CS5800 Algorithms

Module 3. More Divide and Conquer

Master Method For Solving Recurrences (CLRS 4.5)

- "Cookbook" method for solving recurrences of the form
$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$
- Proof is presented in CLRS 4.6, left as optional (not covered in class)
- Intuitive understanding: We compare $f(n)$ with $n^{\log_b a}$
  - If $f(n)$ is smaller (polynomially & asymptotically), then $T(n) = \Theta(n^{\log_b a})$.
  - If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
  - If $f(n)$ is bigger with some more conditions (see Theorem 4.1), then $T(n) = \Theta(f(n))$

Master Method (Revisit)

$$T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$$

Which case?

$$f(n) = n \lg n\,, n^{\log_b a} = n^{\log_2 2} = n$$

So, $f(n)$ is asymptotically bigger than $n^{\log_b a}$. Is it Case 3?

- Try to prove by the substitution method.

Solve $T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$

Master Method (Revisit)

- $T(n) = aT\left(\frac{n}{b}\right) + f(n)$
  - Case 1: If there exists a constant $\epsilon > 0$, such that $f(n) = O(n^{\log_b a - \epsilon})$ (polynomially & asymptotically smaller), then $T(n) = \Theta(n^{\log_b a})$.
  - Case 2: If there exists a constant $k \geq 0$, such that $f(n) = \Theta(n^{\log_b a} \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$
  - Case 3: If there exists a constant $\epsilon > 0$, such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and if $f(n)$ additionally satisfies the *regularity condition*, $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n (polynomially & asymptotically bigger), then $T(n) = \Theta(f(n))$.

    (See Theorem 4.1 and Exercise 4.5-5 for the regularity condition – out of this course.)

Divide-And-Conquer With Searching

Make Every Algorithm Recursive For The Sake Of Algorithm Analysis Only

Binary Search (Exercise 2.3-5)

- Searching a "sorted" array for a value
  - Like looking up a dictionary for a word, or a phone book for a name
- Everyone is expected to write the binary search code (pseudocode or actual language) fluently both recursively and iteratively
  - Remember the "divide-and-conquer" nature
  - And also the "elimination" nature: You can eliminate half of the array once you compare with the middle value
  - What differences & benefits are there in each approach (recursive & iterative)?
    - Keep this question in mind when you experiment the code in the next slides

http://pythontutor.com/

```
def binary_search(array, value):
    return binary_search_recursive(array, 0, len(array) - 1, value)

def binary_search_recursive(array, left, right, value):
    if left > right:
        return -1

    mid = (left + right) / 2
    if value == array[mid]:
        return mid
    if value > array[mid]:
        return binary_search_recursive(array, mid + 1, right, value)

    return binary_search_recursive(array, left, mid - 1, value)

print(binary_search([0,11,22,33,44,55,66,77,88], 77))
```

Iterative version

```
def binary_search_iterative(array, value):
    left = 0
    right = len(array) - 1

    while left <= right:
        mid = (left + right) / 2
        if value == array[mid]:
            return mid
        if value > array[mid]:
            left = mid + 1
        else:
            right = mid - 1
    return -1

print(binary_search_iterative([0,11,22,33,44,55,66,77,88], 77))
```


Q: Given an array [0,11,22,33,44,55,66,77,88] and a searched value 77, what is the correct sequence of (left, right) pairs in the binary search? Assume 0 is the starting index.

a. (1, 8), (4, 8), (6, 8), (7, 7)
b. (0, 8), (0, 4), (0, 2), (0, 1), (0, 0), (0, -1)
c. (0, 8), (5, 8), (7, 8)
d. (0, 8), (4, 8), (6, 9), (7, 7)

Q: Given an array [0,11,22,33,44,55,66,77,88] and a searched value 30, what is the correct sequence of (left, right) pairs in the binary search? Assume 0 is the starting index.

a. (1, 8), (4, 8), (6, 8), (7, 7)
b. (0, 8), (0, 3), (2, 3), (3, 3), (4, 3)
c. (0, 8), (0, 3), (2, 3), (3, 3), (3, 2)
d. (0, 8), (0, 4), (0, 2), (0, 1), (0, 0), (0, -1)

Analysis of Binary Search

- Best case: $T(n) = \Theta(1)$
  - Fixed # operations (1 mid calc op, 1 if, 1 return) when the mid entry is a hit
- Worst case
  - $T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \Theta(1)$
  - $T(0) = \Theta(1)$
- Easier to derive the recurrence from recursive code
- Fewer # steps with iterative code

```
def binary_search_iterative(array, value):
    left = 0
    right = len(array)-1
    while left <= right:
        mid = (left + right) / 2
        if value == array[mid]:
            return mid
        if value > array[mid]:
            left = mid + 1
        else:  # value < array[mid]
            right = mid - 1
    return -1  # No match
```

```
def binary_search_recursive(array, left, right, value):
    if left > right:
        return -1
    mid = (left + right) / 2
    if value == array[mid]:
        return mid
    if value > array[mid]:
        return binary_search_recursive(array, mid+1, right, value)
    # value < array[mid]
    return binary_search_recursive(array, left, mid-1, value)
```

Recursive vs. Iterative Binary Search

- Easier to write recursive code and derive recurrence from it
- Call stack overhead in recursive code : $O(\log n)$ space complexity
- Harder to write iterative code and analyze it
- Faster execution (constant factor speed up) with iterative code, no call stack overhead ($O(1)$ space complexity)
- Quite common to start out writing recursive implementation, then translate it to iterative code for optimization
- Possible variation: What if we need to return the index of the first match when there are multiple matches?

Sidebar: Binary Divide-And-Conquer For Searching Unsorted Array

- With an unsorted array, we can't eliminate the problem size by half, like with a sorted array and binary search
- Had to do sequential search, eliminating problem size by one at a time
- Can we do the binary divide-and-conquer with unsorted array as well?
  - Yes, we can. Can you code that?
  - But we won't get any benefit, as our recurrence will be:
    - $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(1) \rightarrow T(n) = \Theta(n)$, not $\Theta(\log n)$
- Can we derive general solution on $T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^c)$?

Maximum Subarray Problem (CLRS 4.1)

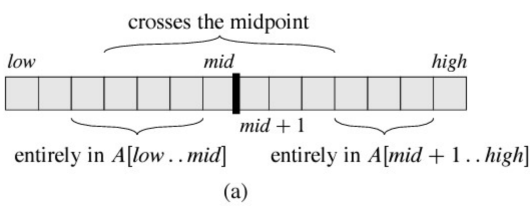
- Given an example array A=[13,-3,-25,20,-3,-16,-23,18,20,-7,12,-5,-22,15,-4,7],
  - What is the subarray whose sum is the maximum of all subarray sums?
  - In this example, it's [18,20,-7,12] with sum 43.
    - You can confirm this yourself by whatever means
  - It doesn't make much difference between finding the max subarray sum (43) and finding the subarray itself ([18,20,-7,12]). Think about why.
  - Interesting only when there are negative numbers in the array.
  - Read CLRS 4.1 for a motivating application of this problem
    - Stock trading to maximize gain when daily change amounts are known.
      - Not a real stock trading technique!

Naïve, Brute-Force Solutions

- Evaluate sums of all A[i..j] for any possible i & j, and find the max.
  - max_subarray_sum = -infinity (or A[1])
  - for i=1 to n (assuming 1-starting array indexing)
    - for j=i to n
      - subarray_sum=0
      - for k=i to j
        - subarray_sum += A[k]
      - If subarray_sum > max_subarray_sum,
        - max_subarray_sum = subarray_sum
  - return max_subarray_sum
- $\Theta(n^3)$: Really naïve, repeating same summation many times
- $\Theta(n^2)$: By separating out summations, storing them in a 2D array, then doing comparisons, we can achieve this improvement.

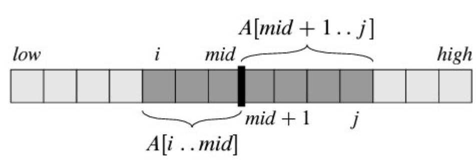
Can We Do Any Better?

- How about binary divide-and-conquer, like earlier?
  - Max subarray sum of A[low..high] is the maximum of:
    - Max subarray sum of A[low..mid] ← Recursively computed
    - Max subarray sum of A[mid+1..high] ← Recursively computed
    - Max of sums of subarrays straddling mid
      - Easier than original problem because this problem is constrained.



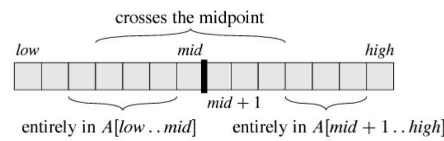
CLRS Fig. 4.4


```
FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
left-sum = -∞
sum = 0
for i = mid downto low
    sum = sum + A[i]
    if sum > left-sum
        left-sum = sum
        max-left = i
right-sum = -∞
sum = 0
for j = mid + 1 to high
    sum = sum + A[j]
    if sum > right-sum
        right-sum = sum
        max-right = j
return (max-left, max-right, left-sum + right-sum)
```



CLRS pp. 71

```
FIND-MAXIMUM-SUBARRAY(A, low, high)
if high == low
    return (low, high, A[low])          // base case: only one element
else mid = ⌊(low + high)/2⌋
    (left-low, left-high, left-sum) =
        FIND-MAXIMUM-SUBARRAY(A, low, mid)
    (right-low, right-high, right-sum) =
        FIND-MAXIMUM-SUBARRAY(A, mid + 1, high)
    (cross-low, cross-high, cross-sum) =
        FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
    if left-sum ≥ right-sum and left-sum ≥ cross-sum
        return (left-low, left-high, left-sum)
    elseif right-sum ≥ left-sum and right-sum ≥ cross-sum
        return (right-low, right-high, right-sum)
    else return (cross-low, cross-high, cross-sum)
```



(a)

CLRS pp. 72

Analysis of Divide-And-Conquer Max Subarray

- $T(1) = \Theta(1)$: Base case. Recursive case is:
- $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + T_{crossing}(n) + \Theta(1)$ where:
  - $T_{crossing}(n) = \Theta(n)$
- $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$: Exactly the same as merge sort
- $T(n) = \Theta(n \log n)$
- Achieved $\Theta(n^2)$ to $\Theta(n \log n)$ improvement
  - Actually we can do better and achieve linear time ($\Theta(n)$)
    - Exercise 4.1-5 in pp. 75.
    - It's not a divide-and-conquer solution, though. It's rather a clever intuition.

Strassen’s Matrix Multiplication Algorithm (CLRS 4.2)

- Multiplying 2 $n \times n$ matrices
  - Study Appendix D if you are not familiar with matrices and operations
- Simple straightforward algorithm, giving $\Theta(n^3)$

```
SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

CLRS pp.75

Simple Divide-And-Conquer Matrix Mult.

- Partition each of A, B, and C into 4 quarters:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \tag{4.9}$$

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. \tag{4.10}$$

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}, \tag{4.11}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}, \tag{4.12}$$
$$C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}, \tag{4.13}$$
$$C_{22} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}. \tag{4.14}$$

- Can we reduce # multiplications some way?

Strassen's Improvement

- Not sure how Strassen found this, but he observed that, by letting:

$$
\begin{aligned}
S_1 &= B_{12} - B_{22}\,, & P_1 &= A_{11} \cdot S_1 &&= A_{11} \cdot B_{12} - A_{11} \cdot B_{22}\,,\\
S_2 &= A_{11} + A_{12}\,, & P_2 &= S_2 \cdot B_{22} &&= A_{11} \cdot B_{22} + A_{12} \cdot B_{22}\,,\\
S_3 &= A_{21} + A_{22}\,, & P_3 &= S_3 \cdot B_{11} &&= A_{21} \cdot B_{11} + A_{22} \cdot B_{11}\,,\\
S_4 &= B_{21} - B_{11}\,, & P_4 &= A_{22} \cdot S_4 &&= A_{22} \cdot B_{21} - A_{22} \cdot B_{11}\,,\\
S_5 &= A_{11} + A_{22}\,, & P_5 &= S_5 \cdot S_6 &&= A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22}\,,\\
S_6 &= B_{11} + B_{22}\,, & P_6 &= S_7 \cdot S_8 &&= A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22}\,,\\
S_7 &= A_{12} - A_{22}\,, & P_7 &= S_9 \cdot S_{10} &&= A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12}\,.\\
S_8 &= B_{21} + B_{22}\,, \\
S_9 &= A_{11} - A_{21}\,, \\
S_{10} &= B_{11} + B_{12}\,.
\end{aligned}
$$

- Submatrix C_{ij} can be represented as follows:

$$C_{11} = P_5 + P_4 - P_2 + P_6\,. \quad C_{12} = P_1 + P_2\,, \quad C_{21} = P_3 + P_4 \quad C_{22} = P_5 + P_1 - P_3 - P_7\,,$$

- Algebraic proofs shown in CLRS pp. 81
- We've now got $7\ \frac{n}{2} \times \frac{n}{2}$ multiplications and 18 additions
  - 18 additions are still $\Theta(n^2)$.
  - Thus new recurrence is

$$T(n) = 7T\left(\frac{n}{2}\right) + \Theta(n^2)$$

  - Solution to this recurrence (using master theorem, which will be presented later) is $T(n) = \Theta\left(n^{\log_2 7}\right) \cong \Theta(n^{2.81})$

Quicksort

Fastest Sorting Algorithm On Average, How To Prove That

Recall Merge Sort

- "Divide": easy. Just divide the input array into two sub-arrays.
- "Conquer": hard. It takes O(n) time complexity.

⇒ What if divide the input array into 3 sub-arrays?
- ⇒ "Divide": O(1)
- ⇒ "Conquer": still O(n)
- ⇒ $T(n) = 3T\left(\frac{n}{3}\right) + O(n)$
- ⇒ Overall, the same time complexity as Merge Sort

⇒ What if divide the input array into $\sqrt{n}$ sub-arrays?
- ⇒ "Divide": O($\sqrt{n}$)
- ⇒ "Conquer": O($n\sqrt{n}$)
- ⇒ $T(n) = \sqrt{n}T(\sqrt{n}) + \mathrm{O}(n\sqrt{n})$
- ⇒ Overall, it is worse than Merge Sort
- ⇒ If we can make the "conquer" part in O(n), the time complexity becomes O($n \log\log n$). But, can we?

Quicksort: Different Kind Of Divide & Conquer

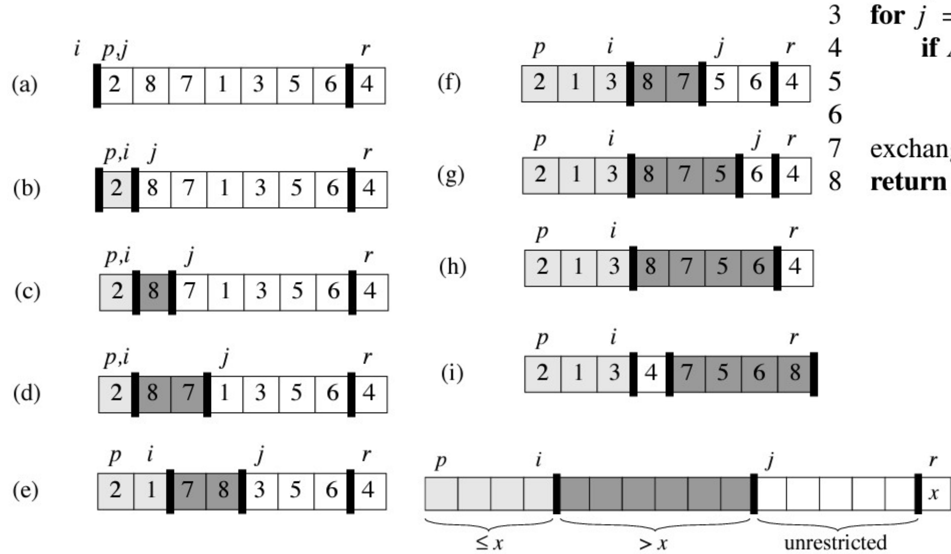
- So far, “divide” was straightforward, and “conquer” was involved.
- In sorting, can we make “conquer” part easy (almost nothing), by doing more on “divide” part?
  - CLRS 7.1 “Divide”: **Partition** (rearrange) the array $A[p..r]$ into two (either one of the two may be empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that:
    - $A[i] \leq A[q]$ for any $p \leq i < q$, and
    - $A[j] > A[q]$ for any $q < j \leq r$.
  - CLRS 7.1 “Conquer”: Then conquering becomes straightforward:
    - Sort $A[p..q-1]$ recursively
    - Sort $A[q+1..r]$ recursively

```
QUICKSORT(A, p, r)
if p < r
    q = PARTITION(A, p, r)
    QUICKSORT(A, p, q - 1)
    QUICKSORT(A, q + 1, r)
```

To sort an entire array A, the initial call is QUICKSORT(A, 1, $A.length$).

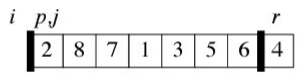
PARTITION(A, p, r) Illustration

```
PARTITION(A, p, r)
x = A[r]
i = p - 1
for j = p to r - 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i + 1
```



PARTITION() Can Be Recursive As Well

- Maybe more overhead, but maybe easier to understand



- If $A[p] \leq A[r]$, return PARTITION($A, p + 1, r$)
- Otherwise, 3-way swap between $A[p]$, $A[r]$, and $A[r - 1]$
  - $A[p]$ to $A[r]$, $A[r]$ to $A[r - 1]$, $A[r - 1]$ to $A[p]$, then return PARTITION($A, p, r - 1$)
  - Definitely more swaps (so more overhead), but still correct (and same asymptotic notation) with easier derivation of the recurrence relation
    - $T(n) = T(n - 1) + \Theta(1) \rightarrow T(n) = \Theta(n)$
- Base case: If $p = r$, return r.

Given the input array A = [13, 19, 9, 5, 12], what is the correct array when the textbook's PARTITION() pseudocode is applied to the array A?

a) [19, 13, 12, 9, 5]

b) [13, 9, 12, 5, 19]

c) [9, 5, 12, 19, 13]

d) [5, 9, 12, 13, 19]

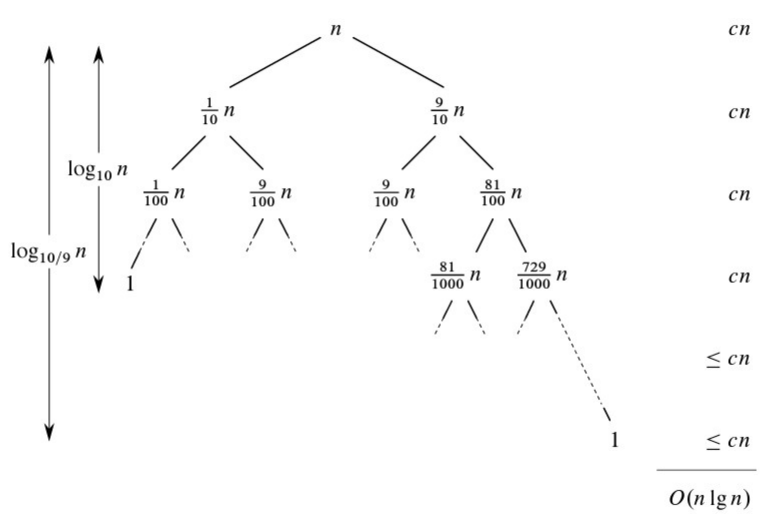
Quicksort Analysis

```
QUICKSORT(A, p, r)
if p < r
    q = PARTITION(A, p, r)
    QUICKSORT(A, p, q − 1)
    QUICKSORT(A, q + 1, r)
```

- From the QUICKSORT() pseudocode, $T_{qsort}(n) = T_{partition}(n) + T_{qsort}(n_1) + T_{qsort}(n_2)$, where $n_1 + n_2 + 1 = n$ and $n_1 \geq 0, n_2 \geq 0$
- It's obvious that $T_{partition}(n) = \Theta(n)$.
- So, $T(n) = T(n_1) + T(n_2) + \Theta(n)$ where $n_1 + n_2 + 1 = n$
- Worst case: $n_1 = 0$ or $n_2 = 0$ all the time (bad split/partition) →
  - $T(n) = T(n-1) + \Theta(n)$ → $T(n) = \Theta(n^2)$
    - When would this happen? What's the insertion/bubble sort performance in that case?
- Best case: $n_1 \cong n_2$ as much as possible (even split) →
  - $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$ → $T(n) = \Theta(n \lg n)$

Balanced Splits, Even Skewed (CLRS Fig. 7.4)

- 9-to-1 splits all the time
- In fact, doesn't matter what x & y in x-to-y splits, as long as x & y are fixed.



Randomized Quicksort (CLRS Section 7.3)

- To avoid worst case as much as possible,
  - Pick the pivot from a random index, not from a fixed one at the end.
  - Still rely on the original PARTITION() after swapping the randomly picked pivot with the original fixed pivot.

```
RANDOMIZED-PARTITION(A, p, r)
i = RANDOM(p, r)
exchange A[r] with A[i]
return PARTITION(A, p, r)
```

```
RANDOMIZED-QUICKSORT(A, p, r)
if p < r
    q = RANDOMIZED-PARTITION(A, p, r)
    RANDOMIZED-QUICKSORT(A, p, q - 1)
    RANDOMIZED-QUICKSORT(A, q + 1, r)
```


Formal Proofs of RANDOMIZED-QUICKSORT Time Complexities (CLRS Section 7.4)

- Lots of algebraic derivations. We won’t focus on those.
- Also random probabilistic analysis and derivations for randomized case. We won’t focus on those either.
- Just read through CLRS Section 7.4 and see how they go.

Brief Summary

- Worst-case: $T(n) = \max_{0 \le q \le n-1} (T(q) + T(n-q-1)) + \Theta(n)$
    - We can show $T(n) \le cn^2$ for some c and large enough n, showing $T(n) = O(n^2)$ (This is done in textbook)
    - Can also show $T(n) \ge cn^2$ for some c and large enough n, showing $T(n) = \Omega(n^2)$ (Exercise 7.4-1)
    - Therefore, worst-case $T(n) = \Theta(n^2)$.
- Average-case (expected running time) of RANDOMIZED-QUICKSORT()
    - Probabilities of possible cases, number of comparisons becoming random variable, derive the expected average of the random variable.
    - $E[X] = \cdots = O(n \log n)$
- In practice, Quick Sort is usually faster than Merge Sort.

Median Finding Algorithm

No Need To Sort Entire Array

Medians and Order Statistics

- Given a set A of n elements,
  - Minimum (first in the ordered sequence), maximum (last), median (mid)
    - If n is even, there could be 2 medians. For simplicity, we mean the lower median.
- General i-th order statistic: The i-th smallest element of A
  - Minimum: A's 1st order statistic, maximum: A's n-th order statistic
  - Median: A's $\lfloor (n+1)/2 \rfloor$-th order statistic
- Algorithm to find the i-th order statistic of A for any given A and i
  - Simple (Naïve): Sort A, return $A[i]$: $O(n \lg n)$
  - Do we really need to sort the entire array? Aren't we doing more than necessary?
  - Finding minimum: Just by scanning the entire array, you can find i-th element in O(n) (In fact, this is the optimal solution – why? Read the textbook).
  - But what about finding i-th order (general algorithm) ?

Average Linear Time Selection Algorithm

- CLRS 9.2 RANDOMIZED-SELECT() (pp. 216)

```
RANDOMIZED-SELECT(A, p, r, i)
if p == r
    return A[p]
q = RANDOMIZED-PARTITION(A, p, r)
k = q - p + 1
if i == k            // the pivot value is the answer
    return A[q]
elseif i < k
    return RANDOMIZED-SELECT(A, p, q - 1, i)
else return RANDOMIZED-SELECT(A, q + 1, r, i - k)
```

Time Complexity Analysis of RAND-SEL

- $T_{select}(n) = T_{partition}(n) + T_{select}(n'), \text{where } 0 \leq n' < n$
- Worst case: T(n) = T(n-1) + O(n)
  - $\Theta(n^2)$, just like quicksort
- Average (expected) case: T(n) = T(n/k) + O(n)
  - Requires random variable analysis, just like in quicksort
    - Assume probabilities, find expected running time, show it's at most $O(n)$
  - Details in CLRS 9.2
    - We don't need to do this all the time.
    - I'd say intuition is more important than formal proof.
      - Think about balanced splits-case (e.g., 2:3), and derive running time, confirm it's $O(n)$.
- Can we achieve $O(n)$ in worst case as well?
  - Surprisingly, yes.
    - Time complexity analysis of SELECT() is more interesting and involved.

Selection algorithm in worst-case linear time

(Main Idea)

- Divide the input array into n/5 groups.
- Find the median for each group (note each group consists of at most 5 elements) – O(1)
- Find the median of n/5 medians (This is done recursively) – T(n/5)
- 3n/10 – 6 (= 3(n/10 -2)is already less than the median of the medians.
- If the median of the medians is the median, return.
- Otherwise, recursively call among 7n/10 + 6 elements. – T(7n/10 + 6)
- Overall running time:
  - T(n) = T(n/5) + T(7n/10 + 6) + O(n) => O(n)