Anelliptic approximation for P-wave phase-velocity in orthorhombic media

Qi Hao, Alexey Stovas, NTNU Norway

Summary

We propose the anelliptic approximation for P-wave phase velocity in an orthorhombic medium. The acoustic approximation is further obtained by the acoustic assumption. A subsequent simplest approximation is also derived by considering the orthorhombic medium with elliptical property in the horizontal plane. We also derive the corresponding approximations for dispersion relation and vertical wavenumber from acoustic approximation of P-wave phase velocity.

Introduction

Orthorhombic media is characterized by three mutually orthogonal planes of mirror symmetry. In each symmetry plane, wave propagation exhibits transversely isotropic (TI) property. Compared with TI media, there are nine independent stiffness parameters in orthorhombic media, which increases difficulties of applying orthorhombic model to practical seismic modeling and inversion. The exact P-wave phase velocity can be calculated from the Christoffel equation (Tsvankin, 1997; Folwer and Lapilli, 2012).

Thomsen parameters are introduced by Tsvankin (1997) to parameterize the anisotropy of orthorhombic media. In his notation, υ_{p0} and υ_{s0} are P- and S-wave vertical velocities; $(\varepsilon_i,\ \delta_i,\ \gamma_i)$, $i=1,\ 2$ are the Thomsen parameters defined in [y, z] and [x, z] planes, respectively; δ_3 is the Thomsen parameter defined in [x, y] plane (x-axis plays the role of symmetry axis). Acoustic assumption is proposed by Alkhalifah (2003) to simplify P-wave seismic processing in orthorhombic media.

To represent our phase-velocity approximation in a compact form, we introduce six new parameters including velocity ratio $\xi=\upsilon_{s0}/\upsilon_{p0}$, factors $r_i=1+2\delta_i$, $i=1,\ 2,\ 3$ and anellipticities $\eta_i=(\varepsilon_i-\delta_i)/(1+2\delta_i)$, $i=1,\ 2$. Factors r_1, r_2 and r_3 are defined in [y, z], [x, z] and [x, y] planes, respectively; η_1 and η_2 are anellipticities in [y, z] and [x, z] planes, respectively.

Anelliptic approximation

We extend the anelliptic approximation proposed by Fomel (2004) to 3D orthorhombic case,

$$\upsilon_p^2(\varphi,\theta)=e(\varphi,\theta)\left(1-s(\varphi)\right)+s(\varphi)\sqrt{e^2(\varphi,\theta)+\frac{2\alpha(\varphi)\upsilon_{p0}^2\upsilon_h^2(\varphi)}{s(\varphi)}\sin^2\theta\cos^2\theta}\ . \qquad (1)$$

where tilt θ is measured from z-axis; azimuth φ is measured from x-axis in [x, y] plane; parameter $s(\varphi)$ is an azimuth-dependent weight; parameter $\alpha(\varphi)$ affects the approximation accuracy off z-axis and [x, y] plane; the elliptic function $e(\varphi,\theta)$ takes the form,

$$e(\varphi,\theta)=\upsilon_{p0}^2\cos^2\theta+\upsilon_h^2(\varphi)\sin^2\theta\ . \qquad (2)$$

Here, $\upsilon_h(\varphi)$ denotes the P-wave velocity in [x, y] plane. The explicit expression for $\upsilon_h(\varphi)$ can be easily obtained according to Tsvankin (1997). Elliptic function (2) guarantees that approximation (1) is exact on z-axis and in [x, y] plane.

To determine the expressions for $s(\varphi)$ and $\alpha(\varphi)$ given in equation (1), we take the first and second order derivatives of $\upsilon_p^2(\varphi,\theta)$ given in equation (1) with respect to $\sin^2\theta$ at $\theta=0$ and fit both derivatives to the corresponding derivatives from exact expression. Consequently, we derive

$$\alpha(\varphi)=\frac{\upsilon_{p0}^2}{\upsilon_h^2(\varphi)}(1+\lambda(\varphi))-1\ , \qquad (3)$$

$$s(\varphi)=-\frac{\alpha^2(\varphi)\upsilon_h^4(\varphi)}{\mu(\varphi)\upsilon_{p0}^4+2\alpha(\varphi)\upsilon_h^4(\varphi)}\ , \qquad (4)$$

where λ and μ are quantities calculated from exact phase velocity,

$$\lambda(\varphi)=\frac{1}{\upsilon_{p0}^2}\left.\frac{d\upsilon_p^2(\theta,\varphi)}{d(\sin^2\theta)}\right|_{\theta=0}\ , \qquad (5)$$

$$\mu(\varphi) = \frac{1}{\upsilon_{p0}^2} \frac{d^2 \upsilon_p^2(\theta,\varphi)}{d(\sin^2\theta)^2}\bigg|_{\theta=0} . \tag{6}$$

We extend the relation between parameter α and anellipticity η for 2D VTI case (Fomel, 2004) to 3D orthorhombic case,

$$\alpha(\varphi) = \frac{1}{1+2\eta(\varphi)} - 1 , \tag{7}$$

where parameter $\eta(\varphi)$ characterizes the strength of anellipticity in the vertical plane with azimuth φ. From equations (3) and (7), we obtain the expression for $\eta(\varphi)$ given by

$$\eta(\varphi) = \frac{\upsilon_h^2(\varphi)}{2\upsilon_{p0}^2(1+\lambda(\varphi))} - \frac{1}{2} . \tag{8}$$

For $\varphi = 0$, $\eta(\varphi)$ becomes η_2 in $[x, z]$ plane; for $\varphi = \pi/2$, $\eta(\varphi)$ is reduced to η_1 in $[y, z]$ plane.

Acoustic approximation

To simply anelliptic approximation (1), we consider the acoustic approximation proposed by Alkhalifah (2003). Under the assumption that on-z-axis S-wave velocity $\upsilon_{s0} = 0$, parameters $\alpha(\varphi)$, $s(\varphi)$ and $\eta(\varphi)$ given in previous section become

$$\alpha(\varphi) = \frac{\upsilon_{p0}^2}{\upsilon_h^2(\varphi)}(r_1 \sin^2\varphi + r_2 cos^2\varphi) - 1 , \tag{9}$$

$$s(\varphi) = -\frac{\alpha^2(\varphi)\upsilon_h^4(\varphi)}{\tilde{\mu}(\varphi)\upsilon_{p0}^4 + 2\alpha(\varphi)\upsilon_h^4(\varphi)} , \tag{10}$$

$$\eta(\varphi) = \frac{\upsilon_h^2(\varphi)}{2\upsilon_{p0}^2(r_1 \sin^2\varphi + r_2 cos^2\varphi)} - \frac{1}{2} , \tag{11}$$

where

$$\tilde{\mu} = 4(r_1^2\eta_1 sin^4\varphi + r_2(-r_1 + \sqrt{r_1 r_2 r_3}(1+2\eta_2))sin^2\varphi cos^2\varphi + r_2^2\eta_2 cos^4\varphi) , \tag{12}$$

$$\upsilon_h^2(\varphi) = \frac{1}{2}\upsilon_{p0}^2(\tilde{\nu}_1 \sin^2\varphi + \tilde{\nu}_2 \cos^2\varphi + \sqrt{(\tilde{\nu}_1 \sin^2\varphi - \tilde{\nu}_2 \cos^2\varphi)^2 + r_3\tilde{\nu}_2^2 \sin^2 2\varphi}) , \tag{13}$$

with

$$\tilde{\nu}_1 = r_1(1+2\eta_1) , \tag{14}$$

$$\tilde{\nu}_2 = r_2(1+2\eta_2) . \tag{15}$$

Here, $\upsilon_h(\varphi)$ denotes phase velocity in $[x, y]$ plane under acoustic assumption. Our numerical example illustrates that parameter $s(\varphi)$ given in equation (10) is almost independent of azimuth φ. Therefore, $s(\varphi) \approx 1/2$ can be taken in practice.

Simplest approximation

To further simplify the results from acoustic approximation, we consider the simplest approximation. We consider the anellipticity η_3 for orthorhombic media (Vasconcelos and Tsvankin, 2006),

$$\eta_3 = \frac{r_1(1+2\eta_1)}{2r_2 r_3(1+2\eta_2)} - \frac{1}{2} . \tag{16}$$

For an orthorhombic medium with elliptical property in $[x, y]$ plane, anellipticity η_3 becomes zero. Parameters $\alpha(\varphi)$, $s(\varphi)$ and $\eta(\varphi)$ given in "acoustic approximation" are reduced to

$$\alpha(\varphi) = -\frac{2(r_1\eta_1 \sin^2\varphi + r_2\eta_2 \cos^2\varphi)}{r_1(1+2\eta_1)\sin^2\varphi + r_2(1+2\eta_2)\cos^2\varphi} , \tag{17}$$

$$s(\varphi) = \frac{(r_1\eta_1 \sin^2\varphi + r_2\eta_2 \cos^2\varphi)^2}{2r_1^2\eta_1^2 \sin^4\varphi + \chi \sin^2\varphi \cos^2\varphi + 2r_2^2\eta_2^2 \cos^4\varphi} , \tag{18}$$

$$\eta(\varphi) = \frac{r_1\eta_1 \sin^2\varphi + r_2\eta_2 \cos^2\varphi}{r_1 \sin^2\varphi + r_2 \cos^2\varphi} . \tag{19}$$

Here, χ given in equation (18) is

$$\chi = r_1 r_2 \left((1+\eta_1+\eta_2+4\eta_1\eta_2) - \sqrt{(1+2\eta_1)(1+2\eta_2)} \right) \ . \tag{20}$$

Parameter $s(\varphi)$ given in equation (18) can be taken to be $1/2$ in practice by an analog to acoustic approximation. Consequently, we derive the simplest approximation,

$$\upsilon_p^2(\varphi,\theta) = \frac{e(\varphi,\theta) + \sqrt{e^2(\varphi,\theta) - 8\upsilon_{p0}^4 \beta(\varphi)\sin^2\theta\cos^2\theta}}{2} \ , \tag{21}$$

where

$$e(\varphi,\theta) = \upsilon_{p0}^2 (\cos^2\theta + (r_1(1+2\eta_1)\sin^2\varphi + r_2(1+2\eta_2)\cos^2\varphi)\sin^2\theta) \ , \tag{22}$$

$$\beta(\varphi) = r_1\eta_1 \sin^2\varphi + r_2\eta_2\cos^2\varphi \ . \tag{23}$$

Dispersion relations

From the acoustic approximation of phase velocity, it follows that the corresponding approximation of dispersion relation is given by

$$\omega^2 = \frac{1}{2}\upsilon_{p0}^2 \left(E(\boldsymbol{k}) + \sqrt{E^2(\boldsymbol{k}) + F(\boldsymbol{k})} \right) \ , \tag{24}$$

where ω denotes angular frequency; vector $\boldsymbol{k} = (k_x, k_y, k_z)$ denotes the P-wave wavenumber; $E(\boldsymbol{k})$ and $F(\boldsymbol{k})$ are functions of wavenumber $\boldsymbol{k}$,

$$E(\boldsymbol{k}) = k_z^2 + G(k_x, k_y) \ , \tag{25}$$

$$F(\boldsymbol{k}) = 4\left[(r_1 k_y^2 + r_2 k_x^2) - G(k_x, k_y) \right] k_z^2 \ , \tag{26}$$

$$G(k_x, k_y) = \frac{1}{2}\left[\tilde{v}_1 k_y^2 + \tilde{v}_2 k_x^2 + \sqrt{(\tilde{v}_1 k_y^2 - \tilde{v}_2 k_x^2)^2 + 4 r_3 \tilde{v}_2 k_x^2 k_y^2} \right] \ . \tag{27}$$

From this approximate dispersion relation, we can also obtain the vertical wavenumber k_z represented in terms of horizontal wavenumber $(k_x,\ k_y)$ given by

$$k_z = \frac{\omega}{\upsilon_{p0}} \sqrt{\frac{\omega^2 - \upsilon_{p0}^2 G(k_x, k_y)}{\omega^2 + \upsilon_{p0}^2 H(k_x, k_y)}} \ , \tag{28}$$

with

$$H(k_x, k_y) = (r_1 k_y^2 + r_2 k_x^2) - G(k_x, k_y) \ . \tag{29}$$

Numerical examples

To test proposed approximations, we select a homogeneous orthorhombic medium. The medium parameters include $\upsilon_{p0} = 3km/s$, $\upsilon_{s0} = 1.2km/s$, $\varepsilon_1 = 0.25$, $\delta_1 = 0.05$, $\gamma_1 = 0.28$, $\varepsilon_2 = 0.15$, $\delta_2 = -0.1$, $\gamma_2 = 0.15$, $\delta_3 = 0.15$. From the left plot in Figure 1, we can see that there are small derivations of $s(\varphi)$ from $1/2$ for acoustic and simplest approximations. It indicates that for both approximations, $s(\varphi)$ can be approximately taken to be $1/2$ in practice. From the right plot in Figure 1, we can see that the anellipticity $\eta(\varphi)$ for acoustic approximation is practically equivalent to the exact one calculated from anelliptic approximation. Next, we compare the accuracy of our approximations with Tsvankin (1997) and Daley and Krebes (2004) approximations. For acoustic and simplest approximations, parameter $s(\varphi)$ is replaced by $1/2$ for simplicity. From Figures 2, we can see that both anelliptic and acoustic approximations have good accuracy compared to other approximations; the simplest approximation has significant errors for large tilt and off-axis azimuths. Last, we analyze the influence of anellipticity parameter η_3 on the accuracy of acoustic and simplest approximations. By comparing plots in Figure 3 and corresponding plots in Figure 2, we can see that acoustic approximation keeps good accuracy no matter that the medium is elliptical or weakly or strongly anelliptical in [x, y] plane; while the simplest approximation is very sensitive to the magnitude of η_3, and valid only for the orthorhombic medium with weak elliptical property in $[x, y]$ plane.

Conclusions

An anelliptic approximation is developed for P-wave phase velocity in orthorhombic media. The simplified formulas are obtained for acoustic and simplest approximations, respectively. The acoustic approximation has good accuracy even for the orthorhombic media with strong anelliptical property in horizontal plane. The simplest approximation is valid only for the orthorhombic media with weak elliptical property in the horizontal plane. The approximations for dispersion relation and vertical wavenumber can apply to reverse time migration and phase-shift migration, respectively.

Acknowledgements

We acknowledge the ROSE project for financial support.

References

Alkhalifah, T., 2003, An acoustic wave equation for orthorhombic anisotropy: Geophysics, 68, 1169-1172.

Daley, P.F. and Krebes E.S., 2004, Alternative linearized expressions for qP, qS1 and qS2 phase velocities in a weakly anisotropic orthorhombic medium: CREWES research report, 16.

Fomel, S., 2004, On anelliptic approximations for qP velocities in VTI media: Geophysical Prospecting, 52, 247-259.

Fowler, P.J., and C. Lapilli, 2012, Generalized pseudospectral methods for modeling and reverse-time migration in orthorhombic media: 74th EAGE expanded abstract, A022.

Tsvankin, I., 1997, Anisotropic parameters and P-wave velocity for orthorhombic media: Geophysics, 62, 1292-1309.

Vasconcelos, I., and I. Tsvankin, 2006, Nonhyperbolic moveout inversion of wide-azimuth P-wave data for orthorhombic media: Geophysical Prospecting, 54, 535-552.

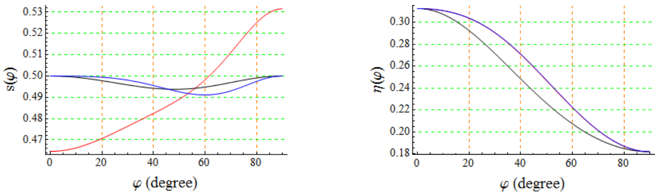

Figure 1: $s(\varphi)$ (left) and $\eta(\varphi)$ (right) calculated from anelliptic (red line), acoustic (blue line) and simplest (black line) approximations.

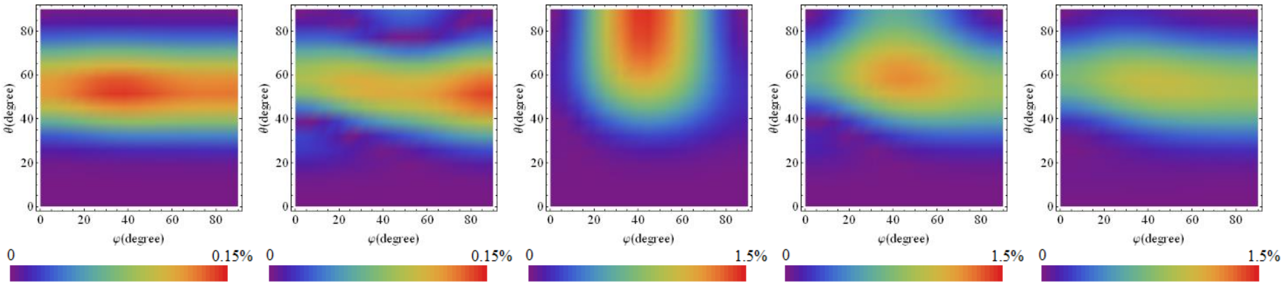

Figure 2: Relative absolute error of several approximations in an orthorhombic medium. From left to right, plots correspond to our anelliptic (equations (1)-(6)), acoustic (equations (1), (9), and (13)-(15)) and simplest (equations (21)-(23)) approximations as well as Tsvankin (1997) and Daley et al. (2004) approximations.

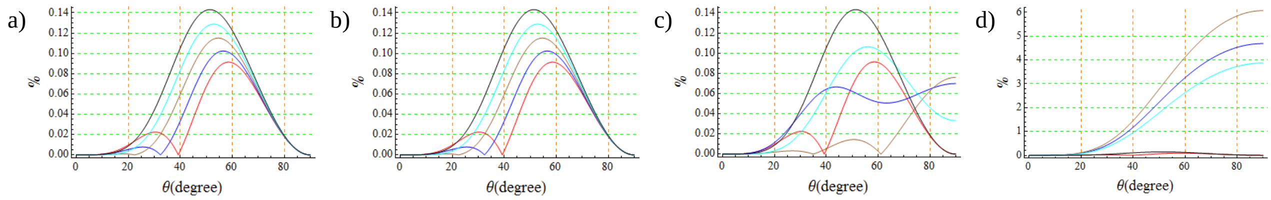

Figure 3: Relative absolute error of acoustic (plots a and c) and simplest approximations (plots b and d) for two orthorhombic media with elliptical (plots a and b) and anelliptical (plot c and d) properties in [x, y] plane, respectively. Error curves correspond to azimuths of 0 (red line), $\pi / 6$ (blue line), $\pi / 4$ (brown line), $\pi / 3$ (cyan line) and $\pi / 2$ (black line), respectively. Only medium parameter δ_3 given in "Numerical examples" is adjusted to be 0.077 and -0.15 for orthorhombic medium with elliptical and strong anelliptical properties, respectively. The corresponding anellipticity η_3 are 0 and 0.32, respectively.