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Analytic Formulae for Wave Normal of P-waves in Orthorhombic Media

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SUMMARY

We present analytic formulae for the wave normal in terms of directions of polarization and Umov-Poynting vectors of P-waves in orthorhombic media. The formulae are useful for extracting incident angle domain common image gathers for orthorhombic media with weak-to-moderate anisotropy. The accuracy of proposed method is illustrated in few numerical examples.

Introduction

Computing the P-wave wave normal (phase propagation direction) from the directions of polarization and Umov-Poynting vectors is an important step in the direction-vector based method (specially the polarization-vector-based and the Umov-Poynting-vector-based methods) for extracting common image gathers in the incident phase angle domain (Jin et al., 2014). The polarization direction describes the direction of the particle displacement. The Umov-Poynting vector describes the directional energy flux density of the wavefield. For elastic media, the direction of Umov-Poynting vector is identical to the group propagation direction of body waves. For general orthorhombic media, no explicit or exact formula exists to express the P-wave wave normal in terms of the polarization direction or the group propagation direction.
In this extended abstract, we consider the orthorhombic media with three symmetry planes orthogonal to the coordinate axes of Cartesian coordinate system. The coordinate system (*x*, *y*, *z*) satisfies right-hand rule, and the vertical axis is *z*-axis. We consider the Thomsen-type notation (Tsvankin, 1994) for orthorhombic media, which includes the velocity υ_{p0} of the vertically propagating P-wave, the velocity υ_{s0} of the vertically propagating S-wave polarized in the *x*-direction, and the Thomsen-style parameters composed in the vector $\boldsymbol{\ell} = (\varepsilon_1, \varepsilon_2, \delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2)^T$, where subscripts 1, 2 and 3 correspond to [*y*, *z*], [*x*, *z*] and [*x*, *y*] planes.

Second-order approximation for P-wave phase velocity

The P-wave phase velocity for an orthorhombic medium can be obtained from the characteristic equation (Cerveny, 2001),

$$\det(\boldsymbol{\Gamma} - \upsilon^2 \mathbf{I}) = 0 , \tag{1}$$

where υ denotes the phase velocity of one wave; $\mathbf{I}$ denotes a three-by-three identity matrix, and elements of matrix $\boldsymbol{\Gamma}$ are given by

$$\begin{aligned} &\Gamma_{11} = A_{11}n_1^2 + A_{66}n_2^2 + A_{55}n_3^2,\ \Gamma_{22} = A_{66}n_1^2 + A_{22}n_2^2 + A_{44}n_3^2,\ \Gamma_{33} = A_{55}n_1^2 + A_{44}n_2^2 + A_{33}n_3^2, \\ &\Gamma_{12} = \Gamma_{21} = (A_{12} + A_{66})n_1 n_2,\ \Gamma_{13} = \Gamma_{31} = (A_{13} + A_{55})n_1 n_3,\ \Gamma_{23} = \Gamma_{32} = (A_{23} + A_{44})n_2 n_3, \end{aligned} \tag{2}$$

where A_{ij} denotes the second-order density-normalized stiffness coefficients.
The second-order Taylor expansion of P-wave phase velocity with respect to Thomsen-type parameters is defined as

$$\upsilon_p^2 = \upsilon_0^2 \left(1 + \sum_{j=1}^{7} b_j^{(1)} \ell_j + \sum_{j,k=1,\, j\le k}^{7} b_{jk}^{(2)} \ell_j \ell_k \right), \tag{3}$$

where ℓ_i denotes the Thomsen parameter, which is the *i*th element of the vector $\boldsymbol{\ell}$ shown at "Introduction" section.
We substitute equation (3) into equation (1) and consider the second-order Taylor expansion of Γ_{ij} in equation (2) with respect to the Thomsen-type parameters. Thus, we can obtain a second-order Taylor expansion of equation (1) with respect to the Thomsen-type parameters. Since all first- and second-order coefficients of the Taylor expansion are equal to zero, we finally derive all coefficients defined in equation (3). The first-order coefficients $b_j^{(1)}$ are expressed by

$$b_1 = 2n_2^4 \ ,\ b_2 = 4n_1^2 n_2^2 + 2n_1^4 \ ,\ b_3 = 2n_2^2 n_3^2 \ ,\ b_4 = 2n_1^2 n_3^2 \ ,\ b_5 = 2n_1^2 n_2^2 \ ,\ b_6 = 0 \ ,\ b_7 = 0 \ . \tag{4}$$

Figure 1 shows the zero and non-zero elements of the second-order coefficients.

$$\begin{array}{ccccccc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & \color{red}{b_{16}} & \color{red}{b_{17}} \\ & b_{22} & b_{23} & b_{24} & b_{25} & \color{red}{b_{26}} & \color{red}{b_{27}} \\ & & b_{33} & b_{34} & b_{35} & \color{red}{b_{36}} & \color{red}{b_{37}} \\ & & & b_{44} & b_{45} & \color{red}{b_{46}} & \color{red}{b_{47}} \\ & & & & b_{55} & \color{red}{b_{56}} & \color{red}{b_{57}} \\ & & & & & \color{red}{b_{66}} & \color{red}{b_{67}} \\ & & & & & & \color{red}{b_{77}} \end{array}$$

Figure 1 *The non-zero (black) and zero (red) elements of* b_{jk} *, where* $j,k = 1,2,3$ *and* $j \ge k$.

Wave normal approximation in terms of polarizatioin direction

The Christoffel equation for anisotropic media reads

$$F_i = \sum_{j=1}^{3} (\Gamma_{ij} - \upsilon^2 \delta_{ij}) g_j = 0 \ , \ i = 1,2,3 \, , \tag{5}$$

where δ_{ij} is the Kronecker delta; Γ_{ij} are given by equation (2). υ and g_i denote the phase-velocity and the components of the unit polarization vector.
We define the second-order expansion of the P-wave wave normal $\boldsymbol{n} = (n_1, n_2, n_3)^T$ with respect to Thomsen-type parameters,

$$n_i = g_i + \sum_{j=1}^{7} c_j^{(i)} \ell_j + \sum_{j,k=1, j \ge k}^{7} c_{jk}^{(i)} \ell_j \ell_k \ , \tag{6}$$

where ℓ_i is explained after equation (3). In equation (6), the first- and second-order coefficients $c_j^{(i)}$ and $c_{jk}^{(i)}$ are determined from equations (3), (5) and (6) . The first-order coefficients are given by

$$\begin{aligned}
&c_1^{(1)} = 2g_1 g_2^4 \zeta \, , \ c_2^{(1)} = -2g_1 (g_2^4 + g_3^2 - g_3^4) \zeta \, , \ c_3^{(1)} = 2g_1 g_2^2 g_3^2 \zeta \, , \ c_4^{(1)} = -g_1 g_3^2 (1 - 2g_1^2) \zeta \, , \\
&c_5^{(1)} = -g_1 g_2^2 (1 - 2g_1^2) \zeta \, , \ c_1^{(2)} = -2g_2^3 (1 - g_2^2) \zeta \, , \ c_2^{(2)} = 2g_2 g_1^2 (g_2^2 - g_3^2) \zeta \, , \ c_3^{(2)} = -g_2 g_3^2 (1 - 2g_2^2) \zeta \, , \\
&c_4^{(2)} = 2g_2 g_1^2 g_3^2 \zeta \, , \ c_5^{(2)} = -g_2 g_1^2 (1 - 2g_2^2) \zeta \, , \ c_1^{(3)} = 2g_3 g_2^4 \zeta \, , \ c_2^{(3)} = 2g_3 g_1^2 (g_1^2 + 2g_2^2) \zeta \, , \\
&c_3^{(3)} = -g_3 g_2^2 (1 - 2g_3^2) \zeta \, , \ c_4^{(3)} = -g_3 g_1^2 (1 - 2g_3^2) \zeta \, , \ c_5^{(3)} = 2g_3 g_1^2 g_2^2 \zeta \, , \ c_6^{(1)} = c_6^{(2)} = c_6^{(3)} = 0 \, , \\
&c_7^{(1)} = c_7^{(2)} = c_7^{(3)} = 0 \, ,
\end{aligned} \tag{7}$$

with

$$\zeta = (1 - \upsilon_{s0}^2 / \upsilon_{p0}^2)^{-1} \ , \tag{8}$$

Some of the second-order coefficients $c_{jk}^{(i)}$ are also zero. Figure 2 shows the zero and non-zero elements of the second-order coefficients.

$c_{11}^{(1)}$	$c_{12}^{(1)}$	$c_{13}^{(1)}$	$c_{14}^{(1)}$	$c_{15}^{(1)}$	$c_{16}^{(1)}$	$c_{17}^{(1)}$
	$c_{22}^{(1)}$	$c_{23}^{(1)}$	$c_{24}^{(1)}$	$c_{25}^{(1)}$	$c_{26}^{(1)}$	$c_{27}^{(1)}$
		$c_{33}^{(1)}$	$c_{34}^{(1)}$	$c_{35}^{(1)}$	$c_{36}^{(1)}$	$c_{37}^{(1)}$
			$c_{44}^{(1)}$	$c_{45}^{(1)}$	$c_{46}^{(1)}$	$c_{47}^{(1)}$
				$c_{55}^{(1)}$	$c_{56}^{(1)}$	$c_{57}^{(1)}$
					$c_{66}^{(1)}$	$c_{67}^{(1)}$
						$c_{77}^{(1)}$

$c_{11}^{(2)}$	$c_{12}^{(2)}$	$c_{13}^{(2)}$	$c_{14}^{(2)}$	$c_{15}^{(2)}$	$c_{16}^{(2)}$	$c_{17}^{(2)}$
	$c_{22}^{(2)}$	$c_{23}^{(2)}$	$c_{24}^{(2)}$	$c_{25}^{(2)}$	$c_{26}^{(2)}$	$c_{27}^{(2)}$
		$c_{33}^{(2)}$	$c_{34}^{(2)}$	$c_{35}^{(2)}$	$c_{36}^{(2)}$	$c_{37}^{(2)}$
			$c_{44}^{(2)}$	$c_{45}^{(2)}$	$c_{46}^{(2)}$	$c_{47}^{(2)}$
				$c_{55}^{(2)}$	$c_{56}^{(2)}$	$c_{57}^{(2)}$
					$c_{66}^{(2)}$	$c_{67}^{(2)}$
						$c_{77}^{(2)}$

$c_{11}^{(3)}$	$c_{12}^{(3)}$	$c_{13}^{(3)}$	$c_{14}^{(3)}$	$c_{15}^{(3)}$	$c_{16}^{(3)}$	$c_{17}^{(3)}$
	$c_{22}^{(3)}$	$c_{23}^{(3)}$	$c_{24}^{(3)}$	$c_{25}^{(3)}$	$c_{26}^{(3)}$	$c_{27}^{(3)}$
		$c_{33}^{(3)}$	$c_{34}^{(3)}$	$c_{35}^{(3)}$	$c_{36}^{(3)}$	$c_{37}^{(3)}$
			$c_{44}^{(3)}$	$c_{45}^{(3)}$	$c_{46}^{(3)}$	$c_{47}^{(3)}$
				$c_{55}^{(3)}$	$c_{56}^{(3)}$	$c_{57}^{(3)}$
					$c_{66}^{(3)}$	$c_{67}^{(3)}$
						$c_{77}^{(3)}$

Figure 2 *The non-zero (black) and zero (red) elements of $c_{jk}^{(1)}$ (left), $c_{jk}^{(2)}$ (middle) and $c_{jk}^{(3)}$ (right), where $j,k = 1,2,3$ and $j \ge k$.*

Wave normal approximation in terms of group propagation direction

Let the unit vector $\boldsymbol{N} = (N_1, N_2, N_3)^T$ express the group propagation direction. It satisfies the following equations (Vavrycuk, 2006) for general anisotropic media,

$$G_i = \upsilon^2 D N_i - \sum_{j,k,l,m=1}^{3} a_{ijkl} D_{jk} n_l n_m N_m = 0 \, , \ i = 1,2,3 \, , \tag{9}$$

with

$$D = D_{11} + D_{22} + D_{33} \ , \tag{10}$$

where υ denotes the phase velocity; a_{ijkl} denotes the fourth-order density-normalized stiffness coefficients; n_i denote the components of phase propagation direction; D_{ij} are given by

$$\begin{aligned}
&D_{11} = (\Gamma_{22} - \upsilon_p^2)(\Gamma_{33} - \upsilon_p^2) - \Gamma_{23}^2, \quad D_{22} = (\Gamma_{11} - \upsilon_p^2)(\Gamma_{33} - \upsilon_p^2) - \Gamma_{13}^2, \\
&D_{33} = (\Gamma_{11} - \upsilon_p^2)(\Gamma_{22} - \upsilon_p^2) - \Gamma_{12}^2, \quad D_{12} = D_{21} = \Gamma_{13}\Gamma_{23} - \Gamma_{12}(\Gamma_{33} - \upsilon_p^2), \\
&D_{13} = D_{31} = \Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \upsilon_p^2), \ D_{23} = D_{32} = \Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \upsilon_p^2).
\end{aligned} \tag{11}$$

For orthorhombic media, Γ_{ij} are given by equations (2).

The second-order perturbation of the phase propagation direction of P-waves is defined as

$$n_i = N_i + \sum_{j=1}^{7} C_j^{(i)} \ell_j + \sum_{j,k=1,j\le k}^{7} C_{jk}^{(i)} \ell_j \ell_k \ , \ i = 1,2,3. \tag{12}$$

where ℓ_i is explained after equation (3). We expand a_{ijkl} in equation (9) with respect to the Thomsen-type parameters. Substituting equations (3) and (12) into equation (9), we can obtain the second-order expansion of equation (9) with respect to the Thomsen-type parameters. The first- and second-order coefficients of the expansion are zero. We finally determine the coefficients defined in equation (12). The first-order coefficients are given by

$$\begin{aligned}
&C_1^{(1)} = 4N_1N_2^4,\ C_2^{(1)} = -4N_1(N_2^4 + N_3^2 - N_3^4),\ C_3^{(1)} = 4N_1N_2^2N_3^2,\ C_4^{(1)} = -2N_1N_3^2(1-2N_1^2),\\
&C_5^{(1)} = -2N_1N_2^2(1-2N_1^2),\ C_1^{(2)} = -4N_2^3(1-N_2^2),\ C_2^{(2)} = 4N_2N_1^2(N_2^2 - N_3^2),\ C_3^{(2)} = -2N_2N_3^2(1-2N_2^2),\\
&C_4^{(2)} = 4N_2N_1^2N_3^2,\ C_5^{(2)} = -2N_2N_1^2(1-2N_2^2),\ C_1^{(3)} = 4N_2^4N_3,\ C_2^{(3)} = 4N_3N_1^2(N_1^2 + 2N_2^2),\\
&C_3^{(3)} = -2N_3N_2^2(1-2N_3^2),\ C_4^{(3)} = -2N_3N_1^2(1-2N_3^2),\ C_5^{(3)} = 4N_3N_1^2N_2^2,\ C_6^{(1)} = C_6^{(2)} = C_6^{(3)} = 0,\\
&C_7^{(1)} = C_7^{(2)} = C_7^{(3)} = 0.
\end{aligned} \tag{13}$$

Some of the second-order coefficients $C_{jk}^{(i)}$ are zero. Figure 3 shows the zero and non-zero elements of the second-order coefficients.

$$\begin{array}{ccccccc}
C_{11}^{(1)} & C_{12}^{(1)} & C_{13}^{(1)} & C_{14}^{(1)} & C_{15}^{(1)} & C_{16}^{(1)} & C_{17}^{(1)}\\
 & C_{22}^{(1)} & C_{23}^{(1)} & C_{24}^{(1)} & C_{25}^{(1)} & C_{26}^{(1)} & C_{27}^{(1)}\\
 & & C_{33}^{(1)} & C_{34}^{(1)} & C_{35}^{(1)} & C_{36}^{(1)} & C_{37}^{(1)}\\
 & & & C_{44}^{(1)} & C_{45}^{(1)} & C_{46}^{(1)} & C_{47}^{(1)}\\
 & & & & C_{55}^{(1)} & C_{56}^{(1)} & C_{57}^{(1)}\\
 & & & & & C_{66}^{(1)} & C_{67}^{(1)}\\
 & & & & & & C_{77}^{(1)}
\end{array}
\quad
\begin{array}{ccccccc}
C_{11}^{(2)} & C_{12}^{(2)} & C_{13}^{(2)} & C_{14}^{(2)} & C_{15}^{(2)} & C_{16}^{(2)} & C_{17}^{(2)}\\
 & C_{22}^{(2)} & C_{23}^{(2)} & C_{24}^{(2)} & C_{25}^{(2)} & C_{26}^{(2)} & C_{27}^{(2)}\\
 & & C_{33}^{(2)} & C_{34}^{(2)} & C_{35}^{(2)} & C_{36}^{(2)} & C_{37}^{(2)}\\
 & & & C_{44}^{(2)} & C_{45}^{(2)} & C_{46}^{(2)} & C_{47}^{(2)}\\
 & & & & C_{55}^{(2)} & C_{56}^{(2)} & C_{57}^{(2)}\\
 & & & & & C_{66}^{(2)} & C_{67}^{(2)}\\
 & & & & & & C_{77}^{(2)}
\end{array}
\quad
\begin{array}{ccccccc}
C_{11}^{(3)} & C_{12}^{(3)} & C_{13}^{(3)} & C_{14}^{(3)} & C_{15}^{(3)} & C_{16}^{(3)} & C_{17}^{(3)}\\
 & C_{22}^{(3)} & C_{23}^{(3)} & C_{24}^{(3)} & C_{25}^{(3)} & C_{26}^{(3)} & C_{27}^{(3)}\\
 & & C_{33}^{(3)} & C_{34}^{(3)} & C_{35}^{(3)} & C_{36}^{(3)} & C_{37}^{(3)}\\
 & & & C_{44}^{(3)} & C_{45}^{(3)} & C_{46}^{(3)} & C_{47}^{(3)}\\
 & & & & C_{55}^{(3)} & C_{56}^{(3)} & C_{57}^{(3)}\\
 & & & & & C_{66}^{(3)} & C_{67}^{(3)}\\
 & & & & & & C_{77}^{(3)}
\end{array}$$

Figure 3 *The non-zero (black) and zero (red) elements of $C_{jk}^{(1)}$ (left), $C_{jk}^{(2)}$ (middle) and $C_{jk}^{(3)}$ (right), where $j,k = 1,2,3$ and $j \ge k$.*

Numerical examples

We express phase propagation direction $\boldsymbol{n} = (n_1, n_2, n_3)^T$, polarization direction $\boldsymbol{g} = (g_1, g_2, g_3)^T$ and group propagation direction $\boldsymbol{N} = (N_1, N_2, N_3)^T$ of P-waves by their polar and azimuthal angles,

$$n_1 = \cos\phi\sin\theta \ , \ n_2 = \sin\phi\sin\theta\,,\ n_3 = \cos\theta \ , \tag{14}$$

$$g_1 = \cos\beta\sin\alpha \ , \ g_2 = \sin\beta\sin\alpha\,,\ g_3 = \cos\alpha \ , \tag{15}$$

$$N_1 = \cos\Phi\sin\Theta \ , \ N_2 = \sin\Phi\sin\Theta\,,\ N_3 = \cos\Theta \ . \tag{16}$$

We consider an orthorhombic model with moderate anisotropy. The model parameters are υ_{p0} =3*km/s,* υ_{s0}=1.5*km/s,* ε_1=0.25, ε_2=0.15, δ_1=0.05, δ_2=−0.1, δ_3=0.15, γ_1=0.28, γ_2=0.15. First, we investigate the deviations of the polarization direction and the group propagation direction from the phase propagation direction. Figures 4 and 5 show that both deviations are very large, which means that we cannot simply approximate phase propagation direction by the polarization direction and the group propagation direction for orthorhombic media. Second, we check the accuracy of wave normal approximation in terms of the polarization direction. Equation (6) describes the second-order approximation. If we consider the terms up to first-order in equation (3), we can obtain the first-order approximation. The comparison between Figures 6 and 7 shows that the second-order terms in equation (6) can significantly improve the accuracy of approximation. Third, we compute the accuracy of wave normal approximation in terms of the group propagation direction. Equation (12) describes the second-order approximation. Comparing Figures 8 and 9, we can see that the second-order approximation improves accuracy in polar and azimuth angles at most about 1.5 and 1.2 degrees.

Conclusions

The proposed formulae for the P-wave wave normal are analytic and second-order accurate in terms of Thomsen-type parameters. The formulae are valid for orthorhombic media with weak to moderate anisotropy.

Acknowledgements

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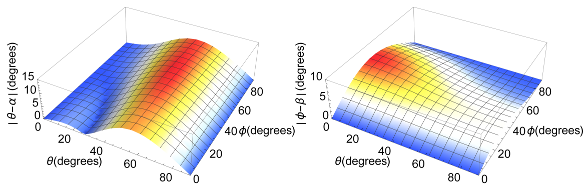

Figure 4 *The deviation of the polarization direction* (α,β) *from the phase propagation direction* (θ,φ).

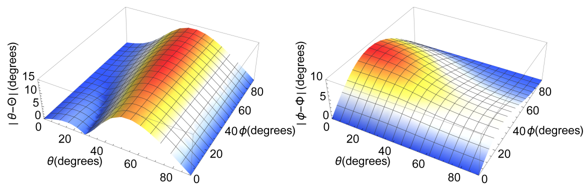

Figure 5 *The deviation of the group propagation direction* (Θ,Φ) *from the phase propagation direction* (θ,φ).

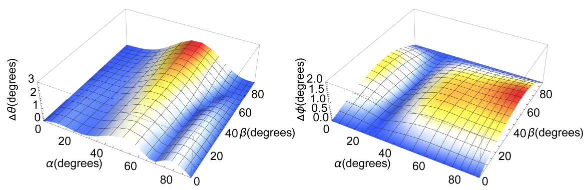

Figure 6 Absolute errors in polar (left) and azimuthal (right) angles of the wave normal for the first-order approximation in terms of the polarization direction (α,β).

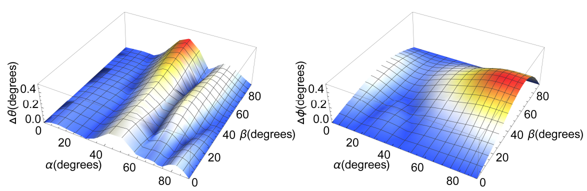

Figure 7 Absolute errors in polar (left) and azimuthal (right) angles of the wave normal for the second-order approximation in terms of the polarization direction (α,β).

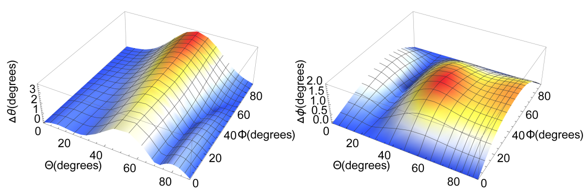

Figure 8 Absolute errors in polar (left) and azimuthal (right) angles of the wave normal for the first-order approximation in terms of the group propagation direction (Θ,Φ).

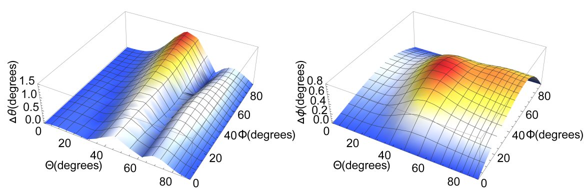

Figure 9 Absolute errors in polar (left) and azimuthal (right) angles of the wave normal for the second-order approximation in terms of the group propagation direction (Θ,Φ).