Northeastern University
CS5800 Algorithms, Fall 2024

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Solve following recurrence is $O(n^2)$ by the substitution method.

$$T(n) = 1 \text{ for } n < 2$$

$$T(n) = T(n-2) + n \text{ for } n \geq 2$$

Proof. We want to prove that $T(n) \leq c \cdot n^2$ for some c.

(Base case) $T(2) = T(0) + 2 = 3$.
We choose $k = 2$ as a base case. Then, for any $c \geq 1$,

$$T(2) = 3 \leq c \cdot 2^2 = 4c$$

.

(String Induction Hypothesis) Suppose that, for all $k < n$, $T(k) \leq c \cdot k^2$.

(Induction step) Now, we have to show that the induction hypothesis holds when $k = n$.

Since $n - 2 < n$, we can apply the I.H. to $T(n-2)$. Thus,

$$T(n) = T(n-2) + n \leq c(n-2)^2 + n$$

$$c(n-2)^2 + n = c(n^2 - 4n + 4) + n = cn^2 - 4cn + 4c + n$$

Now, we want this to be less than cn^2. Then, we have that

$$T(n) = T(n-2) + n \leq c(n-2)^2 + n = cn^2 - 4cn + 4c + n \leq cn^2$$

which proves that

$$T(n) \leq cn^2$$

.

So, we want to explicitly show that the following inequality holds for c.

$$cn^2 - 4cn + 4c + n \leq cn^2$$

.

If we subtract both sides by cn^2, we have

$$n \leq 4cn - 4c$$

.

So,

$$\frac{n}{4(n-1)} \leq c$$

.

Note that the left-hand side is decresing function and always smaller than 1. (When $n = 2$, $\frac{n}{4(n-1)} = \frac{1}{2}$ and as n goes to the infinity, $\frac{n}{4(n-1)}$ goes to $\frac{1}{4}$.)

Thus, for any $c \geq 1$, $\frac{n}{4(n-1)} \leq c$.

Hence, when $c = 1$ (or any $c \geq 1$) ,

$$T(n) \leq cn^2$$

$\square$