Viscoacoustic anisotropic wave equations

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SUMMARY

The wave equation plays a central role in seismic modeling, processing and inversion. The importance of including attenuation in our wave description has prompted the development of viscoacoustic wave equations, especially for *P*-wave modeling in regions prevalent with gas clouds. We present a set of scalar and vector viscoacoustic wave equations for orthorhombic anisotropy. For the Kelvin-Voigt, Maxwell, standard-linear-solid (SLS) and Kjartansson models, we derive their corresponding wave equations in differential form, respectively. We also derive the point-source solution of the scalar wave equation. Numerical examples show the effect of attenuation anisotropy on the point-source acoustic radiation.

INTRODUCTION

Seismic anisotropy can be used to describe the directional variation of wave velocity and attenuation. The acoustic nonattenuating anisotropic wave equations (Alkhalifah, 2000, 2003) have been widely implemented in applied seismology. However, the nonattenuating wave equations lack the ability to describe wave propagation in attenuating anisotropic media, because the attenuation due to energy absorption is an important component in the wave amplitude decay in such media. Incorporating the attenuation anisotropy into the acoustic anisotropic wave equations provides a choice for forward and inverse seismic modeling in fluid-filled anisotropic reservoirs.

The wave equations for specific viscoacoustic transversely isotropic (TI) models have been proposed. Qu et al. (2017) proposed the viscoacoustic TI wave equation in quaisi-differential equation form. da Silva et al. (2019) proposed the viscoacoustic wave equation for the generalized SLS model. These existing viscoacoustic wave equations naturally rely on a specific viscoacoustic model, which means that if we take into account other viscoacoustic models, we have to restart the derivation of the wave equation. Since rocks in the Earth exhibit various viscoacoustic behaviors, we possibly need the viscoacoustic wave equations in a more general form.

In this expanded abstract, we will derive the scalar and vector viscoacoustic wave equations for a general viscoacoustic orthorhombic model. We need to switch between the time-space and frequency-wavenumber domains. The Fourier transform is defined as

$$\hat{f}(\mathbf{k},\omega) = \int\int\int\int f(\mathbf{x},t)e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t,$$

and its inverse is given by

$$f(\mathbf{x},t) = \frac{1}{(2\pi)^4}\int\int\int\int \hat{f}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\mathrm{d}\omega,$$

where the upper and lower bounds ($-\infty$ and ∞) in the integrals are omitted for brevity, $\mathbf{x}=(x,y,z)^T$ and $\mathbf{k}=(k_1,k_2,k_3)^T$ denote the position and wavenumber vectors, respectively, t and ω denote the time and the angular frequency, respectively.

For linear viscoelastic media, the time-domain constitutive equation describing the relation between stress and strain is expressed in a mathematical form by Boltzmann superposition principle (Hudson, 1980). We write the constitutive equation for a general viscoelastic anisotropic medium as

$$\sigma_{ij}(\mathbf{x},t) = \psi_{ijkl}(t)\odot e_{ij}(\mathbf{x},t), \tag{1}$$

where ψ_{ijkl} denote the fourth-rank relaxation functions, e_{ij} denote the strain components, and the operation $\odot$ describes the Riemann-Stieltjes convolution integral defined as

$$g(t)\odot s(t) = \begin{cases} g(0+)s(t)+\int_{0+}^{t}\dot{g}(\tau)s(t-\tau)\mathrm{d}\tau, & \text{if } |g(0+)|<\infty \\ \int_{0+}^{t}\dot{g}(\tau)s(t-\tau)\mathrm{d}\tau, & \text{if } |g(0+)|\to\infty \end{cases}. \tag{2}$$

Here, "0+" means approaching to zero from the positive time axis, the dot on $\dot{g}$ denotes the first temporal derivative, and $g(t)\odot s(t) = s(t)\odot g(t)$.

VISCOACOUSTIC ANISOTROPIC WAVE EQUATIONS

For a general viscoelastic orthorhombic medium, we first combine Tsvankin's (1997) and Zhu and Tsvankin's (2007) notations to describe the density-normalized stiffness coefficients a_{ij} at all frequencies. We then set the shear-wave velocity parameter v_{S0} equal to zero for all frequencies. We next use the modified Alkhalifah's (2003) notation to replace Tsvankin's (1997) notation. Finally, all of these operations give rise to the viscoacoustic orthorhombic version of a_{ij} in terms of the parameters in the modified Alkhalifah's (2003) and Zhu and Tsvankin's (2007) notations as follows:

$$\begin{aligned}
a_{11}(\omega) &= v_{n2}^2(1-2ik_Q(1+\varepsilon_{Q2}))(1+2\eta_2),\\
a_{12}(\omega) &= v_{n1}v_{n2}\xi(1-2ik_Q(1+\varepsilon_{Q2}))\\
&\quad - ik_Q\delta_{Q3}(1+\varepsilon_{Q2})\frac{v_{n2}^3(1+2\eta_2)^2}{v_{n1}\xi},\\
a_{13}(\omega) &= v_{P0}v_{n2}(1-2ik_Q) - ik_Q\delta_{Q2}\frac{v_{P0}^3}{v_{n2}},\\
a_{22}(\omega) &= v_{n1}^2(1-2ik_Q(1+\varepsilon_{Q1}))(1+2\eta_1),\\
a_{23}(\omega) &= v_{P0}v_{n1}(1-2ik_Q) - ik_Q\delta_{Q1}\frac{v_{P0}^3}{v_{n1}},\\
a_{33}(\omega) &= v_{P0}^2(1-2ik_Q),
\end{aligned} \tag{3}$$

with

$$k_Q(\omega) = \mathrm{sgn}(\omega)\frac{A_{P0}}{1-A_{P0}^2}, \quad \xi(\omega) = \sqrt{\frac{(1+2\eta_1)(1+2\eta_2)}{1+2\eta_3}},$$

where the argument ω of the quantities on the right side of these equations is omitted. i denotes the imaginary unit. v_{P0} denotes the vertical velocity in the nonattenuating reference

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medium ($A_{P0}=0$). The subscripts "1", "2" and "3" denote the x-, y- and z-axes normal to the $[y,z]$, $[x,z]$ and $[x,y]$ planes, respectively. v_{n1} and v_{n2} denote the NMO velocities in the $[y,z]$ plane of the reference medium. η_1, η_2 and η_3 denote the anellipticity parameters in the $[y,z]$, $[x,z]$ and $[x,y]$ planes, respectively, of the reference medium. A_{P0} denotes the wavenumber-normalized attenuation coefficient of the vertically propagating P-wave. ε_1 and δ_1 are the attenuation-anisotropy parameters defined in the $[y,z]$ plane. ε_2 and δ_2 are the attenuation-anisotropy parameters defined in the $[x,z]$ plane. δ_3 are the attenuation-anisotropy parameters in the $[x,y]$ plane.

We use the two-index notation ϕ_{ij} to denote the density-normalized relaxation functions in a viscoacoustic orthorhombic medium. We transform the attenuating orthorhombic eikonal equation (Hao and Alkhalifah, 2017) into the dispersion equation, and taking into account the inverse Fourier transform. Finally, we derive the viscoacoustic wave equation in the scalar form,

$$\begin{aligned}\frac{\partial^6 P}{\partial t^6} &= A\odot\frac{\partial^6 P}{\partial t^4\partial x^2}+B\odot\frac{\partial^6 P}{\partial t^4\partial y^2}+C\odot\frac{\partial^6 P}{\partial t^4\partial z^2}\\ &+D\odot\frac{\partial^6 P}{\partial t^2\partial x^2\partial y^2}+E\odot\frac{\partial^6 P}{\partial t^2\partial x^2\partial z^2}\\ &+F\odot\frac{\partial^6 P}{\partial t^2\partial y^2\partial z^2}+G\odot\frac{\partial^6 P}{\partial x^2\partial y^2\partial z^2},\end{aligned}\tag{4}$$

with

$$\begin{aligned}&A=\phi_{11}, B=\phi_{22}, C=\phi_{33}, D=\phi_{12}\odot\phi_{12}-\phi_{11}\odot\phi_{22},\\ &E=\phi_{13}\odot\phi_{13}-\phi_{11}\odot\phi_{33}, F=\phi_{23}\odot\phi_{23}-\phi_{22}\odot\phi_{33},\\ &G=-\phi_{13}\odot\phi_{13}\odot\phi_{22}+2\phi_{12}\odot\phi_{13}\odot\phi_{23}\\ &-\phi_{11}\odot\phi_{23}\odot\phi_{23}-\phi_{12}\odot\phi_{12}\odot\phi_{33}+\phi_{11}\odot\phi_{22}\odot\phi_{33},\end{aligned}$$

where the argument t is omitted from the functions A ... G and ϕ_{ij}, $P=P(\mathbf{x},t)$ denotes the pseudo-pressure wavefield.

Taking into account the equation of motion, the constitutive equation and the relation between stress and displacement, and assuming the density to be constant, we derive the acoustic wave equations in the vector form.

The wave equations in terms of the displacement and the normal stresses are

$$\begin{aligned}\frac{\partial^2 u_x}{\partial t^2} &= \phi_{11}\odot\frac{\partial^2 u_x}{\partial x^2}+\phi_{12}\odot\frac{\partial^2 u_y}{\partial x\partial y}+\phi_{13}\odot\frac{\partial^2 u_z}{\partial x\partial z},\\ \frac{\partial^2 u_y}{\partial t^2} &= \phi_{12}\odot\frac{\partial^2 u_x}{\partial x\partial y}+\phi_{22}\odot\frac{\partial^2 u_y}{\partial y^2}+\phi_{23}\odot\frac{\partial^2 u_z}{\partial y\partial z},\\ \frac{\partial^2 u_z}{\partial t^2} &= \phi_{13}\odot\frac{\partial^2 u_x}{\partial x\partial z}+\phi_{23}\odot\frac{\partial^2 u_y}{\partial y\partial z}+\phi_{33}\odot\frac{\partial^2 u_z}{\partial z^2},\end{aligned}$$

where u_i denote the displacement components.

The wave equations in terms of the normal stresses are

$$\begin{aligned}\frac{\partial^2 \sigma_{xx}}{\partial t^2} &= \phi_{11}\odot\frac{\partial^2 \sigma_{xx}}{\partial x^2}+\phi_{12}\odot\frac{\partial^2 \sigma_{yy}}{\partial y^2}+\phi_{13}\odot\frac{\partial^2 \sigma_{zz}}{\partial z^2},\\ \frac{\partial^2 \sigma_{yy}}{\partial t^2} &= \phi_{12}\odot\frac{\partial^2 \sigma_{xx}}{\partial x^2}+\phi_{22}\odot\frac{\partial^2 \sigma_{yy}}{\partial y^2}+\phi_{23}\odot\frac{\partial^2 \sigma_{zz}}{\partial z^2},\\ \frac{\partial^2 \sigma_{zz}}{\partial t^2} &= \phi_{13}\odot\frac{\partial^2 \sigma_{xx}}{\partial x^2}+\phi_{23}\odot\frac{\partial^2 \sigma_{yy}}{\partial y^2}+\phi_{33}\odot\frac{\partial^2 \sigma_{zz}}{\partial z^2},\end{aligned}\tag{5}$$

where σ_{ii} denote the normal stresses.

The wave equation in terms of the momentum density and the normal stresses are

$$\begin{aligned}&\frac{\partial J_x}{\partial t}=\frac{\partial \sigma_{xx}}{\partial x},\quad \frac{\partial J_y}{\partial t}=\frac{\partial \sigma_{yy}}{\partial y},\quad \frac{\partial J_z}{\partial t}=\frac{\partial \sigma_{zz}}{\partial z},\\ &\frac{\partial \sigma_{xx}}{\partial t}=\phi_{11}\odot\frac{\partial J_x}{\partial x}+\phi_{12}\odot\frac{\partial J_y}{\partial y}+\phi_{13}\odot\frac{\partial J_z}{\partial z},\\ &\frac{\partial \sigma_{yy}}{\partial t}=\phi_{12}\odot\frac{\partial J_x}{\partial x}+\phi_{22}\odot\frac{\partial J_y}{\partial y}+\phi_{23}\odot\frac{\partial J_z}{\partial z},\\ &\frac{\partial \sigma_{zz}}{\partial t}=\phi_{13}\odot\frac{\partial J_x}{\partial x}+\phi_{23}\odot\frac{\partial J_y}{\partial y}+\phi_{33}\odot\frac{\partial J_z}{\partial z},\end{aligned}$$

where $\mathbf{J}=\rho\partial\mathbf{u}/\partial t$ denotes the momentum density.

VISCOACOUSTIC ANISOTROPIC MODELS

We assume that the parameters in the modified Alkhalifah's (2003) and Zhu and Tsvankin's (2007) notations are known at a selected reference angular frequency ($\omega_c>0$), and denote them by putting a bar over them, e.g. $\bar{v}_{P0}=v_{P0}(\omega_c)$. The reference frequency corresponding to ω_c is denoted by f_c. The density-normalized stiffness coefficients at ω_c are denoted by $\bar{a}_{ij}=\bar{a}^R_{ij}-i\bar{a}^I_{ij}$. The elements of the quality factor matrix are denoted by $\bar{Q}_{ij}=\bar{a}^R_{ij}/\bar{a}^I_{ij}$. Using these parameters, we can determine the density-normalized relaxation functions ϕ_{ij} and stiffness coefficients a_{ij} for the following classic viscoacoustic models (Carcione, 2015).

The Kelvin-Voigt model:

$$\begin{aligned}\phi_{ij}(t)&=\eta^{(1)}_{ij}\delta(t)+m^{(1)}_{ij}h(t),\\ a_{ij}(\omega)&=m^{(1)}_{ij}-i\omega\eta^{(1)}_{ij},\end{aligned}$$

with

$$m^{(1)}_{ij}=\bar{a}^R_{ij},\quad \eta^{(1)}_{ij}=\frac{\bar{a}^R_{ij}}{2\pi f_c\bar{Q}_{ij}},$$

where $\delta(t)$ denotes the Dirac delta function, $h(t)$ denotes the Heaviside step function, $m^{(1)}_{ij}$ and $\eta^{(1)}_{ij}$ denote the density-normalized elastic stiffness coefficients, and viscous coefficients.

The Maxwell model:

$$\begin{aligned}\phi_{ij}(t)&=m^{(2)}_{ij}e^{-m^{(2)}_{ij}t/\eta^{(2)}_{ij}}h(t),\\ a_{ij}(\omega)&=\left(\frac{1}{m^{(2)}_{ij}}+\frac{i}{\omega\eta^{(2)}_{ij}}\right)^{-1},\end{aligned}$$

with

$$m^{(2)}_{ij}=\bar{a}^R_{ij}\left(1+\frac{1}{\bar{Q}^2_{ij}}\right),\quad \eta^{(2)}_{ij}=\frac{\bar{a}^R_{ij}}{2\pi f_c}\left(\bar{Q}_{ij}+\frac{1}{\bar{Q}_{ij}}\right).$$

where $m^{(2)}_{ij}$ and $\eta^{(2)}_{ij}$ denote the density-normalized unrelaxed stiffness coefficients and viscous coefficients.

The SLS model:

$$\begin{aligned}\phi_{ij}(t)&=m^{(3)}_{ij}\left(1-\left(1-\frac{\tau_{ij}}{\tau_\sigma}\right)e^{-t/\tau_\sigma}\right)h(t),\\ a_{ij}(\omega)&=m^{(3)}_{ij}\frac{1-i\omega\tau_{ij}}{1-i\omega\tau_\sigma},\end{aligned}$$

with

$$m_{ij}^{(3)} = \bar{a}_{ij}^{R}\left(1-\frac{1}{\bar{Q}_{ij}}\right),\ \tau_{ij} = \tau_\sigma \frac{\bar{Q}_{ij}+1}{\bar{Q}_{ij}-1},\ \tau_\sigma = \frac{1}{2\pi f_c},$$

where $m_{ij}^{(3)}$ denote the density-normalized relaxed stiffness coefficients, τ_{ij} denote the strain relaxation times, and τ_σ denotes the stress relaxation time.

The Kjartansson model:

$$\phi_{ij}(t) = \frac{m_{ij}^{(4)}}{\Gamma(1-2\gamma_{ij})}\left(\frac{t}{t_c}\right)^{-2\gamma_{ij}} h(t),$$

$$a_{ij}(\omega) = m_{ij}^{(4)}\left(\frac{-i\omega}{\omega_c}\right)^{2\gamma_{ij}},$$

with

$$\gamma_{ij} = \arctan\left(\frac{1}{\bar{Q}_{ij}}\right),\ m_{ij}^{(4)} = \frac{\bar{a}_{ij}^{R}}{\cos(\pi\gamma_{ij})},\ \omega_c = \frac{1}{t_c} = 2\pi f_c,$$

where $\Gamma(.)$ denotes the gamma function, $m_{ij}^{(4)}$ denote the density-normalized stiffness coefficients as $\gamma_{ij}\to 0$ for $t\neq 0$ and $\omega\neq 0$, and t_c denotes the reference time (Kjartansson, 1979).

It is noteworthy that in all of the expressions for $a_{ij}(\omega)$ shown in this section, the sign in front of the imaginary unit "i" corresponds to our definition of Fourier transform.

THE WAVE EQUATIONS

For the viscoacoustic anisotropic models shown in the previous section, we substitute the expressions for ϕ_{ij} into the vector wave equations 5 and take into account equation 2. Finally, we derive the wave equations in differential form.

The wave equations for the Kelvin-Voigt model are

$$\begin{aligned}
\frac{\partial^2\sigma_{xx}}{\partial t^2} &= m_{11}^{(1)}\frac{\partial^2\sigma_{xx}}{\partial x^2} + m_{12}^{(1)}\frac{\partial^2\sigma_{yy}}{\partial y^2} + m_{13}^{(1)}\frac{\partial^2\sigma_{zz}}{\partial z^2} \\
&\quad + \eta_{11}^{(1)}\frac{\partial^3\sigma_{xx}}{\partial t\partial x^2} + \eta_{12}^{(1)}\frac{\partial^3\sigma_{yy}}{\partial t\partial y^2} + \eta_{13}^{(1)}\frac{\partial^3\sigma_{zz}}{\partial t\partial z^2},\\
\frac{\partial^2\sigma_{yy}}{\partial t^2} &= m_{12}^{(1)}\frac{\partial^2\sigma_{xx}}{\partial x^2} + m_{22}^{(1)}\frac{\partial^2\sigma_{yy}}{\partial y^2} + m_{23}^{(1)}\frac{\partial^2\sigma_{zz}}{\partial z^2} \\
&\quad + \eta_{12}^{(1)}\frac{\partial^3\sigma_{xx}}{\partial t\partial x^2} + \eta_{22}^{(1)}\frac{\partial^3\sigma_{yy}}{\partial t\partial y^2} + \eta_{23}^{(1)}\frac{\partial^3\sigma_{zz}}{\partial t\partial z^2},\\
\frac{\partial^2\sigma_{zz}}{\partial t^2} &= m_{13}^{(1)}\frac{\partial^2\sigma_{xx}}{\partial x^2} + m_{23}^{(1)}\frac{\partial^2\sigma_{yy}}{\partial y^2} + m_{33}^{(1)}\frac{\partial^2\sigma_{zz}}{\partial z^2} \\
&\quad + \eta_{13}^{(1)}\frac{\partial^3\sigma_{xx}}{\partial t\partial x^2} + \eta_{23}^{(1)}\frac{\partial^3\sigma_{yy}}{\partial t\partial y^2} + \eta_{33}^{(1)}\frac{\partial^3\sigma_{zz}}{\partial t\partial z^2}.
\end{aligned}$$

The wave equations for the Maxwell model are

$$\begin{aligned}
\frac{\partial^2\sigma_{xx}}{\partial t^2} &= m_{11}^{(2)}\frac{\partial^2\sigma_{xx}}{\partial x^2} + m_{12}^{(2)}\frac{\partial^2\sigma_{yy}}{\partial y^2} + m_{13}^{(2)}\frac{\partial^2\sigma_{zz}}{\partial z^2} - \sum_{j=1}^{3} r_{1j},\\
\frac{\partial^2\sigma_{yy}}{\partial t^2} &= m_{12}^{(2)}\frac{\partial^2\sigma_{xx}}{\partial x^2} + m_{22}^{(2)}\frac{\partial^2\sigma_{yy}}{\partial y^2} + m_{23}^{(2)}\frac{\partial^2\sigma_{zz}}{\partial z^2} - \sum_{j=1}^{3} r_{2j},\\
\frac{\partial^2\sigma_{zz}}{\partial t^2} &= m_{13}^{(2)}\frac{\partial^2\sigma_{xx}}{\partial x^2} + m_{23}^{(2)}\frac{\partial^2\sigma_{yy}}{\partial y^2} + m_{33}^{(2)}\frac{\partial^2\sigma_{zz}}{\partial z^2} - \sum_{j=1}^{3} r_{3j},
\end{aligned}$$

where the auxiliary variables r_{ij} satisfy

$$\eta_{ij}^{(2)}\frac{\partial r_{ij}}{\partial t} = \left(m_{ij}^{(2)}\right)^2\frac{\partial^2\sigma_{jj}}{\partial x_j^2} - m_{ij}^{(2)} r_{ij}.$$

Here, the subscripts "i" and "j" are taken as 1, 2 and 3 or x, y and z, the repeated indices do not imply Einstein summation convention, and both $m_{ij}^{(2)}$ and $\eta_{ij}^{(2)}$ are symmetric.

The wave equations for the SLS model are

$$\begin{aligned}
\frac{\partial^2\sigma_{xx}}{\partial t^2} &= \zeta_{11}m_{11}^{(3)}\frac{\partial^2\sigma_{xx}}{\partial x^2} + \zeta_{12}m_{12}^{(3)}\frac{\partial^2\sigma_{yy}}{\partial y^2} + \zeta_{13}m_{13}^{(3)}\frac{\partial^2\sigma_{zz}}{\partial z^2} - w_1,\\
\frac{\partial^2\sigma_{yy}}{\partial t^2} &= \zeta_{12}m_{12}^{(3)}\frac{\partial^2\sigma_{xx}}{\partial x^2} + \zeta_{22}m_{22}^{(3)}\frac{\partial^2\sigma_{yy}}{\partial y^2} + \zeta_{23}m_{23}^{(3)}\frac{\partial^2\sigma_{zz}}{\partial z^2} - w_2,\\
\frac{\partial^2\sigma_{zz}}{\partial t^2} &= \zeta_{13}m_{13}^{(3)}\frac{\partial^2\sigma_{xx}}{\partial x^2} + \zeta_{23}m_{23}^{(3)}\frac{\partial^2\sigma_{yy}}{\partial y^2} + \zeta_{33}m_{33}^{(3)}\frac{\partial^2\sigma_{zz}}{\partial z^2} - w_3,
\end{aligned}$$

where w_i are the auxiliary variables given by

$$\begin{aligned}
\frac{\partial w_1}{\partial t} &= n_{11}\frac{\partial^2\sigma_{xx}}{\partial x^2} + n_{12}\frac{\partial^2\sigma_{yy}}{\partial y^2} + n_{13}\frac{\partial^2\sigma_{zz}}{\partial z^2} - \frac{1}{\tau_\sigma}w_1,\\
\frac{\partial w_2}{\partial t} &= n_{12}\frac{\partial^2\sigma_{xx}}{\partial x^2} + n_{22}\frac{\partial^2\sigma_{yy}}{\partial y^2} + n_{23}\frac{\partial^2\sigma_{zz}}{\partial z^2} - \frac{1}{\tau_\sigma}w_2,\\
\frac{\partial w_3}{\partial t} &= n_{13}\frac{\partial^2\sigma_{xx}}{\partial x^2} + n_{23}\frac{\partial^2\sigma_{yy}}{\partial y^2} + n_{33}\frac{\partial^2\sigma_{zz}}{\partial z^2} - \frac{1}{\tau_\sigma}w_3,
\end{aligned}$$

and $n_{ij} = (1-\zeta_{ij})m_{ij}^{(3)}/\tau_\sigma$ and $\zeta_{ij} = \tau_{ij}/\tau_\sigma$.

The wave equations for the Kjartansson model are

$$\begin{aligned}
\frac{\partial^2\sigma_{xx}}{\partial t^2} &= \tilde{m}_{11}^{(4)}D_t^{2\gamma_{11}}\frac{\partial^2\sigma_{xx}}{\partial x^2} + \tilde{m}_{12}^{(4)}D_t^{2\gamma_{12}}\frac{\partial^2\sigma_{yy}}{\partial y^2} + \tilde{m}_{13}^{(4)}D_t^{2\gamma_{13}}\frac{\partial^2\sigma_{zz}}{\partial z^2},\\
\frac{\partial^2\sigma_{yy}}{\partial t^2} &= \tilde{m}_{12}^{(4)}D_t^{2\gamma_{12}}\frac{\partial^2\sigma_{xx}}{\partial x^2} + \tilde{m}_{22}^{(4)}D_t^{2\gamma_{22}}\frac{\partial^2\sigma_{yy}}{\partial y^2} + \tilde{m}_{23}^{(4)}D_t^{2\gamma_{23}}\frac{\partial^2\sigma_{zz}}{\partial z^2},\\
\frac{\partial^2\sigma_{zz}}{\partial t^2} &= \tilde{m}_{13}^{(4)}D_t^{2\gamma_{13}}\frac{\partial^2\sigma_{xx}}{\partial x^2} + \tilde{m}_{23}^{(4)}D_t^{2\gamma_{23}}\frac{\partial^2\sigma_{yy}}{\partial y^2} + \tilde{m}_{33}^{(4)}D_t^{2\gamma_{33}}\frac{\partial^2\sigma_{zz}}{\partial z^2},
\end{aligned}$$

where $\tilde{m}_{33}^{(4)} = m_{33}^{(4)}/\omega_c^{2\gamma_{ij}}$, and $D_t^{2\gamma_{ij}}$ denotes the fractional temporal derivative of order $2\gamma_{ij}$. The detailed description of fractional derivatives can be found in Oldham and Spanier (1974).

THE POINT-SOURCE ACOUSTIC RADIATION

We put a point-source term $f(\mathbf{x},t) = \frac{\partial^4 s(t)}{\partial t^4}\delta(x)\delta(y)\delta(z)$ into the right side of the scalar wave equation 4, where $s(t)$ is a causal signal. We utilize Buchwald's (1959) method and take into account the steepest descent approximation for the n-fold integral (Bleistein, 2012) to solve the scalar wave equation 4. Finally, we derive the single-frequency asymptotic point-source acoustic radiation:

$$\hat{P}(\mathbf{x},\omega) \approx \hat{s}\,\frac{\exp(i\omega\tau_R - i\nu)}{2\pi r\sqrt{|K|}|\partial\tilde{I}'/\partial\tilde{p}_3'|}\exp(-\omega\tau_I), \qquad (6)$$

with

$$K = \det\left[\frac{\partial^2\tilde{p}_3'}{\partial\tilde{p}_i'\partial\tilde{p}_j'}\right] = \frac{\partial^2\tilde{p}_3'}{\partial\tilde{p}_1'^2}\frac{\partial^2\tilde{p}_3'}{\partial\tilde{p}_2'^2} - \left(\frac{\partial^2\tilde{p}_3'}{\partial\tilde{p}_1'\partial\tilde{p}_2'}\right)^2,$$

$$\nu = \arg\left(\frac{\partial I'}{\partial\tilde{p}_3'}\right) + \frac{1}{2}\arg(K),\quad \tau_R = \frac{r}{V_{ray}},\quad \tau_I = A_{ray}r,$$

where r denotes the radial distance between the observation position and the coordinate origin. $\tilde{I}' = \tilde{I}'(\tilde{\mathbf{p}}')$ describes the P-wave slowness surface in the rotated coordinate system, where the rotated coordinate system is chosen so that the observation point $\mathbf{x}$ is always located at its positive z-axis. $\tilde{\mathbf{p}}' = (\tilde{p}'_1, \tilde{p}'_2, \tilde{p}'_3)^T$ describes the slowness vector at the saddle point. K denotes the Gaussian curvature of the slowness surface. τ_R and τ_I denote the real and imaginary parts of the complex traveltimes. In the heterogeneous case, the complex traveltimes can be obtained by solving the attenuating orthorhombic eikonal equation (Hao and Alkhalifah, 2017). V_{ray} and A_{ray} denote the ray velocity and attenuation (Vavryčuk, 2007), respectively. V_{ray} is always positive, but A_{ray} is positive for $\omega > 0$ and negative for $\omega < 0$, respectively. All of the quantities explained above are dependent on the spatial coordinates $\mathbf{x}$.

NUMERICAL EXAMPLES

The parameters at the reference frequency ($f_c = 40$ Hz) include the velocity-related parameters: $\bar{v}_{P0} = 3.0$ km/s, $\bar{v}_{n1} = 2.846$ km/s, $\bar{v}_{n2} = 3.286$ km/s, $\bar{\eta}_1 = 0.278$, $\bar{\eta}_2 = 0.167$, $\bar{\eta}_3 = 0.229$ and the attenuation-related parameters: $\bar{A}_{P0} = 0.0249$, $\bar{\varepsilon}_{Q1} = 0.66$, $\bar{\delta}_{Q1} = 0.52$, $\bar{\varepsilon}_{Q2} = -0.33$, $\bar{\delta}_{Q2} = 0.98$, $\bar{\delta}_{Q3} = 0.94$. We take into account not only the Kelvin-Voigt, Maxwell, SLS and Kjartansson models, but also a frequency-independent viscoacoustic model and a nonattenuating model. These viscoacoustic models are determined by all of the above parameters, while the nonattenuating model is obtained from the velocity-related parameters. We set $s(t)$ in the source term as a Ricker wavelet with the dominant frequency of 40Hz, where the role of $s(t)$ in the source term can be found before equation 6. We utilize the inverse Fourier transform of equation 6 to calculate the wavefield at a spherical surface with the radius $r = 1$ km. Since the wavefield is centrally symmetric with respect to the source position, we only take into account the polar and azimuthal angles varying from 0 to 90°. Figure 1 shows the directional variation of the waveforms for these viscoacoustic models, which is caused by the comprehensive effect of the velocity and attenuation anisotropies. Figure 2 shows that attenuation significantly decays the wave amplitudes. Figure 3 shows that the waveforms from the SLS model are similar to those from the Kjartansson model, and hence it implies that the generalized SLS model with an adequate number of elements can replace the Kjartansson model to simulate the nearly constant Q wave propagation.

CONCLUSIONS

We utilize the acoustic approximation to derive the scalar and vector viscoacoustic wave equations for a general viscoacoustic model. Given a specific viscoacoustic model, the corresponding wave equations can be obtained from these general wave equations. These wave equations can be used in viscoacoustic anisotropic wavefield modelling, imaging and inversion. The proposed wave equations can be extended to viscoacoustic anisotropic media with lower and/or tilted symmetry.

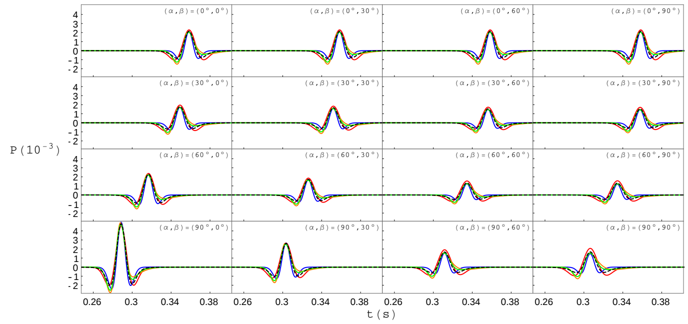

Figure 1: Waveforms observed at the propagation distance $r = 1km$ in the viscoacoustic orthorhombic models. α and β denote the polar and azimuthal angles of source-receiver direction, respectively. The red, blue, orange, green and dashed black lines correspond to the Kelvin-Voigt, Maxwell, SLS, Kjartansson and frequency-independent viscoacoustic models, respectively.

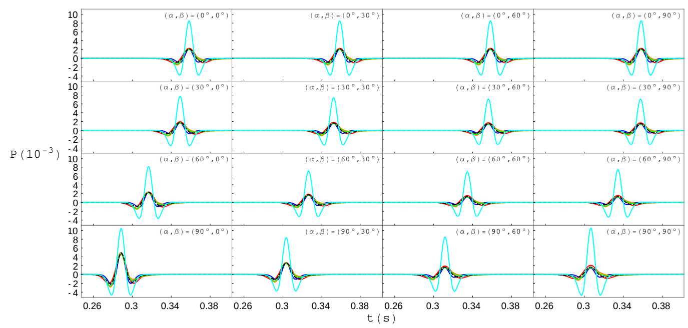

Figure 2: Similar to Figure 1, and the cyan lines correspond to the nonattenuating model. The other colored lines are the same as in Figure 1.

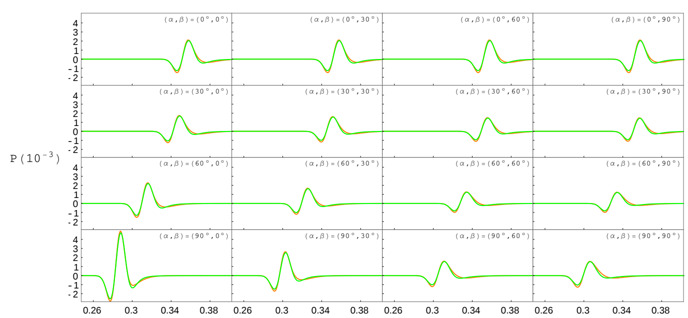

Figure 3: A comparison between the waveforms from the SLS (orange lines) and Kjartansson (green lines) models. These lines are the same as in Figure 1.

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