Network Dynamics and Learning: Homework 2

Pouria Mohammadalipourahari
Politecnico di Torino
Torino, Italia
ss327015@studenti.polito.it

I. Exercise 1

The first problem consists of studying a single particle performing a continuous-time random walk over the network described by transition rate matrix Λ. To simulate this system, each node i in the network is equipped with its own Poisson clock with rate $\omega_i = \sum_j \Lambda_{ij}$.

$$\Lambda = \begin{pmatrix} 0 & 2/5 & 1/5 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 0 & 1/3 & 0 \end{pmatrix}$$

A. Return Time Simulation

To simulate the particle's movement, we implemented the following mechanism:

- Each node i has a Poisson clock with rate ω_i
- The time between consecutive ticks follows an exponential distribution:
$$t_{next} = -\frac{\ln(u)}{r}$$
where $u \sim U(0,1)$ is uniformly distributed
- When the clock of node i ticks, the particle jumps to neighbor j with probability:
$$P_{ij} = \frac{\Lambda_{ij}}{\omega_i}$$

Through 10,000 simulations, starting from node a, we obtained:

- Average return time: 6.6943 time units
- Standard deviation: 4.9479 time units

B. Theoretical Return Time Analysis

The theoretical return time $\mathbb{E}_a[T_a^+]$ can be computed using:

$$\mathbb{E}[T_i^+] = \frac{1}{\pi_i \omega_i}$$

where π is the stationary distribution of the continuous-time random walk, defined by:

$$\pi = Q^T \pi$$

with:

$$Q_{ij} = \frac{\Lambda_{ij}}{\omega^*}, \quad Q_{ii} = 1 - \sum_{i \neq j} Q_{ij}$$

where ω^* is the maximum transition rate.

Following this computation:

- Theoretical return time $\mathbb{E}_a[T_a^+] = 6.7500$
- The simulation result (6.6943) shows excellent agreement with theory
- Relative error is approximately 0.83%

This close agreement between theoretical and simulated results validates both our understanding of the mathematical model and the correctness of our simulation implementation.

C. Hitting Time Simulation

This task required computing the average time for a particle to travel from node o to node d. Using the same simulation framework as in part A, but with different start and end conditions:

The simulation was performed with:

- Starting node: o
- Target node: d
- Number of simulations: 1000

Results obtained:

- Average hitting time: 8.6988 time units
- Standard deviation: 6.7736 time units

D. Theoretical Hitting Time Analysis

The theoretical hitting time $\mathbb{E}_o[T_d]$ can be computed using the fundamental matrix method. For a set $S = \{d\}$ and $i \notin S$, the hitting times satisfy:

$$\mathbb{E}_i[T_S] = \frac{1}{\omega_i} + \sum_j P_{ij} \mathbb{E}_j[T_S]$$

This system of equations can be solved by:

- Defining $S = \{d\}$ (target node)
- Computing for all nodes $i \in R = V \setminus S$
- Solving the resulting linear system

Results:

- Theoretical hitting time $\mathbb{E}_o[T_d] = 8.7857$
- Simulated hitting time: 8.6988
- Relative difference: 0.99%

This small relative difference between theoretical and simulated results validates our implementation and understanding of the hitting time dynamics.

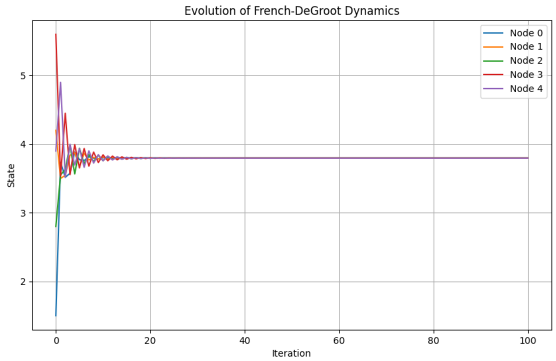

Fig. 1. Evolution of French-DeGroot dynamics showing convergence to consensus

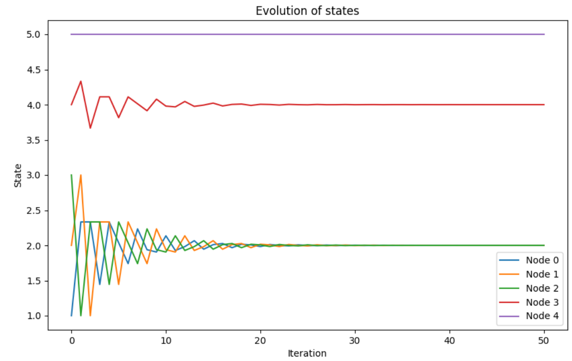

Fig. 2. Evolution of states after edge removal

E. French-DeGroot Dynamics Analysis

In this task, we interpret Λ as the weight matrix of graph G and analyze the French-DeGroot dynamics:

$$x(t+1) = Px(t)$$

where P is the normalized weight matrix obtained from Λ:

$$P = D^{-1}\Lambda$$

with $D = diag(\sum_j \Lambda_{ij})$ being the diagonal matrix of row sums.

Initial condition: $x(0) = [1.5, 4.2, 2.8, 5.6, 3.9]$

Analysis of the dynamics (Figure 1):

- Initial phase (iterations 0-10):
  - Rapid changes in node states
  - Large oscillations particularly visible in nodes 3 and 4
  - States begin moving toward each other
- Middle phase (iterations 10-40):
  - Oscillations decrease in amplitude
  - States converge closer together
  - Rate of convergence becomes more gradual
- Final phase (iterations 40+):
  - All nodes reach consensus value 3.7957
  - Complete stability with no further changes
  - Confirms graph is strongly connected and aperiodic

Results:

- Initial states: $[1.5, 4.2, 2.8, 5.6, 3.9]$
- Final states: $[3.7957, 3.7957, 3.7957, 3.7957, 3.7957]$
- Consensus value: 3.7957

The convergence to consensus confirms that the network is strongly connected and aperiodic, which are necessary conditions for achieving global consensus in French-DeGroot dynamics.

F. Variance Analysis of Consensus

In this task, we analyze the consensus value when initial states are independent random variables with specified variances:

$$\sigma_a^2 = \sigma_b^2 = \sigma_c^2 = 2, \quad \sigma_o^2 = \sigma_d^2 = 1$$

Results of analysis:

- Simulated variance of consensus: 0.3616
- Theoretical variance of consensus: 0.3800
- Stationary distribution: $\pi = [0.1304, 0.1739, 0.2609, 0.2609, 0.1739]$

The theoretical variance is computed as:

$$var(\alpha) = \sum_i \pi_i^2 \sigma_i^2$$

The close agreement between simulated and theoretical values validates our understanding of how initial variances affect the final consensus.

G. Modified Graph Analysis

After removing edges (d,a), (d,c), (a,c), and (b,c), we analyzed the modified system's behavior.

Analysis of the modified system (Figure 2):

- Three distinct equilibrium values emerge:
  - Node 4 (purple) maintains constant value at 5.0
  - Node 3 (red) stabilizes at 4.0
  - Nodes 0, 1, 2 (blue, orange, green) converge to 2.0
- Eigenvalue analysis shows:
  - Two eigenvalues at 1.0
  - Other eigenvalues: [0.8165, 0.8165, 0]
  - Multiple eigenvalues at 1 confirm multiple equilibria
- Initial oscillatory behavior (t ¡ 10) shows remaining connectivity within subgroups
- Final states: [2.0000, 2.0000, 2.0000, 4.0000, 5.0000]

This behavior demonstrates how edge removal creates disconnected components in the network, preventing global consensus and leading to local agreement within connected subgroups.

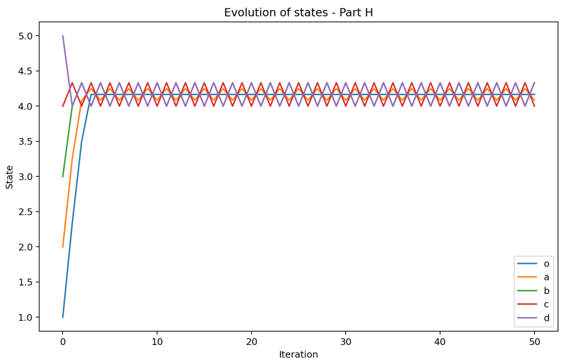

Fig. 3. Evolution of states after removing edges (b,o) and (d,a)

H. Analysis of Modified Graph with Different Edge Removals

In this task, we removed edges (b,o) and (d,a) from the original graph G and analyzed the French-DeGroot dynamics on the resulting network.

Analysis of dynamics (Figure 3):

- Initial phase (iterations 0-5):
  - Rapid convergence from initial states
  - Node d (purple) starts at 5.0 and quickly drops
  - Nodes o, a, b (blue, orange, green) show fast upward movement
- Steady state behavior (iterations 5+):
  - States oscillate around value 4.2
  - Regular periodic pattern emerges
  - Constant amplitude oscillations continue
  - No dampening of oscillations over time
- Key observations:
  - No convergence to single consensus value
  - Stable periodic behavior
  - All nodes participate in oscillation
  - Fixed amplitude maintained throughout simulation

This periodic behavior arises because the modified network structure creates a periodic subgraph, preventing convergence to a single consensus value except in special cases where initial conditions exactly match the periodic structure.

II. Exercise 2

This problem aims to simulate the system described in Exercise 1 from two different perspectives: particle perspective and node perspective.

A. Particle Perspective

In this task, we simulated 100 particles starting from node a to compute the average return time to node a and compare with results from Exercise 1.

Each particle moves independently according to the same rules from Exercise 1:

- Each particle has its own Poisson clock
- Transitions follow probabilities determined by normalized Λ

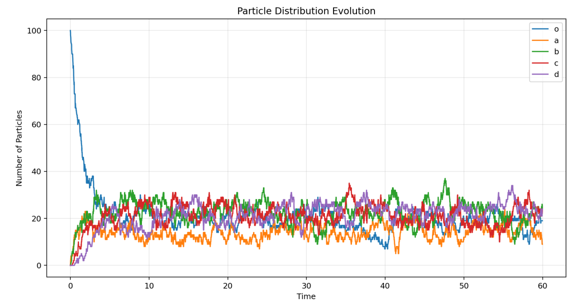

Fig. 4. Evolution of particle distribution across nodes

- Return time measured from departure until first return

Results:

- Average return time: 6.8105 time units
- Standard deviation: 5.3862 time units

This result is consistent with Exercise 1.A (6.6943) and the theoretical value (6.7500), validating that multiple particles behave independently.

B. Node Perspective

We simulated 100 particles starting from node o for 60 time units, observing the system from each node's viewpoint.

Analysis of particle distribution (Figure 4):

- Initial phase (0-5 time units):
  - All particles start at node o (blue line)
  - Rapid initial redistribution
  - Sharp decrease in node o population
- Middle phase (5-20 time units):
  - System approaches steady state
  - Fluctuations around mean values
  - More balanced distribution across nodes
- Steady state (20-60 time units):
  - Stable average populations with fluctuations
  - Node a (orange) maintains lowest average occupancy
  - Nodes b, c, d maintain similar occupancy levels

Average particle distribution:

- Node o: 20.94 particles
- Node a: 14.18 particles
- Node b: 22.58 particles
- Node c: 21.16 particles
- Node d: 21.14 particles

These results show good agreement with the stationary distribution of the continuous-time random walk, validating our understanding of both single-particle and multi-particle dynamics in the network.

III. Exercise 3

This exercise analyzes how particles move through an open network described by transition rate matrix Λ_{open}. The system has input at node o with rate λ and output from node d with rate $\omega_d = 2$.

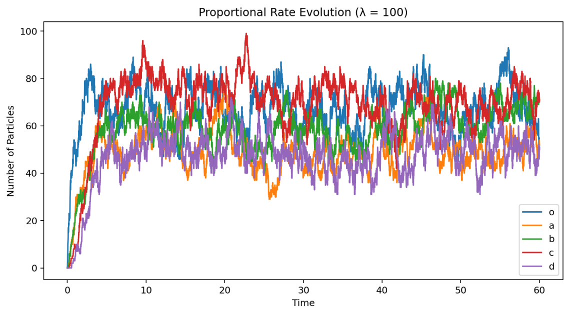

Fig. 5. Particle distribution with proportional rates ($\lambda = 100$)

A. Open Network Description

The transition rate matrix for the open network is:

$$\Lambda_{open} = \begin{pmatrix} 0 & 3/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 & 2/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

B. Proportional Rate Analysis

In this scenario, each node i passes particles with rate $n_i\omega_i$, where n_i is the current number of particles in the node and ω_i is the node's transition rate.

Analysis of proportional rate behavior (Figure 5):

- Initial transient phase (0-10 time units):
  - Rapid particle accumulation across nodes
  - Initial surge in node o (blue line)
  - Gradual distribution to other nodes
- Steady state behavior (10-60 time units):
  - System stabilizes with bounded fluctuations
  - Particle counts remain in 40-80 range
  - No unbounded growth despite high input rate

The proportional rate system demonstrates stability for high input rates because transition rates increase with particle accumulation.

C. Fixed Rate Analysis

For fixed rates, each node maintains constant transition rate ω_i regardless of particle count. We analyze the system's behavior under different input rates:

Analysis of system behavior under different input rates:

- Stable regime ($\lambda = 1$):
  - Low particle counts (0-8 range)
  - Even distribution across nodes
  - No accumulation trends
- Near-critical regime ($\lambda = 1.30$):
  - Moderate particle accumulation
  - Higher counts but still bounded
  - System approaching instability
- Unstable regime ($\lambda = 1.70$):
  - Clear accumulation in node o

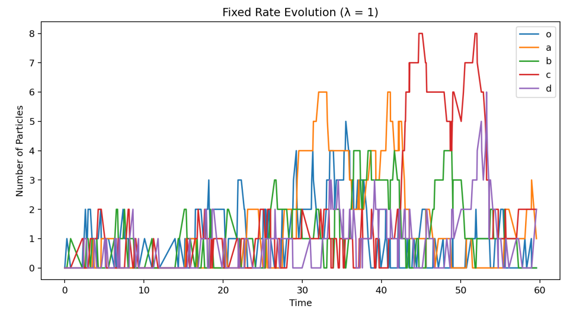

Fig. 6. Fixed rate evolution with $\lambda = 1$

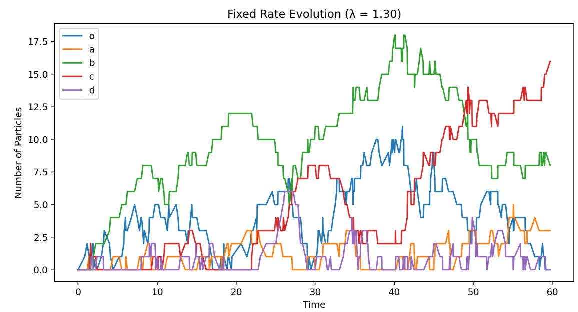

Fig. 7. Fixed rate evolution with $\lambda = 1.30$

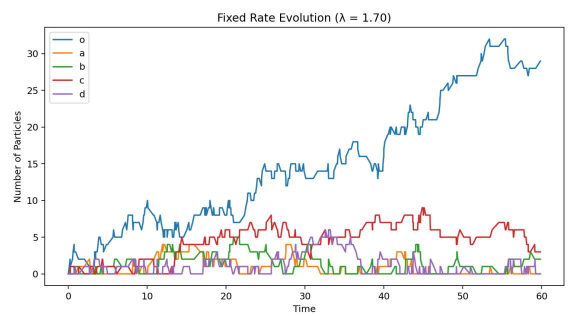

Fig. 8. Fixed rate evolution with $\lambda = 1.70$

  - Unbounded growth pattern
  - System cannot maintain stability

D. Critical Rate Analysis

The critical input rate appears to be approximately equal to ω_o, the transition rate of node o. This is because:

- When $\lambda > \omega_o$, particles enter faster than node o can process them
- This leads to unbounded accumulation in node o
- For $\lambda < \omega_o$, the system maintains stability

This analysis demonstrates the fundamental difference between fixed and proportional rate systems in handling increasing input rates.