Northeastern University
CS5800 Algorithms, Fall 2024

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1. Prove $n^3 + 4n^2 + 5 = O(n^3)$.

Proof. If $n^3 + 4n^2 + 5 = O(n^3)$, there must exist positive constants, c and n_0, such that, for all $n > n_0$, $n^3 + 4n^2 + 5 \leq c \cdot n^3$. Let's choose $c = 2$.
Then, we have $n^3 + 4n^2 + 5 \leq 2n^3$.
We want to prove this inequality holds for all $n > n_0$.
$n^3 + 4n^2 + 5 \leq 2n^3 \iff$
$4n^2 + 5 \leq n^3$
Let's divide both sides by n^2. (Note that we consider only $n > n_0$, so $n \neq 0$).
So, we have $4 + \frac{5}{n^2} \leq n$.
Note that $\frac{5}{n^2} < 1$ if $n > \sqrt{5}$. So, if $n \geq 5$, $4 + \frac{5}{n^2} \leq n$ holds.
Hence, if we choose some number greater than eqaul to 5 for n_0, the inequality holds.
So, let's choose $c = 2$ and $n_0 = 5$.
Then, for any $n \geq 5$, $n^3 + 4n^2 + 5 \leq 2n^3$. □

2. Prove that $\log n = o(n)$.

Proof. Let's use the limit property of o instead of the definition of o.

$$\lim_{n\to\infty} \frac{n}{\log n} = \lim_{n\to\infty} \frac{(n)'}{(\log n)'} (\text{ by L'Hôpital's rule}) = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty$$

Hence, $\log n = o(n)$.

□