Acoustic eikonal solutions for attenuating VTI media

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Summary

We present two approximate methods to solve the acoustic eikonal equation for attenuating transversely isotropic media with a vertical symmetry axis (VTI). Perturbation theory is utilized to derive the perturbation equations for complex-valued traveltimes. The application of Shanks transform further enhances the accuracy of approximation. We derive both analytical and numerical solutions to the acoustic eikonal equation. The analytic solutions from the two methods are valid for homogeneous VTI media with moderate anellipticity and strong attenuation and attenuation-anisotropy. The numerical solutions are applicable for inhomogeneous attenuating VTI media.

Introduction

The attenuation of seismic waves as they propagate in reservoir rocks is closely linked to permeability, fluid content and saturation. Laboratory and field observations have revealed that anisotropic attenuation is a common phenomenon for wave propagation in reservoir rocks. In an attenuating medium, the traveltime of body-waves is complex-valued and it is governed by the complex-valued eikonal equation. The real part of the complex-valued traveltime corresponds to the phase of waves, while the imaginary part admits a decay in the amplitude of waves due to energy absorption.

In this expanded abstract, we propose two methods to solve the acoustic eikonal equation (Hao and Alkhalifah, 2017) for attenuating VTI media. Alkhalifah's (1997, 2000) notation and Zhu and Tsvankin's (2006) notation are combined to describe the P-wave traveltimes in an attenuating VTI medium, where Alkhalifah's notation and Zhu and Tsvankin's notation correspond to the nonattenuating and attenuating parts of the VTI medium, respectively. Alkhalifah's notation includes the P-wave vertical velocity υ_{P0}, the P-wave normal moveout velocity υ_n, and the anellipticity parameter η. Zhu and Tsvankin's notation includes the P-wave vertical attenuation coefficient A_{P0}, the fractional difference ε_Q between the horizontal and vertical attenuation coefficients of homogenous plane P-waves, and the second-order derivative δ_Q of the attenuation coefficient with respect to the phase angle of homogeneous plane P-waves along the vertical direction. The phase angle is measured from the vertical axis.

Acoustic eikonal equation for attenuating VTI media

For 2D attenuating VTI media, the acoustic eikonal equation is given by (Hao and Alkhalifah, 2017)

$$A\tau_{,x}^2 + B\tau_{,z}^2 + C\tau_{,x}^2\tau_{,z}^2 - 1 = 0 \ , \qquad (1)$$

where x and z denote the horizontal and vertical coordinate axes; τ denotes the complex-valued traveltime; the subscript "," followed by x or z denotes the first-order spatial derivative with respect to x or z; coefficients A, B and C are given by

$$A = \upsilon_n^2(1+2\eta)(1-2ik(1+\varepsilon_Q)) \ , \qquad (2)$$

$$B = \upsilon_{P0}^2(1-2ik) \ , \qquad (3)$$

$$\begin{aligned} C = {} & \frac{\upsilon_{P0}^2}{\upsilon_n^2}((1-2ik)\upsilon_n^2 - ik\delta_Q\upsilon_{P0}^2)^2 \\ & - \upsilon_{P0}^2\upsilon_n^2(1+2\eta)(1-2ik)(1-2ik(1+\varepsilon_Q)) \end{aligned} \ , \qquad (4)$$

with

$$k = \frac{A_{P0}}{1-A_{P0}^2} \ , \qquad (5)$$

where i denotes the imaginary unit; υ_{P0}, υ_n, η, A_{P0}, ε_Q and δ_Q are the noattenuating and attenuating parameters explained earlier.

Traveltime perturbation

Method 1

Equation 1 is generally expressed by

$$F(k,\eta,\tau_{,x},\tau_{,z}) = 0 \ , \qquad (6)$$

where F represents the left-hand side of equation 1.

The perturbation parameters ℓ_1 and ℓ_2 are introduced to scale k and η in equation 6,

$$F(\ell_1 k, \ell_2\eta, \tau_{,x}, \tau_{,z}) = 0 \ . \qquad (7)$$

The second-order perturbation solution to equation 7 is defined as

$$\tau = \tau_0 + i\tau_1\ell_1 + \tau_2\ell_2 + \tau_{11}\ell_1^2 + i\tau_{12}\ell_1\ell_2 + \tau_{22}\ell_2^2 \ , \qquad (8)$$

where τ_0, τ_i and τ_{ij} denote the zero-, first- and second-order traveltime coefficients, respectively; the imaginary unit "*i*" in front of τ_1 and τ_{12} guarantee that the governing equations for τ_1 and τ_{12} are always real-valued (as we will see in equations 10 and 11).

We expand equation 7 around $\ell_1 = 0$ and $\ell_2 = 0$ up to second-order and then insert equation 8. This allows us to obtain a series with respect to ℓ_1 and ℓ_2. From the fact that the zero-, first- and second-order expansion coefficients of the series are equal to zero, we derive the following governing equations,

$$\upsilon_n^2\tau_{0,x}^2+\upsilon_{P0}^2\tau_{0,z}^2=1 \ , \tag{9}$$

$$\upsilon_n^2\tau_{0,x}\tau_{i,x}+\upsilon_{P0}^2\tau_{0,z}\tau_{i,z}=f_i(k,\eta,\tau_0) \ , \ i=1,2, \tag{10}$$

$$\upsilon_n^2\tau_{0,x}\tau_{ij,x}+\upsilon_{P0}^2\tau_{0,z}\tau_{ij,z}=f_{ij}(k,\eta,\tau_0,\tau_1,\tau_2) \ , i\le j=1,2, \tag{11}$$

where f_i and f_{ij} are real functions.

Equation 9 governs the P-wave traveltimes in nonattenuating elliptically isotropic media. Once τ_0 is calculated from equation 9, equations 10 and 11 are successively solved to obtain τ_i and τ_{ij}.

For a homogeneous medium, the solutions to equations 9-11 are given by

$$\tau_0=\sqrt{\tau_x^2+\tau_z^2} \ , \tag{12}$$

$$\tau_1=k\upsilon_n^{-2}\tau_0^{-3}(\upsilon_{P0}^2\delta_Q\tau_x^2\tau_z^2+\upsilon_n^2((1+\varepsilon_Q)\tau_x^4+2\tau_x^2\tau_z^2+\tau_z^4)) \ , \tag{13}$$

$$\tau_2=-\eta\tau_0^{-3}\tau_x^4 \ , \tag{14}$$

$$\begin{aligned}\tau_{11}=-\frac{3}{2}k^2\upsilon_n^{-4}\tau_0^{-7}(&\upsilon_{P0}^4\delta_Q^2\tau_x^2\tau_z^2(\tau_x^4-\tau_x^2\tau_z^2+\tau_z^4)\\ &+2\upsilon_n^2\upsilon_{P0}^2\delta_Q\tau_x^2\tau_z^2((1-\varepsilon_Q)\tau_x^4+2(1+\varepsilon_Q)\tau_x^2\tau_z^2+\tau_z^4)\\ &+\upsilon_n^4((1+\varepsilon_Q)^2\tau x^8+4(1+\varepsilon_Q+\varepsilon_Q^2)\tau_x^6\tau_z^2\\ &+2(3+\varepsilon_Q)\tau_x^4\tau_z^4+4\tau_x^2\tau_z^6+\tau_z^8))\end{aligned} \ , \tag{15}$$

$$\begin{aligned}\tau_{12}=-k\eta\upsilon_n^{-2}\tau_0^{-7}\tau_x^4(&\upsilon_n^2(1+\varepsilon_Q)\tau_x^4\\ &+(-3\upsilon_{P0}^2\delta_Q+2\upsilon_n^2(1+4\varepsilon_Q))\tau_x^2\tau_z^2\\ &+(6\upsilon_{P0}^2\delta_Q+\upsilon_n^2(1-2\varepsilon_Q))\tau_z^4)\end{aligned} \ , \tag{16}$$

$$\tau_{22}=\frac{3}{2}\eta^2\tau_0^{-7}\tau_x^6(\tau_x^2+4\tau_z^2) \ , \tag{17}$$

with

$$\tau_x=x/\upsilon_n \ , \quad \tau_z=z/\upsilon_{P0} \ . \tag{18}$$

Setting $\ell_1=\ell_2=1$ in equation 8, we obtain the perturbation solution to the eikonal equation 1,

$$\tau=\tau_0+i\tau_1+\tau_2+\tau_{11}+i\tau_{12}+\tau_{22} \ . \tag{19}$$

Appling Shanks transform (Bender and Orszag, 1978, pp. 369–375) to equation 8 and setting $\ell_1=\ell_2=1$, we derive the other three approximate formulae,

$$\tau=\tau_0+\frac{(i\tau_1+\tau_2)^2}{\tau_2-\tau_{11}-\tau_{22}+i(\tau_1-\tau_{12})} \ , \tag{20}$$

$$\tau=\tau_0+\tau_2+\tau_{22}+\frac{(\tau_1+\tau_{12})^2}{\tau_{11}-i(\tau_1+\tau_{12})} \ , \tag{21}$$

$$\tau=\tau_0+i\tau_1+\tau_{11}+\frac{(\tau_2+i\tau_{12})^2}{\tau_2-\tau_{22}+i\tau_{12}} \ . \tag{22}$$

For equation 20, Shanks transform has been applied in terms of both parameters ℓ_1 and ℓ_2. For equations 21 and 22, Shanks transform has been applied in terms of parameters ℓ_1 and ℓ_2, respectively.

Method 2

In the second method, we use the horizontal velocity υ_h instead of the NMO velocity υ_n to parameterize an attenuating VTI medium. Considering the relation $\upsilon_h=\upsilon_n\sqrt{1+2\eta}$, equations 2 and 4 are rewritten as

$$A=\upsilon_h^2(1-2ik(1+\varepsilon_Q)) \ , \tag{23}$$

$$\begin{aligned}C=&\frac{\upsilon_{P0}^2}{\upsilon_h^2(1+2\eta)}((1-2ik)\upsilon_h^2-ik\upsilon_{P0}^2\delta_Q(1+2\eta))^2\\ &-\upsilon_{P0}^2\upsilon_h^2(1-2ik)(1-2ik(1+\varepsilon_Q))\end{aligned} \ . \tag{24}$$

Substituting equations 23 and 24, the left side of equation 1 is generally expressed by

$$\tilde{F}(k,\eta,\tau_{,x},\tau_{,z})=0 \ . \tag{25}$$

By analogy with method 1, perturbation parameters ℓ_1 and ℓ_2 are introduced to scale k and η in equation 25,

$$\tilde{F}(\ell_1k,\ell_2\eta,\tau_{,x},\tau_{,z})=0 \ . \tag{26}$$

The second-order perturbation solution to equation 7 is defined as

$$\tau=\tilde{\tau}_0+i\tilde{\tau}_1\ell_1+\tilde{\tau}_2\ell_2+\tilde{\tau}_{11}\ell_1^2+i\tilde{\tau}_{12}\ell_1\ell_2+\tilde{\tau}_{22}\ell_2^2 \ , \tag{27}$$

where $\tilde{\tau}_0$, $\tilde{\tau}_i$ and $\tilde{\tau}_{ij}$ denote the zero-, first- and second-order traveltime coefficients, respectively.

The following governing equations for traveltime coefficients are given by

$$\upsilon_h^2\tilde{\tau}_{0,x}^2+\upsilon_{P0}^2\tilde{\tau}_{0,z}^2=1 \ , \tag{28}$$

$$\upsilon_h^2\tilde{\tau}_{0,x}\tilde{\tau}_{i,x}+\upsilon_{P0}^2\tilde{\tau}_{0,z}\tilde{\tau}_{i,z}=f_i(k,\eta,\tilde{\tau}_0) \ , \ i=1,2, \tag{29}$$

$$\upsilon_h^2\tilde{\tau}_{0,x}\tilde{\tau}_{ij,x}+\upsilon_{P0}^2\tilde{\tau}_{0,z}\tilde{\tau}_{ij,z}=\tilde{f}_{ij}(k,\eta,\tilde{\tau}_0,\tilde{\tau}_1,\tilde{\tau}_2) \ , \ i\le j=1,2, \tag{30}$$

where $\tilde{f}_i$ and $\tilde{f}_{ij}$ are real functions.

For a homogeneous medium, the solutions to equations 28-30 are given by

$$\tilde{\tau}_0=\sqrt{\tilde{\tau}_x^2+\tilde{\tau}_z^2} \ , \tag{31}$$

$$\tilde{\tau}_1=\upsilon_h^{-2}\tilde{\tau}_0^{-3}k(\upsilon_{P0}^2\delta_Q\tau_x^2\tau_z^2+\upsilon_h^2((1+\varepsilon_Q)\tau_x^4+2\tau_x^2\tau_z^2+\tau_z^4)) \ , \tag{32}$$

$$\tilde{\tau}_2=\eta\tilde{\tau}_0^{-3}\tilde{\tau}_x^2\tilde{\tau}_z^2 \ , \tag{33}$$

$$\begin{aligned}\tilde{\tau}_{11}=-\frac{3}{2}\upsilon_h^{-4}\tilde{\tau}_0^{-7}k^2(&\upsilon_{P0}^4\delta_Q^2\tilde{\tau}_x^2\tilde{\tau}_z^2(\tilde{\tau}_x^4-\tilde{\tau}_x^2\tilde{\tau}_z^2+\tilde{\tau}_z^4)\\ &+2\upsilon_{P0}^2\upsilon_h^2\delta_Q\tilde{\tau}_x^2\tilde{\tau}_z^2((1-\varepsilon_Q)\tilde{\tau}_x^4+2(1+\varepsilon_Q)\tilde{\tau}_x^2\tilde{\tau}_z^2+\tilde{\tau}_z^4)\\ &+\upsilon_h^4((1+\varepsilon_Q)^2\tilde{\tau}_x^8+4(1+\varepsilon_Q+\varepsilon_Q^2)\tilde{\tau}_x^6\tilde{\tau}_z^2\\ &+2(3+\varepsilon_Q)\tilde{\tau}_x^4\tilde{\tau}_z^4+4\tilde{\tau}_x^2\tilde{\tau}_z^6+\tilde{\tau}_z^8))\end{aligned} \ , \tag{34}$$

$$\begin{aligned}\tilde{\tau}_{12}=k\eta\upsilon_h^{-2}\tilde{\tau}_0^{-7}\tilde{\tau}_x^2\tilde{\tau}_z^2(&\upsilon_h^2((1-3\varepsilon_Q)\tilde{\tau}_x^4\\ &+2(1+3\varepsilon_Q)\tilde{\tau}_x^2\tilde{\tau}_z^2+\tilde{\tau}_z^4)\\ &+\upsilon_{P0}^2\delta_Q(4\tilde{\tau}_x^4-\tilde{\tau}_x^2\tilde{\tau}_z^2+4\tilde{\tau}_z^4))\end{aligned} \ , \tag{35}$$

$$\tilde{\tau}_{22} = -\frac{9}{2}\tilde{\tau}_0^{-7}\eta^2\tilde{\tau}_x^4\tilde{\tau}_z^4 \ , \tag{36}$$

with

$$\tilde{\tau}_x = x / \upsilon_h \ , \quad \tilde{\tau}_z = z / \upsilon_{P0} \ . \tag{37}$$

Setting $\ell_1 = \ell_2 = 1$ in equation 27, we obtain the alternative perturbation solution to the eikonal equation 1,

$$\tau = \tilde{\tau}_0 + i\tilde{\tau}_1 + \tilde{\tau}_2 + \tilde{\tau}_{11} + i\tilde{\tau}_{12} + \tilde{\tau}_{22} \, . \tag{38}$$

Similar to equations 20-22, we derive from equation 27 the other three Shanks formulae,

$$\tau = \tilde{\tau}_0 + \frac{(i\tilde{\tau}_1 + \tilde{\tau}_2)^2}{\tilde{\tau}_2 - \tilde{\tau}_{11} - \tilde{\tau}_{22} + i(\tilde{\tau}_1 - \tilde{\tau}_{12})} \ , \tag{39}$$

$$\tau = \tilde{\tau}_0 + \tilde{\tau}_2 + \tilde{\tau}_{22} + \frac{(\tilde{\tau}_1 + \tilde{\tau}_{12})^2}{\tilde{\tau}_{11} - i(\tilde{\tau}_1 + \tilde{\tau}_{12})} \ , \tag{40}$$

$$\tau = \tilde{\tau}_0 + i\tilde{\tau}_1 + \tilde{\tau}_{11} + \frac{(\tilde{\tau}_2 + i\tilde{\tau}_{12})^2}{\tilde{\tau}_2 - \tilde{\tau}_{22} + i\tilde{\tau}_{12}} \ . \tag{41}$$

For equation 39, Shanks transform has been applied in terms of both parameters ℓ_1 and ℓ_2. For equations 40 and 41, Shanks transform has been applied in terms of parameters ℓ_1 and ℓ_2, respectively.

Fast marching scheme

Let us consider the fast marching scheme to numerically solve the governing equations developed in the previous section.

The first example in the "Numerical examples" section will show that among all of the analytic eikonal solutions shown in methods 1 and 2, equation 41 is the most accurate for a homogeneous attenuating VTI medium. Here, we only discuss the numerical procedure to calculate traveltime from equation 41.

First, we evaluate the zero-order traveltime coefficient $\tilde{\tau}_0$ from equation 28. Since the reference medium is nonattenuating elliptically isotropic, equation 28 can be numerically solved using the existing fast marching scheme (Alkhalifah, 2002, 2011).

Second, we evaluate the first- and second-order traveltime coefficients $\tilde{\tau}_i$ and $\tilde{\tau}_{ij}$ by successively solving equations 29 and 30 along the order of calculating $\tilde{\tau}_0$.

Lastly, we evaluate traveltime τ from equation 41.

For method 1, the procedure shown above only needs to be slightly modified to fit equations 9-11.

Numerical Examples

The comparison between Figures 1 to 4 shows that equation 41 is most accurate for homogenous VTI model. Figure 6 shows the traveltime calculated from equation 41 for the attenuating version of the Marmousi model.

Conclusions

The combination of perturbation theory and an appropriate Shanks transform provides accurate acoustic eikonal solution for homogeneous attenuating VTI models with moderate anellipticity and strong attenuation and attenuation-anisotropy. The proposed numerical scheme is applicable for 2D realistic attenuating VTI models.

Acknowledgments

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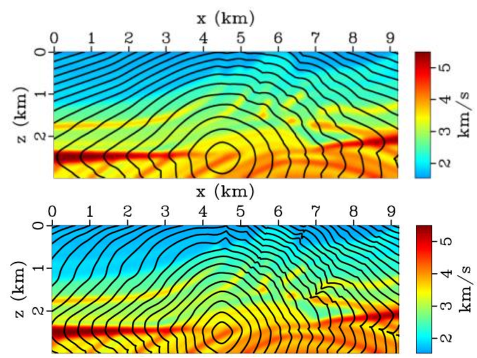

Figure 6: The contours of the real (top) and imaginary (bottom) parts of the traveltimes from the attenuating VTI model illustrated in Figure 3. The point source is located at *x*=4.5km and *z*=2.5km. For the plot on top, the time of the closet contour line around the source is 0.1s, and the time interval between two neighboring contour lines is also 0.1s. For the plot on bottom, the time of the closet contour line around the source is 2ms, and the time interval between two neighboring contour lines is also 2ms. In both plots, the background behind the contour lines shows the υ_{P0} model from Figure 5.

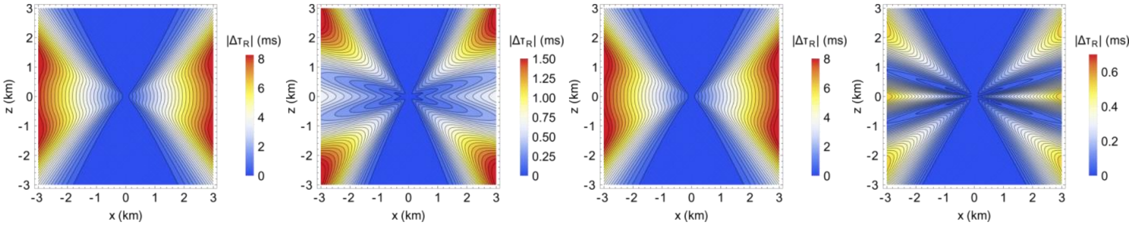

Figure 1: Errors in the real part of complex-valued traveltime. From left to right, the plots correspond to formulae 19, 20, 21 and 22, respectively. The model parameters are $\upsilon_{P0} = 3$km/s , $\upsilon_n = 3.286$km/s , $\eta = 0.167$, $A_{P0} = 0.02498$ (corresponding to $Q_{33} = 20$, where Q_{33} denotes the quality factor of P-waves propagating along the vertical axis), $\varepsilon_Q = -0.33$ and $\delta_Q = 0.98$. These model parameters are taken from Hao and Alkhalifah (2017). The point source is located at *x*=0km and *z*=0km.

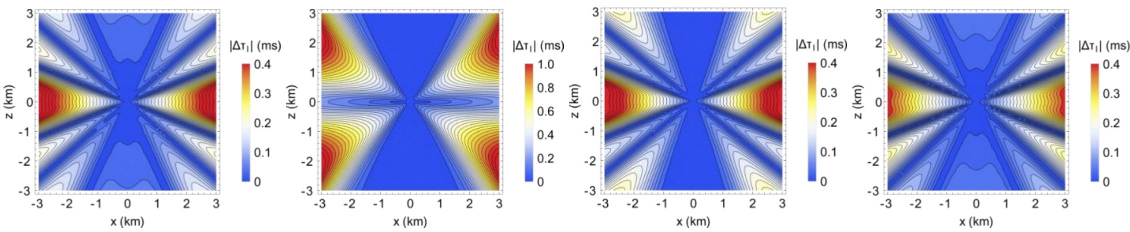

Figure 2: Similar to Figure 1, but for the imaginary part of complex-valued traveltime.

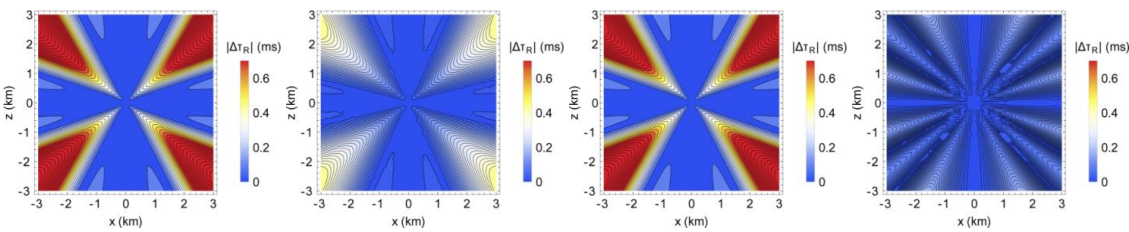

Figure 3: Errors in the real part of complex-valued traveltime. From left to right, the plots correspond to formulae 38, 39, 40 and 41, respectively. The model parameters and the source position are the same as in Figure 1.

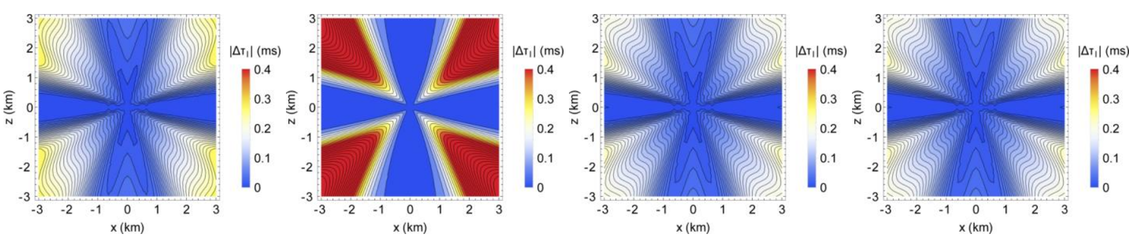

Figure 4: Similar to Figure 3, but for the imaginary part of complex-valued traveltime.

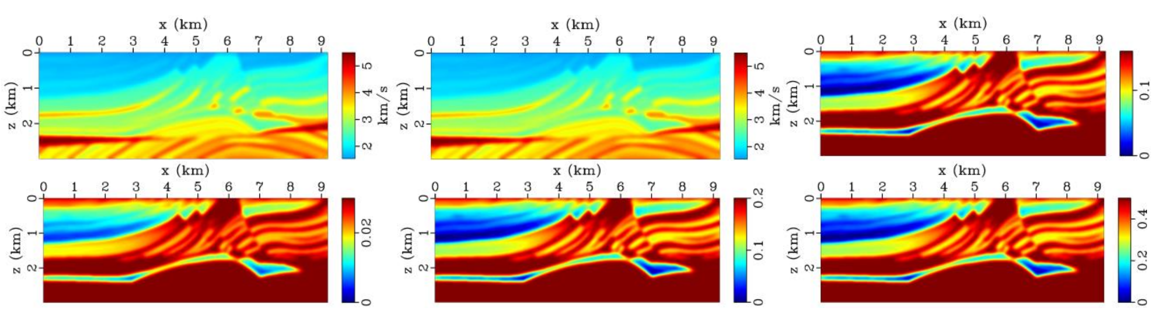

Figure 5: Attenuating VTI Marmousi model. From top left to bottom right, the plots show the models of υ_{P0}, υ_n, η, A_{P0}, ε_Q and δ_Q, respectively.

EDITED REFERENCES

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