Introduction to Derivative Pricing:

An intro to Baxter & Rennie

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- so how should he decide on the odds?

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- How do you price the share of Apple?
- How do you price the call option on Apple share?
- How to price 1 million Apple shares?

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- Arbitrage is a stronger pricing argument than expectation.
- Arbitrage activity in the market will drive price to no-arb pricing.

Betting/Trading strategy

What are the elements that goes into a betting/trading strategy?

- instrument/Rule: what to bet. What are the possible outcomes that would be known later.
- quantity: how much to bet.
- odds/market price: What are the payoffs for the pay for each possible outcome?
- Rule:pre-visibility: Bets off before the outcome is known. No cheating

Martingale: double down stragetgy

For a fair coin toss (odds at 1:1) bet, bet \$1 for head, and double the bet each time tails shows until the 1st head shows. This will give you a sure profit of \$1, as long as you don't go broke before the 1st heads shows.

- Can it be carried out in Marina Bay Sands?
- Why martingale is not an arbitrage?

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- Why martingale is not an arbitrage? — not self-financing

Binomial representation theorem (I)

- A random process is described using binomial tree nodes for all possible future values.
- A probability set for the above tree, called measure.
- Filtration $\mathcal{F}_i$ is the historical path upto i. This can be seen as information set available up to time i.
- Claim X is a function depends on $\mathcal{F}_T$.
- $\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_i)$, converts a claim into a process, i.e. price of the claim at each t.
- A previsible process/trading stragegy
- A martingale measure. ($\mathbb{E}_{\mathbb{P}}(S_j|\mathcal{F}_i) = S_i$, for all $i \leq j$.)

Binomial representation theorem (II)

Theorem

Suppose the measure Q is such that the binomial price process S is a Q-martingale. If N is any other Q-martingale, then there exists a previsible process h such that

$$N_i = N_0 + \sum_{k=1}^{i} h_k \Delta S_k,$$

where $\Delta S_k := S_i - S_{i-1}$ is the change in S from time $i-1$ to i, and h_i is the value of h at the appropriate node at time i.

Binomial representation theorem (III)

- As a result, all claimes can be fully replicated using simple trading strategies and therefore strongly priced by no-arbitrage argument. This is also called market completeness.
- arbitrage free = market complete = existence of unique Equivalence of Martingale Measure
- The theorem can be extended to continuous-time version.

Brownian motion

Definition

The process $W = (W_t : t \geq 0)$ is a $\mathbb{P}$-Brownian motion if and only if:

1. W_t is continuous, and $W_0 = 0$,
2. the value of W_t is distributed, under $\mathbb{P}$, as a normal random variable $N(0, t)$
3. the increment $W_{s+t} - W_s$ is distributed as a normal $N(0, t)$, under $\mathbb{P}$, and is independent of $\mathcal{F}_s$, the history of what the process did up to time s.

Stochastic process

Definition

A stochastic process X is a continous process ($X_t : t \geq 0$) such that X_t can be written as

$$X_t = X_0 + \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds,$$

where σ and μ are random $\mathcal{F}$-previsible process such that $\int_0^t (\sigma_s^2 + |\mu_s|) ds$ is finite for all times t (with probability 1). The differential form of this equation can be written

$$dX_t = \sigma_t dW_t + \mu_t dt.$$

Itô's formula

If X is a stochastic process, satisfying $dX_t = \sigma_t dW + \mu_t dt$, and f is a deterministic twice continously differentiable function, then $Y_t := f(X_t)$ is also a stochastic process and is given by

$$dY_t = \left(\sigma_t f'(X_t)\right) dW_t + \left(\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t)\right) dt.$$

Example

If $X_t = \exp(\sigma W_t)$, then what is dX_t?

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Radon-Nikodym derivative

Definition

Equivalence

Two measures P and Q are equivalent if they operate on the same sample space and agree on what is possible.

$$P(A) > 0 \Leftrightarrow Q(A) > 0.$$

Definition

Given $\mathbb{P}$ and $\mathbb{Q}$ equivalent measures and a time horizon T, we can define a random variable $\frac{d\mathbb{Q}}{d\mathbb{P}}$ defined on $\mathbb{P}$-possible paths, taking positive real values, such that

1. $\mathbb{E}_{\mathbb{Q}}(X_T) = \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}} X_T)$, for all claims X_T knowable by time T.
2. $\mathbb{E}_{\mathbb{Q}}(X_t|\mathcal{F}_s) = \xi_s^{-1}\mathbb{E}_{\mathbb{P}}(\xi_t X_t|\mathcal{F}_s), \; s \leq t \leq T$,

where ξ_t is the process $\mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t)$.

Cameron-Martin-Girsanov theorem

Theorem

If W_t is a $\mathbb{P}$-Brownian motion and γ_t is a $\mathcal{F}$-previsible process satisfying the boundedness condition $\mathbb{E}_{\mathbb{P}} \exp(\frac{1}{2}\int_0^T \gamma_t^2 dt) < \infty$, then there exists a measure $\mathbb{Q}$ such that

1. *$\mathbb{Q}$ is equivalent to $\mathbb{P}$*
2. $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right)$
3. *$\widetilde{W}_t = W_t + \int_0^t \gamma_s ds$ is a $\mathbb{Q}$-Brownian motion.*

In other words, W_t is a drifting $\mathbb{Q}$-Brownian motion with drift $-\gamma_t$ at time t.

Martingale representation theorem

Definition

A stochastic process M_t is a *martingale* with respect to a measure $\mathbb{P}$ if and only if

1. $\mathbb{E}_{\mathbb{P}}(|M_t|) < \infty, \forall t$
2. $\mathbb{E}_{\mathbb{P}}(M_t|\mathcal{F}_s) = M_s, \forall s \leq t.$

Theorem

Suppose that M_t is a $\mathbb{Q}$-martingale process, whose volatility σ_t satisfies the additional condition that it is (with probability one) always non-zero. Then if N_t is any other $\mathbb{Q}$-martingale, there exists an $\mathcal{F}$-previsible process ϕ such that $\int_0^T \phi_t^2 \sigma_t^2 dt < \infty$, a.s., and N can be written as

$$N_t = N_0 + \int_0^t \phi_s dM_s.$$

Further ϕ is (essentially) unique.

Construction of Dynamic Hedging Strategy

Under no-arb pricing, we can fully hedge any contigent claim using following steps:

- The Portfolio (ϕ, ψ), where ϕ_t and ψ_t is the number of units of security and bond we hold at t. ϕ is $\mathcal{F}$-*previsible* (in other words, left continuous).
- Self-financing SDE $dV_t = \phi_t dS_t + \psi_t dB_t$.
- Suppose we are in a market of a riskless bond B and a risky security S with volatility σ_t, and a claim X on events up to time T. A *replication strategy* for X is a self-financing portfolio (ϕ, ψ) such that $\int_0^T \sigma_t^2 \phi_t^2 dt < \infty$ and $V_T = \phi_T S_T + \psi_T B_T = X$.

Black Scholes Formula

We assume a bond price B_t and a stock price S_t that follow

$$S_t = S_0 \exp(\sigma W_t + \mu t)$$
$$B_t = \exp(rt)$$

1. Find a measure Q under which $Z_t := B_t^{-1} S_t$ is a martingale. B_t is therefore the *numeraire*.
2. Find the process $E_t = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$.
3. Find a previsible process ϕ, such that $dE_t = \phi_t dZ_t$
4. We then have

$$V_t = B_t \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$$
$$V_0 = e^{-rT} \mathbb{E}_{\mathbb{Q}}(X)$$

which can be easily solved analytically.

Application to FX: assumptions

We assume constant interest rate for domestic and foreign currency, and the exchange rate expressed in units of domestic currency per 1 foreign currency follows GBM.

$$B_t = e^{rt}$$
$$D_t = e^{ut}$$
$$S_t = S_0 \exp(\sigma W_t + \mu t)$$

for some W_t a $\mathbb{P}$-Brownian motion and constants r, u,σ, and μ, where D_t is the foreign cash bond.

Application to FX: approach

- Choose the numeraire: B_t.
- Make all tradables in this numeraire a martigale at the same time.
  - demostic bond becomes 1.
  - foreign bond is not tradable in domestic ccy
  - fx rate itself is not a tradable
  - foreign bond multiply by the fx spot is. $Z_t = B_t^{-1} S_t D_t$ needs to be a martingale.
- form the martigale process of the derivative value $E_t = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$.
- find the hedging strategy ϕ_t, s.t. $dE_t = \phi_t dZ_t$.

Application to FX: general formula

- original converted foreign bond process in numeraire B_t:

$$Z_t = S_0 \exp(\sigma W_t + (\mu + u - r)t)$$

- we need to make it into a martingale by changing the drift:

$$Z_t = S_0 \exp(\sigma \tilde{W}_t - \frac{1}{2}\sigma^2 t)$$

- therefore the fx process becomes via the same drift change:

$$S_t = S_0 \exp(\sigma \tilde{W}_t + (r - u - \frac{1}{2}\sigma^2)t)$$

- so we have the derivative pricing formula:

$$V_t = B_t \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$$

Application to FX: forward

Let's price a forward $V_T = S_T - K$ using the formula.

$$V_t = B_t \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$$

$$V_0 = e^{-rT}(\mathbb{E}_{\mathbb{Q}}(S_T) - K)$$

$$V_0 = e^{-rT}(S_0 e^{(r-u)T} - K)$$

Application to FX: Call option

Let's price a call option $V_T = (S_T - K)^+$ using the formula.

$$V_t = B_t \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$$
$$V_0 = e^{-rT}(\mathbb{E}_{\mathbb{Q}}(S_T - K)^+$$
$$V_0 = e^{-rT}(F_0 N(d_1) - K N(d_2))$$

Illustration of expectation of lognormal distribution:
$F_T = F_0 e^{\sigma\sqrt{T}\tilde{W}_T - \frac{1}{2}\sigma^2 T}$:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(1_{F_T > K}) &= P(F_T > K) \\
&= P(F_0 e^{\sigma\sqrt{T}\tilde{W}_T - \frac{1}{2}\sigma^2 T} > K) \\
&= P(\tilde{W}_T > \frac{\ln\frac{K}{F_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}) \\
&= P(\tilde{W}_T < \frac{\ln\frac{F_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}) = N(d_2)
\end{aligned}$$

Application to FX: view from foreign side

Let's price a put option from the foreign investor point of view. A call on 1 EUR for 1.25 USD should be the same as 1.25 put on 1 USD for 0.8 EUR. Assuming spot and forward are all 1, and interest rates are all 0. We have

$$C = FN(d_1) - KN(d_2) = N(d_1) - 1.25N(d_2)$$

$$1.25\tilde{P} = 1.25(\tilde{K}N(-\tilde{d}_2) - \tilde{F}N(-\tilde{d}_1)) = N(-\tilde{d}_2) - 1.25N(-\tilde{d}_1)$$

$$d_1 = \frac{\ln\frac{F}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln 0.8 + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -\frac{\ln 1.25 - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -\tilde{d}_2$$

Whether we use domestic measure or foreign one, the pricing and hedging is consistent.

Application to Quantos: Assumptions

Assuming a foreign asset, and fx process both follow lognormal process, with constant interest rates.

$$dS_t = \mu S_t d_t + \sigma_1 S_t dW_1(t)$$
$$dC_t = v C_t d_t + \sigma_2 C_t dW_2(t)$$

dW_1 and dW_2 is correlated at ρ.

Application to Quantos: Numeraire and tradables

- B_t, numeraire.
- C_tD_t, dollar tradable of foreign bond
- C_tS_t, dollar tradable of foreign asset

We need to do a change of measure so that all of the tradables are martingales

$$Y_t = B_t^{-1}C_tD_t$$
$$Z_t = B_t^{-1}C_tS_t$$

Application to Quantos: formula

The result of the change is expected: fx process same as the fx option case, quantoed asset process just change the drift, due to correlation to fx process.

$$dS_t = (u - \rho\sigma_1\sigma_2)S_t d_t + \sigma_1 S_t d\tilde{W}_1(t)$$
$$dC_t = (r - u)C_t d_t + \sigma_2 C_t d\tilde{W}_2(t)$$

The pricing formula is as usual:

$$V_t = B_t \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t),$$

where X is a function of S_T like a forward or call.

Application to Composite

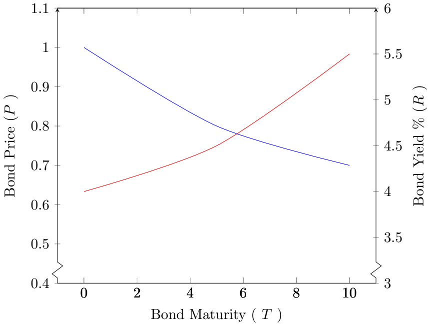
- What is the tradable?
- What is the process?
- Using a single composite process with a simple payoff generates the same answer as using the quanto processes with a composite payoff

Application to Interest Rate

$P(t,T)$ discount bond price with maturity T at t

$R(t,T)$ bond yield (continuous compounding).

$R(t,T) = -\frac{\ln P(t,T)}{T-t}$



Additional IR basics

A bond price can be computed from today's (instantaneous) forward rates or the spot rate.

- $f(t,T) = -\frac{\partial}{\partial T} \ln P(t,T)$
- $r(t) = f(t,t)$
- $P(t,T) = \exp(-\int_t^T f(t,u)du)$
- $R(t,T) = \frac{\ln P(t,T)}{T-t}$
- $P(t,T) = \exp(-(T-t)R(t,T))$

Is it correct to say $P(t,T) = \exp(-\int_t^T r(u)du)$?

HJM 1F

Assuming all forward rates are driving by the same random source, but can have different drift and vol.

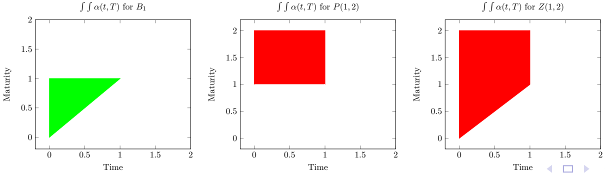
- $df(t,T) = \sigma(t,T)dW_t + \alpha(t,T)dt$
- $f(t,T) = f(0,T) + \int_0^t \sigma(s,T)dW_s + \int_0^t \alpha(s,T)ds, 0 \leq t \leq T$

We skip the technical conditions here, but basically things should be finite.

Short Rate, Bonds, and Numeraire

We can choose any tradable asset as numeraire. For convenience we normally use either the cash bond/money market account B_t, or some discount bond $P(t,T)$.

- $r(t) = f(0,t) + \int_0^t \sigma(s,t)dW_s + \int_0^t \alpha(s,t)ds$
- $B_t = \exp(\int_0^t r_s ds) = e^{\left(\int_0^t (\int_s^t \sigma(s,u)du)dW_s + \int_0^t f(0,u)du + \int_0^t \int_s^t \alpha(s,u)du\, ds\right)}$
- $P(t,T) = \exp(-\int_t^T f(t,u)dt) = e^{-\left(\int_0^t (\int_t^T \sigma(s,u)du)dW_s + \int_t^T f(0,u)du + \int_0^t \int_t^T \alpha(s,u)du\, ds\right)}$
- $Z(t,T) = \frac{P(t,T)}{B_t} = e^{\int_0^t \Sigma(s,T)dW_s - \int_0^T f(0,u)du - \int_0^t \int_s^T \alpha(s,u)du\, ds}$, where $\Sigma(t,T) = -\int_t^T \sigma(t,u)du$.



Change of Numeraire

Using Ito lemma, we have
$dZ(t,T) = Z(t,T)\left(\Sigma(t,T)dW_t + (\frac{1}{2}\Sigma^2(t,T) - \int_t^T \alpha(t,u)du)dt\right)$.
Using Girsanov theorem, we have

$$dZ(t,T) = Z(t,T)\Sigma(t,T)d\tilde{W}_t$$

$$\tilde{W}_t = W_t + \int_0^t \gamma_s ds$$

$$\gamma_t = \frac{1}{2}\Sigma(t,T) - \frac{1}{\Sigma(t,T)}\int_t^T \alpha(t,u)du$$

Note the γ_s is computed from $\alpha(t,T)$ and $\sigma(t,T)$, but cannot not depend on T, so this puts restrictions on the functional form of real world drift and vol.

Rates under $\mathbf{Q}$

We should have 0 by differentiate γ against T, so

$$
\begin{aligned}
\alpha(t,T) &= \sigma(t,T)(\gamma_t - \Sigma(t,T)) \\
df(t,T) &= \sigma(t,T)d\tilde{W}_t - \sigma(t,T)\Sigma(t,T)dt \\
r(t) &= f(0,t) + \int_0^t \sigma(s,t)d\tilde{W}_s - \int_0^t \sigma(s,t)\Sigma(s,t)ds
\end{aligned}
$$

- All bonds have to be made into martingale at the same time, so the market is arbitrage free.
- Real world drift does not matter, and the vol remains the same. Drift of the bonds goes to 0 not the rates.
- Short rate models can be expressed in HJM format (HW, CIR, BK) and can have closed form solution.

Application to Commodity

In commodity, we need to model the forward/futures curve like rates. We know all futures have zero drift.

- Future price equals to forward price, when correlation to margin interest is assumed to be zero.
- Forward price process has zero drift.
- Forward contract $B_t(F_t - K)$ and futures contract $M_t(F_t - K)$ are both tradables. B_t is usual zcb, while M_t is a money market account that earns risk free interest on the balance.
- Forward contract value with zcb as the numeraire $B_t^{-1} B_t F_t$ is a martingale.
- Similarly future contract with either B_t or M_t as numeraire $M_t^{-1} M_t F_t$ is also martingale. In fact, ratio of any two tradables' value is a martingale under complete market no arbitrage.

Assuming a lognormal model for com future, we have

$$\frac{dF(t,T)}{F(t,T)} = \sigma(t,T)d\tilde{W}(t)$$

It is easy to generalize it to 2-factor model:

$$\frac{dF(t,T)}{F(t,T)} = \sigma_1(t,T)d\tilde{W}_1(t) + \sigma_2(t,T)d\tilde{W}_2(t)$$

But good models should be able to both fit/generate realistic scenarios and have parameters that can be controlled with some interpretation. People tends to prefer the model to be specified in a certain way that relates to real world observables (e.g. below Gabillon model), then solve by transform back to a general problem (e.g. Anderson).

$$dF(t,T) = F(t,T)(\sigma_S(t)e^{-\lambda(T-t)}d\tilde{W}_1(t) + \sigma_L d\tilde{W}_2(t))$$