A Fast Sweeping Scheme for P-wave Traveltimes in Attenuating VTI Media

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Summary

High-frequency asymptotic methods, based on solving the eikonal equation, are widely used for many seismic processing and imaging applications. For an attenuating medium, the eikonal equation becomes complex. The real part of the resulting complex traveltime describes the wave propagation while the imaginary part is associated with attenuation effects. During the past decades, several methods have been proposed to numerically solve the eikonal equation for non-attenuating media. However, solving a complex eikonal equation numerically involves several complications. We propose a fast sweeping algorithm to approximate the solution of the acoustic eikonal equation for attenuating transversely isotropic media with a vertical symmetry axis (VTI). We implement a perturbation method to derive the governing equations for the zeroth- and the first-order coefficients of the traveltime expansion. Numerical tests show that the proposed scheme is applicable to VTI media with weak attenuation.

Introduction

In an attenuating medium, the traveltime of body waves is complex-valued and is governed by the complex-valued eikonal equation. The real part of the complex-valued traveltime corresponds to the phase of waves, while the imaginary part admits a decay in the amplitude of waves due to energy absorption. The complex traveltime can be used in attenuation tomography and Kirchhoff imaging with absorption compensation. Hao and Alkhalifah (2017a,b) proposed the acoustic attenuating eikonal equations for transversely isotropic and orthorhombic media.

Numerical techniques to solve the real-valued eikonal equation cannot be applied directly to the complex-valued eikonal equation (Hao and Alkhalifah, 2017b). In this expanded abstract, we propose a fast sweeping based algorithm to approximately solve the acoustic eikonal equation for attenuating transversely isotropic media with a vertical symmetry axis (VTI). The scheme is based on the existing fast sweeping method for non-attenuating anisotropic media (Waheed and Alkhalifah, 2017). Numerical examples are provided to test the fast sweeping scheme for homogeneous and inhomogeneous attenuating VTI models.

Acoustic attenuating VTI eikonal equation

The P-wave traveltime in an attenuating VTI medium is characterized by the parameters v_{P0}, v_n and η from Alkhalifah's (2000) notation and by the parameters A_{P0}, ε_Q and δ_Q from Zhu and Tsvankin's (2006) notation. Here, v_{P0} denotes the vertical velocity of the P-waves in the non-attenuating reference medium, v_n denotes the NMO velocity of the P-wave in the non-attenuating reference medium, η denotes the anellipticity anisotropy parameter, A_{P0} denotes the P-wave attenuation coefficient normalized by the wavenumber, which describes 2π times the decay of the displacement amplitude per wavelength, ε_Q and δ_Q denote the attenuation-anisotropy parameters.

For an attenuating VTI medium, the acoustic eikonal equation is given by (Hao and Alkhalifah, 2017)

$$A\tau_{,x}^2 + B\tau_{,z}^2 + C\tau_{,x}^2\tau_{,z}^2 - 1 = 0, \tag{1}$$

where x and z correspond to the horizontal and vertical axes, τ denotes the complex-valued traveltime, the subscripts "$,x$" and "$,z$" are the first-order derivatives with respect to x and z, respectively, the coefficients A, B and C are expressed in terms of the medium parameters as

$$A = v_n^2(1+\eta)(1-2ik_Q(1+\varepsilon_Q)), \tag{2}$$

$$B = v_{P0}^2(1-2ik_Q), \tag{3}$$

$$C = \frac{v_{P0}^2}{v_n^2}((1-2ik_Q)v_n^2 - ik_Q\delta_Q v_{P0}^2)^2 - v_{P0}^2 v_n^2(1+2\eta)(1-2ik_Q)(1-2ik_Q(1+\varepsilon_Q)), \tag{4}$$

with

$$k_Q = \frac{A_{P0}}{1-A_{P0}^2}, \tag{5}$$

where i is the imaginary unit.

The perturbation method

The attenuating VTI eikonal equation is expressed abstractly as

$$F(\tau, ik_Q) = 0, \tag{6}$$

where the function F represents the left side of equation (1), the product of i and k_Q is taken as the second argument, because they always appear in the product form in the eikonal equation.

We introduce a dimensionless parameter ℓ to scale k_Q in the eikonal equation. This gives rise to the following equation

$$F(\tau(\ell), i\ell k_Q) = 0, \tag{7}$$

where the traveltime τ is a function of the parameter ℓ. Setting $\ell = 1$, equation (7) reduces to the eikonal equation (1) and the resulting τ describes the traveltime in the considered attenuating VTI medium.

The first-order expansion of τ in equation (7) is written as

$$\tau = \tau_0 + i\tau_1\ell, \tag{8}$$

where τ_0 and τ_1 denote the zeroth- and the first-order traveltime coefficients, and τ_0 describes the traveltime in the non-attenuating reference medium ($A_{P0} = 0$). Expanding equation (7) with respect to ℓ leads to the governing equations for τ_0 and τ_1.

$$v_n^2(1+2\eta)\tau_{0,x}^2 + v_{P0}^2\tau_{0,z}^2 - 2\eta v_{P0}^2 v_n^2 \tau_{0,x}^2\tau_{0,z}^2 = 1 \tag{9}$$

$$v_n^2(1+2\eta)\tau_{0,x}\tau_{1,x} + v_{P0}^2\tau_{0,z}\tau_{1,z} = f_1 \tag{10}$$

with

$$f_1 = k_Q v_n^2(1+\varepsilon_Q)\tau_{0,x}^2 + k_Q v_{P0}^2(v_{P0}^2\delta_Q - v_n^2(\varepsilon_Q + 2(2+\varepsilon_Q)\eta)\tau_{0,x}^2\tau_{0,z}^2. \tag{11}$$

Equation (10) is the acoustic eikonal equation for non-attenuating VTI media. The zeroth- and the first-order traveltimes τ_0 and τ_1 can be obtained by successively solving equations (10) and (11). Once τ_0 and τ_1 are known, the traveltime is given by

$$\tau = \tau_0 + i\tau_1. \tag{12}$$

Fast Sweeping Scheme

Referring to Waheed and Alkhalifah (2017), we design a fast sweeping scheme to calculate the traveltime. The 2D spatial coordinates are discretized by regular grids. The horizontal and vertical indexes of the grids are denoted by i and j, and the horizontal and vertical sampling are denoted by Δx and Δz, respectively. The spatial derivatives of τ_0 and τ_1 with respect to x and z are approximated by the upwind finite difference,

$$\tau_{k,x} = \left(\frac{\tau_k^{(i,j)} - \tau_0^{(x)}}{\Delta x}\right)s_x, \qquad \tau_{k,z} = \left(\frac{\tau_k^{(i,j)} - \tau_0^{(z)}}{\Delta z}\right)s_z, \qquad k = 0,1, \tag{13}$$

with

$$\tau_k^{(x)} = \begin{cases} \tau_k^{(i+1,j)}, & \text{if} \quad \tau_0^{(i+1,j)} < \tau_0^{(i-1,j)} \\ \tau_k^{(i-1,j)}, & \text{if} \quad \tau_0^{(i-1,j)} < \tau_0^{(i+1,j)} \end{cases}, \qquad s_x = \begin{cases} +1, & \text{if} \quad \tau_0^{(x)} = \tau_0^{(i+1,j)} \\ -1, & \text{if} \quad \tau_0^{(x)} = \tau_0^{(i-1,j)} \end{cases}, \qquad k = 0,1. \tag{14}$$

Here, we only show the equations for $\tau_k^{(x)}$ and s_x, and the equations for $\tau_k^{(z)}$ and s_z can be obtained in a similar way. In equation (14), the upwind direction of the finite difference is determined fully by τ_0. This guarantees that the gradient direction of τ_1 always forms an acute angle with that of τ_0.

We implement the existing fast sweeping method (Waheed and Alkhalifah, 2017) to calculate the zeroth-order traveltime coefficient. We calculate the first-order traveltime coefficient along the order of updating the zeroth-order traveltime coefficient. The idea of the fast sweeping scheme is summarized in algorithm 1, where for brevity we have omitted details of the iterative fast sweeping method (Waheed and Alkhalifah, 2017) with multiplicative factorization to accurately calculate τ_0.

Numerical Example

Figure 1 shows a comparison of the accurate analytic solution (Hao and Alkhalifah, 2017b) and the numerical method. The traveltime from the numerical method matches with that from the analytic solution. Figure 2 shows an attenuating VTI Marmousi model. Figure 3 shows the real and imaginary parts of the traveltime. While the real part of the traveltime depends on the conventional anisotropic parameters, the imaginary part is further influenced by the attenuation anisotropy parameters.

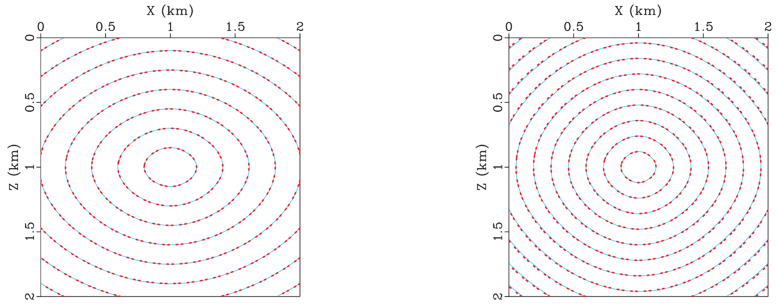

Figure 1: The contours of the (a) real and (b) iamginary parts of the traveltime in a homogeneous attenuating VTI model. The model parameters include $v_{P0} = 3.0$ km/s, $v_n = 3.55$ km/s, $\eta = 0.15$, $A_{P0} = 0.02498$ (the corresponding quality factor $Q_{33} = 20$), $\varepsilon_Q = 0.2$, and $\delta_Q = 0.4$. The cyan lines correspond to the second analytic solution in Hao and Alkhalifah (2017c), and the red dotted lines correspond to the fast sweeping scheme.

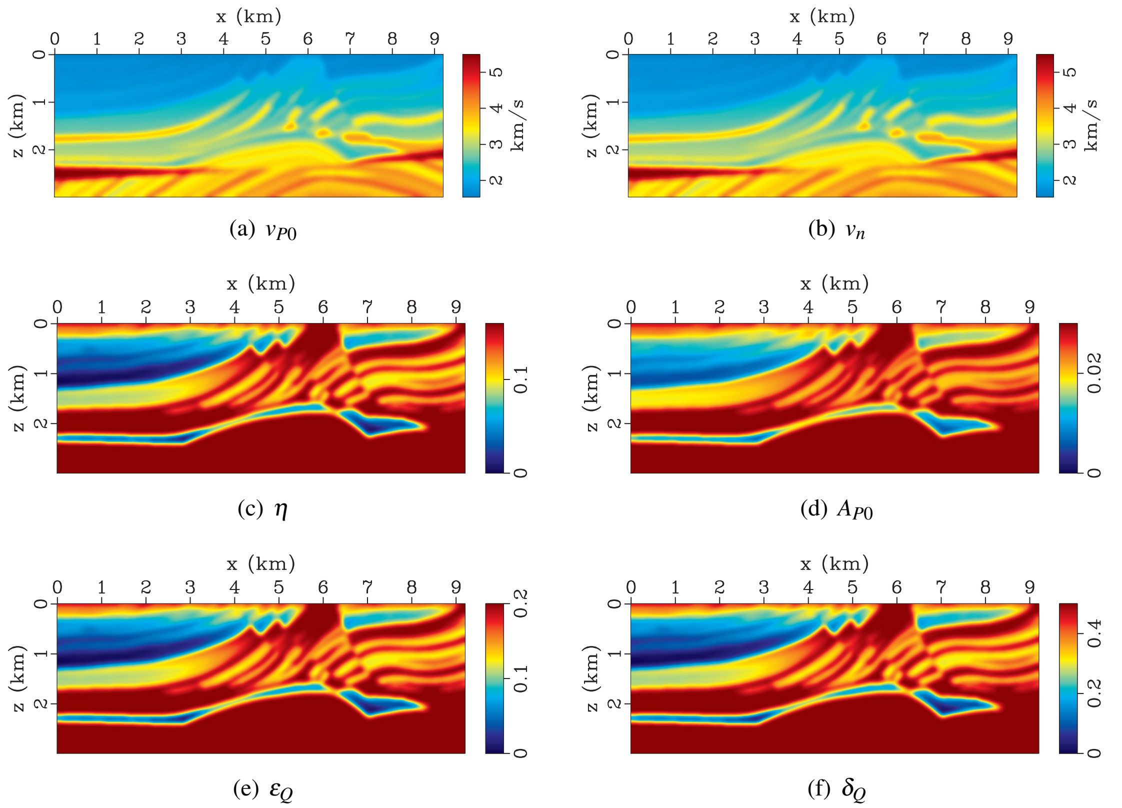

Figure 2: An attenuating VTI Marmousi model.

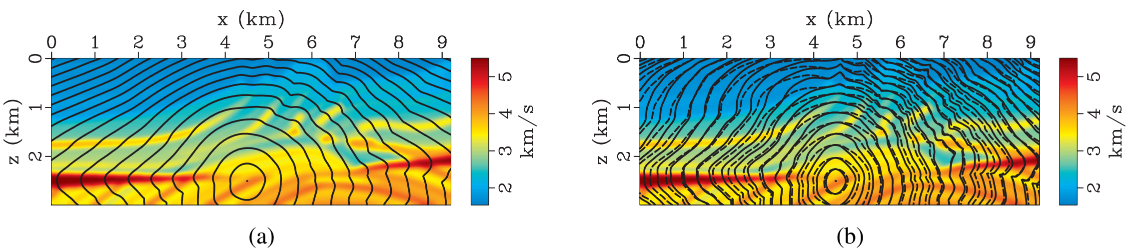

Figure 3: Contours of the (a) real and (b) imaginary parts of the traveltime in the attenuating VTI Marmousi model. The source is located at $x = 4.5$ km and $z = 2.5$ km. The contour intervals in the plots (a) and (b) are 100 ms and 2 ms, respectively. The solid lines correspond to the Marmousi model shown in Figure 2. The dashed lines correspond to the same model but with $\varepsilon_Q = \delta_Q = 0$. Note that in plot (a) the solid and dashed lines are overlaid because the influence of attenuation on the real part of the traveltime is extremely weak (Hao and Alkhalifah, 2017a,b).

Algorithm 1 Pseudo codes of the fast sweeping scheme

```
for $i = 1 : M, j = 1 : N$ do    $(M,N)$: the maximum indexes of the grids
    $\tau_0^{(i,j)} \leftarrow \infty$, $\tau_1^{(i,j)} \leftarrow \infty$
end for
$\tau_0^{(xs,zs)} \leftarrow 0$, $\tau_1^{(xs,zs)} \leftarrow 0$
for $i = 1 : M, j = 1 : N$ do
    Run the function UPDATE
end for
for $i = 1 : M, j = N : 1$ do
    Run the function UPDATE
end for
for $i = M : 1, j = 1 : N$ do
    Run the function UPDATE
end for
for $i = M : 1, j = N : 1$ do
    Run the function UPDATE
end for

function UPDATE
    if $i == xs$ and $j == zs$ then    $(xs,zs)$: the indexes of the source location
        return
    end if
    Solve equation (10) for the zeroth-order traveltime coefficient at the grid $(i, j)$ and store it at $\bar{\tau}_0$
    if $\bar{\tau}_0$ is causal and $\bar{\tau}_0 < \tau_0^{(i,j)}$ then
        $\tau_0^{(i,j)} \leftarrow \bar{\tau}_0^{(i,j)}$
        Calculate $\tau_{0,x}^{(i,j)}$ and $\tau_{0,z}^{(i,j)}$
        Solve equation (11) for $\tau_1^{(i,j)}$
    end if
end function
```

Conclusions

The perturbation method is implemented to split the attenuating eikonal equation into the governing equations for the zeroth- and the first-order traveltime coefficients. The governing equation for the zeroth-order traveltime coefficient is the same as the acoustic eikonal equation for non-attenuating VTI media, which can be solved by existing numerical methods. The governing equation for the first-order traveltime coefficient is solved in the order of updating the zeroth-order traveltime coefficient. The proposed fast sweeping scheme is valid for inhomogeneous VTI media with weak attenuation.

References

Alkhalifah, T. [2000] An acoustic wave equation for anisotropic media. *Geophysics*, **65**(2), 1239–1250.

Hao, Q. and Alkhalifah, T. [2017a] An acoustic eikonal equation for attenuating orthorhombic media. *Geophysics*, **82**(4), WA67–WA81.

Hao, Q. and Alkhalifah, T. [2017b] An acoustic eikonal equation for attenuating transversely isotropic media with a vertical symmetry axis. *Geophysics*, **82**(1), C9–C20.

Hao, Q. and Alkhalifah, T. [2017c] Acoustic eikonal solutions for attenuating VTI media. *87th Annual International Meeting, SEG, Expanded Abstracts*, 289–293.

Waheed, U. and Alkhalifah, T. [2017] A fast sweeping algorithm for accurate solution of the tilted transversely isotropic eikonal equation using factorization. *Geophysics*, **82**(6), WB1–WB8.

Zhu, Y. and Tsvankin, I. [2006] Plane-wave propagation in attenuative transversely isotropic media. *Geophysics*, **71**(2), T17–T30.