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An Approximate Method for the Acoustic Attenuating VTI Eikonal Equation

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Summary

We present an approximate method to solve the acoustic eikonal equation for attenuating transversely isotropic media with a vertical symmetry axis (VTI). A perturbation method is used to derive the perturbation formula for complex-valued traveltimes. The application of Shanks transform further enhances the accuracy of approximation. We derive both analytical and numerical solutions to the acoustic eikonal equation. The analytic solution is valid for homogeneous VTI media with moderate anellipticity and strong attenuation and attenuation-anisotropy. The numerical solution is applicable for inhomogeneous attenuating VTI media.

Introduction

The attenuation of seismic waves as they propagate in reservoir rocks is closely linked to permeability, fluid content and saturation. Laboratory and field observations have revealed that anisotropic attenuation is a common phenomenon for wave propagation in reservoir rocks. In an attenuating medium, the travel time of body-waves is complex-valued and it is governed by the complex-valued eikonal equation. The real part of the complex-valued travel time corresponds to the phase of waves, while the imaginary part affects the decay of amplitude of waves due to energy absorption.

In this expanded abstract, we will propose a method to solve the acoustic eikonal equation (Hao and Alkhalifah, 2017) for attenuating transversely isotropic media with a vertical symmetry axis (VTI). Alkhalifah's (2000) notation and Zhu and Tsvankin's (2006) notation are combined to describe the P-wave traveltimes in an attenuating VTI medium, where Alkhalifah's notation and Zhu and Tsvankin's notation correspond to the nonattenuating and attenuating parts of the VTI medium, respectively. Alkhalifah's notation includes the P-wave vertical velocity υ_{P0}, the P-wave normal moveout velocity υ_n, and the anellipticity parameter η. Zhu and Tsvankin's notation includes the P-wave vertical attenuation coefficient A_{P0}, the fractional difference ε_Q between the horizontal and vertical attenuation coefficients of homogenous plane quasi P-waves, and the second-order derivative δ_Q of the attenuation coefficient with respect to the phase angle of homogeneous plane quasi P-waves along the vertical direction. The phase angle is measured between the phase propagation direction to the vertical axis.

Acoustic eikonal equation for attenuating VTI media

For 2D attenuating VTI media, the acoustic eikonal equation is given by (Hao and Alkhalifah, 2017)

$$A\tau_{,x}^2 + B\tau_{,z}^2 + C\tau_{,x}^2\tau_{,z}^2 - 1 = 0 \ , \tag{1}$$

where x and z denote the horizontal and vertical coordinates axes; τ denotes the complex-valued traveltime; the subscript "," followed by x and z denotes the first-order spatial derivative with respect to x and z; coefficients A, B and C are given by

$$A = \upsilon_n^2(1+2\eta)(1-2ik(1+\varepsilon_Q)) \ , \tag{2}$$

$$B = \upsilon_{P0}^2(1-2ik) \ , \tag{3}$$

$$C = \frac{\upsilon_{P0}^2}{\upsilon_n^2}((1-2ik)\upsilon_n^2 - ik\delta_Q\upsilon_{P0}^2)^2 - \upsilon_{P0}^2\upsilon_n^2(1+2\eta)(1-2ik)(1-2ik(1+\varepsilon_Q)) \ , \tag{4}$$

with

$$k = \frac{A_{P0}}{1-A_{P0}^2} \ , \tag{5}$$

where i denotes the imaginary unit; υ_{P0}, υ_n, η, A_{P0}, ε_Q and δ_Q are the noattenuating and attenuating parameters explained earlier.

The perturbation method

We consider a perturbation method to solve equation 1. Equation 1 is generally expressed by

$$F(k,\eta,\tau_{,x},\tau_{,z}) = 0 , \tag{6}$$

where function F represents the left-hand side of equation 1.

The perturbation parameters ℓ_1 and ℓ_2 are introduced to scale k and η in equation 6,

$$F(\ell_1 k, \ell_2\eta, \tau_{,x}, \tau_{,z}) = 0 . \tag{7}$$

The second-order perturbation solution to equation 7 is defined as

$$\tau = \tau_0 + i\tau_1\ell_1 + \tau_2\ell_2 + \tau_{11}\ell_1^2 + i\tau_{12}\ell_1\ell_2 + \tau_{22}\ell_2^2 \ , \tag{8}$$

where τ_0, τ_i and τ_{ij} denote the zero-, first- and second-order traveltime coefficients, respectively; the imaginary unit "i" in front of τ_1 and τ_{12} guarantee that the governing equations for τ_1 and τ_{12} are always real-valued (as we will see in equations 10 and 11).

We expand equation 7 around $\ell_1 = 0$ and $\ell_2 = 0$ up to second-order and then insert equation 8. This allows us to obtain a series with respect to ℓ_1 and ℓ_2. From the fact that the zero-, first- and second-order expansion coefficients of the series are equal to zero, we derive the following governing equations,

$$\upsilon_n^2 \tau_{0,x}^2 + \upsilon_{P0}^2 \tau_{0,z}^2 = 1 , \tag{9}$$

$$\upsilon_n^2 \tau_{0,x} \tau_{i,x} + \upsilon_{P0}^2 \tau_{0,z} \tau_{i,z} = f_i(k, \eta, \tau_0) , \; i = 1, 2 , \tag{10}$$

$$\upsilon_n^2 \tau_{0,x} \tau_{ij,x} + \upsilon_{P0}^2 \tau_{0,z} \tau_{ij,z} = f_{ij}(k, \eta, \tau_0, \tau_1, \tau_2) , \; i, j = 1, 2 \text{, and } i \le j , \tag{11}$$

where f_i and f_{ij} are real functions.

Equation 9 governs the P-wave traveltimes in nonattenuating elliptically isotropic media. Once τ_0 is calculated from equation 9, equations 10 and 11 are successively solved to obtain τ_i and τ_{ij}.

Setting $\ell_1 = 1$ and $\ell_2 = 1$ in equation 8, we obtain the perturbation solution to the eikonal equation 1,

$$\tau = \tau_0 + i\tau_1 + \tau_2 + \tau_{11} + i\tau_{12} + \tau_{22} . \tag{12}$$

Appling Shanks transform (Bender and Orszag, 1978, pp. 369–375) to equation 8 and setting $\ell_1 = 1$ and $\ell_2 = 1$, we derive the other three approximate formulae,

$$\tau = \tau_0 + \frac{(i\tau_1 + \tau_2)^2}{\tau_2 - \tau_{11} - \tau_{22} + i(\tau_1 - \tau_{12})} , \tag{13}$$

$$\tau = \tau_0 + \tau_2 + \tau_{22} + \frac{(\tau_1 + \tau_{12})^2}{\tau_{11} - i(\tau_1 + \tau_{12})} , \tag{14}$$

$$\tau = \tau_0 + i\tau_1 + \tau_{11} + \frac{(\tau_2 + i\tau_{12})^2}{\tau_2 - \tau_{22} + i\tau_{12}} . \tag{15}$$

For equation 13, Shanks transform has been applied in terms of both parameters ℓ_1 and ℓ_2. For equations 14 and 15, Shanks transform has been applied in terms of parameters ℓ_1 and ℓ_2, respectively.

Homogeneous medium case

We consider a homogeneous attenuating VTI medium, and the source is located at the origin of the Cartesian coordinate system. By defining $\tau_x \equiv x/\upsilon_n$ and $\tau_z \equiv z/\upsilon_{P0}$, the analytic solutions to equations 9-11 are expressed as

$$\tau_0 = \sqrt{\tau_x^2 + \tau_z^2} , \tag{16}$$

$$\tau_1 = k\upsilon_n^{-2} (\tau_x^2 + \tau_z^2)^{-3/2} (\upsilon_{P0}^2 \delta_Q \tau_x^2 \tau_z^2 + \upsilon_n^2 ((1 + \varepsilon_Q)\tau_x^4 + 2\tau_x^2 \tau_z^2 + \tau_z^4)) , \tag{17}$$

$$\tau_2 = -\eta (\tau_x^2 + \tau_z^2)^{3/2} \tau_x^4 , \tag{18}$$

$$\begin{aligned} \tau_{11} = -\frac{3}{2} k^2 \upsilon_n^{-4} (\tau_x^2 + \tau_z^2)^{-7/2} (&\upsilon_{P0}^4 \delta_Q^2 \tau_x^2 \tau_z^2 (\tau_x^4 - \tau_x^2 \tau_z^2 + \tau_z^4) \\ &+ 2\upsilon_n^2 \upsilon_{P0}^2 \delta_Q \tau_x^2 \tau_z^2 ((1 - \varepsilon_Q)\tau_x^4 + 2(1 + \epsilon_Q)\tau_x^2 \tau_z^2 + \tau_z^4) \\ &+ \upsilon_n^4 ((1 + \varepsilon_Q)^2 \tau x^8 + 4(1 + \varepsilon_Q + \varepsilon_Q^2)\tau_x^6 \tau_z^2 + 2(3 + \varepsilon_Q)\tau_x^4 \tau_z^4 + 4\tau_x^2 \tau_z^6 + \tau_z^8)) , \end{aligned} \tag{19}$$

$$\begin{aligned} \tau_{12} = -k\eta \upsilon_n^{-2} (\tau_x^2 + \tau_z^2)^{-7/2} \tau_x^4 (&\upsilon_n^2 (1 + \varepsilon_Q)\tau_x^4 + (-3\upsilon_{P0}^2 \delta_Q + 2\upsilon_n^2 (1 + 4\varepsilon_Q))\tau_x^2 \tau_z^2 \\ &+ (6\upsilon_{P0}^2 \delta_Q + \upsilon_n^2 (1 - 2\varepsilon_Q))\tau_z^4) , \end{aligned} \tag{20}$$

$$\tau_{22} = \frac{3}{2} \eta^2 (\tau_x^2 + \tau_z^2)^{-7/2} \tau_x^6 (\tau_x^2 + 4\tau_z^2) . \tag{21}$$

Consequently, the approximate traveltime is calculated from one of formulae 12-15.

Inhomogeneous medium case

We consider the case of inhomogeneous attenuating VTI media. The fast sweeping method (Zhao, 2004) is adopted to numerically solve the eikonal equation 9. A 2D space is uniformly discretized into the rectangular grid points with the *x*- and *z*-direction indexed by m and n, respectively, and the *x*- and *z*-direction grid spacing are Δx and Δz. The spatial derivatives in equation 9 are approximated by an upwind finite difference,

$$\tau_{0,x}^{(m,n)} \approx \frac{s_x}{\Delta x}\left(\tau_0^{(m,n)} - (\tau_0)_{x\min}\right) \ , \ \tau_{0,z}^{(m,n)} \approx \frac{s_z}{\Delta z}\left(\tau_0^{(m,n)} - (\tau_0)_{z\min}\right), \quad (22)$$

with

$$(\tau_0)_{x\min} = \min(\tau_0^{(m-1,n)}, \tau_0^{(m+1,n)}) \ , \ (\tau_0)_{z\min} = \min(\tau_0^{(m,n-1)}, \tau_0^{(m,n+1)}) \ , \quad (23)$$

$$s_x = \begin{cases} -1, \text{ if } (\tau_0)_{x\min} = \tau_0^{(m+1,n)} \\ +1, \text{ if } (\tau_0)_{x\min} = \tau_0^{(m-1,n)} \end{cases}, \ s_z = \begin{cases} -1, \text{ if } (\tau_0)_{z\min} = \tau^{(m,n+1)} \\ +1, \text{ if } (\tau_0)_{z\min} = \tau^{(m,n-1)} \end{cases}. \quad (24)$$

Substituting equation 22 into equation 9 leads to a quadratic equation in $\tau_0^{(m,n)}$. From this equation, we derive the expression for $\tau_0^{(m,n)}$,

$$\tau_0^{(m,n)} = \frac{1}{a}\left(b + \sqrt{b^2 - a(c-1)}\right) \ , \quad (25)$$

where a, b and c are given by

$$a = \frac{\upsilon_{P0}^2}{\Delta z^2} + \frac{\upsilon_n^2}{\Delta x^2}, \ b = \frac{\upsilon_{P0}^2}{\Delta z^2}(\tau_0)_{z\min} + \frac{\upsilon_n^2}{\Delta x^2}(\tau_0)_{x\min}, \ c = \frac{\upsilon_{P0}^2}{\Delta z^2}(\tau_0)_{z\min}^2 + \frac{\upsilon_n^2}{\Delta x^2}(\tau_0)_{x\min}^2. \quad (26)$$

The expressions for $\tau_i^{(m,n)}$ and $\tau_{ij}^{(m,n)}$ can be derived in a similar way after discretizing equations 10 and 11. The zero-order traveltimes $\tau_0^{(m,n)}$ are calculated along the direction of wavefront expansion. The detailed procedure of this algorithm can be found in Zhao (2004). Once the reference traveltime is known, we may successively solve equations 10 and 11 to calculate $\tau_i^{(m,n)}$ and $\tau_{ij}^{(m,n)}$ along the order of calculating $\tau_0^{(m,n)}$. From the first example in the "Numerical example" section, formula 15 is used to calculate the traveltime.

Numerical examples

Figures 1 and 2 show that formula 15 is most accurate among the proposed formulae for the traveltimes. For the inhomogeneous attenuating VTI model as illustrated in Figure 3, Figure 4 shows that the contour shape of the real part of the traveltimes is different from that of the imaginary part of the traveltimes.

Conclusions

The perturbation method and an appropriate Shanks transform are combined to solve the acoustic attenuating VTI eikonal equation. The analytic eikonal solution is accurate for a VTI medium with moderate anellipticity and strong attenuation and attenuation-anisotropy. The numerical eikonal solution is applicable for 2D inhomogeneous attenuating VTI media.

Acknowledgements

Q. Hao thanks rock and seismic (ROSE) project for support.

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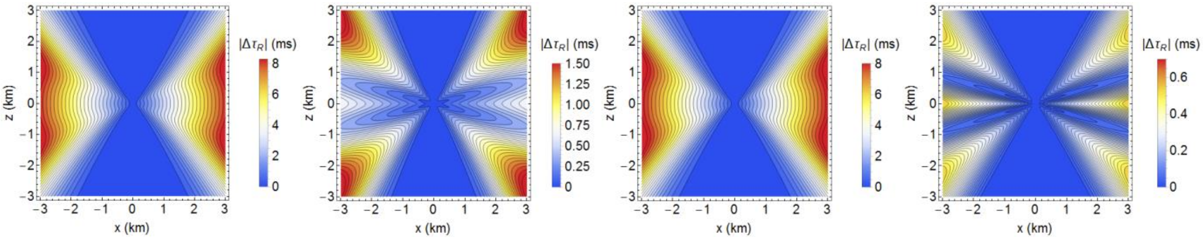

Figure 1. *The errors in the real part of complex-valued traveltime. From left to right, the plots correspond to formulae 12, 13, 14 and 15, respectively. The model parameters are* $\upsilon_{P0} = 3km/s$, $\upsilon_n = 3.286km/s$, $\eta = 0.167$, $A_{P0} = 0.02498$ *(corresponding to* $Q_{33} = 20$, *where* Q_{33} *denotes the quality factor of P-waves propagating along the vertical axis),* $\varepsilon_Q = -0.33$, *and* $\delta_Q = 0.98$. *These model parameters are taken from Hao and Alkhalifah (2017). The point source is located at x=0km and z=0km.*

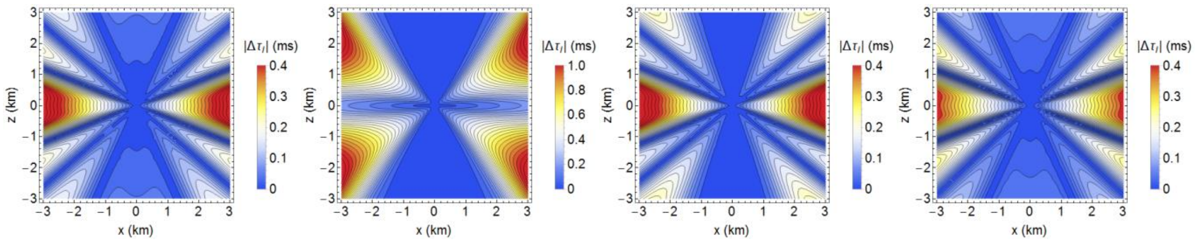

Figure 2. *Similar to Figure 1, but for the imaginary part of complex-valued traveltime.*

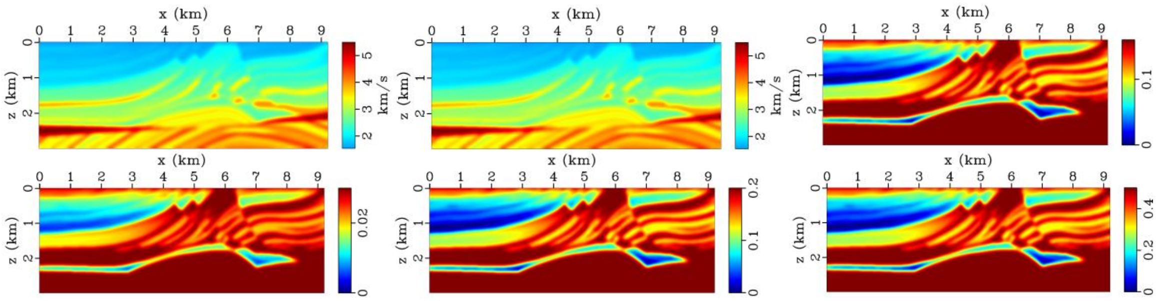

Figure 3. An inhomogeneous attenuating VTI model. From top left to bottom right, the plots show the models of υ_{P0}, υ_n, η, A_{P0}, ε_Q and δ_Q, respectively.

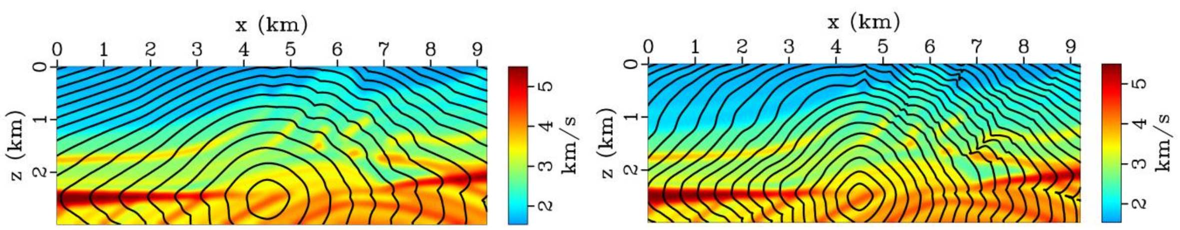

Figure 4. *The contours of the real (left) and imaginary (right) parts of the traveltimes from the attenuating VTI model illustrated in Figure 3. The point source is located at x=4.5km and z=2.5km. For the left plot, the time of the closet contour line around the source is 0.1s, and the time interval between two neighboring contour lines is also 0.1s. For the right plot, the time of the closet contour line around the source is 2ms, and the time interval between two neighboring contour lines is also 2ms. In both plots, the background behind the contour lines shows the* υ_{P0} *model from Figure 3a.*