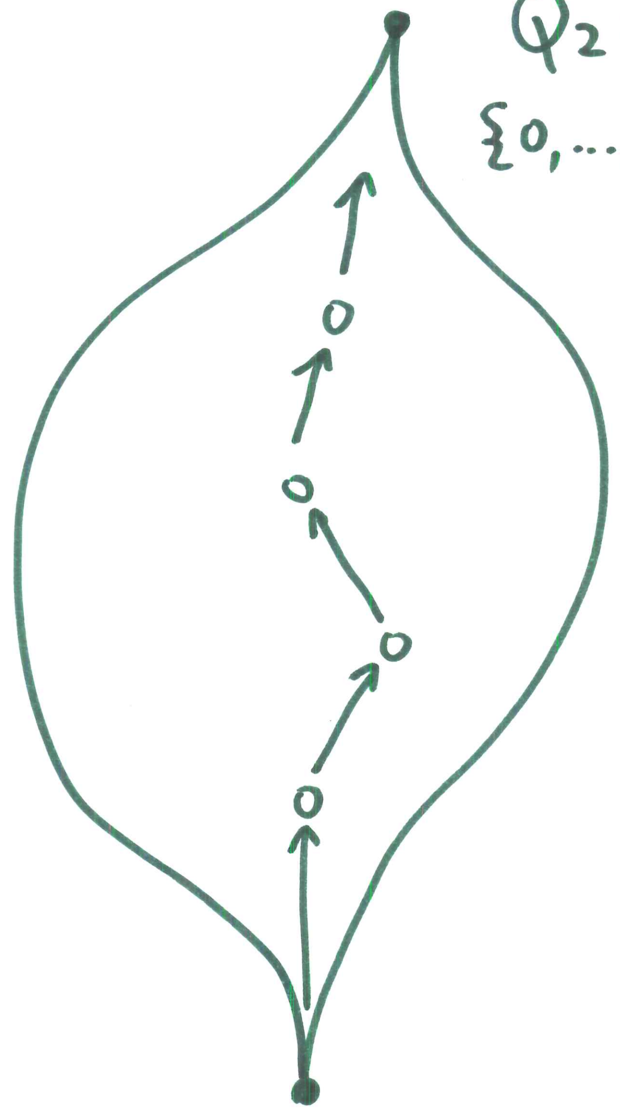

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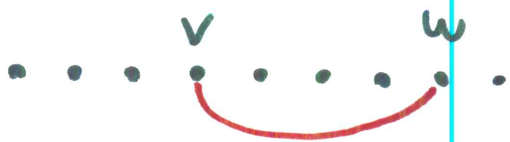

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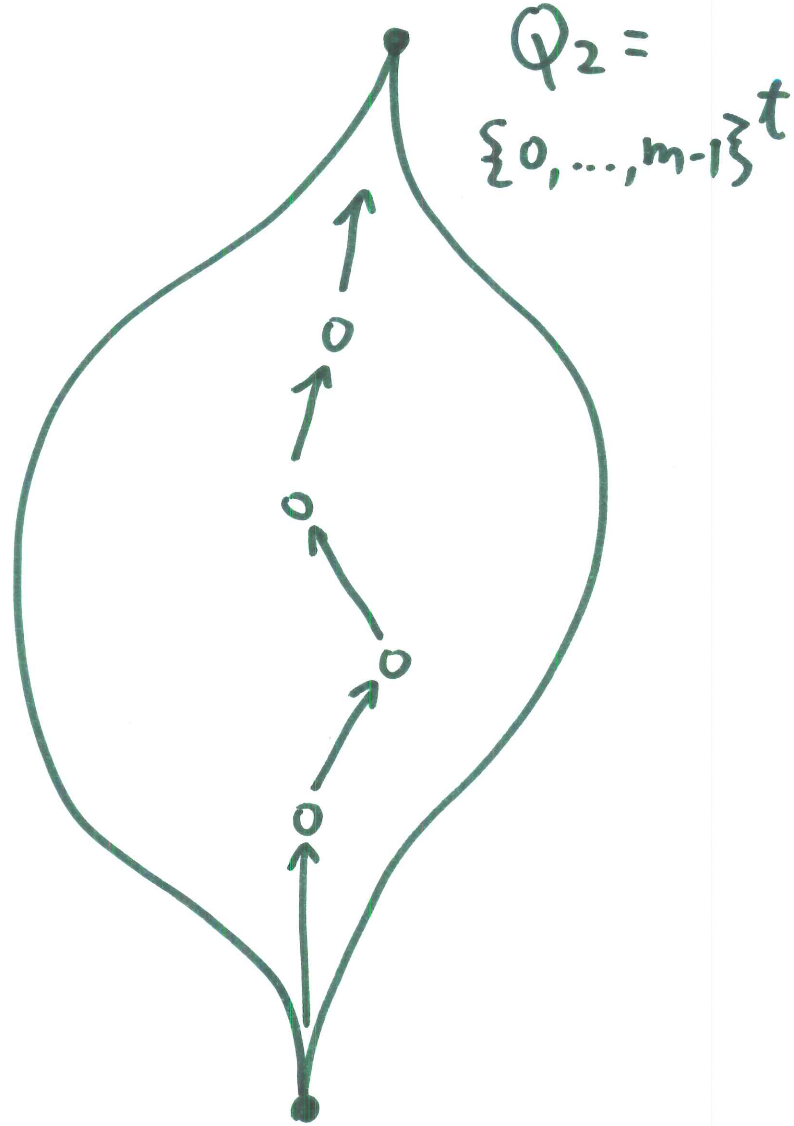

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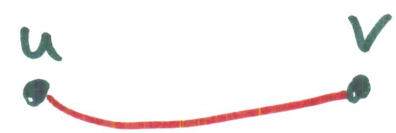

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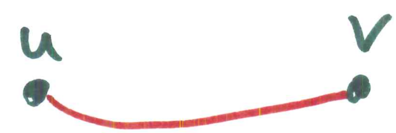

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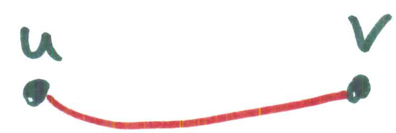

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(So $\nexists$ monochromatic paths of length m.)

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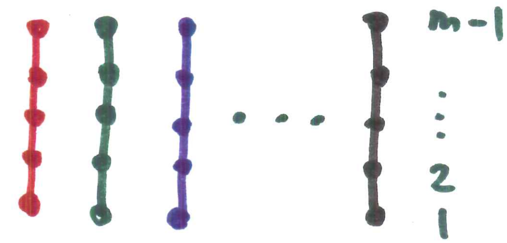

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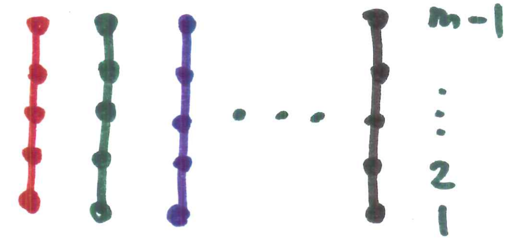
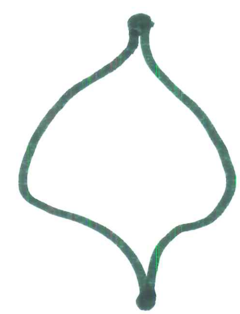


Vertices get labels in $\mathbb{Q}_k$



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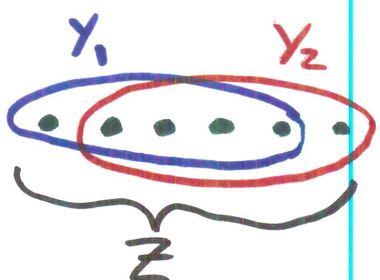

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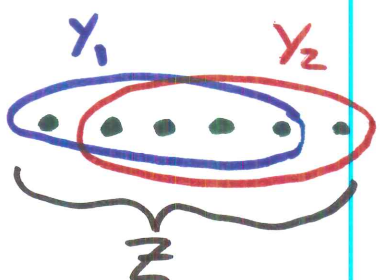

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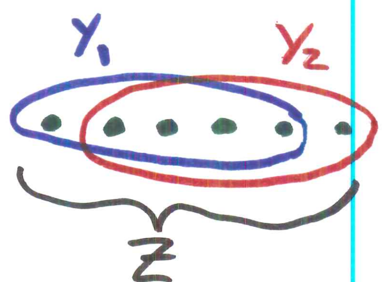

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Labels on edges tell Painter how to colour them.

Builder Strategy

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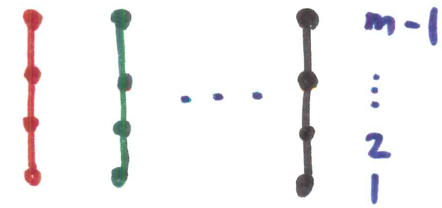

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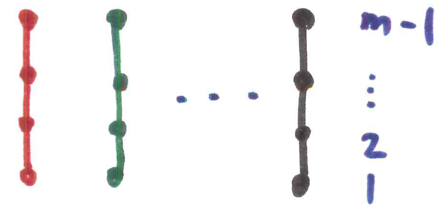

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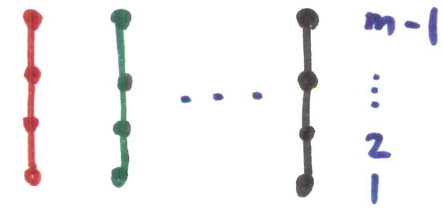

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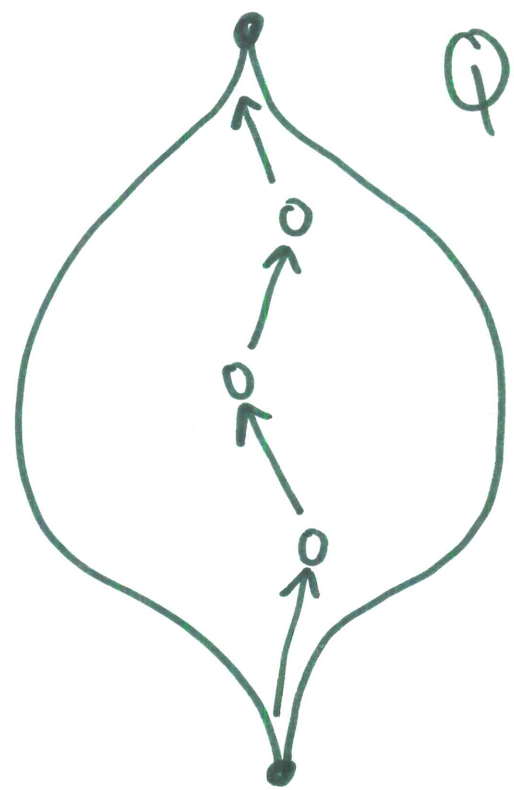
(complicated)

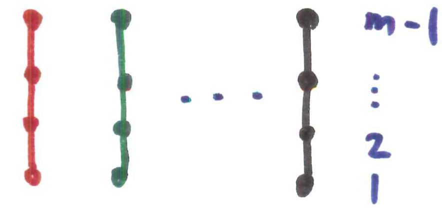
If vertices u, v have $\ell(u) = \ell(v)$,
Builder can increase $\ell(v)$ (v rightmost)
by adding "enough" edges.



Q_K

Builder Strategy

Label edges with labels in Q_1

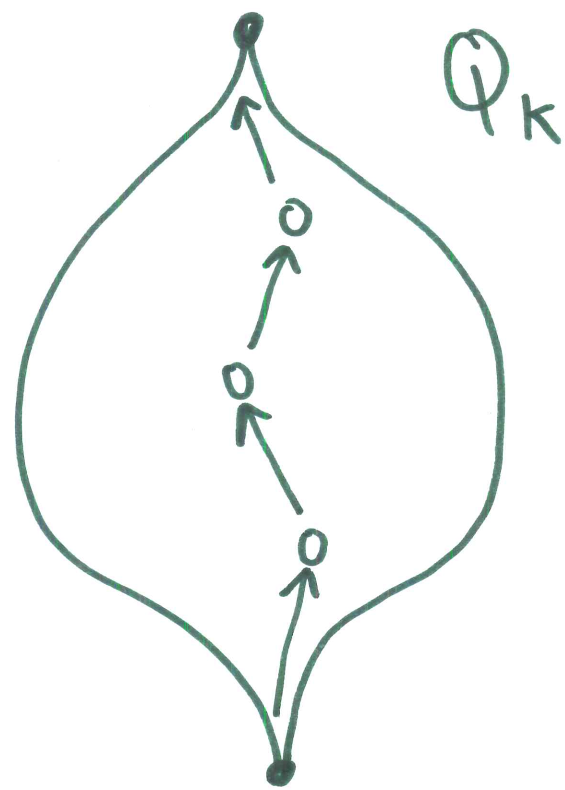


Recursively: compute labels for j-sets, $j = 1 \ldots k-1$
(complicated)

If vertices u, v have $\ell(u) = \ell(v)$,

Builder can increase $\ell(v)$ (v rightmost)

by adding "enough" edges.



Once one vertex reaches the top,

Builder wins in one more step.

Thank You!