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# Analytic Formulae for Vertical Slowness and Tau-P Intercept Time of P-waves in Tilted Orthorhombic Media

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## SUMMARY

We present an analytic formula for the vertical slowness components of down- and up- going P-waves in 3D tilted orthorhombic media. The perturbation method and Shanks transform are used in our derivation. The proposed formula for the vertical slowness components is valid for strongly anelliptic orthorhombic media with tilted symmetry axes. An application of the formula is shown to calculate the tau-p intercept time of P-waves in horizontally layered, tilted orthorhombic media.

## Introduction

Analytic representation of the vertical slowness component of P-waves is useful for ray-tracing, phase shift migration and tau-p domain traveltime calculation for anisotropic media. For general 3D tilted orthorhombic media, however, it is impossible to obtain the exact and analytic formula for the vertical slowness component of P-waves. The perturbation method can help obtain the analytic approximation for the vertical slowness component (e.g. Stovas and Alkhalifah, 2013).

## P-wave slowness surface for tilted orthorhombic media

We define $\boldsymbol{p_v} = (p_{v1}, p_{v2}, q_v)^T$ and $\boldsymbol{p} = (p_1, p_2, q)^T$ as the P-wave slowness vectors for vertical orthorhombic media and tilted orthorhombic media, respectively. A vertical orthorhombic medium under acoustic assumption is parameterized by the P-wave vertical velocity $\upsilon_{p0}$, NMO velocity factors $r_1 = \upsilon_{n1}^2 / \upsilon_{p0}^2$, $r_2 = \upsilon_{n2}^2 / \upsilon_{p0}^2$, anellipticity parameters $\eta_1$, $\eta_2$, $\eta_3$, where subscripts 1, 2, 3 correspond to [*y*, *z*], [*x*, *z*] and [*x*, *y*] planes, respectively; $\upsilon_{n1}$ and $\upsilon_{n2}$ denote NMO velocities. The P-wave slowness surface for vertical orthorhombic medium under acoustic assumption (Alkhalifah, 2003) reads

$$F(p_{v1}, p_{v2}, p_{v3}) = \upsilon_{p0}^2 q_v^2 f_2\left(p_{v1}, p_{v2}\right) - f_1\left(p_{v1}, p_{v2}\right) = 0\,, \tag{1}$$

where

$$\begin{aligned} f_1(p_{v1}, p_{v2}) &= (1 - r_1\xi_1^2 p_{v2}^2 \upsilon_{p0}^2)(1 - r_2\xi_2^2 p_{v1}^2 \upsilon_{p0}^2) - \frac{1}{\xi_3^2} r_1 r_2 \xi_1^2 \xi_2^2 p_{v1}^2 p_{v2}^2 \upsilon_{p0}^4\,, \\ f_2(p_{v1}, p_{v2}) &= 1 + r_1(1-\xi_1^2) p_2^2 \upsilon_{p0}^2 + r_2(1-\xi_2^2) p_1^2 \upsilon_{p0}^2 - r_1 r_2 \Omega p_1^2 p_2^2 \upsilon_{p0}^4\,, \end{aligned} \tag{2}$$

with

$$\Omega = \xi_1^2 + \xi_2^2 - \xi_1^2\xi_2^2 + \frac{\xi_1^2\xi_2^2}{\xi_3^2} - \frac{2\xi_1\xi_2}{\xi_3}\,,\ \xi_1 = \sqrt{1+2\eta_1}\,,\ \xi_2 = \sqrt{1+2\eta_2}\,,\ \xi_3 = \sqrt{1+2\eta_3}\,. \tag{3}$$

The P-wave slowness surface for a vertical orthorhombic medium is linked to the slowness surface for a tilted orthorhombic medium by the following relation,

$$\boldsymbol{p}_v = \boldsymbol{R}_c \boldsymbol{R}_b \boldsymbol{R}_a \boldsymbol{p}\,, \tag{4}$$

where

$$\boldsymbol{R}_a = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \boldsymbol{R}_b = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix},\ \boldsymbol{R}_c = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Here, $\phi$, $\theta$ and $\psi$ are Euler angles that are explained in Lapilli and Fowler (2013).

We introduce the radial horizontal slowness $p_r$ and its azimuth $\varphi$ measured from *x*-axis to express the horizontal slowness components of P-waves in tilted orthorhombic media,

$$p_1 = p_r \cos\varphi\,,\quad p_2 = p_r \sin\varphi\,. \tag{6}$$

Substituting equation (4) with equations (5) and (6) into equation (1), we obtain a sixth-order polynomial equation in the vertical component $q$, which is symbolically denoted by

$$G(q, p_r, \varphi) = 0\,. \tag{7}$$

## Vertical slowness approximation

The second-order perturbation of the vertical slowness component with respect to anellipticity parameters is defined as

$$q(p_r, \varphi) = q_0 + \sum_{i}^{3} q_i \eta_i + \sum_{i,j=1, i\le j}^{3} q_{ij}\eta_i\eta_j\,. \tag{8}$$

Substituting equation (8) into equation (7), we can obtain a new expansion in terms of $\eta_i$, $i=1,2,3$. Since all coefficients in the expansion are zero, we can determine all perturbation coefficients defined in equation (8). Furthermore, we find the second-order perturbation of vertical slowness component,

$$q^{(\pm)}(\tilde{p}_r,\varphi)=\frac{1}{\upsilon_{p0}}\left(\tilde{q}_0^{(\pm)}+\sum_{i=1}^{3}\tilde{q}_i^{(\pm)}\eta_i+\sum_{i,j=1,i\le j}^{3}\tilde{q}_{ij}^{(\pm)}\eta_i\eta_j\right), \tag{9}$$

where the quantity with a symbol "$\sim$" on its top denotes its multiplication with $\upsilon_{p0}$; for instance, $\tilde{p}_r$ is defined as $\tilde{p}_r\equiv\upsilon_{p0}p_r$; the superscripts "+" and "-" correspond to the upper and lower parts of the slowness surface; the zero-order coefficient $\tilde{q}_0$ corresponding to the tilted elliptical orthorhombic media $(\eta_i=0,\ i=1,2,3)$ is given by

$$\tilde{q}_0^{(\pm)}=\frac{A\tilde{p}_r\pm\sqrt{V^2-B\tilde{p}_r^2}}{V^2}, \tag{10}$$

with

$$\begin{aligned}
A&=\frac{1}{2}(r_1-r_2)\sin(2\psi)\sin\theta\sin(\phi-\varphi)-\frac{1}{4}(2-r_1-r_2+(r_1-r_2)\cos(2\psi))\sin(2\theta)\cos(\phi-\varphi),\\
V&=\cos^2\theta+r_2\cos^2\psi\sin^2\theta+r_1\sin^2\theta\sin^2\psi,\\
B&=B_0+B_1\cos(2(\phi-\varphi))+B_2\sin(2(\phi-\varphi)),\\
B_0&=\frac{1}{8}(3r_1+3r_2+2r_1r_2)+\frac{1}{8}(r_1+r_2-2r_1r_2)\cos^2\theta-\frac{1}{8}(r_1+r_2-2r_1r_2+2(r_1-r_2)\cos(2\psi))\sin^2\theta,\\
B_1&=-\frac{1}{8}(r_1-r_2)(3+\cos(2\theta))\cos(2\psi)+\frac{1}{4}(r_1+r_2-2r_1r_2)\sin^2\theta,\\
B_2&=(r_1-r_2)\cos\theta\cos\psi\sin\psi,
\end{aligned} \tag{11}$$

We consider the following definition related to the slowness components for a vertical orthorhombic medium,

$$\begin{aligned}
\tilde{p}_{v1}&\equiv\tilde{p}_r(\cos\theta\cos(\phi-\varphi)\cos\psi-\sin(\phi-\varphi)\sin\psi)-\tilde{q}_0\cos\psi\sin\theta,\\
\tilde{p}_{v2}&\equiv-\tilde{p}_r(\cos\psi\sin(\phi-\varphi)+\sin\psi\cos\theta\cos(\phi-\varphi))+\tilde{q}_0\sin\theta\sin\psi,\\
\tilde{q}_v&\equiv\tilde{p}_r\cos(\phi-\varphi)\sin\theta+\tilde{q}_0\cos\theta,
\end{aligned} \tag{12}$$

where $\tilde{q}_0$ is given by equation (10) with equations (11), the first- and second-order coefficients $\tilde{q}_i$ and $\tilde{q}_{ij}$ in equation are expressed in a compact form as follows.

The first-order coefficients $\tilde{q}_i$, $i=1,2,3$:

$$\tilde{q}_1^{(\pm)}=\mp\frac{r_1\tilde{p}_{v2}^2(1-\tilde{q}_v^2)}{\sqrt{V^2-B\tilde{p}_r^2}},\quad \tilde{q}_2^{(\pm)}=\mp\frac{r_2\tilde{p}_{v1}^2(1-\tilde{q}_v^2)}{\sqrt{V^2-B\tilde{p}_r^2}},\quad \tilde{q}_3^{(\pm)}=\mp\frac{\tilde{q}_v^2-(1-r_1\tilde{p}_{v2}^2)(1-r_2\tilde{p}_{v1}^2)}{\sqrt{V^2-B\tilde{p}_r^2}}. \tag{13}$$

The second-order coefficients $\tilde{q}_{ij}$, $i,j=1,2,3$:

$$\begin{aligned}
&\tilde{q}_{11}^{(\pm)}=\mp\frac{V^2\tilde{q}_1^2-c_{11}\tilde{q}_1-d}{2\sqrt{V^2-B\tilde{p}_r^2}},\ \tilde{q}_{22}^{(\pm)}=\mp\frac{V^2\tilde{q}_2^2-c_{22}\tilde{q}_2-d}{2\sqrt{V^2-B\tilde{p}_r^2}},\ \tilde{q}_{33}^{(\pm)}=\mp\frac{V^2\tilde{q}_3^2-c_{33}\tilde{q}_3-d}{2\sqrt{V^2-B\tilde{p}_r^2}},\\
&\tilde{q}_{12}^{(\pm)}=\mp\frac{V^2\tilde{q}_1\tilde{q}_2-c_{12}\tilde{q}_1-c_{21}\tilde{q}_2+d}{\sqrt{V^2-B\tilde{p}_r^2}},\ \tilde{q}_{13}^{(\pm)}=\mp\frac{V^2\tilde{q}_1\tilde{q}_3-c_{13}\tilde{q}_1-c_{31}\tilde{q}_3+e_{13}}{\sqrt{V^2-B\tilde{p}_r^2}},\\
&\tilde{q}_{23}^{(\pm)}=\mp\frac{V^2\tilde{q}_2\tilde{q}_3-c_{23}\tilde{q}_2-c_{32}\tilde{q}_3+e_{23}}{\sqrt{V^2-B\tilde{p}_r^2}},
\end{aligned} \tag{14}$$

with

$$\begin{aligned}
&c_{11}=4\tilde{p}_{v2}r_1(\tilde{p}_{v2}\tilde{q}_v\cos\theta-(1-\tilde{q}_v^2)\sin\theta\sin\psi),c_{22}=4\tilde{p}_{v1}r_2(\tilde{p}_{v1}\tilde{q}_v\cos\theta+(1-\tilde{q}_v^2)\sin\theta\cos\psi),\\
&c_{33}=-4(\tilde{q}_v\cos\theta+\sin\theta(\tilde{p}_{v2}r_1(1-\tilde{p}_{v1}^2r_2)\sin\psi-\tilde{p}_{v1}r_2(1-\tilde{p}_{v2}^2r_1)\cos\psi)),\\
&c_{12}=c_{32}=\frac{c_{22}}{2},c_{21}=c_{31}=\frac{c_{11}}{2},c_{13}=c_{23}=\frac{c_{33}}{2},\\
&d=r_1r_2\tilde{p}_{v1}^2\tilde{p}_{v2}^2\tilde{q}_v^2,e_{13}=\tilde{p}_{v2}^2r_1(2-2\tilde{p}_{v1}^2r_2-\tilde{q}_v^2(2-\tilde{p}_{v1}^2r_2),e_{23}=p_{v1}^2r_2(2-2p_{v2}^2r_1-\tilde{q}_v^2(2-p_{v2}^2r_1)).
\end{aligned} \tag{15}$$

We apply Shanks transform (Bender and Orszag, 1978, p. 369-375) to accelerate the convergence of equation (9). Consequently, we obtain the formula for the vertical slowness component,

$$q(p_r,\varphi)=\frac{1}{\upsilon_{p0}}\left(Q_0+\frac{Q_1^2}{Q_1-Q_2}\right) , \tag{16}$$

with

$$Q_0\equiv\tilde{q}_0 \ , \ Q_1\equiv\sum_i^3\tilde{q}_i\eta_i \ , \ Q_2\equiv\sum_{i,j=1,i\le j}^3\tilde{q}_{ij}\eta_i\eta_j \ . \tag{17}$$

For a fixed propagation azimuth $\varphi$, the two intersections of the exact slowness surface with the horizontal plane $q=0$, are defined as the boundary of the validity range of radial horizontal slowness of down- and up-going plane waves.

**Tau-p domain intercept time for horizontally layered, tilted orthorhombic media**

Let us consider horizontally layered, tilted orthorhombic media. In this case, the intercept time $\tau$ is expressed by a function of the radial horizontal slowness $p_r$ and its azimuth $\varphi$,

$$\tau(p_r,\varphi)=\sum_{i=1}^N z_i(q^{(+)}(p_r,\varphi)-q^{(-)}(p_r,\varphi)) , \tag{18}$$

where $z_i$ denotes the thickness of the *i*th layer. $q^{(+)}$ and $q^{(-)}$ are calculated by equation (16).

**Numerical Examples**

We design a tilted orthorhombic model with strong anellipticity. The model parameters are $\upsilon_{p0}=3.0km/s$, $r_1=1.2$ and $r_2=1.3$, $\theta=\pi/4$, $\phi=\pi/6$, $\psi=0$, $\eta_1=\eta_2=\eta_3=0.3$. Figure 1 shows that the proposed formula for vertical slowness component is very accurate within the validity range of radial horizontal slowness. Figure 2 shows the approximate intercept times matches well with the exact one. This example indicates that the proposed formula for vertical slowness is applicable for calculating intercept time in tau-p domain for horizontally layered, tilted orthorhombic media.

**Conclusions**

Perturbation method and Shanks transform are combined to derive the formula for vertical slowness component of down- and up-going plane P-waves in tilted orthorhombic media. The proposed formula is exact one for tilted elliptically orthorhombic media. The validity range of proposed formula is determined by the slowness of horizontally propagating plane P-waves in tilted orthorhombic media. The accuracy of the proposed formula is controlled by three anellipticity parameters defined in the symmetry planes of an orthorhombic medium.

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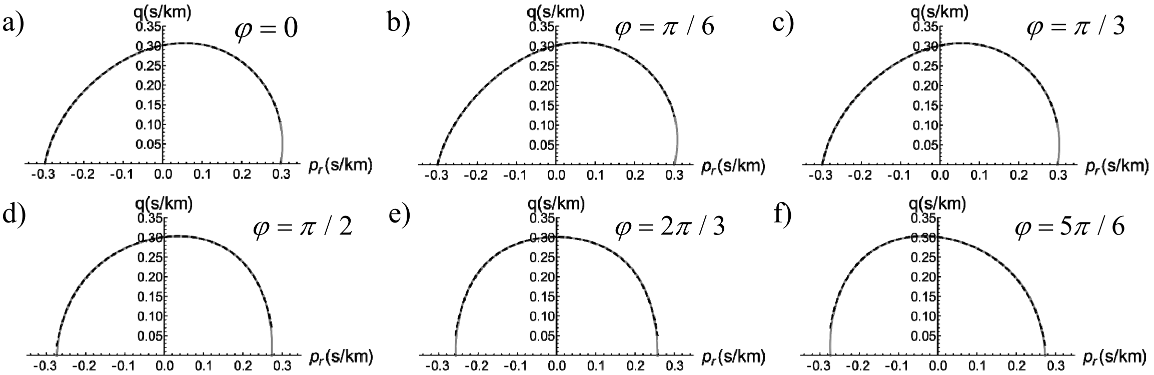


***Figure 1*** *The vertical slowness component of down-going plane P-waves versus the radial horizontal slowness for a tilted orthorhombic model. Solid gray lines and dashed black lines correspond to the exact solution and the proposed formula.* $\varphi$ *denotes the P-wave propagation azimuth.*

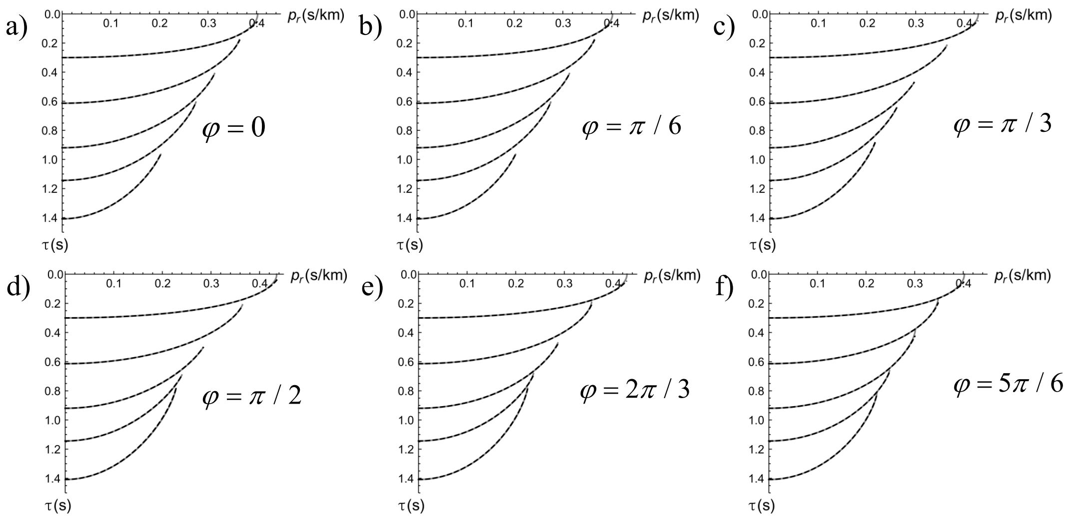


***Figure 2*** *The intercept time versus the radial horizontal slowness for a five-layer orthorhombic model. Dashed black lines correspond to the approximate intercept times calculated by the proposed vertical slowness formula, and solid gray lines correspond to the exact intercept times. respectively.* $\varphi$ *denotes the P-wave propagation azimuth.*

| Layer | $\Delta z$ (km) | $\upsilon_{p0}$ (km/s) | $r_1$ | $r_2$ | $\eta_1$ | $\eta_2$ | $\eta_3$ | $\phi$ | $\theta$ | $\psi$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.3 | 2.0 | 1.1 | 1.2 | 0.1 | 0.2 | 0.1 | 0 | 0 | 0 |
| 2 | 0.4 | 2.5 | 1.2 | 1.1 | 0.05 | 0.1 | 0.1 | $\pi/6$ | 0 | $\pi/3$ |
| 3 | 0.5 | 3.0 | 1.0 | 1.2 | 0.1 | 0.2 | 0.05 | $\pi/4$ | $\pi/6$ | $\pi/4$ |
| 4 | 0.45 | 3.5 | 1.3 | 1.2 | 0.05 | 0.2 | 0.15 | $\pi/3$ | $\pi/4$ | $\pi/6$ |
| 5 | 0.55 | 4.0 | 1.3 | 1.15 | 0.05 | 0.05 | 0.25 | $\pi/6$ | $\pi/3$ | $\pi/3$ |

***Table 1*** *Parameters for a five-layer orthorhombic model.* $\Delta z$ *denotes the layer thickness.*